



هدية GIFT

The Hajós n -property
for
finite p -groups

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for the degree of Doctor Of Philosophy

by

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by

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Declaration

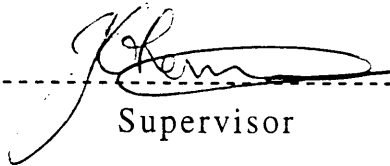
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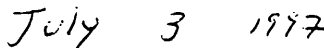
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Dedication

I dedicate this thesis to my

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whose devotion to my well-being and success never did and never
will cease.

Acknowledgement

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1. Dr John Lennox.
2. Dr Sandor Szabo.
3. The University Of Bahrain.

SUMMARY

Let G be a finite abelian group. Let $A_1, \dots, A_i, \dots, A_n$, be subset of G ; then the product $A_1 \dots A_i \dots A_n$, is defined to be the set of all elements $a_1 \dots a_i \dots a_n$, with $a_i \in A_i$ for $1 \leq i \leq n$. $A_1 \dots A_i \dots A_n$ may contain the same element many times. If $A_1 \dots A_i \dots A_n$ contains each element only once and if every element of G is in $A_1 \dots A_i \dots A_n$, then write $A_1 \dots A_i \dots A_n = G$ and call this a factorization of G . A subset A of G is said to be periodic if there exists a nonidentity element g of G such that $gA = A$. A group G is said to have the Hajos n -property if in every factorization $A_1 \dots A_i \dots A_n = G$, at least one of the factors is periodic. In this thesis, a study of finite p -groups with respect to the Hajos n -property is carried out. It is shown that a finite p -group does not have the Hajos n -property, if $p \geq 5$, except when G is cyclic. The cases of 2 and 3-groups do not show a similar pattern, and is more sporadic. However, some partial results are obtained. For example, it is shown that any group of type $(2^a, 2)$ has the Hajos n -property, for all n , while groups of type $(2^3, 2^2)$ does not. Groups of order 3^4 do not have the Hajos 2-property. However, they have the Hajos 3-property.

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CHAPTER ONE

Introduction :

In what follows, we will be dealing with finite abelian groups. The problem we want to pursue is related to the following question: Given such a group G , suppose we factorize G into subsets:

$$G = A_1 \dots A_i \dots A_n$$

(where A_i are not necessarily subgroups) in such a way that the product $A_1 \dots A_i \dots A_n$ is direct and gives G . In this case, we say that G has a factorization $G = A_1 \dots A_i \dots A_n$. More precisely,

$$G = A_1 \dots A_i \dots A_n \dots \dots \dots (1)$$

and every element g of G has a unique representation of the form $g = a_1 \dots a_i \dots a_n$, where $a_i \in A_i$ for $1 \leq i \leq n$. What can we then say about the sets A_i ? The origin of this question comes from a conjecture made in 1896 by H. Minkowski about space filling lattice-like cube systems [1]. In 1938, G. Hajós [4] reformulated this geometric problem into a question about factorizing a finite abelian group. Three years later, Hajós [5] himself proved his reformulation of Minkowski's conjecture in a paper which extended over 40 journal pages. The result is the following:

If G is a finite abelian group such that $G = A_1 \dots A_i \dots A_n$ is a factorization of G and each of the factors is a cyclic subset (see 2.1 for the

definition of a cyclic subset), then at least one of the factors must be a subgroup of G . In 1949, L. Rédei [12] improved and shortened the proof of Hajós' Theorem. However, Rédei was not satisfied with his proof, because it still contained ideas outside group theory, as did Hajós' original proof; for example, both proofs use results on zero-divisors in the integral group ring ZG . Rédei eventually managed to avoid the use of results on group rings in his proof given in [13]. Hajós' Theorem continued getting attention, and many different proofs were given. Although the majority of these later proofs were less natural and even sometimes more complicated, they brought certain gains - improvements to some of the older notions, the introduction of some new ideas and some useful general techniques. The latter include the idea of replaceable factors, simulated sets and the use of character theory. Also, during the process, Hajós' Theorem was generalized. For example, Rédei [11] proved in a Theorem that if the factors in (1) have prime cardinality, and the factorization is normalized, (see 2.3 for the definition of a normalized factorization), then at least one of the factors must be a subgroup of G . The idea of replaceable factors and zero-divisors also proved to be very useful for further investigations. In 1949, Hajós [6] was interested in the question of whether his result is a consequence of the following assertion :

If $G = AB$ is a factorization of the finite group G , then at least one of the factors must be periodic (see 2.4 for the definition). However, it turns out that the above assertion is in general false- there are groups G with factorization $G=AB$, where neither A nor B is periodic. As an example, consider the group $G = \langle x \rangle \times \langle y \rangle \times \langle z \rangle$, where $|x| = |y| = |z| = 5$.

Let $A = \langle x \rangle \cup z \langle y \rangle \cup z^2 \langle y \rangle \cup z^3 \langle y \rangle \cup z^4 \langle y \rangle$ and $B = \{ e, xy, x^2y^2, x^3y^4, x^4y^3 \}$. Then, one can check either directly, or using Lemma 2.13, in Chapter Two, that $AB = G$ is a factorization of G , and that neither A nor B is periodic. We therefore, say that a finite abelian group G has the Hajós 2-property if whenever G is factorized as $G = AB$, then either A or B is periodic. Over a 15 year period (1949 -1964), Hajós, Rédei, De Bruijn and Sands gave a list of groups with the Hajós 2-property. In 1964, Sands [18] posed the problem of which finite cyclic groups satisfy the Hajós n -property, (see 2.5 for the definition). In [18] Sands solved this problem for finite cyclic groups, except one case which was left unsolved and was called the undecided case: the summary of what he did is as follows:

Let G be a cyclic group, with $|G| = p_1^{\alpha_1} \dots p_s^{\alpha_s}$ and suppose G is factorized as $G = A_1 \dots A_1 \dots A_n$. Let $m = \alpha_1 + \dots + \alpha_s$. Then:

- (1) If $2 \leq n \leq m-2$, then G does not have the Hajós n -property.
- (2) If $n = m$, then G has the Hajós n -property.

(We observe that this case is just Rédei's Theorem)

- (3) If $n = m-1$, then, the case was unsolved, and called an undecided case.

In fact, case (3) was 30 years later solved in [1]. In this work, we shall be interested in the question of which non-cyclic p -groups satisfy the Hajós n -property for $n > 2$. However, for the sake of completeness, and also in order to show a flavor of the different techniques that are used in this field, we include in the next chapter, the case of cyclic groups of order p^α , which is a special case of the more general results obtained by A. D. Sands [9], for cyclic groups.

CHAPTER TWO

We begin this chapter with some definitions that are commonly used when dealing with factorization of finite abelian groups.

2.1 Definition [22]

For any element g in a group G and for any natural number n not exceeding $|g|$, the order of g , we define the cyclic subset determined by g and n to be $\{g; n\} = \{g^i : 0 \leq i < n\}$.

2.2 Definition [22]

Let G be a group and A , a subset of G with $|A| \geq 3$. Then A is said to be a simulated subset if there is a subgroup H of G such that :

- (i) $|A| = |H|$ and
- (ii) $|A \cap H| \geq |A| - 1$.

We call H a corresponding subgroup of A .

2.3 Definition [22]

Let G be a group, n a natural number and A_i subsets of G , for $1 \leq i \leq n$. If $G = A_1 \dots A_n$. Then we say that $A_1 \dots A_n$ is a factorization of G if every element g of G has a unique representation of the form $g = a_1 \dots a_n$, where $a_i \in A_i$. If in addition each A_i also contains the identity element e of G , then we say that $A_1 \dots A_n = G$ is a normalized factorization of G .

2.4 Definition [17]

Let G be a group, and A , a subset of G . We say that A is a periodic subset of G if there is a non-identity element g in G such that $gA=A$. We call such an element g a period for A .

REMARK:

We should note that if A is a factor in a normalized factorization of a group G , and A is periodic, then its period must be an element of A .

2.5 Definition [18]

A group G is said to have the Hajós n-property if, whenever $G = A_1 \dots A_i \dots A_n$ is a factorization of G , then at least one of the factors A_i is periodic.

NOTE:

In the proof of the following Proposition, we will use these facts: [7]

(a) If $\Phi_n(x)$ denotes the n th cyclotomic polynomial, then

$\deg \Phi_n(x) = \varphi(n)$, where φ is the Euler's φ -function.

(b) If p is a prime, then $\Phi_p(x) = (1 + x + \dots + x^{p-1})$.

(c) $\Phi_{pn}(x) = \Phi_p(x^{p^{n-1}})$.

(d) The cyclotomic polynomial $\Phi_n(x)$ is irreducible over $\mathbb{Q}$ the field of rational numbers.

2.6 Theorem

Let G be a cyclic group of order p^α , where p is prime, and α , a natural number. Then G has the Hajós n -property for all n , where $1 \leq n \leq \alpha$.

Proof :

Suppose that $A_1 \dots A_i \dots A_n = G$ is a factorization of G . We may assume that this factorization is normalized. For, if it is not normalized, we can normalize it as follows :

Since A_i is not empty for each i , $1 \leq i \leq n$, there is an element $a_i \in A_i$.

Multiplying G by $a = a_1^{-1} \dots a_n^{-1}$, we get that:

$$(a_1^{-1}A_1) \dots (a_n^{-1}A_n) = aG = G \dots \dots \dots (2)$$

It is clear that (2) is a normalized factorization of G . Also if A_i is periodic, then so is $a_i^{-1}A_i$, and conversely. Now G is cyclic, say $G = \langle g \rangle$, for some g , and there exist positive integers $a(i, j)$, where $1 \leq i \leq n$, and $1 \leq j \leq s_i$, such that :

$$A_1 = \{ e, g^{a(1, 1)}, \dots, g^{a(1, s_1)} \}$$

.....

$$A_i = \{ e, g^{a(i, 1)}, \dots, g^{a(i, s_i)} \}$$

.....

$$A_n = \{ e, g^{a(n, 1)}, \dots, g^{a(n, s_n)} \}$$

Now consider the following polynomials associated with G and $A_1, \dots, A_i, \dots, A_n$:

$$G(x) = (1 + x + \dots + x^{p^\alpha - 1})$$

$$A_1(x) = (1 + x^{a(1, 1)} + \dots + x^{a(1, s_1)})$$

$$A_i(x) = (1 + x^{a(i,1)} + \dots + x^{a(i,s_i)})$$

$$A_n(x) = (1 + x^{a(n,1)} + \dots + x^{a(n,s_n)})$$

Then it is easy to see that we have the following polynomial congruence relation:

$$G(x) \equiv A_1(x) \dots A_n(x) \pmod{x^{p^\alpha} - 1}.$$

Next look at the p^α -th cyclotomic polynomial;

$$\Phi_{p^\alpha}(x) = (1 + x^{p^{\alpha-1}} + x^{2p^{\alpha-1}} + \dots + x^{(p-1)p^{\alpha-1}})$$

Note that $\Phi_{p^\alpha}(x)$ divides $x^{p^\alpha} - 1$, but :

$$x^{p^\alpha} - 1 = (x - 1)(1 + x + \dots + x^{p^\alpha-1}).$$

It follows that $\Phi_{p^\alpha}(x)$ also divides $G(x)$.

Since $G(x) \equiv A_1(x) \dots A_n(x) \pmod{x^{p^\alpha} - 1}$, we get that:

$\Phi_{p^\alpha}(x)$ divides $A_1(x) \dots A_n(x)$, and since $\Phi_{p^\alpha}(x)$ is irreducible over $\mathbb{Q}$ there is some i , $1 \leq i \leq n$, such that $\Phi_{p^\alpha}(x)$ divides $A_i(x)$.

Claim

A_i is periodic with $g^{p^\alpha-1}$ as a period.

Proof of claim

Since $\Phi_{p^\alpha}(x)$ divides $A_i(x)$, we get that :

$A_i(x) = \Phi_{p^\alpha}(x) B(x)$ for some non-zero polynomial $B(x) \in \mathbb{Q}[x]$.

First we observe that :

$$\begin{aligned} 0 \leq \deg B(x) &= \deg A_i(x) - \deg \Phi_{p^\alpha}(x) \\ &\leq p^\alpha - 1 - (p-1)p^{\alpha-1} \\ &= p^\alpha - 1 - p^\alpha + p^{\alpha-1} \\ &= p^{\alpha-1} - 1. \end{aligned}$$

That is, $0 \leq \deg B(x) \leq p^{\alpha-1} - 1$.

Now, suppose $B(x) = b_0 + b_1x + b_2x^2 + \dots + b_mx^m$.

Multiplying : $B(x) \Phi_{p^\alpha}(x) =$

$$\begin{aligned} &(b_0 + b_1x + b_2x^2 + \dots + b_mx^m)(1 + x^{p^\alpha-1} + x^{2p^\alpha-1} + \dots + x^{(p-1)p^\alpha-1}) \\ &= (b_0 + b_1x + b_2x^2 + \dots + b_mx^m) + \\ &(b_0x^{p^\alpha-1} + b_1xx^{p^\alpha-1} + b_2x^2x^{p^\alpha-1} + \dots + b_mx^mx^{p^\alpha-1}) + \end{aligned}$$

$$\dots\dots\dots +$$

$$(b_0 x^{(p-1)p^{\alpha-1}} + b_1 x x^{(p-1)p^{\alpha-1}} + b_2 x^2 x^{(p-1)p^{\alpha-1}} + \dots + b_m x^m x^{(p-1)p^{\alpha-1}}).$$

Suppose $x^i x^j p^{\alpha-1} = x^k x^l p^{\alpha-1}$, where $0 \leq i, k \leq m$, and $0 \leq j, l \leq p-1$. Therefore, we must have $i + j p^{\alpha-1} = k + l p^{\alpha-1}$, from which it follows that $(i - k) = (l - j) p^{\alpha-1}$. Now, recall that $m \leq p^{\alpha-1} - 1$. Hence, $|i - k| < p^{\alpha-1}$. So that we must have $l - j = 0$. Consequently, $l = j$, and also $i = k$. Therefore, the nonzero terms of $B(x)$ occur among the nonzero terms of $A_i(x)$,

$$\text{Now } x^{p^{\alpha-1}} A_i(x) = x^{p^{\alpha-1}} \Phi_{p^{\alpha}}(x) B(x)$$

$$= \Phi_{p^{\alpha}}(x) B(x) = A_i(x) \bmod (x^{p^{\alpha}} - 1).$$

Finally, using the original correspondence, we get that $g^{p^{\alpha-1}} A_i = A_i$. So A_i is periodic with period $g^{p^{\alpha-1}}$. □

NOTE :

We say that a group G is of type $(p^{\alpha_1}, \dots, p^{\alpha_i}, \dots, p^{\alpha_s})$, if G is the direct product of cyclic groups of orders $p^{\alpha_1}, \dots, p^{\alpha_i}, \dots, p^{\alpha_s}$.

2.7 Groups of type (p^α, p^β)

We next want to consider groups of type (p^α, p^β) , where $\alpha \geq 1$, $\beta \geq 1$, and as before p is a prime and for the rest of this chapter, $p \geq 5$. We first record some facts that we shall need.

(a) Rédei's Theorem : [20]

In any normalized factorization of a finite abelian group by subsets of prime cardinality, at least one of the factors is a subgroup.

(b) We will also need some results from character theory for finite abelian groups. However, since we shall not need the theory in its full generality, and for the sake of completeness we shall develop here the ideas that we need : We begin by defining a character for a cyclic group, and then extend the result to any finite abelian group, using The Fundamental Theorem of Finite Abelian Groups (FTOFAg).

So let $G = \langle x \rangle$ be a cyclic group of order n , and recall that a complex number ξ is a primitive n th root of unity if $\xi^n = 1$ and if $\xi^m \neq 1$ whenever $m < n$. A character of G is a homomorphism $\chi : G \rightarrow \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers, defined by $\chi(x^r) = \xi^r$, where ξ is a primitive n -th root of unity. In fact, we should observe that once we specify $\chi(x)$, then χ is defined for all the elements of G . The principal character is the homomorphism taking the constant value 1. It is immediate that the group $G = \langle x \rangle$, where $|x| = n$, will have exactly n characters, and the set $S = \{\chi_i : 1 \leq i \leq n\}$, becomes a group under pointwise multiplication. Next, if G is the direct product of two cyclic groups, say $G = \langle x \rangle \times \langle y \rangle$, then using the above observation, it follows that

any character χ of G arises as a homomorphism $\chi : G \rightarrow \mathbb{C}$, given by $\chi(x^i y^j) = \psi(x^i) \eta(y^j)$, where ψ and η are characters of the groups $\langle x \rangle$ and $\langle y \rangle$ respectively. Finally, using The Fundamental Theorem of Finite Abelian Groups (FTOAG), we can extend this notion to any finite abelian group. Now, since we have as many characters for a group as that of its order, it follows that the character table of a group

$G = \{g_1, g_2, \dots, g_n\}$ of order n is an $n \times n$ matrix, say :

$$M = \begin{bmatrix} \chi_1(g_1) & \chi_1(g_2) & \dots & \chi_1(g_n) \\ \chi_2(g_1) & \chi_2(g_2) & \dots & \chi_2(g_n) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_n(g_1) & \chi_n(g_2) & \dots & \chi_n(g_n) \end{bmatrix} \dots\dots\dots(3)$$

Let us agree that χ_1 denotes the principal character and $g_1 = e$, the identity element of G ,

2.8 Lemma [15]

The columns of the matrix M above are orthogonal.

Next, we need the following Definitions and Lemma .

2.9 Definition [15]

If χ is a character and A , a subset of G , then $\chi(A)$ denotes :

$$\sum_{a \in A} \chi(a)$$

2.10 Definition [15]

Let A be a subset of a group G , then the set of all characters of G for which $\chi(A) = 0$, is called the annihilator of A and is denoted by :

$$\text{Ann}(A) = \{ \chi : \chi(A) = 0 \}.$$

Using the fact that the columns of the matrix in (3) are orthogonal, we get the following

2.11 Lemma [15]

Let A and B be subsets of a group G and suppose that for each character χ of G , $\chi(A) = \chi(B)$, then $A = B$.

Now, suppose $AB = G$ is a factorization of G . Then it follows that for any character χ of G , we have $\chi(G) = \chi(AB)$. For the principal character, this is equivalent to the equation $|G| = |A||B|$, while for the other characters, we get that $\chi(A)\chi(B) = 0$.

2.12 Corollary [15]

Suppose that A and B are subsets of G , such that $\chi(G) = \chi(AB)$ for all characters χ of G , then it follows that $G = AB$ is a factorization of G .

We now have the following general result, which we record as a Lemma, for easy reference.

2.13 Lemma [15]

If $|G| = |A_1| \dots |A_i| \dots |A_n|$, and each non-principal character of G is in $\text{Ann}(A_1) \cup \dots \cup \text{Ann}(A_i) \cup \dots \cup \text{Ann}(A_n)$, then $G = A_1 \dots A_i \dots A_n$ is a factorization of G .

We are now ready to continue investigating groups of type (p^α, p^β) . Please recall that for the time being, we are assuming that the prime $p \geq 5$. Observe that if $\alpha = \beta = 1$, then G has the Hajós 1 and 2-property, by Rédei's Theorem (a) above. So consider the case where either $\alpha > 1$ or $\beta > 1$. We first look at the case $\alpha = 2$ and $\beta = 1$. That is G is of type (p^2, p) . The following Theorem is essentially due to De Bruijn, [3].

2.14 Proposition

If G is of type (p^2, p) , then G does not have the Hajós 2-property.

Proof :

We show that in this case G does not have the Hajós 2-property by exhibiting a factorization AB of G , in which neither of the factors is periodic. Say $G = \langle x \rangle \times \langle y \rangle$, where $|x| = p^2$ and $|y| = p$.

Let $A = \langle x^p \rangle \cup x\langle y \rangle \cup x^2\langle y \rangle \cup \dots \cup x^{p-1}\langle y \rangle$ and

$$B = \{ e, (x^p y), (x^p y)^2, \dots, (x^p y)^{p-3}, x^{p(p-2)} y^{p-1}, x^p (p-1) y^{p-2} \}.$$

Remark:

Please note that for us to be able to swap the last two elements in B , it should

contain at least three elements, namely $p \geq 3$. But when $p = 3$, then $B = \{ e, x^3 y^2, x^6 y \} = \langle x^3 y^2 \rangle$. So B is a subgroup, hence periodic. Therefore, to ensure a chance for non-periodicity of B (as we shall see), we need $p \geq 5$.

We will now use Lemma 2.13, making case-by-case inspection to show that $AB = G$ is a factorization of G . Now the possible orders of $\chi(x)$ are 1, p , and p^2 , while the possible orders of $\chi(y)$ are 1 and p only. So we have 6 different cases to consider. The following table summarizes the 6 possible cases :

Case	Order of $\chi(x)$	Order of $\chi(y)$
1	1	1
2	1	p
3	p	1
4	p	p
5	p^2	1
6	p^2	p

Case (1) : $\chi(x) = 1$ and $\chi(y) = 1$: this is straightforward since in this case, χ is the principal character, and so $|G| = |A| |B|$.

Case (2) : $\chi(x) = 1$ and $\chi(y) = \xi$, where ξ is a primitive p -th root of unity.

$$\chi(B) = 1 + \xi + \xi^2 + \dots + \xi^{p-1} = 0.$$

Case (3) : $\chi(x) = \xi$ and $\chi(y) = 1$, where ξ is a primitive p -th root of unity :

$$\begin{aligned}
\chi(A) &= p + p\xi + p\xi^2 + p\xi^3 + \dots + p\xi^{p-1} \\
&= p(1 + \xi + \xi^2 + \xi^3 + \dots + \xi^{p-1}) \\
&= 0.
\end{aligned}$$

Case (4) : $\chi(x) = \xi$ and $\chi(y) = \lambda$, where ξ and λ are primitive p -th roots of unity :

$$\chi(B) = 1 + \lambda + \lambda^2 + \dots + \lambda^{p-1} = 0.$$

Case (5) : $\chi(x) = \rho$ and $\chi(y) = 1$, where ρ is a primitive p^2 -th roots of unity :

$$\chi(B) = 1 + \rho^p + \rho^{2p} + \dots + \rho^{(p-1)p} = 0, \text{ since } \rho^p \text{ is a}$$

primitive p -th root of unity.

Case (6) : $\chi(x) = \rho$ and $\chi(y) = \xi$, where ρ is a primitive p^2 -th root of unity, and ξ is a primitive p -th root of unity.

Note that in this case $\chi(\langle y \rangle) = 0$, and $\chi(\langle x^p \rangle) = 0$. Consequently, we get that $\chi(A) = 0$. Therefore, by Lemma 2.13, $AB = G$ is a factorization of G . It now only remains to show that neither A nor B is periodic.

A is not periodic :

Suppose A is periodic. So there is an element $a \neq e$ in A that is a period for A . But then all the non-identity powers of a are also periods for A . Now $|a| = p$ or p^2 . It is impossible to have $|a| = p^2$, because then A has to be a subgroup, since A has p^2 elements. So we may assume that $|a| = p$. This is also impossible. For if $|a| = p$, then a must be an element of the subgroup $\langle x^p \rangle$. This is because all the other elements of A have order p^2 . But note that $\langle x^p \rangle y$ is not contained in A . Therefore A is not periodic.

B is not periodic :

Suppose B is periodic. So there is an element $b \neq e$ in B such that $bB = B$. But then all the non-identity powers of b are also periods for B. Now $|b| = p^2$ or p. To have $|b| = p^2$ is impossible, since then we would have too many elements. So we may assume that $|b| = p$. Then B must be a subgroup. But B is clearly not a subgroup. For instance, $(x^p y)$ is in B, however, $(x^p y)^{p-1} = x^{p(p-1)} y^{p-1}$ is not in B. $\square$

Before discussing the general case for α and β , we need to state the following Lemma and then a Theorem on the factorization by periodic subsets.

2.15 Lemma [20]

Let G be a group, and $\{g; n\}$ the cyclic subset determined by g and n. If $n = rs$, where $1 < r, s < n$, then $\{g; n\}$ can be factorized as $\{g; n\} = \{g; r\} \{g^r; s\}$.

2.16 Theorem [24]:

Let G be a finite abelian group such that G has factorization $G = AD$. Suppose $A = A_1 \dots A_i \dots A_n$ is a factorization of the subset A and assume further that A is periodic. If for each i, $1 \leq i \leq n$, either :

- (i) $|A_i| = p_i$ is a prime and there is at most one prime divisor of $|A_i|$ which is smaller than p_i , or

(ii) A_i is a simulated subset.

Then one of the factors A_i is a subgroup of G .

2.17 Theorem

If G is of type of type (p^α, p^β) , where $1 \leq \alpha \leq \beta$, and $\beta \geq 2$, then G does not have the Hajós n -property for $2 \leq n \leq \alpha + \beta - 1$.

Proof :

Say $G = \langle x \rangle \times \langle y \rangle$, where $|x| = p^\alpha$ and $|y| = p^\beta$, where $1 \leq \alpha \leq \beta$, and $\beta \geq 2$.

Using Lemma 2.15, we can write $\langle x \rangle$ as :

$$\begin{aligned} \langle x \rangle &= \\ &\{x; p\} \{x^p; p\} \{x^{p^2}; p\} \dots \{x^{p^i}; p\} \dots \{x^{p^{\alpha-2}}; p\} \{x^{p^{\alpha-1}}; p\} \\ &= A_1 A_2 \dots A_i \dots A_{\alpha-1} A_\alpha \text{ where } A_i = \{x^{p^{i-1}}; p\} \text{ and } 1 \leq i \leq \alpha. \end{aligned}$$

Similarly, we can write $\langle y \rangle$ as :

$$\begin{aligned} \langle y \rangle &= \\ &\{y; p\} \{y^p; p\} \{y^{p^2}; p\} \dots \{y^{p^j}; p\} \dots \{y^{p^{\beta-2}}; p\} \{y^{p^{\beta-1}}; p\} \\ &= B_1 B_2 \dots B_j \dots B_{\beta-1} B_\beta \text{ where } B_j = \{y^{p^{j-1}}; p\} \text{ and } 1 \leq j \leq \beta. \end{aligned}$$

Therefore :

$$\begin{aligned} G &= A_1 A_2 \dots A_i \dots A_{\alpha-1} A_\alpha B_1 B_2 \dots B_j \dots B_{\beta-1} B_\beta \\ &= A_1 A_2 \dots A_i \dots A_{\alpha-1} B_1 B_2 \dots B_j \dots B_{\beta-2} A_\alpha B_{\beta-1} B_\beta \end{aligned}$$

$$= A_1 A_2 \dots A_i \dots A_{\alpha-1} B_1 B_2 \dots B_j \dots B_{\beta-2} \langle x^{p^{\alpha-1}} \rangle \langle y^{p^{\beta-2}} \rangle$$

$$= A_1 A_2 \dots A_i \dots A_{\alpha-1} B_1 B_2 \dots B_j \dots B_{\beta-2} H$$

where $H = \langle x^{p^{\alpha-1}}, y^{p^{\beta-2}} \rangle$ is a subgroup of type (p, p^2) .

Now by Proposition 2.14, H itself can be factorized as $H = EF$, where $|E| = p$, $|F| = p^2$, with neither E nor F periodic. Hence :

$$G = A_1 A_2 \dots A_i \dots A_{\alpha-1} B_1 B_2 \dots B_j \dots B_{\beta-2} EF \dots \dots \dots (\dagger)$$

Clearly if $n = 1$ or $n = \alpha + \beta$, then G has the Hajós n -property.

Now observe that none of the factors in $(\dagger)$ is periodic, We use Theorem 2.16, to show that G does not have the Hajós n -property for any n , where $2 \leq n \leq \alpha + \beta - 1$.

If $n = \alpha + \beta - 1$, then G does not have the Hajós n -property, since none of the factors in $(\dagger)$ is periodic.

For $n < \alpha + \beta - 1$, the result follows from Theorem 2.16 above by regrouping the factors appropriately :

for example : if $n = \alpha + \beta - 2$, let $A = A_1 A_2$, and

$$D = A_3 \dots A_i \dots A_{\alpha-1} B_1 B_2 \dots B_j \dots B_{\beta-2} EF$$

If we assume that A is periodic, then the Theorem says that either A_1 or A_2 must be a subgroup of G , which is obviously not the case. By repeating the same argument for any n , $2 \leq n \leq \alpha + \beta - 1$, the result follows. $\square$

Now, we plan to generalize the result in Proposition 2.17 to groups of type $(p^{\alpha_1}, \dots, p^{\alpha_i}, \dots, p^{\alpha_s})$. As before, we begin with some specific cases. The following Theorem is also in its essence, due to De Bruijn, [3].

2.18 Proposition

If G is of type (p, p, p) , then G does not have the Hajós 2-property.

Proof :

We exhibit a factorization AB of G , in which none of the factors is periodic. Say $G = \langle x \rangle \times \langle y \rangle \times \langle z \rangle$, and let :

$$A = \langle x \rangle \cup z \langle y \rangle \cup z^2 \langle y \rangle \cup \dots \cup z^{p-1} \langle y \rangle, \text{ and}$$

$$B = \{ e, (xy), (xy)^2, \dots, (xy)^{p-3}, x^{p-2}y^{p-1}, x^{p-1}y^{p-2} \}.$$

Now, the possible orders of $\chi(x)$, $\chi(y)$, and $\chi(z)$ are 1, and p . So in total we have the 8 cases to consider. The following table summarizes the 8 possible cases.

Case	Order of $\chi(x)$	Order of $\chi(y)$	Order of $\chi(z)$
1	1	1	1
2	1	1	p
3	1	p	1
4	1	p	p
5	p	1	1
6	p	1	p
7	p	p	1
8	p	p	p

In the following each of ξ , η , and ρ is a primitive p -th root of unity.

Case (1): this is straightforward since in this case, χ is the principal character, and so $|G| = |A| |B|$.

Case (2):

$$\begin{aligned}
 \text{Let } \chi(x) = \xi, \text{ then } \chi(A) &= p + p\rho + p\rho^2 + \dots + p\rho^{p-1} \\
 &= p(1 + \rho + \rho^2 + \dots + \rho^{p-1}) \\
 &= 0.
 \end{aligned}$$

Case (3):

$$\text{Let } \chi(y) = \eta, \text{ then } \chi(B) = (1 + \eta + \eta^2 + \dots + \eta^{p-1}) = 0.$$

Case (4):

Let $\chi(y) = \eta$, then $\chi(B) = (1 + \eta + \eta^2 + \dots + \eta^{p-1}) = 0$.

Case (5):

Let $\chi(y) = \eta$, then $\chi(B) = (1 + \eta + \eta^2 + \dots + \eta^{p-1}) = 0$.

Case (6):

Let $\chi(x) = \xi$, and $\chi(z) = \rho$, then

$$\chi(B) = (1 + \xi + \xi^2 + \dots + \xi^{p-1}) = 0.$$

Case (7):

Let $\chi(x) = \xi$, and $\chi(y) = \eta$, then

$$\begin{aligned}\chi(A) &= (1 + \xi + \xi^2 + \dots + \xi^{p-1}) + \\ &\quad p(1 + \eta + \eta^2 + \dots + \eta^{p-1}) \\ &= 0 + 0 = 0.\end{aligned}$$

Case (8):

Let $\chi(x) = \xi$, $\chi(y) = \eta$, and $\chi(z) = \rho$, then,

$$\begin{aligned}\chi(A) &= (1 + \xi + \xi^2 + \dots + \xi^{p-1}) \\ &\quad + \rho(1 + \eta + \eta^2 + \dots + \eta^{p-1}) \\ &\quad + \rho^2(1 + \eta + \eta^2 + \dots + \eta^{p-1})\end{aligned}$$

$$+ \dots\dots\dots$$

$$+ \rho^{p-1} (1 + \eta + \eta^2 + \dots + \eta^{p-1})$$

$$= 0.$$

Since $|G| = |A| |B| = p \cdot p^2 = p^3$, Lemma 2.13 implies that $AB = G$ is a factorization of G . We now need to show that neither A nor B is periodic.

A is not periodic :

If A were periodic, then a period of A must be an element of A , since A contains the identity. It is obvious that none of the elements of A can be a period for A .

B is not periodic :

If B were periodic, then it would follow that B must be a subgroup. Clearly, B is not a subgroup. For example, xy is in B , but $(xy)^{p-1} = x^{p-1} y^{p-1}$ is not in B .

□

Before we proceed to the next Proposition, we state the following Definition and a subsequent Lemma.

2.19 Definition [14]

Let H be a subgroup of G , then a (right) transversal of H in G (or a complete set of (right) coset representatives modulo H) is a subset T of G consisting of one element from each right coset of H in G .

2.20 Lemma [14]

Let H be a subgroup of a group G and T a subset of G . Then T is a right transversal to H in G if and only if HT is a factorization of G .

We shall also need the following

2. 21 Lemma [22]

Let A be a simulated subset of the finite abelian group G , and let H be a corresponding subgroup of G . If A is periodic, then $A = H$.

2. 22 Proposition :

If G is of type $(p, p, \dots, p)$, say an s - tuple, where $s \geq 3$, then G does not have the Hajós n -property for $2 \leq n \leq s - 1$.

Proof :

Note that the case $s = 3$, was taken care of in Proposition 2.17, so we shall assume that $s \geq 4$. Say $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \dots \times \langle x_s \rangle$. First factorize G in the following way :

Proof :

Write $K = \{ k_1, k_2, \dots, k_r \}$ and $T = \{ t_1, t_2, \dots, t_r \}$, where $t_i = k_i h_i$. Say $t_i = t_j \pmod{H}$ for $i \neq j$. So $k_i h_i = k_j h_j \pmod{H}$, which means that $k_i h_i (k_j h_j)^{-1} = h$ for some h in H . Hence $k_i k_j^{-1} h_i h_j^{-1} = k_i h_i h_j^{-1} k_j^{-1} = h$. This means that $k_i k_j^{-1} \in H$. Therefore we must have $k_i = k_j \pmod{H}$, and this contradicts the fact that K is a right transversal of H in G . Hence the proof of the claim. Finally by Lemma 2.19, HT is a factorization of G . Clearly if $n = 1$ or $n = s$, then G has the Hajós n -property. Now we note that none of the factors in $G = A B S_1 S_2 S_3 \dots S_i \dots S_t$ is periodic. For, neither A nor B is periodic by Proposition 2.17, and none of the S_i can be periodic by Lemma 2.21, (since clearly none of them is a subgroup). Now, the fact that G does not have the Hajós n -property for $2 \leq n \leq s-1$, is a consequence of Theorem 2.16. □

Finally, we have the most general result in the next Theorem.

2.23 Theorem

If G is of type $(p^{\alpha_1}, p^{\alpha_2}, \dots, p^{\alpha_i}, \dots, p^{\alpha_s})$, where $s \geq 3$, then G does not have the Hajós n -property for $2 \leq n \leq \alpha_1 + \alpha_2 + \dots + \alpha_s - 1$.

Proof :

Let $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \dots \times \langle x_i \rangle \times \dots \times \langle x_s \rangle$, where $|x_i| = p^{\alpha_i}$, and $1 \leq i \leq s$.

Write each $\langle x_i \rangle$ as $\langle x_i \rangle =$

$$\{x_i; p\} \{x_i^p; p\} \{x_i^{p^2}; p\} \dots \{x_i^{p^{j-1}}; p\} \dots \{x_i^{p^{\alpha_i-2}}; p\} \{x_i^{p^{\alpha_i-1}}; p\}$$

$$= \prod_{j=1}^{\alpha_i} \{x_i^{p^{j-1}}; p\} = A_{i,j},$$

where of course, $1 \leq i \leq s$, and $1 \leq j \leq \alpha_i$.

Thus we can write G as

$$G = A_{1,1} A_{1,2} \dots A_{1,\alpha_1-1} A_{2,1} A_{2,2} \dots A_{2,\alpha_2-1} \dots A_{s,1} A_{s,2} \dots A_{s,\alpha_s-1} H \dots (\dagger),$$

where $H = \langle x_1^{p^{\alpha_1-1}} \rangle \times \langle x_2^{p^{\alpha_2-1}} \rangle \dots \times \dots \times \langle x_i^{p^{\alpha_i-1}} \rangle \dots \times \langle x_s^{p^{\alpha_s-1}} \rangle$ is a

subgroup of type $(p, p, \dots, p)$. Now note that by Proposition 2.22, H can be factorized into non-periodic subsets, and the rest of the factors in $(\dagger)$ are clearly non-periodic subsets. Clearly for $n = 1$ or s , G has the Hajós n -property. The fact that G does not have the Hajós n -property for $2 \leq n \leq \alpha_1$

$+ \alpha_2 + \dots + \alpha_s - 1$, follows from Theorem 2.16. $\square$

CHAPTER THREE

Introduction :

In Chapter two, we assumed throughout that $p \geq 5$, except when G was cyclic. Now we will try to remedy that situation by filling the gap in the cases of $p = 3$ and $p = 2$. That is we will study groups of type :

$$(3^{\alpha_1}, 3^{\alpha_2}, \dots, 3^{\alpha_i}, \dots, 3^{\alpha_s}) \dots\dots\dots(1)$$

and groups of type :

$$(2^{\alpha_1}, 2^{\alpha_2}, \dots, 2^{\alpha_i}, \dots, 2^{\alpha_t}) \dots\dots\dots(2)$$

This chapter is divided into three sections. It will contain results about the groups in case (1). Results about the groups in case (2) are the contents of the next chapter. So, in this chapter, we will study 3-groups with respect to the Hajós n -property. Observe that in the cyclic case, G will have the Hajós n -property for all relevant n , by Theorem 2.6. Similarly if G is of type $(3, 3)$, then Rédei's Theorem gives us a similar result. Another easy case is when G is of type $(3, 3, 3)$ or $(3^2, 3)$. In this case, we also claim that G has the Hajós n -property for $n = 1, 2$ and 3 . Of course when $n = 1$ or $n = 3$, G clearly has the Hajós n -property. Hence, we only need detail the case when $n = 2$.

Note that in this case, one of the factors must have order 3. So assume G is factorized as $G = AB$, and assume $|A| = 3$. Let $A = \{e, u, v\}$, where $u \neq e$ and $v \neq e$.

Then $G = AB = \{e, u, v\} B = B \cup uB \cup vB \dots \dots \dots (3)$

Multiplying both sides of equation (3) by u results in :

$$G = uG = uAB = uB \cup u^2B \cup uvB \dots \dots \dots (4)$$

Comparing equations (3) and (4), we get that :

$$B \cup vB = u^2B \cup uvB \dots \dots \dots (5)$$

We claim that : $vB \cap uvB = \emptyset$.

Suppose this is not the case. Then there is an $x \in vB \cap uvB$. So $x = vb_1 = uvb_2$ for some b_1 and b_2 in B . But from the factorization $G = AB$, this means that $u = e$. And this contradicts our assumption that $u \neq e$. The preceding argument implies that $B = uvB$ and $vB = u^2B$. Hence, either B is periodic, or $uv = e$ and $v = u^2$, in which case A is periodic. So the first nontrivial case of 3-groups occurs when $|G| = 3^4$. Thus, we start the next section with a study of groups of order 3^4 .

Case	Order of $\chi(x)$	Order of $\chi(y)$	$\chi(A)$	$\chi(B)$
1	1	1	.	.
2	1	3		0
3	1	9		0
4	3	1	0	
5	3	3	0	
6	3	9		0
7	9	1	0	
8	9	3	0	
9	9	9	0*	0*

(*) : All the other cases are straightforward, except this case, for which we include an explanation: Suppose $\chi(x) = \xi$ and $\chi(y) = \eta$, where ξ and η are primitive 9-th roots of unity.

Then :

$$\begin{aligned}
\chi(A) &= 1 + \xi + \xi^2 + \xi^3 + \xi^4 + \xi^6 + \xi^7 + \xi^5 \eta^3 + \xi^8 \eta^6 \\
&= (1 + \xi^3 + \xi^6) + \xi (1 + \xi^3 + \xi^6) + \xi^2 (1 + \xi^3 \eta^3 + \xi^6 \eta^6) \\
&= \xi^2 (1 + \xi^3 \eta^3 + \xi^6 \eta^6).
\end{aligned}$$

§1. In this section, we make some constructions which show that groups of order 3^4 do not have the Hajós 2-property, except of course when G is cyclic, which by Theorem 2.6, has the Hajós n -property for $n = 1, 2, 3$, and 4 . Altogether, there are 5 non-isomorphic groups of order 3^4 to deal with. Since the cyclic case is taken care of, there remain four cases as follows : These are groups of type $(3^3, 3), (3^2, 3^2), (3^2, 3, 3)$, and $(3, 3, 3, 3)$.

3.1.1 Proposition

If G is of type $(3^3, 3)$, then G does not have the Hajós 2-property.

Proof :

We show that in this case G does not have the Hajós 2-property by exhibiting a factorization AB of G , in which neither of the factors is periodic. Say $G = \langle x \rangle \times \langle y \rangle$, where $|x| = 3^3 = 27$ and $|y| = 3$.

Let $A = \langle x^9 y \rangle \cup x \langle x^9 y \rangle \cup x^2 \langle x^9 y^2 \rangle$ and

$$B = \langle x^9 \rangle \cup x^3 \langle y \rangle \cup x^6 \langle y \rangle$$

Now the possible orders of $\chi(x)$ are 1, 3, 9 and 27, and the possible orders of $\chi(y)$ are 1 and 3 only. So we have 8 different cases to consider. We use Lemma 2.13 to show that $AB = G$ is a factorization of G . The following table summarizes all the cases :

Case	Order of $\chi(x)$	Order of $\chi(y)$	$\chi(A)$	$\chi(B)$
1	1	1	.	.
2	1	3	0	
3	3	1	0	
4	3	3	0	
5	9	1		0
6	9	3	0	
7	27	1	0	
8	27	3		0

It only remains to show that neither A nor B is periodic.

A is not periodic :

First note that if g is a period for A, then any non-identity power of g is also a period for A. Now the elements of G all have order a multiple of 3. So the only candidates for a period are elements of order 3. We check elements of A of order 3 for periodicity. Clearly the only elements of order 3 in A are x^9y and $x^{18}y$, and none of these is a period for A :

$$(x^9y)(x^2) = (x^{11}y) \notin A \text{ and}$$

$$(x^{18}y)(x^2) = (x^{20}y) \notin A.$$

$$\begin{aligned}
\chi(B) &= 1 + \eta + \eta^2 + \eta^3 + \eta^4 + \eta^6 + \eta^7 + \xi^6 \eta^5 + \xi^3 \eta^8 \\
&= (1 + \eta^3 + \eta^6) + \eta (1 + \eta^3 + \eta^6) + \eta^2 (1 + \xi^3 \eta^6 + \xi^6 \eta^3) \\
&= \eta^2 (1 + \xi^3 \eta^5 + \xi^6 \eta^3).
\end{aligned}$$

Now ξ and η are both primitive 9-th roots of unity. Hence:

$$\eta = \xi, \xi^2, \xi^4, \xi^5, \xi^7 \text{ or } \xi^8.$$

Straightforward calculations show that in the cases $\eta = \xi, \xi^4$ or ξ^7 , we get that $\chi(A) = 0$, while in the remaining cases, $\chi(B) = 0$. Now it only remains to show that neither A nor B is periodic.

A is not periodic:

The only candidates for a period of A are x^3 and x^6 , and none of them is a period for B:

$$\begin{aligned}
(x^3)(x^5 y^3) &= (x^8 y^3) \notin A \text{ and} \\
(x^6)(x^8 y^6) &= (x^5 y^6) \notin A.
\end{aligned}$$

B is not periodic:

The only candidates for a period of B are y^3 and y^6 , and none of them is a period for B:

$$(y^3) (x^6 y^5) \notin B \text{ and}$$

$$(y^6) (x^3 y^8) \notin B.$$

□

3.1.3 Proposition

If G is of type $(3^2, 3, 3)$, then G does not have the Hajós 2-property.

Proof :

Say $G = \langle x \rangle \times \langle y \rangle \times \langle z \rangle$, where $|x| = 9$, and $|y| = |z| = 3$.

Let $A = \langle x^3 \rangle \cup x \langle y \rangle \cup x^2 \langle y \rangle$ and

$$B = \langle x^3 y \rangle \cup z \langle x^3 y^2 \rangle \cup z^2 \langle x^3 y \rangle.$$

Now the possible orders of $\chi(x)$ are 1, 3, 9, while the possible orders of $\chi(y)$ and $\chi(z)$ are 1 and 3 only. So altogether we have 12 different cases to consider. The following table summarizes all of these cases.

Case	Order of $\chi(x)$	Order of $\chi(y)$	Order of $\chi(z)$	$\chi(A)$	$\chi(B)$
1	1	1	1	.	.
2	1	1	3		0
3	1	3	1		0
4	1	3	3		0
5	3	1	1	0	
6	3	1	3		0
7	3	3	1	0	
8	3	3	3	0	
9	9	1	1		0
10	9	1	3		0
11	9	3	1	0**	
12	9	3	3	0**	

(**) : All the other cases are straightforward, except these cases, for which we use a similar argument as in Proposition 3.1.2.

It only remains to show that neither A nor B is periodic.

A is not periodic:

The only candidates for a period of A are x^3 and x^6 , and none of them is a period for A :

$$(x^3)(xy) = (x^4y) \notin A \text{ and} \\ (x^6)(xy) = (x^7y) \notin A.$$

B is not periodic:

In this case, each non-identity elements of B is a candidate for a period of B.

However, none of them is a period :

$$(x^3y)(y) = (x^3y^2) \notin B, \quad (x^6yz)(y) = (x^6y^2z) \notin B, \\ (x^6y^2)(y) = (x^6) \notin B, \quad (x^3yz^2)(y) = (x^3y^2z^2) \notin B, \\ (z)(y) = (yz) \notin B, \quad (x^6y^2z^2)(y) = (x^6z^2) \notin B, \\ (x^3y^2z)(y) = (x^3z) \notin B.$$

□

3.1.4 Proposition:

If G is of type (3, 3, 3, 3), then G does not have the Hajós 2-property.

Proof :

Say $G = \langle x \rangle \times \langle y \rangle \times \langle u \rangle \times \langle v \rangle$, where $|x| = |y| = |u| = |v| = 3$.

Let $A = (\langle x, y \rangle \setminus \{x^2y, x^2y^2\}) \cup \{x^2yv, x^2y^2v^2\}$ and

$$B = (\langle u, v \rangle \setminus \{u^2v, u^2v^2\}) \cup \{yu^2v^2, y^2u^2v\}.$$

Now, the possible orders of $\chi(x)$ are 1 and 3, only.

Similarly for $\chi(y)$, $\chi(u)$, $\chi(v)$. So altogether there are 16 different cases to

consider. Let $\chi(x) = \alpha$,

$$\chi(y) = \beta,$$

$$\chi(u) = \gamma,$$

$$\chi(v) = \delta,$$

where α, β, γ , and δ are primitive 3rd roots of unity.

$$\begin{aligned} \text{Then: } \chi(A) &= \alpha^2 \beta \delta + \alpha^2 \beta^2 \delta^2 - \alpha^2 \beta - \alpha^2 \beta^2 \\ &= \alpha^2 \beta (\delta + \beta \delta^2 - 1 - \beta). \end{aligned}$$

$$\begin{aligned} \chi(B) &= \beta \gamma^2 \delta^2 + \beta^2 \gamma^2 \delta - \gamma^2 \delta - \gamma^2 \delta^2 \\ &= \gamma^2 \delta (\beta \delta + \beta^2 - 1 - \delta). \end{aligned}$$

Now: If $\beta = 1$, we get $\chi(B) = 0$. This takes care of 8 cases.

If $\delta = 1$, we get $\chi(A) = 0$. This takes care of 4 more cases.

Otherwise $\beta \neq 1$ and $\delta \neq 1$, then either $\beta = \delta$ or $\beta = \delta^2$.

If $\beta = \delta$, then $\chi(A) = 0$. This takes care of 4 cases.

If $\beta = \delta^2$, then $\chi(B) = 0$. This takes care of even 4 more cases.

The following table summarizes all of these cases.

Case	Order of $\chi(x)$	Order of $\chi(y)$	Order of $\chi(u)$	Order of $\chi(v)$	$\chi(A)$	$\chi(B)$
1	1	1	1	1	.	.
2	1	1	1	3		0
3	1	1	3	1	0	0
4	1	1	3	3		0
5	1	3	1	1	0	
6	1	3	1	3		0
7	1	3	3	1	0	
8	1	3	3	3		0
9	3	1	1	1	0	0
10	3	1	1	3		0
11	3	1	3	1	0	0
12	3	1	3	3		0
13	3	3	1	1	0	
14	3	3	1	3	0	
15	3	3	3	1	0	
16	3	3	3	3	0	

We next check that neither A nor B is periodic.

A is not periodic:

All the non-identity elements of A are candidates for a period of A. However,

none of them is a period :

$$(x)(xy) = x^2y \notin A, (y)(x^2) = x^2y \notin A, (x^2)(y) = x^2y \notin A,$$

$$(xy)(xy) = x^2y^2 \notin A, (y^2)(x^2) = x^2y^2 \notin A,$$

$$(xy^2)(x) = x^2y^2 \notin A, (x^2yv)(x^2yv) = xy^2v^2 \notin A,$$

$$\text{and } (x^2y^2v^2)(x^2y^2v^2) = xyv \notin A.$$

B is not periodic:

All the non-identity elements of B are candidates for a period of B. However, none of them is a period :

$$(u)(uv) = u^2v \notin B, (v)(u^2) = u^2v \notin B, (u^2)(v) = u^2v \notin B,$$

$$(uv)(uv) = u^2v^2 \notin B, (v^2)(u^2) = u^2v^2 \notin B,$$

$$(uv^2)(u) = u^2v^2 \notin B, (yu^2v)(yu^2v) = uy^2v^2 \notin B, \text{ and}$$

$$(y^2u^2v^2)(y^2u^2v^2) = uyv \notin B.$$

□

§2. In this section, we show that groups of order 3^4 have the Hajós 3-property. In what follows, we will need the following.

In some instances, we will make the use of the group ring ZG . To a subset A of G , we assign an element $\overline{A} = \sum_{a \in A} a$. As a consequence, we extend group

characters to ring homomorphisms from ZG to the set of complex numbers C . Also, we define $\text{Ann}(\overline{A}) = \{\chi : \chi(\overline{A}) = 0\}$.

3.2.1 Definition [11]

If A and A' are subsets of finite abelian group G such that for every subset B of G , whenever $AB = G$ is a factorization of G , then $A'B = G$ is also a factorization of G , we say that A is replaceable by A' .

3.2.2 Lemma [1]

The simulated subset $A = \{e, g, g^2, \dots, g^{r-2}, g^{r-1}u\}$, where $r > 2$, of a finite abelian group is replaceable by :

$$A' = \{e, g, g^2, \dots, g^{r-2}, g^{r-1}u^i\},$$

for each i .

3.2.3 Theorem [15]

Let G be a group, A , a subset of G , H and K subgroups of G . Then:

$\text{Ann}(H) \cap \text{Ann}(K) \subset \text{Ann}(A) \Leftrightarrow$ there exist subsets S_1 and S_2 of G such that $A = HS_1 \cup KS_2$, where the union is disjoint and the products (HS_1) and

3.2.4 Lemma [20]

Let A be a subset of a finite abelian p -group such that $|A| = p$, and $e \in A$. Let k be relatively prime to p . Then A is replaceable by A' , where $A' = \{e, g^k, g^{2k}, \dots, g^{(p-1)k}\}$, for any $g \in A - \{e\}$.

3.2.5 Proposition

If G is of type $(3, 3, 3, 3)$, then G has the Hajós 3-property.

Proof :

The proof will consist of several steps. Suppose G is factorized as $G = ABC$, where $|A| = 9$, and $|B| = |C| = 3$. We show that one of the factors A , B , or C must be periodic. Suppose that none of the factors is periodic. It is immediate that B and C must be simulated subsets of G . Say $B = \{e, x, x^2b\}$, and $C = \{e, y, y^2c\}$. By Lemma 3.2.2, we can replace B and C in the factorization $ABC = G$ by $\langle x \rangle$ and $\langle y \rangle$, respectively, to get the factorization $G = A \langle x \rangle \langle y \rangle$. It is clear that x and y are independent elements of G . Furthermore, by Lemma 2.20, A is a right transversal of $K = \langle x, y \rangle$ in G . We augment the set $\{x, y\}$ to $\{x, y, u, v\}$ to get a set of generators for G .

Then, A will consist of the following elements :

$$\begin{array}{lll} e k_{00} & v k_{01} & v^2 k_{02} \\ u k_{10} & uv k_{11} & uv^2 k_{12} \\ u^2 k_{20} & u^2 v k_{21} & u^2 v^2 k_{22} \end{array}$$

where $k_{ij} \in K$, and $k_{00} = e$.

From $G = ABC$, we get:

$$\overline{G} = \overline{A} (e + x + x^2b) \overline{C} \dots\dots\dots(1)$$

After replacement, we get :

$$\overline{G} = \overline{A} (e + x + x^2) \overline{C} \dots\dots\dots(2)$$

Subtracting (1) from (2), we get :

$$0 = \overline{A} (x^2 - x^2b) \overline{C}$$

So: $0 = \overline{A} (e - b) \overline{C}$

Treating C in a similar fashion, we get :

$$0 = \overline{A} (e - b) (e + y + y^2c) \dots\dots\dots(3)$$

$$0 = \overline{A} (e - b) (e + y + y^2) \dots\dots\dots(4)$$

Subtracting (3) from (4), we get:

$$0 = \overline{A} (e - b) (y^2 - y^2c)$$

So: $0 = \overline{A} (e - b) (e - c) \dots\dots\dots(5)$

Claim :

$$\text{Ann} (\langle b \rangle) \cap \text{Ann} (\langle c \rangle) \subset \text{Ann} (A)$$

Proof of claim :

Suppose that χ is a character of G with $\chi \in \text{Ann} (\langle b \rangle) \cap \text{Ann} (\langle c \rangle)$.

Applying χ to equation (5) , we get that :

$$0 = \chi (\overline{A}) (1 - \chi (b)) (1 - \chi (c))$$

Now if $\chi \notin \text{Ann} (A)$, then $\chi (\overline{A}) \neq 0$. Hence either $\chi (b) = 1$ or $\chi(c) = 1$.

Suppose $\chi(b) = 1$. Then $\chi(\langle b \rangle) = \chi(e + b + b^2)$

$$= \chi(e) + \chi(b) + \chi(b)^2$$

$$= 1 + 1 + 1^2$$

$$= 3 \neq 0.$$

Hence $\chi \notin \text{Ann} (\langle b \rangle)$. Similarly if $\chi(c) = 1$, then we would conclude that $\chi \notin \text{Ann} (\langle c \rangle)$. So in either case, we will have a contradiction. Thus the claim is proved.

Therefore by Theorem 3.2.3, there exist subsets U and V of G such that :

$$A = U \langle b \rangle \cup V \langle c \rangle \dots\dots\dots(6)$$

where the union is disjoint and the products $(U \langle b \rangle)$ and $(V \langle c \rangle)$ are direct.

To show that one of the factors A , B or C is periodic, we will have to distinguish several cases according to x, y, b, c .

(I) $b, c \in K = \langle x, y \rangle$:

In this case, it is clear that $BC = K$ is a factorization of K . And since $|B| = |C| = 3$, Rédei's Theorem gives that either B or C is periodic.

(II) x, y, b , and c are independent:

In this case, we may assume b and c to be u and v respectively. Thus, $B = \{e, x, x^2u\}$ and $C = \{e, y, y^2v\}$, and so:

$$A = U \langle u \rangle \cup V \langle v \rangle$$

Now, if either $U = \emptyset$ or $V = \emptyset$, then A is periodic. Suppose $U \neq \emptyset$, and $V \neq \emptyset$.

Now, $e \in A$. So either $e \in U$ or $e \in V$. We may assume that $e \in U$, since the roles of u and v are symmetric. Then $e \langle u \rangle = u^0 v^0 k_{00} \langle u \rangle \subset A$.

It follows that: $k_{00} = k_{10} = k_{20} = e \dots \dots \dots (*)$

Let $u^i v^j k_{ij} \in V \langle v \rangle$. Now, observe that $u^i v^j k_{ij} \langle v \rangle \subset A$, from which it follows $k_{i0} = k_{i1} = k_{i2}$. But from (*) above, we already have $k_{i0} = e$. Therefore $k_{i0} = k_{i1} = k_{i2} = e$. Hence, $u^i v^j k_{ij} \langle v \rangle = u^i v^0 k_{i0} \langle v \rangle = u^i e \langle v \rangle = u^i \langle v \rangle$. This implies that $u^i \in V \langle v \rangle$. But, $u^i \in U \langle u \rangle$. And this contradicts the fact that $U \langle v \rangle \cap V \langle v \rangle = \emptyset$. Hence, $V = \emptyset$. Therefore in this case A is periodic.

(III) x, y , and b are independent & $c \in \langle x, y, b \rangle$:

Since x, y , and b are independent elements, we may assume b to be u in the set of generators $\{x, y, u, v\}$. Hence, $c = x^\alpha y^\beta u^\gamma$, for some integers α, β , and γ , where $0 \leq \alpha, \beta, \gamma \leq 2$.

Now $B = \{e, x, x^2u\}$ and $C = \{e, y, y^2x^\alpha y^\beta u^\gamma\}$. We further split this into cases according to α, β and γ as follows:

1. $\alpha = \beta = \gamma = 0$:

In this case, $C = \{e, y, y^2\}$. So C is a subgroup, and hence periodic.

2. $\alpha = \beta = 0, \gamma \neq 0$:

In this case, $C = \{e, y, y^2u^\gamma\}$. By Lemma 3.2.2, we may assume $\gamma = 1$. Therefore, $C = \{e, y, y^2u\}$, and $A = U \langle u \rangle \cup V \langle u \rangle$. So, A is periodic.

3. $\alpha = 0, \beta \neq 0, \gamma = 0$:

In this case, $C = \{e, y, y^2y^\beta\}$. This is impossible.

4. $\alpha = 0, \beta \neq 0, \gamma \neq 0$:

In this case, $C = \{e, y, y^2y^\beta u^\gamma\}$. We may assume $\beta = 1$. Therefore, $C = \{e, y, u^\gamma\}$. Suppose $t \in U$. The product $t \langle u \rangle BC$ is direct, and $t \langle u \rangle BC = t \langle u, x \rangle C$. This cannot be, since $u^\gamma \in C$, and $u^\gamma \in \langle u, x \rangle$.

5. $\alpha \neq 0, \beta = 0, \gamma = 0$:

In this case, $C = \{e, y, y^2x^\alpha\}$. We may assume $\alpha = 1$. Therefore $C = \{e, y, y^2x\}$. Suppose $V \neq \emptyset$. So there is an element $t \in V$. Now, $t \langle x \rangle V \subset A$. The product $t \langle x \rangle B$ is direct, since the product ABC is direct. But we have, $x \in \langle x \rangle$ and $x \in B$. Thus, we get a contradiction.

6. $\alpha \neq 0, \beta = 0, \gamma \neq 0$:

In this case, $C = \{e, y, y^2 x^\alpha u^\gamma\}$. We may assume that $\gamma = 1$. Hence, $C = \{e, y, y^2 x^\alpha u\}$, and $A = U \langle u \rangle \cup V \langle x^\alpha u \rangle$.

Obviously, if either U or $V = \emptyset$, then A is periodic, and we are done.

Otherwise both U and V are non-empty. We show that this cannot happen. In

fact, we show that $V = \emptyset$. Say there is an element $u^i v^j k_{ij} \in V$, for some j , $0 \leq j \leq 2$.

Computing ABC , we will see that ABC will contain $u^i v^j k_{ij} \langle x^\alpha u \rangle BC$. Let us detail $\langle x^\alpha u \rangle B$; Direct computations give:

$$\langle x^\alpha u \rangle B = \{e, x^\alpha u, x^{2\alpha} u^2\} \{e, x, x^2 u\} =$$

e	$x^\alpha u$	$x^{2\alpha} u^2$
x	$x^{\alpha+1} u$	$x^{2\alpha+1} u^2$
$x^2 u$	$x^{\alpha+2} u^2$	$x^{2\alpha+2}$

Recall that the possible values for α are 0, 1 and 2 only. But we are assuming that $\alpha \neq 0$, which means that either $\alpha = 1$ or $\alpha = 2$. We show that these values for α are impossible by observing the following:

The presence of $x^2 u$ and $x^{\alpha+1} u$, gives that $\alpha \neq 1$, while the presence of $x^2 u$ and $x^\alpha u$ gives that $\alpha \neq 2$. Therefore, we conclude that V must be empty, and hence A is periodic.

7. $\alpha \neq 0, \beta \neq 0, \gamma = 0$:

In this case, $C = \{e, y, y^2 x^\alpha y^\beta\}$. We may assume $\beta = 1$.

Therefore, $C = \{e, y, x^\alpha\}$. Suppose $U \neq \emptyset$. So there is an element $t \in U$. Now, $t \langle u \rangle U \subset A$. The product $t \langle u \rangle BC$ is direct. It is clear that $\langle u \rangle BC = \langle u, x \rangle C$. This is a contradiction, since $x^\alpha \in \langle u, x \rangle$ and $x^\alpha \in C$.

8. $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$:

Here again, we may assume that $\beta = 1$. Therefore, $C = \{e, y, x^\alpha u^\gamma\}$. Say $t \in U$. Now, $t \langle u \rangle U \subset A$. The product $t \langle u \rangle BC$ is direct. It is clear that $\langle u \rangle BC = \langle u, x \rangle C$. This is a contradiction, since $x^\alpha u^\gamma \in \langle u, x \rangle$ and $x^\alpha u^\gamma \in C$. $\square$

3.2.6 Proposition :

If G is of type $(3^3, 3)$, then G has the Hajós 3-property.

Proof :

Suppose G is factorized as $G = ABC$, where $|A| = 9$, and $|B| = |C| = 3$. We show that one of the factors A , B or C is periodic. The proof, as in the previous Proposition, will consist of several steps and different cases, as follows :

(1) First of all, we observe that, we may assume that B is either cyclic or simulated :

If there is an element $b \in B \setminus \{e\}$ with $|b| \neq 3$, then by Lemma 3.2.3, we can replace B by $\{e, b, b^2\}$. On the other hand, if each element of B is of order 3, then B is of the form $\{e, b, b^2 u\}$. Similarly, we may assume that C is either cyclic or simulated. Hence :

$$\begin{aligned} B &= \{e, b, b^2\} \text{ or } B = \{e, b, b^2 u\} \\ C &= \{e, c, c^2\} \text{ or } C = \{e, c, c^2 v\} \end{aligned}$$

(2) We distinguish three cases, as follows:

	B	C
I	cyclic	cyclic
II	simulated	simulated
III	cyclic	simulated

Case (I) Both B and C are cyclic:

Assume that both B and C are cyclic. So $B = \{e, b, b^2\}$, and $C = \{e, c, c^2\}$. Say $G = \langle x \rangle \times \langle y \rangle$, where $|x| = 27$, and $|y| = 3$. Now, if $|b| = 3$, or $|c| = 3$, then we are done. Therefore, we assume that $|b| \geq 9$, and $|c| \geq 9$. Let $b = x^\alpha y^\beta$ and $c = x^\gamma y^\delta$, where $0 \leq \alpha, \gamma \leq 26$, and $0 \leq \beta, \delta \leq 2$. Now,

$$G = ABC = A \{e, b, b^2\} \{e, c, c^2\}.$$

We use a similar argument, as in the previous Proposition, using characters to get the following equations:

$$\text{First, we have: } G = ABC = A \{e, b, b^2\} C \dots \dots \dots (1)$$

If we multiply both sides of this equation by b, we get:

$$G = A \{b, b^2, b^3\} C \dots \dots \dots (2)$$

Now from (1), we get:

$$\overline{G} = \overline{A} (e + b + b^2) \overline{C} \dots\dots\dots(3)$$

and from (2), we get:

$$\overline{G} = \overline{A} (b + b^2 + b^3) \overline{C} \dots\dots\dots(4)$$

Subtracting (4) from (3), we get:

$$0 = \overline{A} (e - b^3) \overline{C} \dots\dots\dots(5)$$

Treating C in a similar way, we get the following equations:

$$\overline{G} = \overline{A} (e - b^3) (e + c + c^2) \dots\dots\dots(6)$$

$$\overline{G} = \overline{A} (e - b^3) (c + c^2 + c^3) \dots\dots\dots(7)$$

Subtracting (7) from (6), we get:

$$0 = \overline{A} (e - b^3) (e - c^3) \dots\dots\dots(8)$$

Now, an argument similar to that used in Theorem 3.2.5, will show that

$A = U \langle b^3 \rangle \cup V \langle c^3 \rangle$, for some subsets U and V of G.

Since, $b^3 = x^{3\alpha}$ and $c^3 = x^{3\gamma}$, we get that $A = U \langle x^{3\alpha} \rangle \cup V \langle x^{3\gamma} \rangle$.

Consider the lattice of subgroups of $\langle x \rangle$.

It looks as follows:

$$\begin{array}{c}
.<x> \\
| \\
.<x^3> \\
| \\
.<x^9> \\
| \\
.<e>
\end{array}$$

Hence, either $\langle x^{3\alpha} \rangle \subset \langle x^{3\gamma} \rangle$ or $\langle x^{3\gamma} \rangle \subset \langle x^{3\alpha} \rangle$.

In the first case, $x^{3\alpha}$ is a period for A, while in the latter case $x^{3\gamma}$ is a period for A. Therefore, in this case, A is periodic.

Case (II) Both B and C are simulated:

Assume that both B and C are simulated. So $B = \{e, b, b^2u\}$, and $C = \{e, c, c^2v\}$, where $|b| = |c| = 3$. Now, if either $u = e$, or $v = e$, then we are done. So, we assume that $|u| = |v| = 3$. Note that the elements of G of order 3 (along with the identity element e) form a unique subgroup $K = \langle x^9, y \rangle$ of G. Now, $K = \{e, b, b^2u\} \{e, c, c^2v\}$ is a factorization of K. Rédei's Theorem gives that one of the factors B or C is periodic.

Case (III) B is cyclic and C is simulated:

Assume that B is cyclic, and C is simulated.

So, that $B = \{e, b, b^2\}$, and $C = \{e, c, c^2v\}$. Therefore:

$$G = A \{e, b, b^2\} \{e, c, c^2 v\} \dots \dots \dots (9)$$

Next, we derive the equation,

$$0 = \overline{A} (e - b^3) (e - v) \dots \dots \dots (10)$$

as follows:

$G = A \{e, b, b^2\} C$, gives:

$$\overline{G} = \overline{A} \{e + b + b^2\} \overline{C} \dots \dots \dots (11)$$

$G = A \{b, b^2, b^3\} C$, gives:

$$\overline{G} = \overline{A} \{b + b^2 + b^3\} \overline{C} \dots \dots \dots (12)$$

Subtracting (12) from (11), gives:

$$0 = \overline{A} (e - b^3) \overline{C} \dots \dots \dots (13)$$

We also, have:

$$\overline{G} = \overline{A} (e - b^3) (e + c + c^2) \dots \dots \dots (14)$$

and:

$$\overline{G} = \overline{A} (e - b^3) (e + c + c^2 v) \dots \dots \dots (15)$$

Finally, subtracting (15) from (14), gives:

$$0 = \overline{A} (e - b^3) (c^2 - c^2 v) \dots \dots \dots (16)$$

This implies that, $0 = \overline{A} (e - b^3) (e - v)$, which is equation (10).

From this, it follows that there are subsets U and V of G such that $A = U \langle b^3 \rangle \cup V \langle v \rangle$. Now, if either U or V is empty, we get that A is periodic, and we are done. So assume that both U and V are nonempty. If $|b| = 27$, then, $|b^3| = 9$, and so $|U \langle b^3 \rangle| \geq 9$. Hence, in this case, V is empty. Thus, we may assume that $|b| = 9$. Then, $\langle b^3 \rangle = \langle x^9 \rangle$.

If $v \in \langle x^9 \rangle$, then $\langle v \rangle = \langle x^9 \rangle$, and then we get that A is periodic. Thus, we may assume that $v \notin \langle x^9 \rangle$. We now recall that since both B and C are simulated, by Lemma 3.2.2, we also have the factorization:

$$G = A \{e, b, b^2\} \{e, c, c^2\}.$$

We compute, $A \{e, c, c^2\}$:

$$\begin{aligned} A \{e, c, c^2\} &= A \langle c \rangle = (U \langle x^9 \rangle \cup V \langle v \rangle) \langle c \rangle \\ &= U \langle x^9 \rangle \langle c \rangle \cup V \langle v \rangle \langle c \rangle. \end{aligned}$$

Since the product $AB \langle c \rangle$ is direct, so are the products $\langle x^9 \rangle \langle c \rangle$ and $\langle v \rangle \langle c \rangle$, and they must both equal to the subgroup K of G . Now, observe that the set $\{x^9, v\}$ also generates the subgroup K . There are two choices for c . Namely, either $c = x^9 v$ or $c = x^9 v^2$. So,
either (a) $C = \{e, x^9 v, x^{18} v^2 v\}$
or (b) $C = \{e, x^9 v^2, x^{18} v v\}$.

If $U \neq \emptyset$, then there is an element s in U . So $s \langle x^9 \rangle C$ is a part of ABC . We compute $\langle x^9 \rangle C$, in both cases :

Case(a) :

$\langle x^9 \rangle C = \{e, x^9 x^{18}\} \{e, x^9 v, x^{18}\}$. This is impossible, since we have $x^{18} \in \langle x^9 \rangle$ and $x^{18} \in C$ also.

Case (b) :

$\langle x^9 \rangle C = \{e, x^9, x^{18}\} \{e, x^9 v^2, x^{18} v^2\}$
 $= \{e, x^9, x^{18}, x^9 v^2, x^{18} v^2, v^2, x^{18} v^2, v^2, x^9 v^2\}$.

This too, is obviously impossible. □

3.2.7 Proposition:

If G is of type $(3^2, 3^2)$, then G has the Hajós 3-property.

Proof:

Suppose G is factorized as $G = ABC$, where $|A| = 9$, and $|B| = |C| = 3$. We show that one of the factors A, B or C is periodic. Let $G = \langle x \rangle \times \langle y \rangle$, where $|x| = |y| = 9$. We may assume that B is either cyclic, or simulated. Namely, either $B = \{e, b, b^2\}$, where $|b| = 9$, or $B = \{e, b, b^2 u\}$, where $|b| = |u| = 3$. Similarly, C is either cyclic, or simulated. So either $C = \{e, c, c^2\}$, where $|c| = 9$, or $C = \{e, c, c^2 v\}$, where $|c| = |v| = 3$. We tabulate all the possibilities for B and C below :

	B	C
I	cyclic	cyclic
II	simulated	simulated
III	cyclic	simulated

Case (I) Both B and C are cyclic:

Assume that both B and C are cyclic. So $B = \{e, b, b^2\}$, and $C = \{e, c, c^2\}$. Now, if $|b| = 3$, or $|c| = 3$, then we are done. So, we assume that $|b| = |c| = 9$. Let $K = \langle b \rangle$. Now, if $c \in K$, then $BC = K$ is a factorization of K and so by Rédei's Theorem, we get that either B or C is a subgroup, and hence periodic. Therefore, we may assume $c \notin K$, which implies that the elements b and c are independent. We may choose x and y to be b and c respectively. So $B = \{e, x, x^2\}$, and $C = \{e, y, y^2\}$.

Claim

The factorization $G = A \{e, x, x^2\} \{e, y, y^2\}$, implies that:

$$\text{either } A \subset \langle x^3, y \rangle \text{ or } A \subset \langle x, y^3 \rangle.$$

Proof of claim

Let $a_1, a_2 \in A$. Consider $a_1 a_2^{-1} = x^\alpha y^\beta$. Suppose 3 does not divide α and 3 does not divide β . Now $a_1 \cdot e \cdot e = a_1 = a_2 x^\alpha y^\beta$.

But we also have the factorization $G = A \{e, x^\alpha, x^{2\alpha}\} \{e, y^\beta, y^{2\beta}\}$ of G. Hence, we must have $x^\alpha = e$ and $y^\beta = e$. Therefore, 9 divides α and 9 divides β , from

which it follows that 3 divides α and 3 divides β . Therefore, the assumption that 3 does not divide both α and β leads to a contradiction. Consequently, we conclude that we must have either 3 divides α or 3 divides β . Now if it happens that 3 divides α for each $a \in A$, then $A \subset \langle x^3, y \rangle$. Similarly, if 3 divides β for each $a \in A$, then $A \subset \langle x, y^3 \rangle$. Thus, we may assume that there are elements :

$a_1 = x^{\alpha_1} y^{\beta_1} \in A$, where 3 divides α_1 , and 3 does not divide β_1 .

$a_2 = x^{\alpha_2} y^{\beta_2} \in A$, where 3 does not divide α_2 , and 3 divides β_2 .

But, then 3 does not divide $\alpha_1 - \alpha_2$ and 3 does not divide $\beta_1 - \beta_2$. Now, observe that $a_1 a_2^{-1} = x^{\alpha_1 - \alpha_2} y^{\beta_1 - \beta_2}$. And this is a contradiction as was shown before. This ends the proof of the claim. Next, let $M = \langle x^3, y \rangle$, and let $N = \langle x, y^3 \rangle$.

Now, if $A \subset \langle x^3, y \rangle = M$, then clearly $A \setminus \{e, y, y^2\}$ is a factorization of M , because, $\{e, y, y^2\} \subset M$, and $|A \setminus \{e, y, y^2\}| = |M| = 27$.

From this, we obtain the equation:

$$0 = \overline{A} (e - y^3) \dots \dots \dots (19)$$

which in turn, implies that A is periodic, with period y^3 .

Similarly, if $A \subset \langle x, y^3 \rangle = N$, then we obtain the equation:

$$0 = \overline{A} (e - x^3) \dots \dots \dots (20)$$

which gives that A is periodic with period x^3 .

Case (II) Both B and C are simulated:

Assume that both B and C are simulated. So $B = \{e, b, b^2u\}$, and $C = \{e, c, c^2v\}$, where $|b| = |c| = |u| = |v| = 3$. The elements of order 3, (along with the identity, of course) form a unique subgroup, say L. In fact, $L = \langle x^3, y^3 \rangle$. Now, $BC = L$ is a factorization of L, and hence by Rédei's Theorem, we get that either B or C is periodic.

Case (III) B is cyclic and C is simulated:

Assume that B is cyclic and C is simulated. So $B = \{e, b, b^2\}$, and $C = \{e, c, c^2v\}$.

From the factorization $G = ABC$, we obtain the equation:

$$0 = \overline{A} (e - b^2) (e - v) \dots\dots\dots(21)$$

from which it follows that:

$$A = U \langle b^3 \rangle \cup V \langle v \rangle \dots\dots\dots(22)$$

Now, if $v \in \langle b^3 \rangle$, then $\langle b^3 \rangle = \langle v \rangle$, and so A is periodic with period v. Therefore, we may assume that b^3 and v are independent elements of the subgroup $L = \langle x^3, y^3 \rangle$. Now, since C is simulated, we also have the factorization $G = A\{e, b, b^2\} \{e, c, c^2\}$.

We compute:

$$A \{e, c, c^2\} = (U \langle b^3 \rangle \cup V \langle v \rangle) \langle c \rangle$$

$$= U \langle b^3 \rangle \langle c \rangle \cup V \langle v \rangle \langle c \rangle \dots (23)$$

Equation (23) gives that both $\langle b^3 \rangle \langle c \rangle$ and $\langle v \rangle \langle c \rangle$ must be L. So c is independent from b^3 and v . We have two choices for c . Namely, b^3v and b^3v^2 . Thus:

$$\begin{aligned} \text{either (a) } C &= \{e, b^3v, b^6v^2v\} = \{e, b^3v, b^6\} \\ \text{or (b) } C &= \{e, b^3v^2, b^6vv\} = \{e, b^3v^2, b^6v^2\} \end{aligned}$$

Note that if $U = \emptyset$, then we are done. So assume that $U \neq \emptyset$, then there is an element $t \in U$. We compute $t \langle b^3 \rangle C$, in each case:

Case (a) :

$$t \langle b^3 \rangle C = t \{e, b^3, b^6\} \{e, b^3v, b^6\}$$

In this case, the presence of b^6 (twice) gives us a contradiction.

Case (b) :

$$t \langle b^3 \rangle C = t \{e, b^3, b^6\} \{e, b^3v^2, b^6v^2\}$$

$$= t \{e, b^3, b^6, b^3v^2, b^6v^2, v^2, b^6v^2, v^2, b^3v^2\}.$$

In this case, again we also have a contradiction, since b^3v^2 is represented twice. □

3.2.8 Proposition

If G is of type $(3^2, 3, 3)$, then G has the Hajós 3-property.

Proof:

Suppose G is factorized as $G = ABC$, where $|A| = 9$, and $|B| = |C| = 3$.

We show that one of the factors A, B or C is periodic. Let $G = \langle x \rangle \times \langle y \rangle \times \langle z \rangle$, where $|x| = 9$, and $|y| = |z| = 3$. We may assume that B is either cyclic, or simulated. Namely, either $B = \{e, b, b^2\}$, where $|b| = 9$, or $B = \{e, b, b^2u\}$, where $|b| = |u| = 3$. Similarly, C is either cyclic, or simulated. So either $C = \{e, c, c^2\}$, where $|c| = 9$, or $C = \{e, c, c^2v\}$, where $|c| = |v| = 3$. We tabulate all the possibilities for B and C below :

	B	C
I	cyclic	cyclic
II	simulated	simulated
III	cyclic	simulated

Case (I) Both B and C are cyclic:

Assume that both B and C are cyclic. So $B = \{e, b, b^2\}$, and $C = \{e, c, c^2\}$. We may assume $|b| = |c| = 9$. Let $b = x^\alpha y^\beta z^\gamma$ and $c = x^\delta y^\epsilon x^\xi$, where $0 \leq \alpha, \delta \leq 8$, and $0 \leq \beta, \gamma, \epsilon, \xi \leq 2$. Now from the factorization $G = ABC = \{e, b, b^2\} \{e, c, c^2\}$, we get the equation:

$$0 = \overline{A} (e - b^3) (e - c^3) \dots \dots \dots (24)$$

Hence, from (24), we obtain:

$$A = U \langle b^3 \rangle \cup V \langle c^3 \rangle \dots \dots \dots (25)$$

Observe that $b^3 = x^{3\alpha}$ and $c^3 = x^{3\delta}$.

Next we consider the lattice of subgroups of the group $\langle x \rangle$.

It looks as follows:

$$\begin{array}{c} \langle x \rangle \\ | \\ \langle x^3 \rangle \\ | \\ \langle e \rangle \end{array}$$

It is clear that $\langle x^3 \rangle = \langle b^3 \rangle = \langle c^3 \rangle$, and hence A is periodic with period x^3 .

Case (II) Both B and C are simulated:

Assume that both B and C are simulated. Hence, $B = \{e, b, b^2u\}$

and $C = \{e, c, c^2v\}$, where $|b| = |c| = |u| = |v| = 3$. Now, since

$G = A\{e, b, b^2\}\{e, c, c^2\} = A\langle b \rangle \langle c \rangle = A\langle b, c \rangle$ is also a factorization of G ,

we find that b and c are independent elements. If it happens that $u, v \in \langle b, c \rangle$,

then $BC = \langle b, c \rangle$ is a factorization of BC , and since $|B| = |C| = 3$, Rédei's

Theorem gives that either B or C is periodic. So in this case we are done.

Therefore, we assume that b, c and u are independent elements and

$v = b^\alpha c^\beta u^\gamma$, where $0 \leq \alpha, \beta, \gamma \leq 2$. Now $B = \{e, b, b^2u\}$ and $C = \{e, c, c^2b^\alpha c^\beta u^\gamma\}$ and so:

$$A = U\langle u \rangle \cup V\langle b^\alpha c^\beta u^\gamma \rangle \dots\dots\dots(26)$$

We further divide this into subcases as follows:

$$\underline{\alpha = \beta = \gamma = 0 :}$$

In this case, $C = \{e, c, c^2\}$. So C is a subgroup, and hence periodic.

$$\underline{\alpha \neq 0, \beta = 0, \gamma = 0 :}$$

In this case, $C = \{e, c, c^2 b^\alpha\}$, and we may assume that $\alpha = 1$.

Now $A = U \langle u \rangle \cup V \langle b^\alpha \rangle$. In this case, $V = \emptyset$, for suppose not. Then there must exist an element, say $t \in V$. But note that since $t \langle b \rangle B$ is part of the factorization ABC , we get that $t \langle b \rangle B \subset ABC$, from which we get a contradiction, because then, $b \in \langle b \rangle$, and $b \in B$.

$$\underline{\alpha \neq 0, \beta = 0, \gamma \neq 0 :}$$

We may assume that $\gamma = 1$. Hence, $C = \{e, c, c^2 b^\alpha u\}$, and therefore, $A = U \langle u \rangle \cup V \langle b^\alpha u \rangle$. Now, if either U or $V = \emptyset$, we of course, get that A is periodic, and we are done. Otherwise both U and V are both non-empty. We show that this cannot happen. In fact, we show that $V = \emptyset$. If this is not the case, then there is an element $t \in V$. Now observe that $t \langle b^\alpha u \rangle B$ is a part of ABC . We compute $\langle b^\alpha u \rangle B$. Computations give:

$$\langle b^\alpha u \rangle B = \{e, b^\alpha u, b^{2\alpha} u^2\} \{e, b, b^2 u\} =$$

$$\begin{array}{lll} e & b^\alpha u & b^{2\alpha} u^2 \\ b & b^{\alpha+1} u & b^{2\alpha+1} u^2 \\ b^2 u & b^{\alpha+2} u^2 & b \end{array}$$

Recall that the possible values for α are 0, 1 and 2 only. But we are assuming that $\alpha \neq 0$, this means that either $\alpha = 1$ or $\alpha = 2$.

We show that these values for α are impossible by observing the following :
The presence of $b^2 u$ and $b^\alpha u$, gives that $\alpha \neq 2$. Therefore, the only remaining possibility is that $\alpha = 1$. But this last value is ruled out by the presence of $b^{2\alpha+1} u$ and $b^{\alpha+2} u$.

$\alpha \neq 0, \beta \neq 0, \gamma \neq 0$:

Here again, we may assume that $\beta = 1$. So $C = \{e, c, c^2 b^\alpha c u^\gamma\}$. and therefore $A = U \langle u \rangle \cup V \langle b^\alpha c^\beta u^\gamma \rangle$. We show that in this case, $U = \emptyset$. For, suppose not. Then, there is an element $t \in U$. Note that $t \langle u \rangle B$ is a part of ABC . We compute $\langle u \rangle BC$. Easy computations give:

$$\begin{aligned} \langle u \rangle BC &= \langle b, u \rangle \{e, c, c^2 b^\alpha c u^\gamma\} \\ &= \langle b, u \rangle \{e, c, b^\alpha u^\gamma\} \end{aligned}$$

But this is a contradiction, since we have, $b^\alpha u^\gamma \in \langle b, u \rangle$ and $b^\alpha u^\gamma \in C$ also.

Case (III) B is cyclic and C is simulated:

Assume that B is cyclic and C is simulated.

Therefore, $B = \{e, b, b^2\}$ and $C = \{e, c, c^2 v\}$. An argument analogous to that in the previous cases, gives us the equation :

$$A = U \langle b^3 \rangle \cup V \langle v \rangle \dots\dots\dots(27)$$

If $v \in \langle b^3 \rangle$, then $\langle v \rangle = \langle b^3 \rangle$ and so A is periodic in this case with period b^3 . Therefore, we assume that v and b^3 are independent elements. Suppose $U \neq \emptyset$. Say, $t \in U$. The product $t \langle b^3 \rangle C$ is direct since it is part of the product

ABC. Now, if $c \in \langle b^3 \rangle$, then $c = b^{3i}$ for some i , $1 \leq i \leq 2$. This is a contradiction, since $c \in \langle b^3 \rangle$, and $c \in C$. Thus, we may assume that $c \notin \langle b^3 \rangle$. We consider two cases according to whether $v \in \langle b^3, c \rangle$ or $v \notin \langle b^3, c \rangle$. If $v \in \langle b^3, c \rangle$, then $v = b^{3\alpha} c^\beta$. Now, if $\beta = 0$, then A is periodic with period b^3 . Otherwise, $\beta \neq 0$. We may assume that $\beta = 1$. So, $C = \{e, c, c^2 b^{3\alpha} c\} = \{e, c, b^{3\alpha}\}$. Say $t \in U$, then $t \langle b^3 \rangle C$ is direct since it is part of the product ABC . This leads to contradiction, since $b^{3\alpha} \in \langle b^3 \rangle$, and $b^{3\alpha} \in C$. Thus, we may assume that b^3, c and v are pairwise independent. From the factorization $G = ABC$, it follows that:

$$0 = \overline{A} \overline{C} (e - b^3) \dots\dots\dots(28)$$

and so AC is periodic with period b^3 . Therefore, $V \langle v \rangle C$ is periodic with period b^3 . Observe that :

$$\begin{aligned} \langle v \rangle C &= \langle v \rangle \{e, c, c^2 v\} \\ &= \langle v \rangle e \cup \langle v \rangle c \cup \langle v \rangle c^2 v \\ &= \langle v \rangle \{e, c, c^2\} \\ &= \langle v \rangle \langle c \rangle \\ &= \langle v, c \rangle. \end{aligned}$$

Let $t \in V$. Now, $\langle b^3 \rangle \subset V \langle v, c \rangle$. Now, $V \langle v, c \rangle$ is a union of cosets modulo $\langle v, c \rangle$ and the elements of $\langle b^3 \rangle$ are incongruent modulo $\langle v, c \rangle$. Therefore, $27 = |\langle b^3 \rangle \langle v, c \rangle| \leq |V \langle v, c \rangle| = 9|V|$. Thus, $|V| \geq 3$. And this contradicts our assumption that U is also not empty. $\square$

§3. In this section, we prove a Theorem which generalizes the results in Theorems 3.2.5 - 3.2.8, about groups of order 3^4 to groups of order p^4 , where p is any prime. cf. [21].

3.3.1 Lemma [1]

Let $G = ABC$ be a factorization of the finite abelian group G , where the factors B and C are one of the following :

$$\begin{aligned} B &= \{e, b, b^2, \dots, b^{r-1}\}, & C &= \{e, c, c^2, \dots, c^{s-1}\}, \\ B &= \{e, b, b^2, \dots, b^{r-2}, b^{r-1}u\}, & C &= \{e, c, c^2, \dots, c^{s-2}, c^{s-1}v\}, \\ B &= \{e, b, b^2, \dots, b^{r-1}\}, & C &= \{e, c, c^2, \dots, c^{s-2}, c^{s-1}v\}. \end{aligned}$$

Then, there exist subsets U, V of G such that A can be represented in the following forms respectively :

$A = U \langle b^r \rangle \cup V \langle c^s \rangle$, $A = U \langle u \rangle \cup V \langle v \rangle$, $A = U \langle b^r \rangle \cup V \langle v \rangle$. where the unions are disjoint and the products are direct.

3.3.2 Lemma [1]

The simulated subset $A = \{e, g, g^2, \dots, g^{r-2}, g^{r-1}u\}$, where $r > 2$ of a finite abelian group is replaceable by :

$$A' = \{e, g, g^2, \dots, g^{i-1}, g^i u, g^{i+1}, \dots, g^{r-1}\},$$

for each i , $1 \leq i \leq r-1$.

3.3.3 Lemma [1]

Let G be a finite abelian group and let $A = \{e, g, g^r, \dots, g^{r-1}\}$ be a cyclic subset of G . Then A is replaceable by :

$$A' = \{e, g^i, g^{2i}, \dots, g^{(r-1)i}\},$$

for each i , where i is relatively prime to r .

3.3.4 Theorem

Let p be a prime, and G an abelian group of order p^4 . Let $G = ABC$ be a factorization of G , where, $|A| = p^2$, $|B| = |C| = p$. Furthermore, assume that the factors B and C are cyclic or simulated. Then one of A , B or C is periodic.

Proof :

First of all, we note that G can be one of the following types :

1. (p^4) .
2. (p^3, p) .
3. (p^2, p^2) .
4. (p^2, p, p) .
5. (p, p, p, p) .

We further, distinguish each of the cases in 1 - 5, as follows :

- (a) B, C are both cyclic.
- (b) B, C are both simulated.
- (c) B is cyclic and C is simulated.

Case 1 : G is of type (p^4) .

This case is settled by Theorem 2.6.

Case 2 : G is of type (p^3, p) .

Let $G = \langle x \rangle \times \langle y \rangle$, where $|x| = p^3$ and $|y| = p$.

(a) Both B and C are cyclic.

In this case $B = \{e, b, b^2, \dots, b^{p-1}\}$, and

$$C = \{e, c, c^2, \dots, c^{p-1}\}.$$

If $b^p = e$, or $c^p = e$, then we are done. So we may assume that $|b| \geq p^2$, and $|c| \geq p^2$. Now, by Lemma 3.3.1, there exist subsets U, V of G such that $A = U\langle b^p \rangle \cup V\langle c^p \rangle$. Let $b = x^\alpha y^\beta$ and $c = x^\gamma y^\delta$. Then, $b^p = x^{p\alpha}$ and $c^p = x^{p\gamma}$. Also, the subgroups of $\langle x^p \rangle$ form a chain. Hence, $\langle x^{p^2} \rangle \subset \langle x^{p\alpha} \rangle \cap \langle x^{p\gamma} \rangle = \langle b^p \rangle \cap \langle c^p \rangle$. Thus, x^{p^2} is a period of A .

(b) Both B and C are simulated.

In this case $B = \{e, b, b^2, \dots, b^{p-2}, b^{p-1}u\}$, and

$$C = \{e, c, c^2, \dots, c^{p-2}, c^{p-1}v\},$$

Here, $|b| = |c| = p$.

Now, if either $u = e$ or $v = e$, we are done. So, we may assume that $|u| \geq p$, and $|v| \geq p$. By Lemma 3.2.2, we may assume that $|u| = |v| = p$. The elements of G of order p generate the subgroup $K = \langle x^{p^2}, y \rangle$ of order p^2 . Also, $K = BC$ is a factorization of K . Hence, by Rédei's Theorem, one of the factor B or C is a subgroup of K .

(c) B is cyclic and C is simulated.

In this case $B = \{e, b, b^2, \dots, b^{p-1}\}$, and

$$C = \{e, c, c^2, \dots, c^{p-2}, c^{p-1}v\},$$

Here, we may assume that $|b| \geq p^2$, and $|c| = |v| = p$. By Lemma 3.3.1, $A =$

$U\langle b^p \rangle \cup V\langle v \rangle$, for some subsets U, V of G . Now, if either $U = \emptyset$ or $V = \emptyset$, then we are done. So, we may assume that $U \neq \emptyset$ and $V \neq \emptyset$. Furthermore, if $|b| = p^3$, then $|U||\langle b^p \rangle| \geq p^2$, and hence $V = \emptyset$. So, we assume that $|b| = p^2$. Therefore, $\langle b^p \rangle = \langle x^{p^2} \rangle$.

If $v \in \langle x^{p^2} \rangle$, then $v = x^{p^2}$, and so A is periodic.

If $v \notin \langle x^{p^2} \rangle$, then replace C by $\langle c \rangle$ in the factorization $G = ABC$. So now, G has the factorization $G = AB\langle c \rangle$.

Consider $A\langle c \rangle$:

$$A\langle c \rangle = (U\langle x^{p^2} \rangle \cup V\langle v \rangle)\langle c \rangle = U\langle x^{p^2} \rangle\langle c \rangle \cup V\langle v \rangle\langle c \rangle.$$

The product $AB\langle c \rangle$ is direct, so the products $\langle x^{p^2} \rangle\langle c \rangle$ and $\langle v \rangle\langle c \rangle$ must also be direct. Furthermore, $\langle x^{p^2} \rangle\langle c \rangle = K = \langle v \rangle\langle c \rangle$. Note that x^{p^2} and v generate K . Now, by Lemma 3.3.2, we can replace c in C by c^i for each i , $1 \leq i \leq p-1$. Now, there are $p-1$ choices for c . Namely, $x^{p^2} v^i$, $1 \leq i \leq p-1$. Now:

$$C = \{e, x^{p^2} v^1, x^{2p^2} v^{2i}, \dots, x^{(p-2)p^2} v^{(p-2)i}, x^{(p-1)p^2} v^{(p-1)i+1}\}.$$

Now, for j , $1 \leq j \leq p-2$, the congruence $(j+1)i \equiv 1 \pmod{p}$ is solvable.

So, there exists an integer j such that $ji \equiv (p-1)i+1 \pmod{p}$. Let $t \in U$.

The product $t\langle x^{p^2} \rangle C$ is direct, since it is part of ABC .

Computing, $\langle x^{p^2} \rangle C$, we see that the sets

$$\langle x^{p^2} \rangle x^{jp^2} v^{ji} = \langle x^{p^2} \rangle v^{ji} \quad \text{and}$$

$$\langle x^{p^2} \rangle x^{(p-1)p^2} v^{(p-1)i+1} = \langle x^{p^2} \rangle v^{(p-1)i+1}$$

are parts of $\langle x^{p^2} \rangle \subset C$, and they contain the same elements. This contradicts the factorization.

Case 3: G is of type (p^2, p^2) .

Let $G = \langle x \rangle \times \langle y \rangle$, where $|x| = |y| = p^2$.

(a) Both B and C are cyclic.

In this case $B = \{e, b, b^2, \dots, b^{p^2-1}\}$, and

$$C = \{e, c, c^2, \dots, c^{p^2-1}\}.$$

If $b^{p^2} = e$, or $c^{p^2} = e$, then we are done. So we may assume that

$|b| = p^2$, and $|c| = p^2$. Let $L = \langle b \rangle$.

If $c \in L$, then $L = BC$ is a factorization of L , and so by Rédei's Theorem, one of the factor B or C is a subgroup of L .

So, we may assume that $c \notin L$. We may choose x, y to be b, c respectively. So, $B = \{e, x, x^2, \dots, x^{p^2-1}\}$, and $C = \{e, y, y^2, \dots, y^{p^2-1}\}$.

We show that if $G = ABC$ is a normalized factorization of G , then,

$$\text{either } A \subset \langle x^{p^2}, y \rangle \text{ or } A \subset \langle x, y^{p^2} \rangle \dots\dots\dots (*)$$

We show $(*)$, as follows :

Suppose a and $a' \in A$. Let $a' a^{-1} = x^\alpha y^\beta$.

If p does not divide α and p does not divide β , then from $a' = a x^\alpha y^\beta$, we get a contradiction to the factorization $G = A B' C'$ of G , where :

$$B' = \{e, x^\alpha, x^{2\alpha}, \dots, x^{(p-1)\alpha}\}, \text{ and}$$

$$C' = \{e, x^\beta, x^{2\beta}, \dots, x^{(p-1)\beta}\}.$$

Note that this replacement is possible, by Lemma 3.3.3.

If we let $a = x^\alpha y^\beta \in A$, then the previous argument with $a' = e$, gives us that either p divides α or p divides β .

Now, if p divides α for each $a \in A$, then $A \subset \langle x^p, y \rangle$.

Similarly, if p divides β for each $a \in A$, then $A \subset \langle x, y^p \rangle$.

Thus, we may assume that there exist $a = x^\alpha y^\beta \in A$, $a' = x^{\alpha'} y^{\beta'} \in A$, such that: p divides α , p does not divide β ,

and: p does not divide α' , p divides β' .

But then p does not divide $\alpha' - \alpha$, $\beta' - \beta$.

Hence, $a' a^{-1} = x^{\alpha' - \alpha} y^{\beta' - \beta}$, and this is a contradiction.

Now, let $M = \langle x^p, y \rangle$, and $N = \langle x, y^p \rangle$.

If $A \subset \langle x^p, y \rangle = M$, then $M = AC$ is a factorization of M , and so,

$$\overline{A} (e - y^p) = 0.$$

Hence, A is periodic with period y^p .

If $A \subset \langle x, y^p \rangle = N$, then $N = AB$ is a factorization of N , and so,

$$\overline{A} (e - x^p) = 0.$$

Hence, A is periodic with period x^p .

(b) Both B and C are simulated.

In this case $B = \{e, b, b^2, \dots, b^{p-2}, b^{p-1}u\}$, and

$$C = \{e, c, c^2, \dots, c^{p-2}, c^{p-1}v\},$$

Here, $|b| = |c| = p$, and we may assume $|u| = |v| = p$. The elements of G of order p

generate the subgroup $K = \langle x^p, y^p \rangle$ of order p^2 . Now, $K = BC$ is a factorization of K . Hence, by Rédei's Theorem, one of the factor B and C is a subgroup of K .

(c) B is cyclic and C is simulated.

In this case $B = \{e, b, b^2, \dots, b^{p-1}\}$, and

$$C = \{e, c, c^2, \dots, c^{p-2}, c^{p-1}v\},$$

Here, we may assume that $|b| \geq p^2$, and $|c| = |v| = p$. By Lemma 3.3.1, $A = U\langle b^p \rangle \cup V\langle v \rangle$, for some subset U, V of G . Now, if either $U = \emptyset$ or $V = \emptyset$, then we are done. So, we may assume that $U \neq \emptyset$ and $V \neq \emptyset$. Furthermore, if $v \in \langle b^p \rangle$, then $\langle v \rangle = \langle b^p \rangle$, and so A is periodic. Thus, we may assume that $v \notin \langle b^p \rangle$. Now, b^p, v generate $K = \langle x^p, y^p \rangle$. In the factorization $G = ABC$, replace C by $\langle c \rangle$ to get the factorization $G = AB\langle c \rangle$. We compute $A\langle c \rangle$:

$$A\langle c \rangle = (U\langle b^p \rangle \cup V\langle v \rangle)\langle c \rangle = U\langle b^p \rangle\langle c \rangle \cup V\langle v \rangle\langle c \rangle.$$

Now, both $\langle b^p \rangle\langle c \rangle$ and $\langle v \rangle\langle c \rangle$ must be direct and equal to K . Note that b^p, v generate $K = \langle x^p, y^p \rangle$. Using Lemma 3.3.2, c can be replaced in C by c^i for each i , $1 \leq i \leq p-1$. Now, there are $p-1$ choices for c . Namely, $b^p v^i$, $1 \leq i \leq p-1$. Then:

$$C = \{e, b^p v^i, b^{2p} v^{2i}, \dots, b^{(p-2)p} v^{(p-2)i}, b^{(p-1)p} v^{(p-1)i+1}\}.$$

Now, for j , $1 \leq j \leq p-2$, the congruence $(j+1)i \equiv 1 \pmod{p}$ is solvable.

So, there exists an integer j such that $ji \equiv (p-1)i+1 \pmod{p}$.

Let $t \in U$. The product $t\langle b^p \rangle C$ is direct, since it is part of ABC .

Computing, $\langle b^p \rangle C$, We see that the sets

$$\begin{aligned} \langle b^p \rangle b^{jp} v^{ji} &= \langle b^p \rangle v^{ji} \quad \text{and} \\ \langle b^p \rangle b^{(p-1)p} v^{(p-1)i+1} &= \langle b^p \rangle v^{(p-1)i+1} \end{aligned}$$

are parts of $\langle b^p \rangle C$, and they contain the same elements. And this contradicts the factorization.

Case 4: G is of type (p^2, p, p) .

Let $G = \langle x \rangle \times \langle y \rangle \times \langle z \rangle$, where $|x| = p^2$, $|y| = |z| = p$.

(a) Both B and C are cyclic.

In this case $B = \{e, b, b^2, \dots, b^{p-1}\}$, and

$$C = \{e, c, c^2, \dots, c^{p-1}\}.$$

Here, we may assume that $|b| = p^2$, and $|c| = p^2$. Using Lemma 3.3.1, we get that $A = U \langle b^p \rangle \cup V \langle c^p \rangle$, for some subset U, V of G . Now, if either $U = \emptyset$ or $V = \emptyset$, then we are done. So, we may assume that $U \neq \emptyset$ and $V \neq \emptyset$.

Let $b = x^\alpha y^\beta z^\gamma$ and $c = x^\delta y^\epsilon z^\mu$. Now, $b^p = x^{p\alpha}$ and $c^p = x^{p\delta}$. The subgroups of $\langle x^p \rangle$ form a chain. So $\langle x^p \rangle = \langle b^p \rangle = \langle c^p \rangle$. Thus, A is periodic with period x^p .

(b) Both B and C are simulated.

In this case $B = \{e, b, b^2, \dots, b^{p-2}, b^{p-1}u\}$, and

$$C = \{e, c, c^2, \dots, c^{p-2}, c^{p-1}v\},$$

Here, $|b| = |c| = p$, and by Lemma 3.2.2, we may assume $|u| = p$, and $|v| = p$. Replacing B, C by $\langle b \rangle, \langle c \rangle$ in the factorization

$G = ABC$, we have $G = A \langle b \rangle \langle c \rangle$. Thus, the product $\langle b \rangle \langle c \rangle$ is direct. If $u, v \in \langle b, c \rangle$, then $BC = \langle b, c \rangle$, is a factorization, and so by Rédei's Theorem, we are done. We assume that $u \notin \langle b, c \rangle$, and $v \in \langle b, c, u \rangle$. So, $v = b^\alpha c^\beta u^\gamma$. Then,

$$\begin{aligned} B &= \{e, b, b^2, \dots, b^{p-2}, b^{p-1}u\}, \\ C &= \{e, c, c^2, \dots, c^{p-2}, c^{p-1}b^\alpha c^\beta u^\gamma\}, \text{ and} \\ A &= U \langle u \rangle \cup V \langle b^\alpha c^\beta u^\gamma \rangle \end{aligned}$$

We assume that $U \neq \emptyset$ and $V \neq \emptyset$, since otherwise, we are done.

Assume $\beta = 0$. If $\gamma = 0$, then $\alpha \neq 0$. By Lemma 3.2.2, we may assume that $\alpha =$

1. Let $t \in V$. The product $t \langle b \rangle B$ is direct since, the product ABC is direct. On the other hand, $b \in \langle b \rangle$ and $b \in B$. This is a contradiction. If $\gamma \neq 0$, then by Lemma 3.2.2, again, we may assume that $\gamma = 1$. Now, if $\alpha = 0$, then A is periodic with period u . So, assume that $\alpha \neq 0$. Let $t \in V$. The product $t \langle b^\alpha u \rangle B$ is direct. The elements $b^{(p-1)\alpha + p-1}, e, b, b^2, \dots, b^{p-2}$ belong to $\langle b^\alpha u \rangle B$. Therefore, $(p-1)\alpha + p-1$ is

not congruent to $0, 1, 2, \dots, p-2 \pmod{p}$. Hence $(p-1)\alpha + p-1 = p-1 \pmod{p}$. Thus, $\alpha = 0$. But we know this is not the case. Secondly, assume $\beta \neq 0$. By Lemma 3.2.2, we may assume that $\beta = 1$. Now, $C = \{e, c, c^2, \dots, c^{p-2}, b^\alpha u^\gamma\}$. Let $t \in U$. The product $t \langle u \rangle BC$ is direct. Clearly, $\langle u \rangle BC = \langle b, u \rangle C$. But this is a contradiction, since $b^\alpha u^\gamma \in \langle b, u \rangle$ and $b^\alpha u^\gamma \in C$.

(c) B is cyclic and C is simulated.

In this case $B = \{e, b, b^2, \dots, b^{p-1}\}$,

$$C = \{e, c, c^2, \dots, c^{p-2}, c^{p-1}v\},$$

We may assume that $|b| = p^2$, and $|c| = |v| = p$. By Lemma 3.3.1, $A = U \langle b^p \rangle \cup V \langle v \rangle$, for some subset U, V of G . Now, if either $U = \emptyset$ or $V = \emptyset$, then we are done. So, we may assume that $U \neq \emptyset$ and $V \neq \emptyset$. Let $t \in U$. The product $t \langle b^p \rangle C$ is direct, since it is part of the product ABC . Furthermore, if $c \in \langle b^p \rangle$, then $c = b^{pi}$, for some i , where $1 \leq i \leq p-1$. This is cannot happen, since $c \in \langle b^p \rangle$, $c \in C$. Thus, we may assume that $c \notin \langle b^p \rangle$. We distinguish two cases as follows : $v \in \langle b^p, c \rangle$ or $v \notin \langle b^p, c \rangle$.

Suppose $v \in \langle b^p, c \rangle$, so that $v = b^{p\alpha} c^\beta$.

If $\beta = 0$, then A is periodic with period b^p .

If $\beta \neq 0$, then by Lemma 3.2.2, we may assume $\beta = 1$. Now,

$$C = \{e, c, c^2, \dots, c^{p-2}, c^{p-1} b^{p\alpha} c\} = \{e, c, c^2, \dots, c^{p-2}, b^{p\alpha}\}.$$

Let $t \in U$. The product $t \langle b^p \rangle C$ is direct, since it is part of the product ABC . But $b^{p\alpha} \in \langle b^p \rangle$, and $b^{p\alpha} \in C$, leads to a contradiction. Consider the case when $v \notin \langle b^p, c \rangle$. From the factorization $G = ABC$, we get :

$$\overline{AC} (e - b^p) = 0.$$

and so AC is periodic with period b^p . Consequently, $V \langle v \rangle C$ is periodic with b^p . (Observe that $\langle v \rangle C = \langle c, v \rangle$). Let $t \in V$. Now, $\langle b^p \rangle C \subset V \langle v, c \rangle$, and $V \langle v, c \rangle$ is a union of cosets modulo $\langle v, c \rangle$, and the elements of $\langle b^p \rangle$ are incongruent modulo $\langle v, c \rangle$, it follows that $p^3 = |\langle b^p \rangle \langle v, c \rangle| \leq |V \langle v, c \rangle| = p^2 |V|$. Thus, $|V| \geq p$, and so we get the contradiction that $U = \emptyset$.

Case 5: G is of type (p, p, p, p) .

In this case, each non-identity element of G is of order p . Hence, a cyclic subset of G is a subgroup of G . Thus, the only case we should consider is when both B and C are simulated.

B and C are simulated.

In this case $B = \{e, b, b^2, \dots, b^{p-2}, b^{p-1}u\}$, and

$$C = \{e, c, c^2, \dots, c^{p-2}, c^{p-1}v\},$$

Here, $|b| = |c| = p$, and by Lemma 3.2.2, we may assume $|u| = p$, and $|v| = p$. Replacing B, C by $\langle b \rangle, \langle c \rangle$ in the factorization $G = ABC$, we get $G = A\langle b \rangle\langle c \rangle$. This gives us that, the product $\langle b \rangle\langle c \rangle$ is direct. So, the group $\langle b, c, u, v \rangle$ is of type

$$(p, p), (p, p, p), \text{ or } (p, p, p, p).$$

$\langle b, c, u, v \rangle$ is of type (p, p) :

In this case, $u, v \in \langle b, c \rangle$. Furthermore, $BC = \langle b, c \rangle$ is a factorization and so by Rédei's Theorem, we are done.

$\langle b, c, u, v \rangle$ is of type (p, p, p) :

In this case, we may assume $u \notin \langle b, c \rangle$, and $v \in \langle b, c, u \rangle$.

Hence, $v = b^\alpha c^\beta u^\gamma$. So,

$$B = \{e, b, b^2, \dots, b^{p-2}, b^{p-1}u\},$$

$$C = \{e, c, c^2, \dots, c^{p-2}, c^{p-1}b^\alpha c^\beta u^\gamma\}, \text{ and}$$

$$A = U\langle u \rangle \cup V\langle b^\alpha c^\beta u^\gamma \rangle.$$

We may assume that $U \neq \emptyset$ and $V \neq \emptyset$, since otherwise, we are done.

Assume first that $\beta = 0$. Now, if $\gamma = 0$, then $\alpha \neq 0$, and by Lemma 3.2.2, we may assume that $\alpha = 1$. Let $t \in V$. The product $t \langle b \rangle B$ is direct since, the product ABC is direct. On the other hand, $b \in \langle b \rangle$ and $b \in B$. This is a contradiction.

If $\gamma \neq 0$, then by Lemma 3.2.2, again, we may assume that $\gamma = 1$.

Now, if $\alpha = 0$, then A is periodic with period u . So, assume that $\alpha \neq 0$. Let $t \in V$. The product $t \langle b^\alpha u \rangle B$ is direct. The elements $b^{(p-1)\alpha + p-1}, e, b, b^2, \dots, b^{p-2}$ belong to $\langle b^\alpha u \rangle B$.

Therefore, $(p-1)\alpha + p-1$ is not congruent to $0, 1, 2, \dots, p-2 \pmod{p}$, and hence $(p-1)\alpha + p-1 \equiv p-1 \pmod{p}$. Thus, $\alpha = 0$. And this contradicts our assumption about α . Now, assume $\beta \neq 0$. By Lemma 3.2.2, we may assume that $\beta = 1$. Now, $C = \{e, c, c^2, \dots, c^{p-2}, b^\alpha u^\gamma\}$. Let $t \in U$. The product $t \langle u \rangle BC$ is direct. Clearly, $\langle u \rangle BC = \langle b, u \rangle C$. But this is a contradiction, since $b^\alpha u^\gamma \in \langle b, u \rangle$ and $b^\alpha u^\gamma \in C$.

$\langle b, c, u, v \rangle$ is of type (p, p, p, p) :

From the factorization $G = A \langle b \rangle \langle c \rangle$, it follows that A is a complete set of coset representatives modulo $\langle b, c \rangle$. Hence,

$$A = \{ u^i v^j a_{ij}; 0 \leq i, j \leq p-1, a_{ij} \in \langle b, c \rangle \}$$

We may assume that $e \in A$, so that $a_{00} = e$. We also have

$$A = U \langle u \rangle \cup V \langle u \rangle.$$

Suppose $U \neq \emptyset$ and $V \neq \emptyset$. Note that the roles of u and v are symmetric. So, we may assume that $e \in U$. Then $\langle u \rangle \subset A$. This implies that $a_{i0} = e$, for each i , $0 \leq i \leq p - 1$. Let $u^i v^j a_{ij} \in V \langle v \rangle$. Now, $u^i v^j a_{ij} \langle v \rangle \subset A$. This gives that the a_{ij} 's are equal for each j , where $0 \leq j \leq p - 1$. But $a_{i0} = e$, for each i . Thus $a_{ij} = e$, for each j . Therefore, $u^i v^j a_{ij} \langle v \rangle = u^i \langle v \rangle$. Hence, $u^i \in V \langle v \rangle$, for each i . However, $u^i \in U \langle u \rangle$. And this is a contradiction since $U \langle u \rangle$ and $V \langle v \rangle$ are disjoint. This completes the proof. $\square$

CHAPTER FOUR

In the previous Chapter, we considered some 3-groups with respect to the Hajós-property. Even though some important results were obtained, the picture is far from complete, of course, by contrast with the situation in Chapter two. In this Chapter, we study some 2-groups with respect to the Hajós-property.

In the literature, groups which have the Hajós 2-property are called 'good' groups, and those which do not are called 'bad' groups. For example, in Theorem 7, [17], A. Sands pursuing a problem set forth by de Bruijn shows that groups of type $(2^\alpha, 2)$ are 'good' groups, while groups of type $(2^\alpha, 2, 2)$, where $\alpha \geq 4$ are 'bad'. We begin our study of 2-groups with the following Theorem which will prove very useful in the sequel.

4.1 Theorem

Let G be a group of type $(2^\alpha, 2, \dots, 2)$, where the factors of order 2 occur, say s times, and suppose G is factorized as :

$$G = B A_1 \dots A_i \dots A_n$$

where $|A_1| = \dots = |A_n| = 2$. Then at least one of the factors is periodic.

Proof :

Let $G = H \times K$, where $H = \langle h \rangle$, with $|h| = 2^\alpha$, and $K = \langle k_1, k_2, \dots, k_i, \dots, k_s \rangle$, where $|k_i| = 2$ for $1 \leq i \leq s$. Choose a character χ of G which is faithful on H .

Let $\chi(h) = \rho$, where ρ is a primitive 2^α -th root of unity. Clearly χ is not

principal on G . Hence $\chi(G) = 0$. Thus, we get:

$$\chi(B) \chi(A_1) \dots \chi(A_i) \dots \chi(A_n) = 0 \dots \dots \dots (1)$$

Now, for each i , $A_i = \{e, a_i\}$, where:

$$a_i \neq e, a_i = h^{\alpha_i} k_1^{\beta_{i1}} k_2^{\beta_{i2}} \dots k_s^{\beta_{is}}$$

where $0 \leq \alpha_i < 2^\alpha$ and $0 \leq \beta_{ij} < 2$. So $\chi(A_i) = 1 + \chi(a_i)$ for each i .

From (1), we get that :

either (a) $\chi(A_i) = 0$, for some i .

or (b) $\chi(B) = 0$.

Consider case (a) : In this case, we have, $0 = 1 + \chi(a_i)$. So $0 = 1 + \rho^{\alpha_i}$ ($\neq 1$), from which it follows that $\rho^{\alpha_i} = \pm 1$. Now, if $\rho^{\alpha_i} = +1$, then 2^α divides α_i . But $0 \leq \alpha_i \leq 2^{\alpha-1}$. Hence, $\alpha_i = 0$. While, if $\rho^{\alpha_i} = -1$, then $\rho^{2\alpha_i} = +1$, which implies that 2^α divides $2\alpha_i$, which in turn implies that $2^{\alpha-1}$ divides α_i . Hence either $\alpha_i = 0$ or $\alpha_i = 2^{\alpha-1}$. So in either case, we get that $|a_i|$ divides 2, and since $a_i \neq e$, it follows that $|a_i| = 2$. Therefore, A_i is a subgroup of G , and so it is periodic.

(b) Let $\chi(B) = 0$. We show that in this case B is periodic with period $h^{2^{\alpha-1}}$. To show this, let $g = h^{2^{\alpha-1}}$, and observe the following equivalent statements:

$$\begin{aligned} Bg = B &\Leftrightarrow \overline{B} (g - e) = 0, \text{ in } ZG. \\ &\Leftrightarrow \chi (\overline{B} (g - e)) = 0, \forall \chi. \end{aligned}$$

Now, χ is EITHER : (1) faithful on H.

OR : (2) not faithful on H.

In case (1), we have $\chi(B) = 0$, by assumption. Hence:

$$\chi(\overline{B} (g - e)) = \chi(\overline{B}) \chi(g - e) = 0.$$

In case (2), we have $\text{Ker}(\chi) \neq \{e\}$. Now, notice that $|g| = 2$. Hence, $\langle g \rangle \subset \text{Ker}(\chi)$. Therefore, $\chi(g) = 1$, and so $\chi(g - e) = 0$. As a consequence, we get that $\chi(\overline{B}) \chi(g - e) = 0$. $\square$

The following two results are clear.

4.2 Theorem

Let G be a group of type $(2^\alpha, 2, \dots, 2)$, and let $n = \alpha + s$, where s is the number of times the cyclic factors of order 2 occurs. Then G has the Hajós n -property.

Proof :

This is a straight-forward application of Rédei's Theorem. $\square$

4.3 Theorem

Let G be a group of type $(2^\alpha, 2, \dots, 2)$, and let $n = \alpha + s$, where s is the number of times the cyclic factors of order 2 occurs. Then G has the Hajós $(n - 1)$ -property.

Proof :

This follows from Theorem 4.1. □

We now, recall that in [17], A. Sands, shows that groups of type $(2^\alpha, 2)$ are 'good' groups. That is, groups of this type have the Hajós 2 - property. For reference, we include it as

4.4 Theorem [17]

If G is a group of type $(2^\alpha, 2)$, then G has the Hajós 2 - property.

We now, try to exploit the situation in the above Theorem, by making it a starting point for our studies of 2-groups with respect to the Hajós n -property.

4.5 Theorem

Let G be a group of type $(2^3, 2)$, then G has the Hajós n - property, for all relevant n ; namely for $n = 1, 2, 3, 4$.

Proof :

The case $n = 1$ is trivial.

The case $n = 2$ follows from Theorem 4.4.

The case $n = 3$ follows from Theorem 4.1.

The case $n = 4$ follows from Rédei's Theorem. □

4.6 Theorem

Let G be a group of type $(2^4, 2)$, then G has the Hajós n - property, for all relevant n ; namely for $n = 1, 2, 3, 4, 5$.

Proof :

The case $n = 1$ is trivial.

The case $n = 2$ follows from Theorem 4.4.

The case $n = 4$ follows from Theorem 4.1.

The case $n = 5$ follows from Rédei's Theorem.

So we only need detail the cases $n = 3$.

Suppose G is factorized as $G = ABC$.

We observe that if $|A| = 8$, and $|B| = |C| = 2$, then by Theorem 4.1, we get that one of the factors is periodic. Suppose $|A| = 4$, $|B| = 4$, $|C| = 2$. Then $G = A(BC)$, is also a factorization of G . Hence, by Theorem 4.4, either A or BC is periodic. Now, if A is periodic, then there is nothing to do. So we may and shall assume that BC is periodic. Let $g \neq e$, be a period for BC . Then we get:

$$\overline{B} \overline{C} (e - g) = 0 \dots \dots \dots (2)$$

Observe that $C = \{e, c\}$. Now, if it happens that $|c| = 2$, then, C is a subgroup, and hence C is periodic, and again, we are done. So we may assume that $|c| \neq 2$. Hence, $c^2 \neq e$.

Now, we rewrite equation (2) as :

$$\overline{B} (e + c) (e - g) = 0 \dots \dots \dots (3)$$

Multiplying equation (3) by $(e - c)$, we obtain the following equation :

$$\overline{B} (e - c^2) (e - g) = 0 \dots \dots \dots (4)$$

Therefore, by Theorem 3.2.4, we get that there exist subsets X and Y of G such that :

$$B = X \langle c^2 \rangle \cup Y \langle g \rangle \dots \dots \dots (5)$$

Next, we consider the factorization $G = B(AC)$. Again, by Theorem 4.4, we get that either B or AC is periodic. Again, if B is periodic, there is nothing to do. So we may and shall assume that AC is periodic. Let $h \neq e$ be a period for AC . From this, we get the equation :

$$\overline{A} \overline{C} (e - h) = 0 \dots \dots \dots (6)$$

Again, a similar argument as before, gives us the equation :

$$\overline{A} (e - c^2) (e - h) = 0 \dots \dots \dots (7)$$

And as before, Theorem 3.2.4, implies that there exist subsets U and V of G such that :

$$A = U \langle c^2 \rangle \cup V \langle h \rangle \dots \dots \dots (8)$$

Putting , (5) and (8) together, we obtain the equation :

$$G = [X \langle c^2 \rangle \cup Y \langle g \rangle] [U \langle c^2 \rangle \cup V \langle h \rangle] \{e, c\} \dots \dots (9)$$

Now, if both $X \neq \emptyset$ and $U \neq \emptyset$, then we must have $c^2 \in A$ and $c^2 \in B$. But this is a contradiction to the factorization of G as $G = ABC$. So either $X = \emptyset$ or $U = \emptyset$. But then, we get that either A or B is periodic. □

□

The situation in the previous two Theorems is not a coincidence. In fact, we have the following

4.7 Theorem

Let G be a group of type $(2^\alpha, 2)$, then G has the Hajós n -property, for all relevant n .

Proof:

Suppose $G = \langle a, b \rangle$, where $|a| = 2^\alpha$ and $|b| = 2$. We may assume $\alpha \geq 2$. Let $m = 2^{\alpha-1}$. Say $G = A_1 \dots A_i \dots A_n$ is a factorization of G . Then, for any character χ of G , $\chi(G) = 0$, implies that $\chi(A_i) = 0$, for some i , $1 \leq i \leq n$.

Consider χ_1, χ_2 of G such that :

$$\chi_1(a) = \rho, \chi_1(b) = 1.$$

$$\chi_2(a) = \rho, \chi_2(b) = -1.$$

Here, ρ is a primitive 2^α -th root of unity.

We consider two cases :

$$(1) \chi_1(A_1) = 0, \text{ and } \chi_2(A_1) = 0.$$

$$(2) \chi_1(A_1) = 0, \text{ and } \chi_2(A_2) = 0.$$

We will show that case (1) implies that A_1 is periodic, while, in case (2), either A_1 or A_2 is periodic.

Consider case (1) :

We observe that $\chi \in \text{Ann}(\langle a^m \rangle) \Leftrightarrow \chi\{e, a^m\} = 0, \Leftrightarrow 1 + \chi(a^m) = 0, \Leftrightarrow \chi(a^m) = -1, \Leftrightarrow \chi(a) = \rho$. Hence, in this case :

$\text{Ann}(\langle a^m \rangle) \subset \text{Ann}(A_1)$, and so by Theorem 3.2.3, $A_1 = \langle a^m \rangle S$, for some

subset S of G . Hence, in this case A_1 is periodic with period a^m .

Consider case (2) :

In this case, we show that:

$$\text{Ann} (\langle a^m \rangle) \cap \text{Ann} (\langle a^m b \rangle) \subset \text{Ann} (A_1), \text{ and}$$

$$\text{Ann} (\langle a^m \rangle) \cap \text{Ann} (\langle b \rangle) \subset \text{Ann} (A_2).$$

We have $\chi \in \text{Ann} (\langle a^m \rangle) \cap \text{Ann} (\langle a^m b \rangle) \Leftrightarrow \chi (a^m) = -1 = \chi (a^m b), \Leftrightarrow \chi (a) = \rho$, and $\chi (b) = 1$.

Similarly, $\chi \in \text{Ann} (\langle a^m \rangle) \cap \text{Ann} (\langle b \rangle) \Leftrightarrow \chi (a^m) = -1 = \chi (b), \Leftrightarrow \chi (a) = \rho$, and $\chi (b) = -1$.

Therefore, by Theorem 3.2.3 :

$$A_1 = \langle a^m \rangle X \cup \langle a^m b \rangle Y \text{ and}$$

$$A_2 = \langle a^m \rangle U \cup \langle b \rangle V$$

for some subset X, Y, U, V of G .

Since, $A_1 A_2$ is direct, it follows that either X or U must be empty.

If X is empty, then A_1 is periodic with period $a^m b$, while if U is empty, then A_2 is periodic with period b . □

4.8 Theorem

Let G be a group of type $(2^2, 2^2)$, then G has the Hajós n - property, for all relevant n ; namely for $n = 1, 2, 3, 4$.

Proof :

The case $n = 1$ is trivial.

The case $n = 2$ follows from Theorem 8 in [15].

The case $n = 4$ follows from Rédei's Theorem.

So, we only need detail the case $n = 3$:

Suppose $G = ABC$ is a factorization of G . Then $|A| = 4$, and $|B| = |C| = 2$. Now, $G = A(BC)$ is also a factorization of G . Hence, by Theorem 8 in [15], again we get that either A or BC is periodic. If A is periodic, we are done. So assume that BC is periodic. Say $g \neq e$ is a period of BC . We may assume that $|g| = 2$.

Let $B = \{e, b\}$, and $C = \{e, c\}$. Then $BC = \{e, b, c, bc\}$.

Now, if $g = b$, then B is a subgroup, and hence B is periodic.

While if $g = c$, then C is a subgroup, and hence C is periodic.

Finally, if $g = bc$, then we get that :

Either (1) $b^2c = b$ and $bc^2 = c$, both of which imply that $bc = e$.

This impossible.

Or (2) $b^2c = c$ and $bc^2 = b$, from which follows that B and C are subgroups of G . □

4.9 Theorem

Let G be a group of type $(2^3, 2, 2)$, then G has the Hajós n - property, for all relevant n ; namely for $n = 1, 2, 3, 4, 5$.

Proof :

The case $n = 1$ is trivial.

The case $n = 2$ follows from Theorem 6, in [17].

The case $n = 4$ follows from Theorem 4.1.

The case $n = 5$ follows from Rédei's Theorem.

Therefore, only the case $n = 3$, needs elaboration. Here are the details.

Suppose G is factorized as $G = ABC$.

We observe that if $|A| = 8$, and $|B| = |C| = 2$, then by Theorem 4.1, we get that one of the factors is periodic.

Suppose, $|A| = 4$ and $|B| = 4$, and $|C| = 2$. An argument, similar to that used in the proof of Theorem 4.6, will lead to the equation :

$$B = X \langle c^2 \rangle \cup Y \langle g \rangle \dots\dots\dots (*)$$

where $c^2 \neq e$, and $g^2 = e$.

Now, if either $X = \emptyset$ or $Y = \emptyset$, then we get that B is periodic. Suppose not.

Now, we observe that $|B| = 4$, $|\langle g \rangle| = 2$, and $|\langle c^2 \rangle| \geq 2$. Therefore, the only possibility is that $|X| = |Y| = 1$, and $c^4 = e$. But then, $BC = x \{e, c^2\} \{e, c\} \cup y \{e, g\} \{e, c\}$. Now note that g is a period of $y \{e, g\} \{e, c\}$. But g is also a period of BC . Therefore, g is a period of $x \{e, c^2\} \{e, c\}$. Now, the only candidates for such a period are c, c^2, c^3 . But g is a period of order 2. Hence, $g = c^2$. Therefore $(*)$ implies that B is periodic with period c^2 . $\square$

4.10 Theorem

Let G be a group of type $(2^2, 2, 2, 2)$, then G has the Hajós n - property, for all relevant n ; namely for $n = 1, 2, 3, 4, 5$.

Proof :

The case $n = 1$ is trivial.

The case $n = 2$ follows from Theorem 6, in [17].

The case $n = 4$ follows from Theorem 4.1.

The case $n = 5$ follows from Rédei's Theorem.

The case $n = 3$ can be handled exactly as in the previous Theorem. □

Continuing our investigations, we now attempt to generalize some of the results to groups of type $(2^{\alpha_1}, \dots, 2^{\alpha_i}, \dots, 2^{\alpha_s})$.

We start with the following :

4.11 Theorem

Let G be a group of type $(2^2, 2^2, 2)$, then G does not have the Hajós 3-property.

Proof :

Let $G = \langle x, y, z \rangle$, where $|x| = |y| = 4$ and $|z| = 2$.

Let $A = \{e, x^2, xy^2, x^3y^2, z, y^2z, x^2yz, x^2y^3z\}$

$B = \{e, x\}$

$C = \{e, y\}$

We will show that $G = ABC$ is a factorization of G with none of A , B or C periodic. Now, the possible orders of $\chi(x)$ are 1, 2, and 4 only. Similarly, with that of $\chi(y)$. While the possible orders of $\chi(z)$ are 1 and 2 only. So altogether there are 18 different cases to consider.

Let $\chi(x) = \alpha$,

$\chi(y) = \beta$,

$\chi(z) = \gamma$.

Then :

$$\chi(A) = 1 + \alpha^2 + \alpha \beta^2 + \alpha^3 \beta^2 + \gamma + \beta^2 \gamma + \alpha^2 \beta \gamma + \alpha^2 \beta^3 \gamma,$$

$$\chi(B) = 1 + \alpha,$$

$$\chi(C) = 1 + \beta.$$

Inserting the appropriate values of α , β and γ into these equations, one can show that in each case, at least one of $\chi(A)$, $\chi(B)$ or $\chi(C)$ is zero. The following table summarizes all of these cases.

Case	Order of $\chi(x)$	Order of $\chi(y)$	Order of $\chi(z)$	$\chi(A)$	$\chi(B)$	$\chi(C)$
1	1	1	1	.	.	.
2	1	1	2	0		
3	1	2	1			0
4	1	2	2			0
5	1	4	1	0		
6	1	4	2	0		
7	2	1	1		0	
8	2	1	2		0	
9	2	2	1		0	0
10	2	2	2		0	0
11	2	4	1		0	
12	2	4	2		0	
13	4	1	1	0		

14	4	1	2	0		
15	4	2	1			0
16	4	2	2			0
17	4	4	1	0		
18	4	4	2	0		

By Lemma 2.13, we get that $G = ABC$ is a factorization of G . Hence, it only remains to show that none of A , B or C is periodic. It is obvious that neither B nor C is periodic. To show that A is not periodic, we argue as follows: If A were periodic, then there would exist a nonidentity element a in A , which serves as a period for A . But then all the nonidentity powers a^i of a are also periods for A . Observe that if $|a| = 4$, then $|a^2| = 2$. So we may assume that the order of any period of A is 2. And since $e \in A$, we only need check the elements of A for periodicity. But x^2 and z are the only elements of A of order 2, and it is easy to see that none of them is a period for A . Hence, we conclude that A is not periodic. $\square$

4.12 Theorem

Let G be a group of type $(2^3, 2^2)$, then G does not have the Hajós 3-property.

Proof :

Let $G = \langle x, y \rangle$, where $|x| = 8$, and $|y| = 4$.

Let $A = \{e, x^4, x^2y^2, x^6y^2, x, x^5y, xy^2, x^5y^3\}$

$$B = \{e, x^2\}$$

$$C = \{e, y\}$$

We will show that $G = ABC$ is a factorization of G with none of A , B or C periodic.

Now, the possible orders of $\chi(x)$ are 1, 2, 4 and 8 only. While the possible orders of $\chi(y)$ are 1, 2 and 4 only. So altogether there are 12 different cases to consider.

Let $\chi(x) = \alpha$,

$$\chi(y) = \beta.$$

$$\text{Then } \chi(A) = 1 + \alpha^4 + \alpha^2 \beta^2 + \alpha^6 \beta^2 + \alpha + \alpha^5 \beta + \alpha \beta^2 + \alpha^5 \beta^3,$$

$$\chi(B) = 1 + \alpha^2,$$

$$\chi(C) = 1 + \beta.$$

Inserting the appropriate values of α , and β into these equations, one can show that in each case, at least one of $\chi(A)$, $\chi(B)$ or $\chi(C)$ is zero. The following table summarizes all of these cases.

Case	Order of $\chi(x)$	Order of $\chi(y)$	$\chi(A)$	$\chi(B)$	$\chi(C)$
1	1	1	.	.	.
2	1	2			0
3	1	4	0		
4	2	1	0		
5	2	2			0

6	2	4	0		
7	4	1		0	
8	4	2		0	0
9	4	4		0	
10	8	1	0		
11	8	2	0		0
12	8	4	0		

By Lemma 2.13, we get that $G = ABC$ is a factorization of G . Hence, it only remains to show that none of A , B or C is periodic. It is obvious that neither B nor C is periodic. To show that A is not periodic, we argue as follows: If A were periodic, then there would exist a nonidentity element a in A , which serves as a period for A . But then all the nonidentity powers a^i of a are also periods for A . We may assume that the order of any period of A is 2. And since $e \in A$, we only need check the elements of A for periodicity. But x^2 is the only elements of A of order 2, and it is easy to see that it is not a period for A . Hence, we conclude that A is not periodic. $\square$

4.13 Theorem

Let G be a group of type $(2^3, 2, 2, 2)$, then G does not have the Hajós 3-property.

Proof :

Let $G = \langle x, y, u, v \rangle$, where $|x| = 8, |y| = |u| = |v| = 2$.

Let $A = \{e, x^4, x^2y, x^6y, x, xy, x^5u, x^5uy\}$

$B = \{e, u, v, uv\}$

$C = \{e, x^2\}$

We will show that $G = ABC$ is a factorization of G with none of A, B or C periodic.

Now, the possible orders of $\chi(x)$ are 1, 2, 4 and 8, while the possible orders of $\chi(y)$, $\chi(u)$ and $\chi(v)$ are 1 and 2 only. So altogether there are 32 different cases to consider.

Let $\chi(x) = \alpha, \chi(y) = \beta, \chi(u) = \gamma, \chi(v) = \delta$.

Then :

$$\chi(A) = 1 + \alpha^4 + \alpha^2\beta + \alpha^6\beta + \alpha + \alpha\beta + \alpha^5\gamma + \alpha^5\beta\gamma,$$

$$\chi(B) = 1 + \gamma + \delta + \beta\gamma\delta,$$

$$\chi(C) = 1 + \alpha^2.$$

Inserting the appropriate values of α, β, γ , and δ into these equations, one can show that in each case, at least one of $\chi(A), \chi(B)$ or $\chi(C)$ is zero.

The following table summarizes all of these cases.

Case	Order of $\chi(x)$	Order of $\chi(y)$	Order of $\chi(u)$	Order of $\chi(v)$	$\chi(A)$	$\chi(B)$	$\chi(C)$
1	1	1	1	1	.	.	.
2	1	1	1	2		0	
3	1	1	2	1		0	

4	1	1	2	2		0	
5	1	2	1	1	0		
6	1	2	1	2	0		
7	1	2	2	1	0		
8	1	2	2	2		0	
9	2	1	1	1	0		
10	2	1	1	2		0	
11	2	1	2	1		0	
12	2	1	2	2		0	
13	2	2	1	1		0	
14	2	2	1	2		0	
15	2	2	2	1		0	
16	2	2	2	2		0	
17	4	1	1	1			0
18	4	1	1	2			0
19	4	1	2	1			0
20	4	1	2	2			0
21	4	2	1	1			0
22	4	2	1	2			0
23	4	2	2	1			0

24	4	2	2	2		0	0
25	8	1	1	1	0		
26	8	1	1	2		0	
27	8	1	2	1		0	
28	8	1	2	2	0		
29	8	2	1	1	0		
30	8	2	1	2	0		
31	8	2	2	1	0		
32	8	2	2	2		0	

By Lemma 2.13, we get that $G = ABC$ is a factorization of G . Hence, it only remains to show that none of A , B or C is periodic. It is obvious that neither B nor C is periodic. To show that A is not periodic, we argue as before: So we may assume that the order of a period of A is 2. And since $e \in A$, we only need check the elements of A for periodicity. But x^4 is the only elements of A of order 2, and it is easy to see that it is not a period for A . Hence, we conclude that A is not periodic. $\square$

4.14 Theorem

Let G be a group of type $(2^2, 2, 2, 2, 2)$, then G does not have the Hajós 3-property.

Proof :

Let $G = \langle x, y, u, v, w \rangle$, where $|x| = 8, |y| = |u| = |v| = |w| = 2$.

Let $A = \{e, x^2, xy, x^3y, w, yw, x^2uw, x^2uyw\}$

$B = \{e, u, v, uv\}$

$C = \{e, x\}$

Then $G = ABC$ is a factorization of G with none of A, B or C periodic. $\square$

4.15 Theorem

Let G be a group of type $(2, 2, 2, 2, 2, 2, 2)$, then G does not have the Hajós 3-property.

Proof :

Let $G = \langle x, y, u, v, w, z, t \rangle$, where $|x| = |y| = |u| = |v| = |w| = |z| = |t| = 2$.

Let $A = \{e, x, yv, xyv, t, tv, txw, txwv\}$

$B = \{e, y, u, xyu\}$

$C = \{e, w, z, wzv\}$

Then $G = ABC$ is a factorization of G with none of A, B or C periodic. $\square$

Next, we will need the following two Lemmas.

4.16 Lemma [16]

If A and B are non-periodic subsets of a group G and if A is contained in a subgroup H of G such that the product HB is direct, then the set AB is also non-periodic.

4.17 Lemma [16]

If H is a proper subgroup of a group G , then G has a factorization of the form $G = HB$, with B non-periodic, except when H is a subgroup of index 2 in an elementary abelian 2-group.

Using the above two lemmas, we obtain we get the following

4.18 Theorem

Let H be a proper subgroup of a finite abelian group G . If H does not have the Hajós n -property, then G itself does not have the Hajós n and $(n+1)$ -property.

Proof :

Since H does not have the Hajós n -property, it has a factorization of the form $H = A_1 A_2 \dots A_i \dots A_n$, where none of the A_i is periodic. Now by Lemma 4.17, G has a factorization of the form $G = HA_{n+1}$, with A_{n+1} non-periodic, so that :

$$G = A_1 A_2 \dots A_i \dots A_n A_{n+1}$$

with none of the factors periodic. From this it follows G does not have the Hajós $(n+1)$ -property. Now, we write G as :

$$G = A_1 A_2 \dots A_i \dots B$$

where $B = A_n A_{n+1}$, and since Lemma 4.16 implies that B is non-periodic, we conclude that G does not have the Hajós n -property. □

Putting all this together, we get this next Theorem.

4.19 Theorem

Let G be a group of type of type $(2^{\alpha_1}, 2^{\alpha_2}, 2^{\alpha_3}, \dots, 2^{\alpha_i}, \dots, 2^{\alpha_s})$, where $s \geq 3$, $\alpha_1 \geq 3$, $\alpha_2 \geq 2$, and $\alpha_3 \geq 1$. Then G does not have the Hajós 3 and 4-property.

Proof:

G has a subgroup H of type $(2^2, 2^2, 2)$, which by Theorem 4.11, does not have the Hajós 3-property. The result now follows from Theorem 4.18. $\square$

For the next Theorem, we need the following two results.

4.20 Lemma [17]

Let G be a group, and suppose G is factorized as $G = AB$, where $|A| = 4$, and two elements of A have a common square. Then either A or B is periodic.

4.21 Theorem [15]

Let G be a finite abelian non-cyclic p -group. If a direct product of subsets of G of prime order is periodic, then one of the subsets is a subgroup of G .

Algebraic Form Of Keller's Conjecture [6]

Let G be a finite abelian group, and suppose G has a factorization of the form $G = BA_1 \dots A_n$, where:

$$A_i = \{e, a_i, a_i^2, \dots, a_i^{k_i - 1}\}.$$

Then :

$$BB^{-1} \cap \{ a_1^{k_1}, \dots, a_n^{k_n} \} \neq \emptyset.$$

4.22 Theorem [10 & 23]

If G is a group of type $(p^\alpha, p^\beta, p, \dots, p)$, then G satisfies Keller's conjecture.

4.23 Theorem

Let G be a group of type $(2^2, 2^2, 2)$. Suppose G is factorized as :

$$G = ABCD$$

where $|A| = 4$, and $|B| = |C| = |D| = 2$. Then one of the factors is periodic.

Proof :

Let $G = \langle x, y, z \rangle$, where $|x| = |y| = 4$, and $|z| = 2$. Now we can write G as :

$$G = A \{e, b\} \{e, c\} \{e, d\}$$

Now, if one of b, c , or d has order 2, then we are done, since then one of B, C or D is a subgroup of G , and hence periodic. So we may assume that $|b| = |c| = |d| = 4$. But then, by Theorem 4.22, we obtain :

$$AA^{-1} \cap \{b^2, c^2, d^2\} \neq \emptyset$$

Say, $b^2 \in AA^{-1}$. That is, $b^2 = uv^{-1}$, where $u, v \in A$. Hence, $u^2 = v^2$. Now, by Lemma 4.20, either A or BCD is periodic. If A is periodic, we are done. So assume BCD is periodic. In this case, Theorem 4.21 implies that one of the factors B, C or D is a subgroup of G , which cannot happen unless one of b, c , or d is of order 2. But this contradicts our assumption that none of them is of order 2. Hence, in either case, we get that one of the factors is periodic. $\square$

The previous Theorem can be generalized as follows.

4.24 Theorem

Let G be a group of type $(2^2, 2^2, 2, \dots, 2)$. Suppose G is factorized as :

$$G = B A_1 \dots A_i \dots A_n$$

where $|B| = 4$, and $|A_1| = |A_2| = \dots = |A_n| = 2$. Then one of the factors is periodic.

Proof :

Exactly similar to the proof of Theorem 4.23.

□

CHAPTER FIVE

Epilogue:

Factorization of abelian groups is interconnected with many other areas of mathematics, such as combinatorics, geometry and number theory, [18]. In the introduction to this work, we specifically mentioned Minkowski's conjecture, and its reformulation into Group Theory by G. Hajós. Hence, I found it appropriate to include in this last chapter some of those ideas involved, and finally state the conjecture in both of its forms. To do that, we need the following.

5.1 Definition [20]

A set of points L is a lattice in $\mathbb{R}^n$ if it is a group under vector addition and each of its points is the center of a ball that contains no other points of L . The most famous lattice is $\mathbb{Z}^n$, where $\mathbb{Z}$ is the set of integers. It is called the standard lattice.

5.2 Definition [20]

Tiling a space, whatever its dimension, means filling the entire space by cubes, without any overlapping.

5.3 Definition [20]

A lattice tiling is a tiling for which the centers of the cubes form a regular,

grid-like lattice of points, like the locations of carbon atoms in a crystal of diamond.

Now, we can state the:

Geometric Form of Minkowski's conjecture [20]

In a lattice tiling of an n -dimensional space by unit cubes, there must be a pair of cubes which share a complete $(n - 1)$ -dimensional face.

In 1930, Ott-Henrich Keller conjectured that :

No matter what the dimension d , if you tile a d -dimensional space with d -dimensional "cubes" of equal size, then you wind up with cubes that have an entire $(d - 1)$ -dimensional "side" in common. We observe that Keller was just generalizing the above conjecture of Minkowski. In 1940, Oskar Perron showed that Keller's conjecture was true up to dimension 6, but beyond that the conjecture lay unresolved for about fifty years. Recently, Logarias and Shor have found an explicit counterexample to Keller's conjecture in a 10-dimensional space.

In 1938, G. Hajós reformulated Minkowski's conjecture into finite abelian groups, as follows :

Algebraic Form of Minkowski's conjecture [20]

In any factorization of a finite abelian group by cyclic subsets, at least one of the factors is a subgroup.

Three years later, Hajós proved the algebraic form of Minkowski's conjecture, hence showing that the original conjecture for lattice-tiling is true in all dimensions.

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