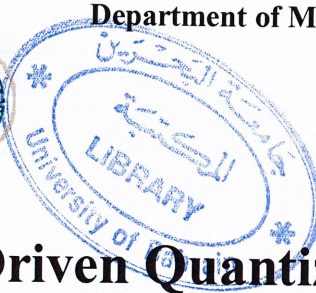


UNIVERSITY OF BAHRAIN

College of Science

Department of Mathematics



Transient Spectra of Driven Quantized Harmonic Oscillator

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Master Degree in Mathematics

Submitted by

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University of Bahrain
Graduate Studies Council



Thesis Discussion Committee Decision

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The Discussion Committee met on wednesday 7/1/15 in Hall no. 541
(hall/room no.) 2000. Upon discussing the student whose details are stated above in
public, the discussion committee recommends the following:

The student awarded grade (A-) In the thesis subject MATHS 599

- ☒ Without making any changes to the thesis
☐ After making the modifications attached with this decision

Discussion Committee Members Approval:

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- Dr/ M. Abolatta (Internal Examiner)
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ABSTRACT

Exact analytical results are derived for the pulsed driven quantized harmonic oscillator (HO) with arbitrary pulse shape. Specifically, for a sequential $\sin^2$ – pulse shape, the transient fluorescent spectrum is obtained in the three cases of initial states of the HO, namely, the vacuum, number and coherent states. Of special interest, with initial coherent state the dominant central Lorentzian spectrum is surrounded by weak oscillations. This is due to the larger number of sequential pulses which get amplified asymmetrically in the non-resonant case, as a result of balancing and interference processes between the initial excitation and the strength of the exciting pulse. Further pronounced oscillations is noticed with 'hole burning' structure in the central peak due to non-zero phase of the initial amplitude of the coherent state.

To My Family

ACKNOWLEDGMENT

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LIST OF SYMBOLS

Symbol	Meaning
$\dagger$	Hermitian conjugate
$\hat{a}$	annihilation (lowering) operator
$\hat{a}^\dagger$	creation (raising) operator
$\hat{H}$	Hamiltonian operator
$ >$	quantum state ket vector
$< $	quantum state bra vector
$[\hat{A}, \hat{B}]$	quantum commutator of operators $\hat{A}$ and $\hat{B}$

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CHAPTER 1

INTRODUCTION

- 1.1 Statement of the Problem
- 1.2 Background and History
- 1.3 Justification of the Problem
- 1.4 Theoretical Framework
- 1.5 Methods and Procedures
- 1.6 Proposed Chapter Outline

CHAPTER 1

INTRODUCTION

1.1 Statement of the Problem

The main work in this thesis concerns an analytical and computational study, within the framework of quantum theory, of the interaction of a single mode quantized harmonic oscillator (HO) with short pulsed-laser field of various shapes, in the absence of any dissipation processes. Specifically, we calculate the Fourier transient spectrum of the scattered radiation starting from a quantized Hamiltonian for the combined system (HO + pulsed driving field). The corresponding Heisenberg operator equations of motion for the HO variables are solved analytically for arbitrary pulse shape. This is followed by the calculation of the auto-correlation functions for the HO operator amplitude variables. The transient spectrum given as the Fourier transform of the auto-correlation function is calculated for arbitrary initial state of the HO.

The resulting transient spectrum is examined computationally and displayed graphically for various shapes of the pulsed-driving field and system parameters. Physical interpretation and comparison with earlier results are provided.

1.2 Background and History

Within the mathematics (Clapham and Nicholson 2009) or physics (Daintith 2009) scientific vocabulary, the field of quantum mechanics (QM) is concerned with the study of behavior or structure of microscopic entities (e.g., atoms, molecules) as well discreteness (quantization) of phys-

ical quantities, such as energy and momentum of both matter and radiation. Max Planck is the sole founder of the concept of energy quantization in 1900 (Planck 1900). Specifically, the energy of the black body radiation field is not a continuous variable (as in the classical radiation theory), but has its own discrete values which are proportional to the radiation frequency (ω): each value of the energy is an integer multiple of ($\hbar\omega$), Rigorously, the radiation energy values of a single quantized mode of frequency (ω) are $(n + \frac{1}{2})\hbar\omega$; $n=0,1,2,\dots$ where $\hbar = \frac{h}{2\pi}$ ($= 1.0545717\dots * 10^{-34} Js$) is the reduced universal Planck's constant. The minimum energy value is ($\frac{1}{2}\hbar\omega$), called the quantized zero-point energy, compared with minimum zero energy value within the classical radiation theory.

Each bundle of the radiation energy is called "quanta of radiation" or simply "photon". The quantization (or discreteness) of the radiation energy introduced by Max Planck was *dictated* by the interpretation of experiments done with black body radiation (Planck 1900), that could not be explained, otherwise, through the classical radiation theory.

Following Max Planck quantization concept of the radiation energy, and in conformity, the discrete nature of atomic energy (and momentum) was adopted by Albert Einstein (Einstein 1905) to explain the famous photoelectric effect: radiation (photons) are absorbed / emitted by matter in bundles.

Theoretical study of interaction of radiation and matter has led to one of the greatest discovery of light sources, namely, the laser as a coherent collimated source of light in (1960's) [e.g, (Band 2006)]. Nowadays, the advanced laser sources as continuous wave (cw) or short-pulsed laser of femto-second scale play an active and central role in both theoretical and experimental research in areas such as, non-linear optics, quantum information processing and quantum computation.

Study of radiation-matter or radiation-radiation interaction within the

framework of QM reveals information about the resulting scattered radiation and its emitted frequencies, as well the energy change in the original system due to such interaction.

The proposed topic in the thesis deals with the study of the scattered radiation due to the interaction of a single mode of the quantized radiation field (modeled as quantized HO) with short-laser pulses of various shapes. The HO represents in some limiting case, many excited atoms in a very small volume (λ^3 ; λ = radiation wavelength $\sim 10^{-5}cm$ in the optical range), (Gross and Haroche 1982). Mathematically speaking, all atoms are assumed to be on the same single site (Dicke 1954).

1.3 Justification of the problem

Spectral study of the scattered radiation is a one way to gain information about its frequency distribution, as well to know the response of the HO with respect to the frequency of the driving laser pulse. Availability of intense short laser pulses (Gibbon 2007) makes it possible to test our analytical and computational results. The HO as an approximation to the many atomic excited system, offers a practical technological model, where exciting such a model with short laser pulses (of various shapes) create coherent superposition states which is a basic concept towards quantum computing and quantum information networking (Ekert and Zozsa 1996, Zoller et al 2005).

In essence, the scattered radiation due to the proposed interacting model (HO + pulsed laser) offers a produced light signal with specific spectral distribution that can be used as a carrier signal of information.

1.4 Theoretical Framework

The proposed model for study (quantized HO + pulsed laser field) is to be investigated within the general framework of quantum mechanics (QM). The single mode (i.e. frequency) quantized HO has an ∞ -number of discrete energy levels, while the exciting pulse is treated classically (i.e. its energy is a continuous variable). The interaction Hamiltonian (i.e. total energy) description is formulated in terms of the HO quantized amplitude variables. The adopted analytical formula for the scattered transient spectrum (Eberly and Wodkiewicz 1977) is based on the quantum aspect of the driven system, namely, HO mode. This is termed as Fourier transform of the quantum auto-correlation function of the HO amplitude variables.

So, the two quantum aspects of study in the proposed model are:

- (i) The quantized (discrete) nature of the energy eigenvalues of the HO.
- (ii) The quantum (non-classical) auto-correlation function which relates emission and absorption of "photons" at time delay intervals.

1.5 Methods and Procedures

Within the mathematical framework of QM, the steps of calculations towards the model differential equations (DEs) are itemized as follow:

- (i) Identify the quantized Hamiltonian (total energy) of the coupled model, together with the commutator algebra for the HO operator variables.
- (ii) Accordingly, the Heisenberg differential equations of motion for the HO operators are derived in the general case of exciting pulse shape.
- (iii) These operator DEs are linear and are solved exactly in the special case of squared-sine pulse shape.

- (iv) Calculations of the transient spectrum will be formulated analytically.
- (v) Computational display of the spectrum will be presented for various system parameters and initial conditions.
- (vi) Comparison of results with previous cases of pulse shapes.

1.6 Proposed Chapter Outline

In chapter (2) to follow, entitled (Literature Review), we present brief review to the pulsed driven systems, namely, of 2-level atom and of HO systems.

In chapter (3), entitled (Methodology: Quantized Hamiltonian and Model Equations), we present, through quantum mechanical formulation, the derivation of the model DEs for the proposed model (HO + pulsed field) under study. The DEs for the HO variable operators are derived and solved exactly for arbitrary pulse shape.

In chapter (4), entitled (Results: Squared-Sine Pulse Case), we specialize the investigation to the case of (classical) squared-sine pulse shape. Analytical solutions of the DEs and calculations of the transient spectrum will be derived. Computational results will be displayed for different system parameters. This is followed by conclusion and suggested future work.

CHAPTER 2

LITERATURE REVIEW

CHAPTER 2

LITERATURE REVIEW

Interest in the current interacting model (HO + pulsed laser), apart from its fundamental nature, is activated by two main factors:

- (i) The calculated theoretical scattered spectrum due to single 2-level dissipative atom interacting with a single mode (CW) laser by Mollow in (1969) (Mollow 1969) and its experimental verification in (1974) (Schuda, Stroud and Herscher 1974) in what is known "Mollow spectrum": the spectral shape consists of triplet Lorentzians in the strong laser case due to the splitting of the two atomic energy levels to doublets (pitchfork shape).
- (ii) Availability of short laser pulses of various shapes in the laboratories. (cf (Gibbon 2007)).

For the case of non-dissipative 2-level atom excited by pulsed laser of various shapes (rectangular, exponential, Gaussian, ...) atomic dynamics has been investigated at large in (Shore 1990). The transient spectrum for the same model has been investigated analytically and computationally in the case of hyperbolic secant (Rzazewski and Florjanczyk 1984, Florjanczyk, Rzazewski and Zakerzewski 1985, Haus, Lewenstein and Rzazewski 1984) single mode rectangle (Rodgers and Swain 1991, Hassan, Joshi and Al-Madhari 2008) and multi-mode rectangular (Joshi and Hassan 2002) and exponential (Batarfi 2012) pulse shapes.

Unlike the much work with the 2-level atom, the pulsed driven HO system has not received much attention regarding the spectrum analysis. The work in (Joshi and Hassan 2002) studied in addition to the 2-level atom case, the scattered spectrum of HO model excited by a multi-mode (symmetrically displaced on both sides of the HO frequency) rectangular

pulse, where the main results are:

- (1) The larger the number of exciting modes is the more prominent oscillatory nature of the transient spectrum.
- (2) For large (∞) number of modes, within a restriction of pulse duration, the dominant central peak is reduced ("hole burning" effect) with increased symmetrical side peaks, as the area of the pulse increases.

The pulsed-driven HO system can be viewed in many ways, e.g :

- (i) A fundamental example of radiation-radiation interaction where the HO represents a single mode of the quantized electromagnetic (em) field in a Fock (number) or coherent state representation.
- (ii) A limiting case of many excited atoms contained in a very small volume $O(\lambda^3)$; $\lambda = 10^{-5}$ cm, is the optical radiation wave length- as mentioned in section 1.2.
- (iii) Recent study shows that nano-biosensors detecting biomolecules are physically justified to be *modeled* as a perturbed HO (cf Okamoto et al. 2013, Snyder et al. 2014).

Here, we extend the work in (Joshi and Hassan 2002) by exciting the HO with the classical sequential squared-sine pulse.

CHAPTER 3

METHODOLOGY: QUANTIZED HAMILTONIAN AND MODEL EQUATIONS

- 3.1 Quantum Mechanics Framework
- 3.2 Quantized Harmonic Oscillator
 - 3.2.1 The Hamiltonian
 - 3.2.2 The Eigenvalue Problem
 - 3.2.3 Normalization
 - 3.2.4 Matrix Representation
 - 3.2.5 The Coherent State
- 3.3 Hamiltonian Model and Heisenberg Equations
- 3.4 Transient Spectrum

CHAPTER 3

METHODOLOGY: QUANTIZED HAMILTONIAN AND MODEL EQUATIONS

In this chapter, we briefly review in section 3.1 the assumptions and framework of quantum mechanics (QM) in terms of Dirac's formulation (Dirac 1981). The specific eigenvalue problem of the quantized harmonic oscillator (HO) is reviewed in section 3.2 for completeness of presentation and notations to be used in the following chapter. The interaction Hamiltonian model of HO with arbitrary pulse shape together with Heisenberg equations and its general solution is given in section 3.3 . The expressions for the transient spectrum in terms of the auto-correlation function is given in section 3.4 .

3.1 Quantum Mechanics Framework

As noted in chapter 1, observed experiments with blackbody radiation does not fit with classical mechanics criteria. Hence, the need to adopt the quantum mechanical (QM) description of the HO.

The basic assumptions to describe a dynamical system within Dirac's framework of QM (Dirac 1981) are the following:-

- (i) Observable dynamical variables (like, energy, momentum, coordinate,..) are represented by Hermitian operators (i.e. Hermitian matrices).
- (ii) The state of a dynamical system (the momentum (p) and coordinate (q)) is represented by state vector: ket vector $|p, q\rangle$ or its adjoint bra vector $\langle p, q|$ (i.e. column or row matrix, respectively).

The order of the state vector and the Hermitian operator and the expressions of their matrix elements depend on the degrees of freedom of the system and the eigenvalues of the concerned Hermitian operator.

For example, for a system of 2-degrees of freedom:

The ket vector $|a\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and the corresponding bra vector $\langle a|$ (mirror image of $\langle a|$) $= (|a\rangle)^\dagger = [a_1^* \quad a_2^*]$

where $(\dagger)$ stands for Hermitian conjugate (i.e. complex conjugate of the transposed of the ket vector $|a\rangle$).

If the operator $\hat{O} = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix}$, where the elements O_{ij} are complex numbers in general, then its Hermitian conjugate is given by

$$\hat{O}^\dagger = \begin{bmatrix} O_{11}^* & O_{21}^* \\ O_{12}^* & O_{22}^* \end{bmatrix}.$$

The operator $\hat{O}$ is called Hermitian operator if $\hat{O} = \hat{O}^\dagger$ and as known any Hermitian matrix has real eigenvalues.

We now summarize the mathematical operations and its concept in dealing with operators, ket and bra vectors:-

(1) if $|a\rangle, |b\rangle$ are possible state (ket) vectors of a system, then the superposition state,

$$|c\rangle = \alpha_1 |a\rangle + \alpha_2 |b\rangle \quad (3.1)$$

is also a state of the system, where $\alpha_{1,2}$ are complex numbers. The equivalent bra vector to (3.1) is

$$\langle c| = \alpha_1^* \langle a| + \alpha_2^* \langle b| \quad (3.2)$$

(2) The inner (direct) product $(\langle c|)(|c\rangle) \equiv \langle c|c\rangle$, called bracket, is real and positive. If $\langle c|c\rangle = 1$ then the state $|c\rangle$ is said to be normalized. If not, it can be normalized by multiplying it by $\langle c|c\rangle^{-\frac{1}{2}}$. Note, the norm of $|c\rangle$ is $\sqrt{\langle c|c\rangle} \geq 0$ and its length is $\langle c|c\rangle$.

(3) Two different states $|a_1\rangle, |a_2\rangle$ are called orthogonal iff the bracket

$$\langle a_1|a_2\rangle \equiv \langle a_2|a_1\rangle^* = 0 \quad .$$

Note, if $|c\rangle$ in equation (3.1) is normalized where $|a\rangle, |b\rangle$ are orthogonal, then $|\alpha_1|^2 + |\alpha_2|^2 = 1$.

The quantities $|\alpha_1|^2, |\alpha_2|^2$ are the probabilities that the system exists in the states $|a\rangle, |b\rangle$, respectively.

(4) A linear operator $\hat{O}$ means an association (or transformation) from $|a\rangle \rightarrow |b\rangle$; (as the action of $\hat{O}$ on $|a\rangle$ generates $|b\rangle$)

$$\hat{O}|a\rangle = |b\rangle \quad (3.3)$$

(providing the matrix product on the left hand side is allowed).

(5) The outer product of two normalized states $|a\rangle, |b\rangle$ is defined as,

$$|b\rangle\langle a| = (|a\rangle\langle b|)^\dagger. \quad (3.4)$$

It is an operator, since operating on arbitrary $|c\rangle$,

$$(|b\rangle\langle a|)|c\rangle = \langle a|c\rangle |b\rangle, \quad (3.5)$$

produces the state $|b\rangle$ multiplied by the complex number $\langle a|c\rangle$.

The special operator $|a\rangle\langle a|$ is called projection operator, since operating on $|a\rangle$ produces (projects) the same normalized state $|a\rangle$.

(6) The eigenvalue equation for a Hermitian operator $\hat{O}$ is,

$$\hat{O}|a_n\rangle = a_n|a_n\rangle \quad (3.6)$$

where the real numbers $\{a_n\}$ are the eigenvalues of $\hat{O}$ associated with the orthogonal eigenvectors $\{|a_n\rangle\}$.

If the set $\{|a_n\rangle\}$ is a basic set of eigenvectors (i.e each ket vector $|a_n\rangle$ is not a linear combination of other ket vectors), then,

$$\sum_n |a_n\rangle\langle a_n| = \hat{I} \quad (3.7)$$

where $\hat{I}$ is the identity (Hermitian) operator. Equation (3.7), expresses the fact that the sum of all projection operators associated with the basic state vectors is the identity $\hat{I}$. It is the quantum counter-part of the classical statement that, the sum of probabilities of the system being in all possible states is equal (1).

(7) Any arbitrary state $|M\rangle$ can be expressed as linear combination of the complete basic set $\{|a_n\rangle\}$,

$$|M\rangle = \sum_n c_n |a_n\rangle \quad (3.8)$$

where c_n are complex numbers.

(8) The quantum expectation (average) of operator $\hat{O}$ in the state $|a\rangle$, is the bracket,

$$\langle \hat{O} \rangle = \langle a | \hat{O} | a \rangle \quad (3.9)$$

and it is real if $\hat{O}$ is Hermitian operator. Also, the bracket expression,

$$\langle b | \hat{O} | a \rangle \quad (3.10)$$

is called a transformation function from the state $(\hat{O}|a\rangle)$ to the state $|b\rangle$.

The Quantum Mechanics dynamical behavior in Heisenberg's picture [Dirac 1981, Liboff 2003] deals with operators ($\hat{O}$) varying with time while state vectors kept fixed. Heisenberg equation of motion for the operator $\hat{O}$ is, (see Appendix (A)),

$$i\hbar \frac{d\hat{O}}{dt} = [\hat{O}, \hat{H}] \quad (3.11)$$

where the commutator,

$$[\hat{O}, \hat{H}] = \hat{O}\hat{H} - \hat{H}\hat{O} \quad (3.11a)$$

and $\hat{H}$ is the total Hamiltonian (energy) operator of the system.

3.2 Quantized Harmonic Oscillator

Our universe is composed of matter and radiation. Radiation is absorbed and emitted by matter, or equivalently, matter absorbs and emits radiation. Light, and radiation in general, is a wave in its simplest shape and is represented by a simple harmonic motion or simple harmonic oscillator (SHO). The SHO is the simplest, yet basic example of free (i.e. undamped) periodic motion and plays a central role in the quantum theo-

retical description of radiation. This stems from the fact that the energy of the radiation of the electromagnetic field in free space is equivalent to that of infinite collection of independent harmonic oscillators (e.g. (Liboff 2003)).

Within the Newtonian classical description of a single harmonic oscillator (HO) of unit mass in one dimension and of frequency (ω) the Hamiltonian (total energy) is given by (e.g. (Liboff 2003, Scherrer 2006)),

$$H_c = \frac{1}{2}(p^2 + \omega^2 q^2) \quad (3.12)$$

where p, q are the momentum and coordinate (displacement) dynamical variables, respectively. The first term in (3.12) represents the kinetic energy while the second term represents the potential energy (due to a force proportional to displacement). Classically speaking, the energy H_c of the HO is a continuous variable of zero minimum value.

3.2.1 The Hamiltonian

For our review concern, the starting point is the corresponding QM Hamiltonian of the HO - as opposed to the classical one in equation (3.12), which has the form (Dirac 1981, Liboff 2003),

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2) \quad (3.13)$$

where the momentum and coordinate operators $\hat{p}$ and $\hat{q}$ satisfy the commutation relation (Dirac 1981, Liboff 2003),(see Appendix (B)),

$$[\hat{q}, \hat{p}] = i\hbar \quad (3.14)$$

where $\hbar$ is the (reduced) Planck's constant (see chapter (1)).

Note that, both $\hat{p}$ and $\hat{q}$ are Hermitian operators, i.e. $\hat{p} = \hat{p}^\dagger, \hat{q} = \hat{q}^\dagger$ and hence $\hat{H} = \hat{H}^\dagger$ is Hermitian operator.

To solve the eigenvalue problem associated with the quantized harmonic oscillator (i.e. energy eigenvalues and its corresponding state vectors), it is convenient to introduce the two non-Hermitian operators $\hat{a}, \hat{a}^\dagger$ (annihilation and creation operators)(Dirac 1981),

$$\begin{aligned}\hat{a} &= \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}) \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p}).\end{aligned}\tag{3.15}$$

Hence we get $\hat{p}, \hat{q}$ in the form,

$$\begin{aligned}\hat{p} &= i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a}) \\ \hat{q} &= \sqrt{\frac{\hbar}{2\omega}}(\hat{a}^\dagger + \hat{a}).\end{aligned}\tag{3.16}$$

From equations (3.14),(3.16) the commutator relation for the operators $\hat{a}, \hat{a}^\dagger$ is

$$[\hat{a}, \hat{a}^\dagger] = 1 \quad .\tag{3.17}$$

The quantized Hamiltonian of the HO, equation (3.13) has then the form,

$$\begin{aligned}\hat{H} &= \frac{\hbar\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \\ &= \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \quad .\end{aligned}\tag{3.18}$$

Notice, in (3.18) the constant term ($\frac{1}{2}\hbar\omega$) is called the zero-point (i.e. minimum) energy value of the HO (as we show below).

To find the energy eigenvalues and eigenvectors of the quantized HO, define the Hermitian number operator

$$\hat{N} = \hat{a}^\dagger \hat{a} = \hat{N}^\dagger . \quad (3.19)$$

Hence, the Hamiltonian (3.18) has the form,

$$\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2}) . \quad (3.20)$$

Thus, finding the eigenvalues and eigenvectors of the operator $\hat{N}$ lead to the same (after adding the term $\frac{1}{2}$ to the eigenvalues) for $\hat{H}$.

3.2.2 The Eigenvalue Problem

The energy eigenvalue problem, within the framework of Quantum Mechanics, is to solve the equation

$$\hat{N}|n\rangle = n|n\rangle \quad (3.21)$$

where (n) is the eigenvalue of the Hermitian number operator $\hat{N}$ with corresponding eigenket $|n\rangle$. In other words, we need to know n and $|n\rangle$ according to (3.21).

Using the commutator identity for any three operators $\hat{A}, \hat{B}, \hat{C}$:

$$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}] \quad (3.22)$$

and the commutator (3.17), one gets the following commutator relations,

$$[\hat{N}, \hat{a}] = -\hat{a}, \quad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger \quad (3.23a)$$

$$[\hat{N}, \hat{a}^2] = -2\hat{a}^2, \quad [\hat{N}, \hat{a}^{\dagger 2}] = 2\hat{a}^{\dagger 2} \quad (3.23b)$$

and in general (using mathematical induction) for positive integer (m),

$$[\hat{N}, \hat{a}^m] = -m\hat{a}^m, \quad [\hat{N}, \hat{a}^{\dagger m}] = m\hat{a}^{\dagger m}. \quad (3.23c)$$

Now, we proceed to solve the eigenvalue problem (3.21) with (3.23) (cf (Louisell 1973)).

First, from eq. (3.23a) we have:

$$\begin{aligned} \hat{N}\hat{a} &= \hat{a}(\hat{N} - 1) \\ \hat{N}\hat{a}^\dagger &= \hat{a}^\dagger(\hat{N} + 1) \end{aligned} \quad (3.24)$$

Operating both sides of eqs. (3.24) on $|n\rangle$ and use of (3.21) we get

$$\begin{aligned} \hat{N}\hat{a}|n\rangle &= \hat{a}(\hat{N} - 1)|n\rangle = (n - 1)\hat{a}|n\rangle \\ \hat{N}\hat{a}^\dagger|n\rangle &= \hat{a}^\dagger(\hat{N} + 1)|n\rangle = (n + 1)\hat{a}^\dagger|n\rangle \end{aligned} \quad (3.25)$$

Second, from eq. (3.23b) we have,

$$\begin{aligned} \hat{N}\hat{a}^2 &= \hat{a}^2(\hat{N} - 2) \\ \hat{N}\hat{a}^{\dagger 2} &= \hat{a}^{\dagger 2}(\hat{N} + 2) \end{aligned} \quad (3.26)$$

and upon operating both sides of (3.26) on $|n\rangle$ and use of (3.21) we get,

$$\begin{aligned} \hat{N}\hat{a}^2|n\rangle &= (n - 2)\hat{a}^2|n\rangle \\ \hat{N}\hat{a}^{\dagger 2}|n\rangle &= (n + 2)\hat{a}^{\dagger 2}|n\rangle. \end{aligned} \quad (3.27)$$

From (3.23c) and continuing along the same lines for $m > 2$, we reach similar results like (3.25), (3.27), namely ,

$$\begin{aligned}\hat{N}\hat{a}^m|n\rangle &= (n-m)\hat{a}^m|n\rangle \\ \hat{N}\hat{a}^{\dagger m}|n\rangle &= (n+m)\hat{a}^{\dagger m}|n\rangle .\end{aligned}\tag{3.28}$$

Thus from (3.21), if $|n\rangle$ is an eigenstate of $(\hat{N})$ with eigenvalue (n) , then by (3.25) $(\hat{a}|n\rangle), (\hat{a}^\dagger|n\rangle)$ are eigenstates of $\hat{N}$ with eigenvalues $(n-1), (n+1)$, respectively. As well, by (3.27), $(\hat{a}^2|n\rangle), (\hat{a}^{\dagger 2}|n\rangle)$ are eigenstates of $\hat{N}$ with eigenvalues $(n-2), (n+2)$, respectively.

Proceeding, using (3.28), we then get two ∞ - set of eigenkets and eigenvalues (differ by 1) of the number operator $\hat{N}$ as follows,

Table (1)

First set of eigenvalues and eigenkets of the operator $\hat{N}$

Eigenkets	$ n\rangle$	$\hat{a} n\rangle$	$\hat{a}^2 n\rangle \dots$
Eigenvalues	n	$(n-1)$	$(n-2)\dots$

(3.29)

and

Table (2)

Second set of eigenvalues and eigenkets of the operator $\hat{N}$

Eigenkets	$ n\rangle$	$\hat{a}^\dagger n\rangle$	$\hat{a}^{\dagger 2} n\rangle \dots$
Eigenvalues	n	$(n+1)$	$(n+2)\dots$

(3.30)

Since $\hat{N}$ is Hermitian operator, then (n) is real and by assumption the length of the bracket, $(\langle n|n \rangle) > 0$ for non-zero state $|n \rangle$.

From (3.21), multiplying on the left by $\langle n|$, we get

$$\begin{aligned} \langle n|\hat{N}|n \rangle &= \langle n|\hat{a}^\dagger \hat{a}|n \rangle \\ &\equiv (\langle n|\hat{a}^\dagger)(\hat{a}|n \rangle) \end{aligned} \quad (3.31)$$

$$= n \langle n|n \rangle \quad (3.31a)$$

Since the right side of (3.31) is the length of the ket vector $(\hat{a}|n \rangle)$ which is non-negative. Thus, from (3.31a), $n \geq 0$ and hence the eigenvalues of $\hat{N}$ are real and non-negative.

If $n = 0$, then

$$\hat{a}|0 \rangle = 0 \quad (3.32)$$

and the norm of the state $|0 \rangle$ is zero.

If $n \neq 0$ then the length of $\hat{a}|n \rangle$ is given by (3.31). By (3.29) if (n) is not an integer, the eigenvalues will be negative and this leads to contradiction (see after (3.31a)).

Hence, the eigenvalues (n) of $(\hat{N})$ are *real and positive integers or zero with no upper bound*, as seen from (3.30): $n = 0, 1, 2, \dots$

Thus from (3.20) and (3.21) the energy eigenvalue problem,

$$\hat{H}|n\rangle = \hbar\omega(\hat{N} + \frac{1}{2})|n\rangle \quad (3.33)$$

show that the energy eigenvalues are discrete, $\hbar\omega(n + \frac{1}{2})$; $n = 0, 1, 2, \dots$. The eigenstates $\{|n\rangle\}$ are called the number (or photon) energy states, and the state $|0\rangle$ is called the vacuum energy state of eigenvalue $\frac{1}{2}\hbar\omega$, which is the minimum value (called quantum zero-point energy) as opposed to the zero classical minimum value.

The quantized (discrete) energy level diagram (full lines) for the HO is shown in Figure 3.1 together with the classical continuous energy (dashed lines).

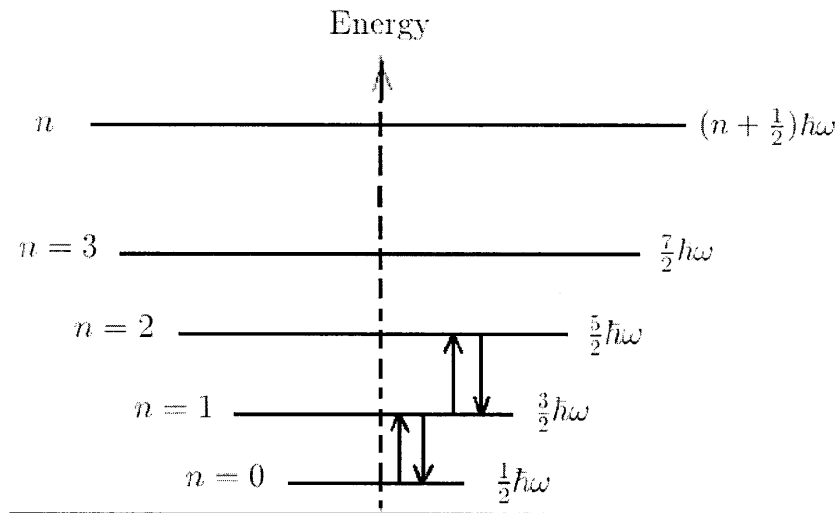


Figure 3.1

The quantized energy levels of the HO

$$E_n = (n + \frac{1}{2})\hbar\omega; n = 0, 1, 2, \dots$$

3.2.3 Normalization

The length of the state vector ($\hat{a}^\dagger|n\rangle$) is

$$\langle n|\hat{a}\hat{a}^\dagger|n\rangle = \langle n|(1 + \hat{a}^\dagger\hat{a})|n\rangle$$

$$= (n+1) \langle n|n \rangle .$$

Since $n \geq 0$, $\langle n|n \rangle > 0$ then $\hat{a}^\dagger |n \rangle \neq 0$, i.e. not zero vector. From (3.21) and (3.25) both $|n \rangle$ and $\hat{a}|n \rangle$ are eigenkets of the operator $\hat{N}$. So, we may write,

$$\hat{a}|n \rangle = C_n |n-1 \rangle \quad (3.34)$$

with C_n is constant. The length of this equation gives;

$$\langle n|\hat{a}^\dagger \hat{a}|n \rangle = |C_n|^2 \langle n-1|n-1 \rangle$$

$$= n \langle n|n \rangle$$

and for $\langle n|n \rangle = 1$ for all (n) we have $C_n = \sqrt{n}$ (up to an arbitrary phase).

So, (3.34) gives

$$\hat{a}|n \rangle = \sqrt{n} |n-1 \rangle, \quad n > 0. \quad (3.35)$$

Similarly, we get

$$\hat{a}^\dagger |n \rangle = \sqrt{n+1} |n+1 \rangle . \quad (3.36)$$

The orthonormality and completeness relation for the states $\{|n \rangle\}$ are,

$$\langle n|\acute{n} \rangle = \delta_{n\acute{n}} \quad (3.37a)$$

$$, \sum_{n=0}^{\infty} |n\rangle \langle n| = \hat{I} \quad (3.37b)$$

where $\delta_{nn'}$ is the Kronecker delta symbol,

$$\delta_{nn'} = \begin{cases} 1 & , n = n' \\ 0 & , n \neq n'. \end{cases}$$

The relation (3.35), (3.36) shows the action of the annihilation and creation operators, $(\hat{a})$ and $(\hat{a}^\dagger)$, respectively, on the state $|n\rangle$ is to transform to lower and higher states $|n-1\rangle$ and $|n+1\rangle$, respectively.

3.2.4 Matrix Representation

The matrix elements of the operators $\hat{a}, \hat{a}^\dagger$ and $\hat{H}$ in the $|n\rangle$ states are given with the use of (3.32), (3.33), (3.35), (3.36) as follows,

$$\begin{aligned} \langle n' | \hat{a} | 0 \rangle &= 0 \\ \langle n' | \hat{a} | n \rangle &= \sqrt{n} \delta_{n', n-1} \\ \langle n' | \hat{a}^\dagger | n \rangle &= \sqrt{n+1} \delta_{n', n+1} \\ \langle n' | \hat{H} | n \rangle &= \hbar \omega (n + \frac{1}{2}) \delta_{n', n} \end{aligned} \quad (3.38)$$

Since the basic states $\{|n\rangle\}; n = 0, 1, 2, \dots$ is an ∞ -set then these operators have square matrix representation of ∞ -order,

$$\hat{a} = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\hat{a}^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ \sqrt{1} & 0 & 0 & 0 & \dots \\ 0 & \sqrt{2} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

$$\hat{H} = \hbar\omega \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \dots \\ 0 & \frac{3}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{5}{2} & 0 & \dots \\ 0 & 0 & 0 & \frac{7}{2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

3.2.5 The Coherent State

According to the relations (3.35), (3.36) absorbing or emitting a photon within the quantum theory changes the basic state $|n\rangle \rightarrow |n \mp 1\rangle$.

Another state (which is a linear combination of the basic states $|n\rangle$) is close to the classical description (that is no change of state upon absorbing or emitting a photon) and most suitable to describe ideal laser light quantum mechanically is the coherent state $|\alpha\rangle$. Coherency means monochro-

maticity. In other words, the laser radiation has a single frequency or a narrow frequency band.

Mathematically, it is defined as an eigenstate of the non-Hermitian annihilation operator $\hat{a}$, (cf (Louisell 1973)),

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \quad (3.39)$$

with α is complex parameter (in general).

To find $|\alpha\rangle$ in terms of the basic states $\{|n\rangle\}; n = 0, 1, \dots$ we set,

$$|\alpha\rangle = \sum_{n=0}^{\infty} b_n |n\rangle \quad (3.40)$$

$$\equiv \sum_{n=0}^{\infty} |n\rangle \langle n|\alpha\rangle \quad (3.40a)$$

where b_n is constant and in (3.40a) we have used the completeness relation (3.37b) for the states $|n\rangle$.

Thus $b_n = \langle n|\alpha\rangle$, which is to be determined as follows. Operating with $(\hat{a})$ on (3.40) and use (3.39) we get

$$\begin{aligned} \hat{a}|\alpha\rangle &= \sum_{n=1}^{\infty} b_n \sqrt{n} |n-1\rangle \\ &= \sum_{n=0}^{\infty} b_{n+1} \sqrt{n+1} |n\rangle \end{aligned} \quad (3.41a)$$

$$\equiv \sum_{n=0}^{\infty} \alpha b_n |n\rangle \quad (3.41b)$$

Comparing the coefficients of both series, we have the difference equation,

$$b_{n+1} \sqrt{n+1} = \alpha b_n \quad (3.42)$$

of solution,

$$b_n = \frac{\alpha^n}{\sqrt{n!}} b_0, \quad n = 1, 2, \dots \quad (3.43)$$

To get b_0 we assume $\langle \alpha | \alpha \rangle = 1$.

Hence,

$$\begin{aligned} 1 &= |b_0|^2 \sum_{n, \dot{n}=0}^{\infty} \frac{(\alpha^*)^{\dot{n}} (\alpha)^n}{\sqrt{\dot{n}! n!}} \langle \dot{n} | n \rangle \\ &= |b_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \end{aligned}$$

where the relation (3.37a) has been used. Thus,

$$1 = |b_0|^2 e^{|\alpha|^2}$$

which gives,

$$b_0 = e^{-\frac{1}{2}|\alpha|^2} \quad (3.44)$$

up to a phase (which can be absorbed in α).

So, by (3.43), (3.44), we have

$$b_n = e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} \quad (3.45)$$

$$\text{and} \quad |b_n|^2 \equiv |\langle n | \alpha \rangle|^2 = \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2} \quad (3.46)$$

represents a Poisson distribution with mean $|\alpha|^2$ over the basic number

states $|n\rangle$.

Hence, by (3.40), (3.45), the expression for $|\alpha\rangle$ is,

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle . \quad (3.47)$$

Further relations in the $|\alpha\rangle$ – state from (3.39) are:

$$\langle \alpha | \hat{a}^\dagger = \alpha^* \langle \alpha | \quad (3.48a)$$

$$\langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = |\alpha|^2 \quad (3.48b)$$

$$\langle \alpha | \hat{a} | \alpha \rangle = \alpha \quad (3.48c)$$

$$\langle \alpha | \hat{a}^\dagger | \alpha \rangle = \alpha^* . \quad (3.48d)$$

The relation (3.48b) states that the average photon number in the $|\alpha\rangle$ – state is $|\alpha|^2$ (opposed to $\langle n \rangle$ in the $|n\rangle$ – state) while the relations (3.48c,d) show that the HO quantum average amplitudes $\langle \hat{a} \rangle, \langle \hat{a}^\dagger \rangle$ are non-zero in the $|\alpha\rangle$ – state (opposed to zero in the $|n\rangle$ – state).

3.3 Hamiltonian Model and Heisenberg Equations

The Hamiltonian model representing the interaction of a single HO with a classical pulse is of the following form (Joshi and Hassan 2002) (in units of $\hbar = 1$),

$$H = \omega_0 \hat{a}^\dagger \hat{a} + \Omega_o f(t) (\hat{a}^\dagger e^{-i\omega_L t} + \hat{a} e^{i\omega_L t}) \quad (3.49)$$

where ω_0 = HO frequency , ω_L = oscillating frequency of the pulse en-

velope, $f(t)$ = pulse shape function (not operator), Ω_0 = strength (Rabi frequency) of pulse.

The HO operators $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators, respectively, obeying the commutator algebra (3.17),

The first term of the Hamiltonian (3.49) is the unperturbed Hamiltonian of the HO (eq (3.18) apart from the constant term $\frac{1}{2}\hbar\omega$, which does not affect the dynamics) while the rest of terms (in Ω_0) represent the interaction Hamiltonian between the HO and the classical pulse. (Derivation of this interaction Hamiltonian given in (Louisell 1973) and refs therein is beyond our presentation in this thesis work).

Heisenberg equations of motion (equation (3.11)) for the operators $\hat{a}$ and $\hat{a}^\dagger$ according to the Hamiltonian (3.49) and the commutator algebra (3.17) are,

$$\begin{aligned}\dot{\hat{a}} &= -i\omega_0\hat{a} - i\Omega_0 f(t)e^{-i\omega_L t} \\ \dot{\hat{a}^\dagger} &= i\omega_0\hat{a}^\dagger + i\Omega_0 f(t)e^{i\omega_L t}\end{aligned}\tag{3.50}$$

The operator solutions of these linear differential equations are easily obtained for arbitrary pulse shape $f(t)$,

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega_0 t} - i\Omega_0 e^{-i\omega_0 t} \int_0^t f(t')e^{-i\Delta t'} dt' \tag{3.51}$$

$$= e^{-i\omega_0 t} \hat{A}(t) \tag{3.51a}$$

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega_0 t} + i\Omega_0 e^{i\omega_0 t} \int_0^t f(t')e^{i\Delta t'} dt' \tag{3.52}$$

$$= e^{i\omega_0 t} \hat{A}^\dagger(t) \tag{3.52a}$$

where,

$$\hat{A}(t) = \hat{a}(0) - i\Omega_0 \int_0^t f(t') e^{-i\Delta t'} dt' \quad (3.53)$$

and $\Delta = (\omega_L - \omega_0)$ is a frequency detuning parameter.

The HO photon number operator $\hat{n}(t) \equiv \hat{a}^\dagger(t)\hat{a}(t)$ is given by,

$$\begin{aligned} \hat{n}(t) &= (\hat{a}^\dagger(0)e^{i\omega_0 t} + i\Omega_0 e^{i\omega_0 t} \int_0^t f(t') e^{i\Delta t'} dt') \\ &\quad \cdot (\hat{a}(0)e^{-i\omega_0 t} - i\Omega_0 e^{-i\omega_0 t} \int_0^t f(t') e^{-i\Delta t'} dt') \\ &= \hat{a}^\dagger(0)\hat{a}(0) + \Omega_0^2 |I(t)|^2 - i\Omega_0 \hat{a}^\dagger(0)I(t) + i\Omega_0 \hat{a}(0)I^*(t) \end{aligned} \quad (3.54)$$

where,

$$I(t) = \int_0^t f(t') e^{-i\Delta t'} dt' . \quad (3.55)$$

The averaged photon number $n(t) = \langle \hat{n}(t) \rangle$ of the HO, in an arbitrary state $| \rangle$ is then given by,

$$n(t) = \langle \hat{a}^\dagger(0)\hat{a}(0) \rangle + \Omega_0^2 |I(t)|^2 - i\Omega_0 (\langle \hat{a}^\dagger(0) \rangle I(t) - c.c.) \quad (3.56)$$

where *c.c* stands for complex conjugate.

The first term in (3.56) represents the initial average photon number of the HO, while the second term in Ω_0^2 represents the intensity of the (classical) pulse. The last term in (3.56) represents the change in the photon number $n(t)$ (i.e. energy) due to the exchange of initial amplitude $\langle \hat{a}^\dagger(0) \rangle$ of

the HO with driving pulse.

The averaged photon number $n(t)$, eq. (3.56), takes the following forms in the three specific initial conditions for the HO,

(a) In the zero (vacuum) number state $|0\rangle$,

$$\langle \hat{a}^\dagger(0)\hat{a}(0) \rangle = 0, \quad \langle \hat{a}^\dagger(0) \rangle = 0,$$

and (3.56) reduced to

$$n_0(t) = \Omega_0^2 |I(t)|^2. \quad (3.57a)$$

(b) In the number state $|n\rangle$ ($n \neq 0$),

$$\langle \hat{a}^\dagger(0)\hat{a}(0) \rangle = n_0, \quad \langle \hat{a}^\dagger(0) \rangle = 0,$$

and (3.56) reduced to

$$n(t) = n_0 + \Omega_0^2 |I(t)|^2. \quad (3.57b)$$

(c) In the Coherent state $|\alpha\rangle$,

$$\langle \hat{a}^\dagger(0)\hat{a}(0) \rangle = |\alpha|^2, \quad \langle \hat{a}^\dagger(0) \rangle = \alpha^*,$$

and (3.56) takes the general form,

$$n_\alpha(t) = |\alpha|^2 + \Omega_0^2 |I(t)|^2 + 2\Omega_0 \text{Im}(\alpha^* I(t)). \quad (3.57c)$$

3.4 Transient Spectrum

Information about the scattered (fluorescent) radiation due to the interaction of the HO with the driving pulse is achieved through the transient spectrum function which is proportional to the counting rate of the emitted photons. This is given by (Joshi and Hassan 2002, Eberly and Wodkiewicz 1977),

$$S(t, D, \Gamma) = 2\Gamma \int_0^t dt_1 \int_0^t dt_2 e^{(-\Gamma + iD)(t-t_1)} e^{(-\Gamma - iD)(t-t_2)} < \hat{A}^\dagger(t_1) \hat{A}(t_2) > \quad (3.58)$$

where $D = \omega - \omega_0$ is the frequency mismatch between the frequency (ω) of the instrument photon detector and the frequency (ω_0) of the HO and Γ is the detector's width (i.e. Γ^{-1} is the life-time or time-duration of the detected photon in the instrument). The function $< \hat{A}^\dagger(t_1) \hat{A}(t_2) >$, with $\hat{A}^\dagger$ (and its adjoint) given in (3.53), is the auto-correlation function for the shift (creation and annihilation) operators. It represents a measure of quantum average of correlation between the action of creating a photon ($\hat{A}^\dagger(t_1)$) at time (t_1) and annihilation ($\hat{A}(t_2)$) at time (t_2). In the language of statistical mechanics (cf (Reif 2008, Patheria 1996)), the integration of the quantum expectation value $< \hat{A}^\dagger(t_1) \hat{A}(t_2) >$ over both times (t_1), (t_2) represents a kind of ensemble average over the single actions at (t_1), (t_2).

From (3.53), (3.55) we have ,

$$\begin{aligned} \hat{A}(t_2) &= \hat{a}(0) - i\Omega_0 I(t_2) \\ \hat{A}^\dagger(t_1) &= \hat{a}^\dagger(0) + i\Omega_0 I^*(t_1). \end{aligned} \quad (3.59)$$

Thus,

$$\begin{aligned}
& \langle \hat{A}^\dagger(t_1)\hat{A}(t_2) \rangle = \langle \hat{a}^\dagger(0)\hat{a}(0) \rangle + \Omega_0^2 I^*(t_1)I(t_2) \\
& \quad - i\Omega_0 \langle \hat{a}^\dagger(0)I(t_2) \rangle \\
& \quad + i\Omega_0 \langle \hat{a}(0)I^*(t_1) \rangle
\end{aligned} \tag{3.60}$$

where $I(t)$ is given by (3.55).

Inserting (3.60) into (3.58), we have the following form for the spectrum,

$$\begin{aligned}
S(t, D, \Gamma) = & 2\Gamma e^{-2\Gamma t} [n(0)|J_1(t)|^2 \\
& + \Omega_0^2 |J_2(t)|^2 \\
& - 2\Omega_0 \text{Im}(\bar{a}(0)J_1^*(t)J_2(t))]
\end{aligned} \tag{3.61}$$

where,

$$\begin{aligned}
J_1(t) &= \int_0^t e^{(\Gamma - iD)t'} dt' \\
&= \frac{e^{(\Gamma - iD)t} - 1}{\Gamma - iD}
\end{aligned} \tag{3.62a}$$

$$J_2(t) = \int_0^t e^{(\Gamma - iD)t'} I^*(t') dt' \tag{3.62b}$$

and $n(0) = \langle a^\dagger(0)a(0) \rangle$, $\bar{a}(0) = \langle \hat{a}(0) \rangle$.

Effect of pulse shape, through $I(t)$ in $J_2(t)$, is shown in the last two terms in (3.61), while the first term is only associated with the initial average photon number, $n(0)$, of the HO.

Three cases of initial conditions of the HO, to be considered, namely:

(a) The HO is initially in the zero (vacuum) number state $|0\rangle$, $n(0) = 0$, $\bar{a}(0) = 0$, so (3.61) is reduced to,

$$S_0(t, D, \Gamma) = 2\Gamma\Omega_0^2 e^{-2\Gamma t} |J_2(t)|^2 . \quad (3.63a)$$

(b) The HO is initially in the number state $|n\rangle$ ($n \neq 0$), $n(0) = n_0$, $\bar{a}(0) = 0$, and (3.61) is reduced to,

$$S_{n_0}(t, D, \Gamma) = 2\Gamma e^{-2\Gamma t} (n_0 |J_1(t)|^2 + \Omega_0^2 |J_2(t)|^2) . \quad (3.63b)$$

(c) The HO is initially in the coherent state $|\alpha\rangle$, $n(0) = |\alpha|^2$, $\bar{a}(0) = \alpha$, and (3.61) is of the general form,

$$\begin{aligned} S_\alpha(t, D, \Gamma) = & 2\Gamma e^{-2\Gamma t} [|\alpha|^2 |J_1(t)|^2 \\ & + \Omega_0^2 |J_2(t)|^2 \\ & - 2\Omega_0 \text{Im}(\alpha J_1^*(t) J_2(t))] . \end{aligned} \quad (3.63c)$$

The general expression (3.57) for the averaged photon number of the HO and (3.63) for the corresponding transient spectrum are examined analytically and computationally in the following chapter in the special case of pulse shapes, namely, squared-sine pulse.

CHAPTER 4

RESULTS: SQUARED-SINE PULSE CASE

- 4.1 The Pulse Shape
- 4.2 Averaged Photon Number
- 4.3 Transient Spectrum
 - 4.3.1 Vacuum State
 - 4.3.2 Number State
 - 4.3.3 Coherent State
- 4.4 Conclusion and Future Work

CHAPTER 4

RESULTS: SQUARED-SINE PULSE CASE

4.1 The Pulse Shape

The case of the pulse field of envelope form,

$$f(t) = \sin^2(n\omega_0 t); \quad n = 1, 2, \dots \quad (4.1)$$

$$= \sin^2(n\tau); \quad \tau = \omega_0 t$$

represents a pulsed laser field of sequential n -pulse in the period $[0, n\pi]$ with (π) is a one pulse duration (see Figure 4.1). Note, in view of the interaction terms, $f(t)e^{\pm i\omega_L t}$ in eq (3.49), $n\omega_0$ is known as the beating frequency of the oscillating pulse, while ω_L , as noted below eq (3.4a), is the frequency of the pulse envelope. In electrical engineering context, this is known as rectified full-wave or "hum", which allows more electrical current to flow in one direction than the other (i.e. it converts alternating current, a.c., to direct current, d.c.).

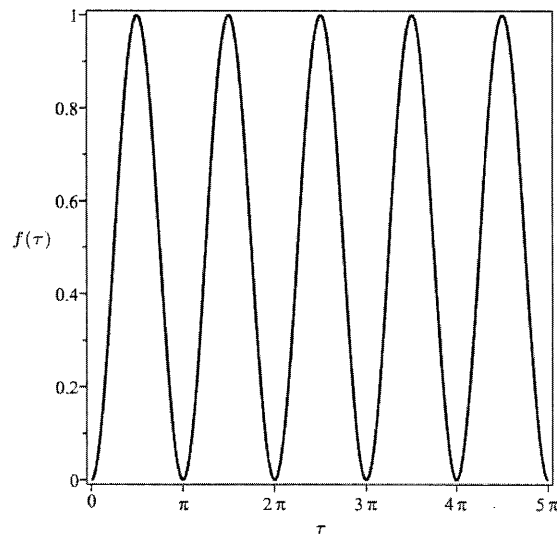


Figure 4.1
Plot of $f(\tau)$ against τ in period $[0, 5\pi]$

With the pulse form (4.1), the integration (3.55) is now given as follows,

$$\begin{aligned}
I(t) &= \int_0^t f(t') e^{-i\Delta t'} dt' \\
&= \int_0^t \sin^2(\omega_n t') e^{-i\Delta t'} dt' \quad ; \omega_n = n\omega_0 \\
&= \begin{cases} \frac{1}{2} \left(t - \frac{\sin(2\omega_n t)}{2\omega_n} \right) & , \Delta = 0 \\ I_r + iI_i & , \Delta \neq 0 \end{cases} \quad (4.2a) \\
&\quad (4.2b)
\end{aligned}$$

where, for $\Delta \neq 0$,

$$I_r = \frac{\sin\Delta t}{2\Delta} - \frac{1}{4} \left(\frac{\sin(2\omega_n - \Delta)t}{2\omega_n - \Delta} + (\Delta \rightarrow -\Delta) \right) \quad (4.3a)$$

$$I_i = \frac{\cos\Delta t - 1}{2\Delta} + \frac{1}{4} \left(\frac{\cos(2\omega_n - \Delta)t - 1}{2\omega_n - \Delta} - (\Delta \rightarrow -\Delta) \right) \quad (4.3b)$$

$$\text{and} \quad |I(t)|^2 = I_r^2 + I_i^2. \quad (4.3c)$$

4.2 Average Photon Number

There are two main cases to examine:

(A) The two cases of averaged photon numbers $n(t)$ *with initial condition in the vacuum* $|0\rangle$ *and number* $|n\rangle$ *states of eq (3.57 a,b) can be com-*

binned in one form, namely,

$$n(t) = n_0 + \Omega_0^2 |I(t)|^2$$

where $n_0 = 0$ in the $|0\rangle$ state, $n_0 \neq 0$ in the $|n\rangle$ state. Define the scaled quantity for the photon number,

$$\begin{aligned} N_1(t) &= \frac{n(t) - n_0}{\Omega_o'^2} \\ &= |I(\tau)|^2 \end{aligned} \tag{4.4}$$

where $\Omega_o' = \Omega_0/\omega_0$ is the normalized Rabi frequency and the normalized quantity $|I(\tau)|^2 = \omega_0^2 |I(t)|^2$.

In the resonant case ($\Delta = 0$), Figure 4.2(a), $N_1(t)$ shows a slowly monotonic growing behavior. This means that at exact resonance, having started initially the HO in a fixed $|n\rangle$ state; $n \geq 0$, the interaction with the pulse causes the HO to shift to higher energy states. This is unlike the non-resonant case ($\Delta \neq 0$, Figure 4.2(b)) where the oscillatory behavior is a result of the dispersion, i.e. mismatch between the frequency ω_0 of the HO and that of the pulse envelope (ω_L).

Note for small $|\Delta_1| = |\frac{\Delta}{\omega_0}| = 1$ (Figure 4.2(b)) each single pulse has duration of 2π , whereas with larger $|\Delta_1| = 10$ the oscillation increases with each pulse of duration π , as that of the original applied pulse (Figure 4.1).

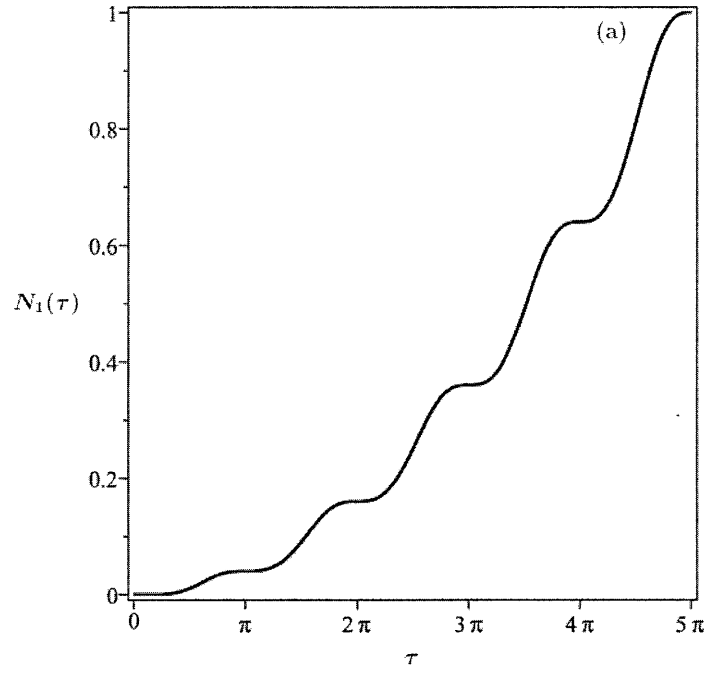


Figure 4.2(a)
Plot of $N_1(\tau)$ against τ for $\Delta_1 = 0$

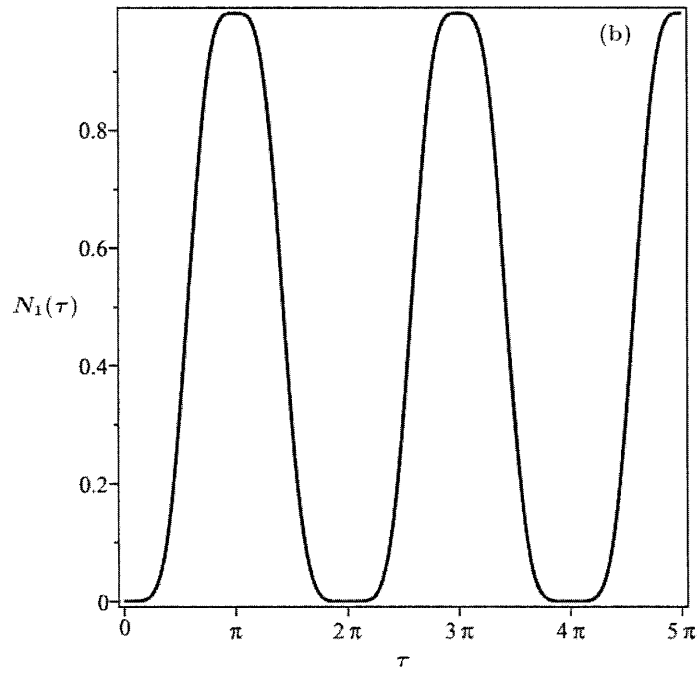


Figure 4.2(b)
Plot of $N_1(\tau)$ against τ for $|\Delta_1| = 1$

(B) In the case of *initial coherent state* $|\alpha\rangle$, the averaged photon number $n_\alpha(t)$ is given by (3.57c). Define the scaled quantity,

$$N_\alpha(t) = |\alpha|^2 + \Omega_0'^2 |I(\tau)|^2 + 2\Omega_0' \text{Im}(\alpha^* I(\tau)). \quad (4.5)$$

For *real* $\alpha = 1, \Omega_0' = 0.1$ and resonant case ($\Delta = 0$, Figure 4.3(a)) $N_\alpha(\tau)$ resembles that of Figure 4.2(a) in vacuum or number state. In the non-resonant case and small $\Delta_1 = \pm 1$, Figure 4.3(a) $N_\alpha(\tau)$ shows the oscillatory behavior with however different phase that depends on the sign of Δ_1 (compare Figure 4.3(a) with Figure 4.2(b)) due to the effect of the initial amplitude (α).

Increasing $\Delta_1 = 10$ (Figure 4.3(b)) has the effect of increasing the envelope oscillations (each pattern of period π) as well the amplitudes of the beat oscillations. The case of $\Delta_1 = -10$, is the inverse of Figure 4.3(b).

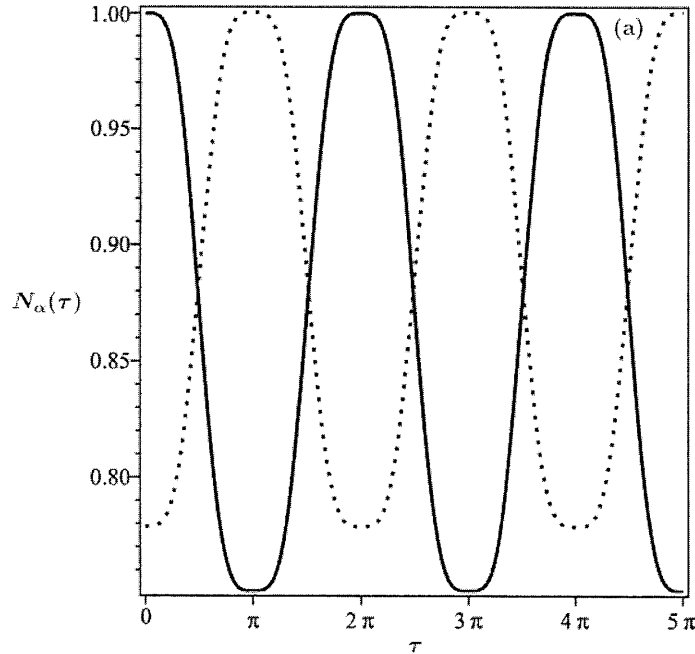


Figure 4.3(a)

Plot of $N_\alpha(\tau)$ with real $\alpha = 1, \Omega_0' = 0.1$ and for

$\Delta_1 = 1$ (full line), -1 (dotted line).

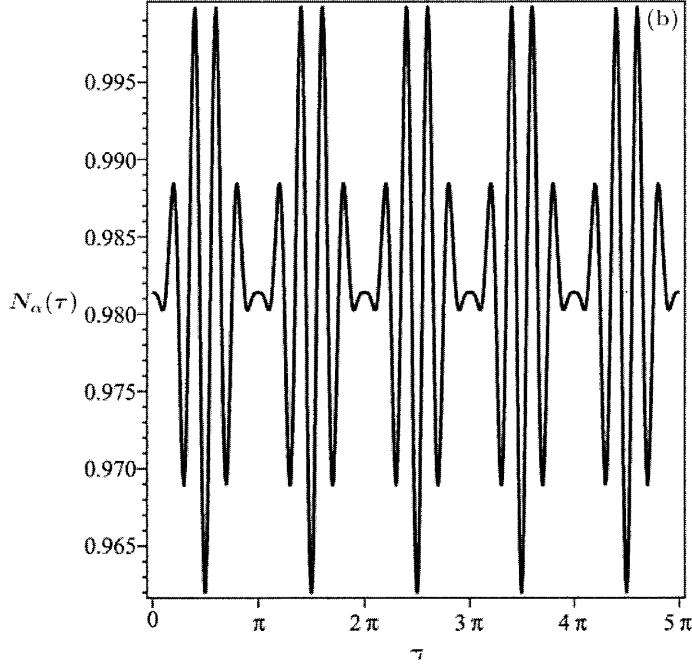


Figure 4.3(b)

As Figure 4.3(a) but for $\Delta_1 = 10$

For complex $\alpha = |\alpha|e^{i\theta}$ ($0 < \theta < \pi$) with $|\alpha| = 1, \Omega'_0 = 0.1$, in the resonant case ($\Delta = 0$), Figures 4.4(a,b) show the effect of the coherent phase $\theta = \frac{\pi}{4}, \frac{\pi}{2}$: in both cases $N_\alpha(t)$ shows monotonic decrease with lesser rate for $\theta = \frac{\pi}{4}$ compared with that for $\theta = \frac{\pi}{2}$. In the non-resonant case ($\Delta \neq 0$) the various coherent phase θ has a pronounced effect: for small $|\alpha| = 1, \theta = \frac{\pi}{4}$ (Figure 4.4(b)) the oscillation peaks of period 2π has its peaks and minima skewed. For $\theta = \frac{\pi}{2}$ (Figure 4.4(c)) the skewness occurs only in the peaks structure. With larger $|\alpha| = 10, \theta = \frac{\pi}{2}$, the skewness turns to pronounced envelope and beating oscillation (Figure 4.4(d)).

The regularity of these oscillations occurs with phase $\theta = 0$ or π (i.e real α).

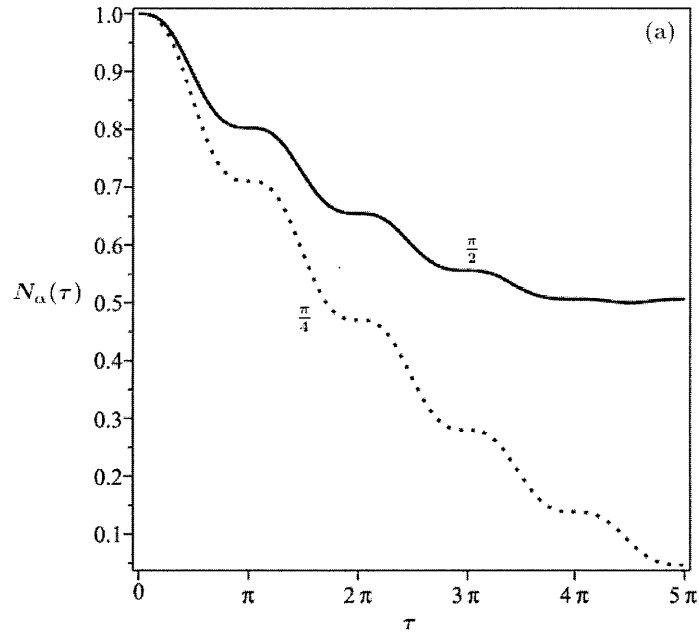


Figure 4.4(a)

Plot of $N_\alpha(\tau), \Delta_1 = 0, \theta = \frac{\pi}{2}, \frac{\pi}{4}$

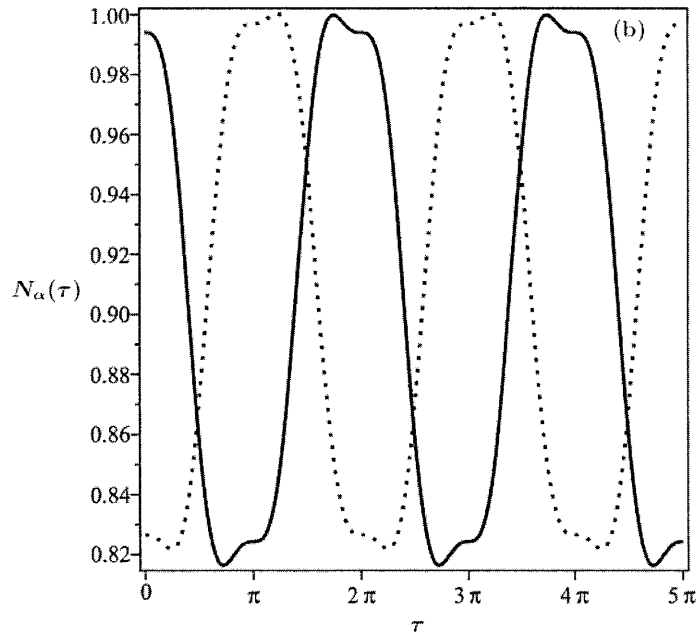


Figure 4.4(b)

Plot of $N_\alpha(\tau)$ for $\Delta_1 = 1$ (full line), -1 (dotted line), $\theta = \frac{\pi}{4}$.

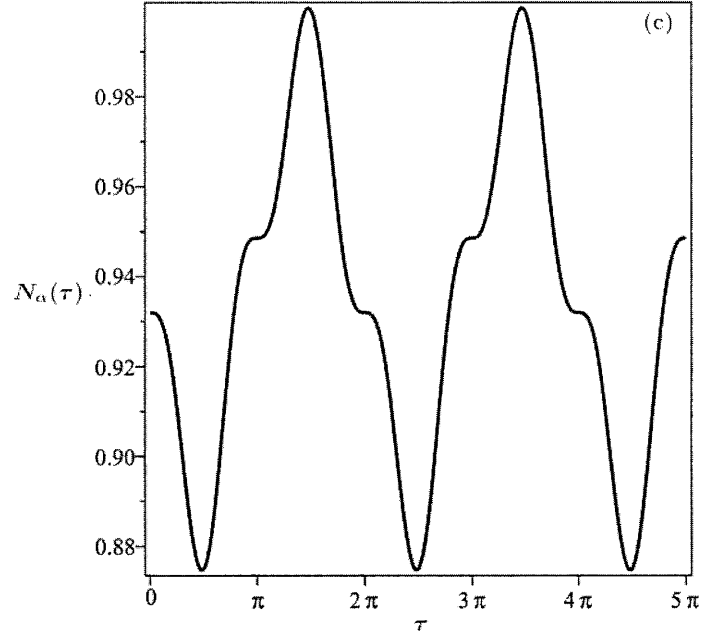


Figure 4.4(c)

As Figure 4.4(b) but for $|\Delta_1| = 1, \theta = \frac{\pi}{2}$

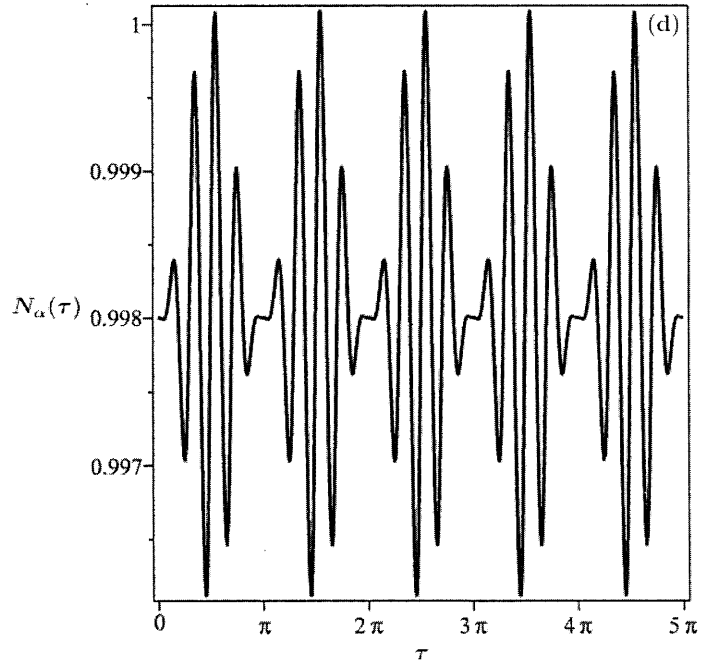


Figure 4.4(d)

As Figure 4.4(c) but for $\Delta_1 = 10$

4.3 Transient Spectrum :

In the general case of the HO in the initial $|\alpha\rangle$ state, the transient scattered spectrum is given by eq (3.63c), namely,

$$\begin{aligned}
 S_\alpha(t, D, \Gamma) = & 2\Gamma e^{-2\Gamma t} [|\alpha|^2 |J_1(t)|^2 \\
 & + \Omega_0^2 |J_2(t)|^2 \\
 & - 2\Omega_0 \text{Im}(\alpha J_1^*(t) J_2(t))] \quad (4.6)
 \end{aligned}$$

with,

$$\begin{aligned}
 J_1(t) &= \frac{e^{(\Gamma-iD)t} - 1}{(\Gamma-iD)}, \\
 J_2(t) &= \int_0^t e^{(\Gamma-iD)t'} I^*(t') dt'
 \end{aligned}$$

$$I^*(t) = \begin{cases} \frac{1}{2} \left(t - \frac{\sin(2\omega_n t)}{2\omega_n} \right) & ; \Delta = 0 \\ I_r - iI_i & ; \Delta \neq 0 \end{cases}$$

and I_r, I_i are given by (4.3a,b). Hence, $J_2(t)$ is given explicitly for $\Delta = 0, \Delta \neq 0$ as follows,

$$\begin{aligned}
 J_{2,\Delta=0}(t) &= \frac{1}{2(\Gamma-iD)} \left[\left(t - \frac{1}{(\Gamma-iD)} \right) e^{(\Gamma-iD)t} + \frac{1}{(\Gamma-iD)} \right] \\
 &+ \frac{i}{8\omega_n} \left[\frac{(e^{(\Gamma-i(D-2\omega_n))t} - 1)}{(\Gamma-i(D-2\omega_n))} - \text{same, } \omega_n \rightarrow -\omega_n \right] \quad (4.7a)
 \end{aligned}$$

and,

$$\begin{aligned}
J_{2,\Delta \neq 0}(t) = & \frac{-i}{2\Delta} \left[\frac{e^{(\Gamma-i(D-\Delta))t} - 1}{\Gamma - i(D-\Delta)} - \frac{e^{(\Gamma-iD)t} - 1}{\Gamma - iD} \right] \\
& + \left[\frac{-i}{4(2\omega_n - \Delta)} \left(\frac{e^{(\Gamma-i(D+2\omega_n-\Delta))t} - 1}{\Gamma - i(D+2\omega_n-\Delta)} - \frac{e^{(\Gamma-iD)t} - 1}{\Gamma - iD} \right) \right. \\
& \left. + \text{same, } \omega_n \rightarrow -\omega_n \right] .
\end{aligned} \tag{4.7b}$$

Now, we consider the following three cases of initial state of the HO.

4.3.1 Vacuum State

In this case $\alpha, |\alpha|^2 \rightarrow 0$, and the spectrum formula (4.6) reduces to

$$S_0(t, D, \Gamma) = 2\Gamma e^{-2\Gamma t} \Omega_0^2 |J_2(t)|^2. \tag{4.8}$$

In general, the analytical expression for the spectrum is too lengthy to write down. In the special case of resonance ($\Delta = 0$), the expression is given in the Appendix (C). A general look at this expression shows that the spectrum structure comprises Lorentzians, dispersive-line shapes and their narrowed shapes centered at $D = 0, \pm 2\omega_n$ with interference terms and oscillating weight coefficients: $\cos Dt, \sin Dt, \cos D_{\pm}t, \sin D_{\pm}t$ with $D_{\pm} = D \pm 2\omega_n$.

The normalized spectrum $S_0(D) = S_0(t, D, \Gamma) / \max(S_0(t, D, \Gamma))$ which is independent of the pulse strength Ω_0 , is plotted against the normalized detector's detuning $D' = D/\Gamma$ for fixed normalized detector's width $\Gamma' = \Gamma/\omega_0$ and detection time $\tau = \omega_0 t$. For $\tau = \pi, \Gamma' = 0.04, \Delta' = 0$ (resonance case) the spectrum is symmetric and develops from a single Lorentzian

at $D' = 0$ for $n = 1$ (single pulse), Fig (4.5a), to a Lorentzian with surrounded two small side bands for $n = 2$ (two pulses), Figure 4.5(b), and to a Lorentzian surrounded by oscillations and very weak side bands at larger $|D'| \simeq 250$ with larger number of pulses ($n=5$), Figure 4.5(c). These oscillations are due the larger number of sequential pulses over the period $[0, \pi]$. For larger $\tau = 10\pi; n = 1$ (So, 10 sequential pulses in the period $[0, 10\pi]$), the spectrum $S_0(D)$ shows a dominant single peak at $D' = 0$ with ultra-weak side bands at large $|D'| \simeq 50$ (Figure 4.6).

In the off-resonance case $\Delta' = 5, -5$, the main peak shifts to the right (left) (Figure 4.7(a)) with weak asymmetrical side bands and dispersive-line shapes (Figure 4.7(b)).

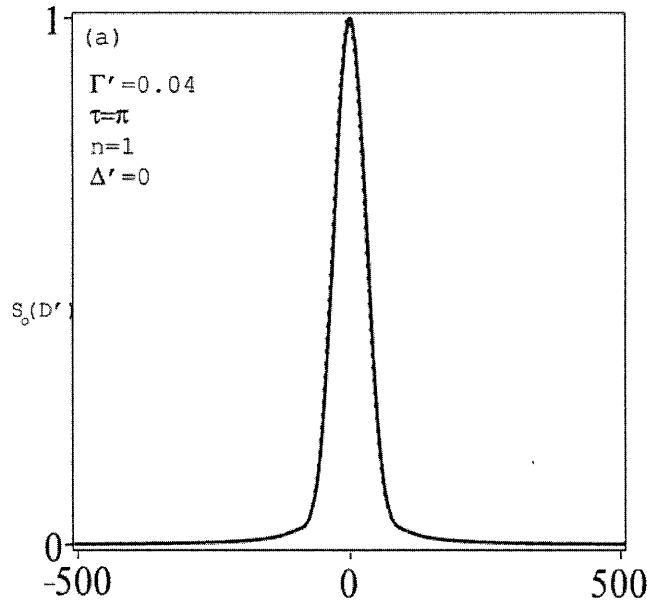


Figure 4.5(a)

The normalized spectrum with initial vacuum state $S_0(D')$ versus the normalized detector's detuning parameter D' at exact resonance

($\Delta' = 0$) and normalized detection time $\tau = \pi$ for a single pulse ($n = 1$).

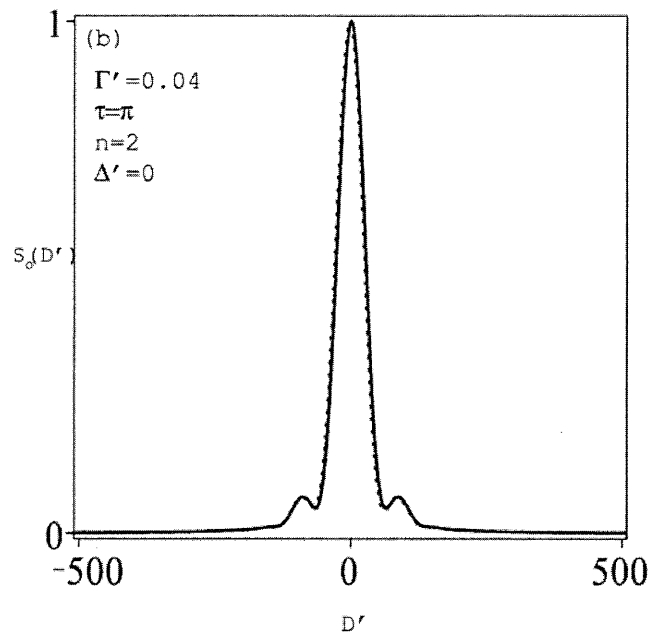


Figure 4.5(b)

As Figure 4.5(a) but for two pulses ($n = 2$)

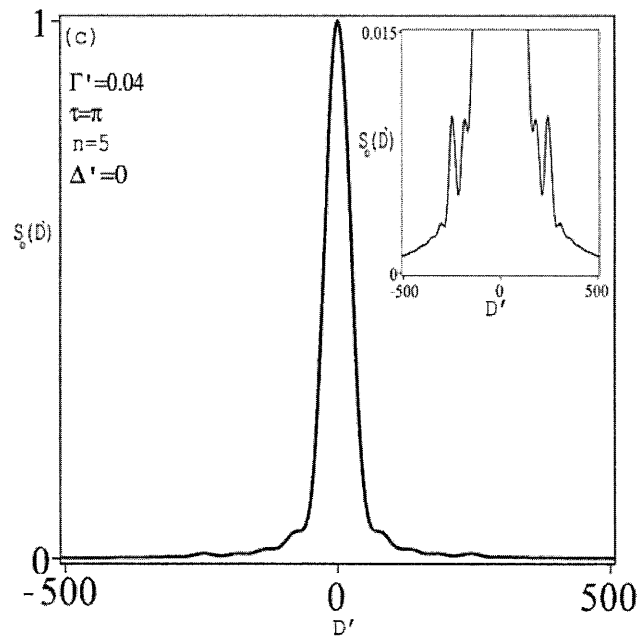


Figure 4.5(c)

As Figure 4.5(a) but for ($n = 5$)

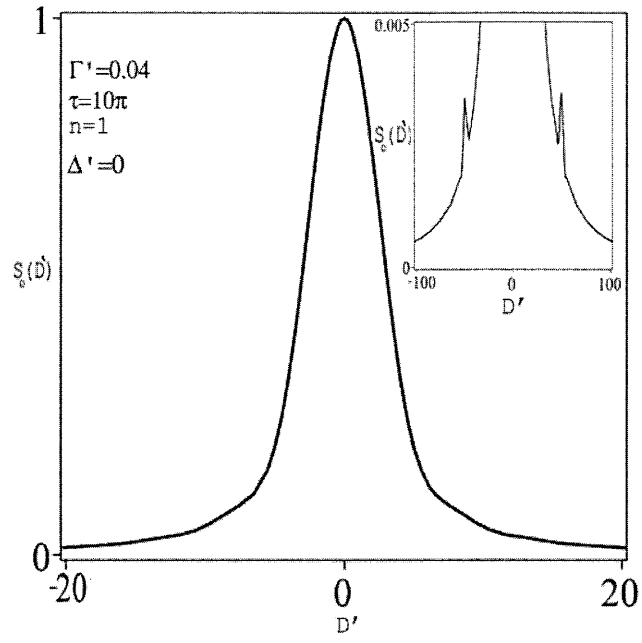


Figure 4.6

As Figure 4.5(a) but for $\tau = 10\pi$

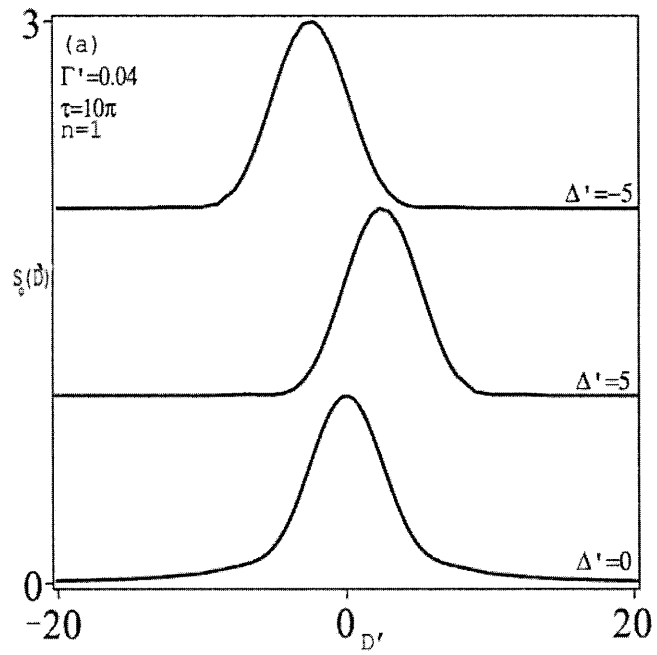


Figure 4.7(a)

As Figure 4.5(a) but for $\tau = 10\pi$ and various $\Delta' = 0, 5, -5$

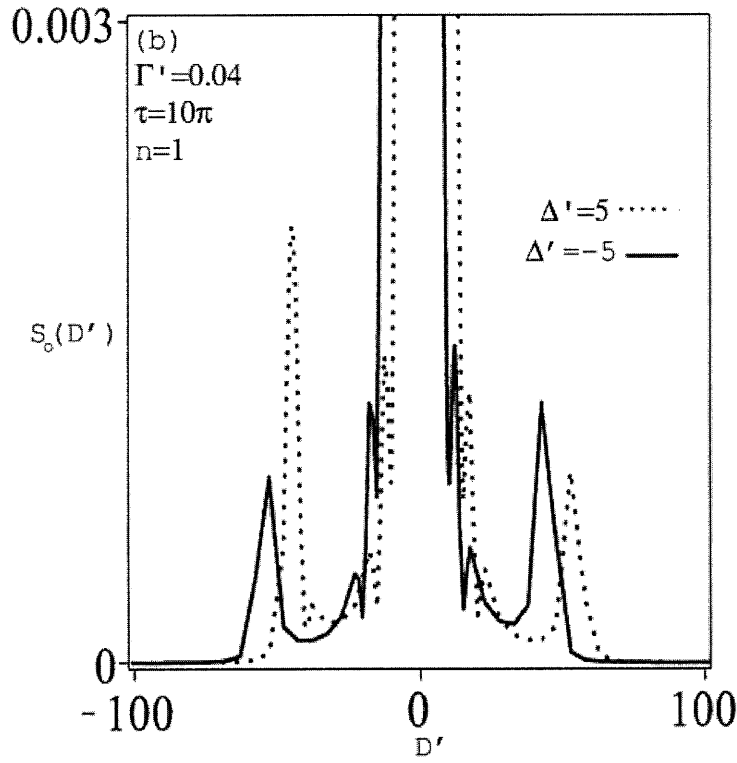


Figure 4.7(b)

The zooming of the spectrum $S_0(D')$ in Figure 4.7(a) for the cases $\Delta' = 5, -5$

4.3.2 Number State

In this case $\alpha \rightarrow 0, |\alpha|^2 \rightarrow n_0$ and the spectrum formula (4.6) has the form,

$$S_{n_0}(t, D, \Gamma) = 2\Gamma e^{-2\Gamma t} [n_0 |J_1(t)|^2 + \Omega_0^2 |J_2(t)|^2] \quad (4.9)$$

which has an additive term in $n_0 |J_1(t)|^2$ (to the previous case of eq (4.8)) that comprises the same structure line-shapes as that of the second term in $\Omega_0 |J_2(t)|^2$. The balance between the two terms in (4.9), namely in n_0 and Ω_0 , is illustrated in Fig (4.8) in the off-resonance case ($\Delta' \neq 0$) for equal weights $n_0 = \Omega_0'' = 1, (\Omega_0'' = \frac{\Omega_0}{\Gamma})$ where the normalized spectrum $S_{n_0}(D')$

is essentially symmetric and the oscillations around the central peak are relatively prominent. For stronger pulse strength $\Omega_0'' > n_0$, the spectrum becomes asymmetric near the tail of the central peak.

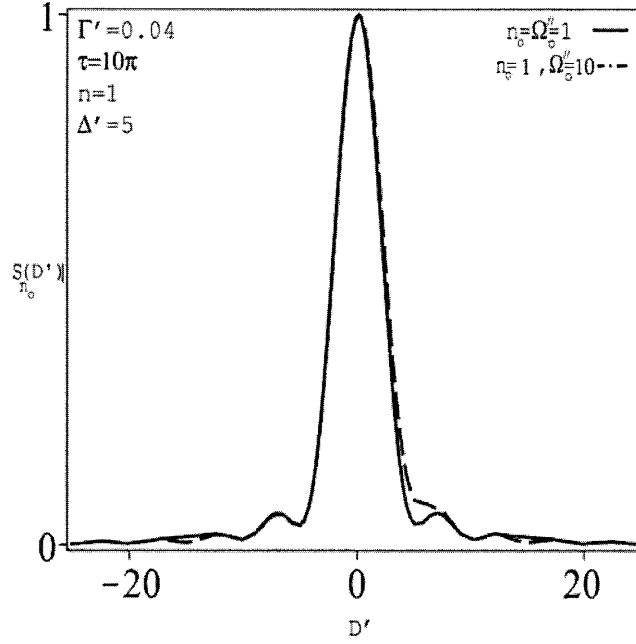


Figure 4.8

The normalized spectrum $S_{n_0}(D')$ with initial number state at $\tau = 10\pi, n = 1, \Delta' = 5$ and for different pair values of (n_0, Ω_0'') .

4.3.3 Coherent State

In this case, the spectrum is given by the full expression in (4.6), where the last term represents an inference between the initial amplitude parameter (α) and the normalized pulse strength (Ω_0'').

For fixed parameters $\Gamma' = 0.04, \tau = 10\pi, n = 1$ we notice the following structures for the normalized spectrum $S_\alpha(D')$:

(a) For real α , the balance between the pair of parameters (α, Ω_0) is shown at resonance ($\Delta' = 0$) in Figure 4.9(a), where $S_\alpha(D')$ is symmetric with the weak oscillations fade away for $\Omega_0'' > \alpha$. The corresponding results in the non-resonant case ($\Delta' \neq 0$) are shown in Figure 4.9(b,c): asymmetry with additional smaller peak at $D' = \Delta' = 5$ is emerged for $\Omega_0'' > \alpha$ (Figure 4.9(b)). The opposite value of $\Delta' = -5$ only causes little asymmetry in the weak oscillations with $\Omega_0'' > \alpha$ (Figure 4.9(c)). For fixed pair values of (Ω_0'', α) with $\Omega_0'' > \alpha$, changing the detuning parameter $|\Delta'|$ causes either shifted symmetrical, (Figure 4.10(a)), or asymmetrical, (Figure 4.10(b)), splitting in the main peak that depends on how small the initial value of the parameter α .

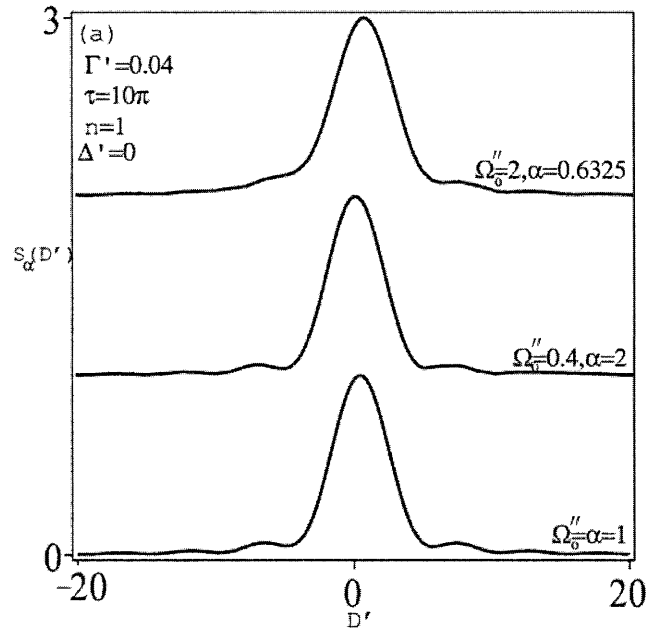


Figure 4.9(a)

The normalized spectrum $S_\alpha(D')$ with initial coherent state

α ($\alpha = \text{real}$) for the same data of Figure 4.6 and

different values of (Ω_0'', α)

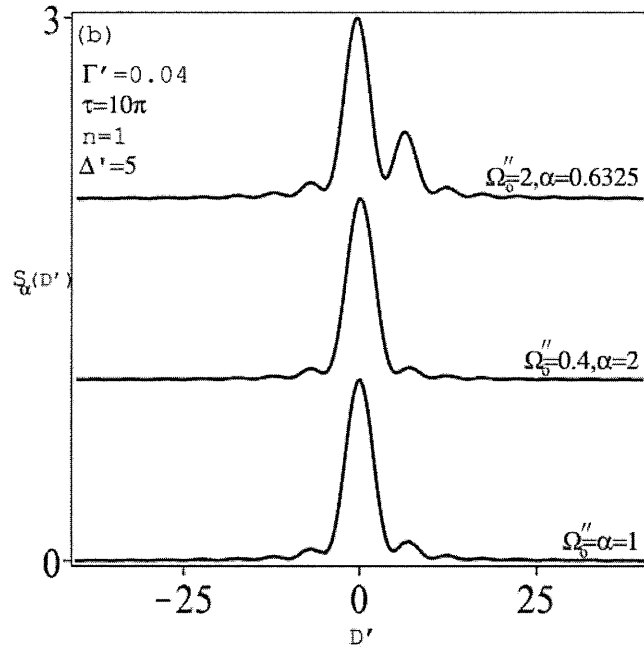


Figure 4.9(b)

As Figure 4.9(a) but for $\Delta' = 5$

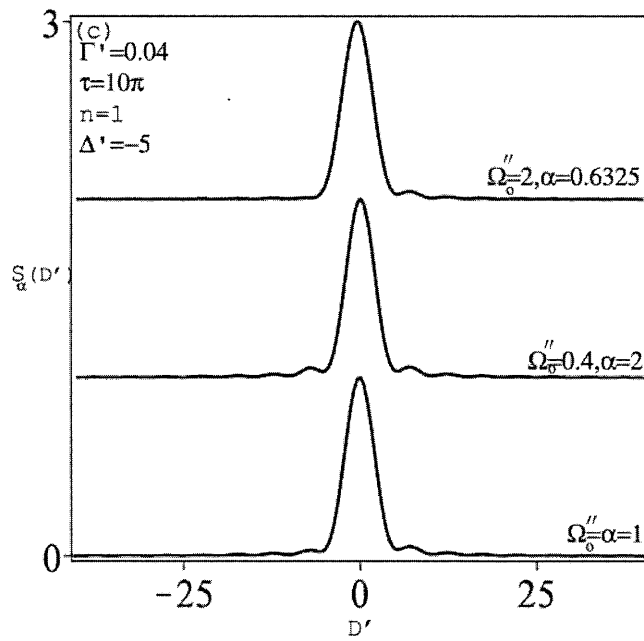


Figure 4.9(c)

As Figure 4.9(a) but for $\Delta' = -5$

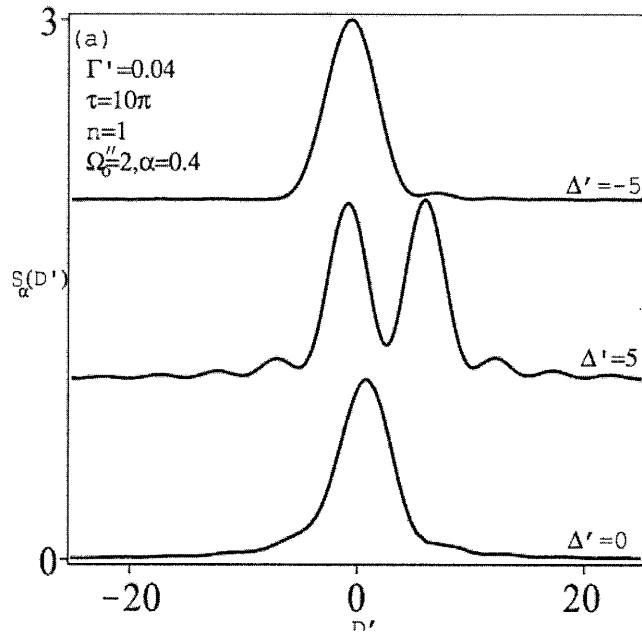


Figure 4.10(a)

As Figure 4.9(a) but for $(\Omega_0'', \alpha) = (2, 0.4)$ and various $\Delta' = 0, 5, -5$

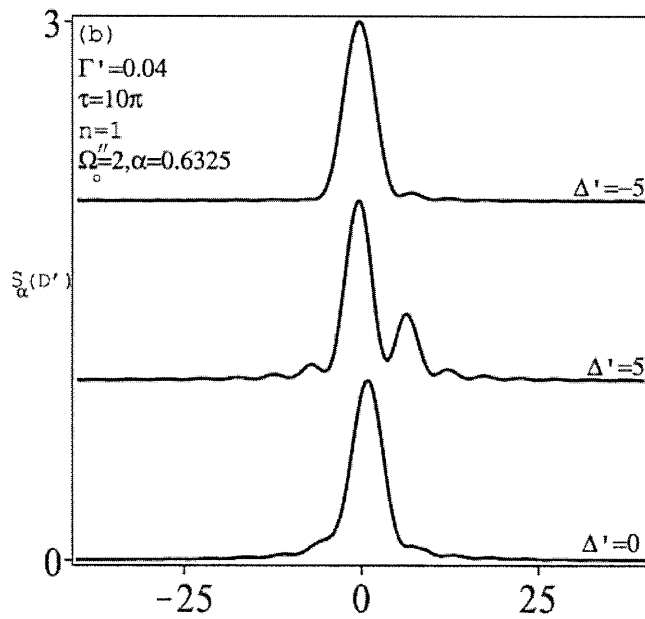


Figure 4.10(b)

As Figure 4.10(a) but with $\alpha = 0.6325$

(b) For complex $\alpha = |\alpha|e^{i\theta}$, the phase parameter θ has a pronounced effect compared with the case of real $\alpha(\theta = 0)$ of Figure 4.9. This best shown in Figure 4.11 for $\theta = \frac{\pi}{2}$ and $\Omega_0'' = 5, |\alpha| = 0.6325$ and varied values of $|\Delta'| = 0, 0.1, 2$.

At exact resonance ($\Delta' = 0$) the dominant single peak (for $\theta = 0$) splits into two peaks (hole burning) with pronounced oscillations in the spectrum wings. In the non-resonant case ($|\Delta'| \neq 0$), the hole burning structure becomes asymmetric with lesser oscillations and washes away with larger values of $|\Delta'| > 1$.

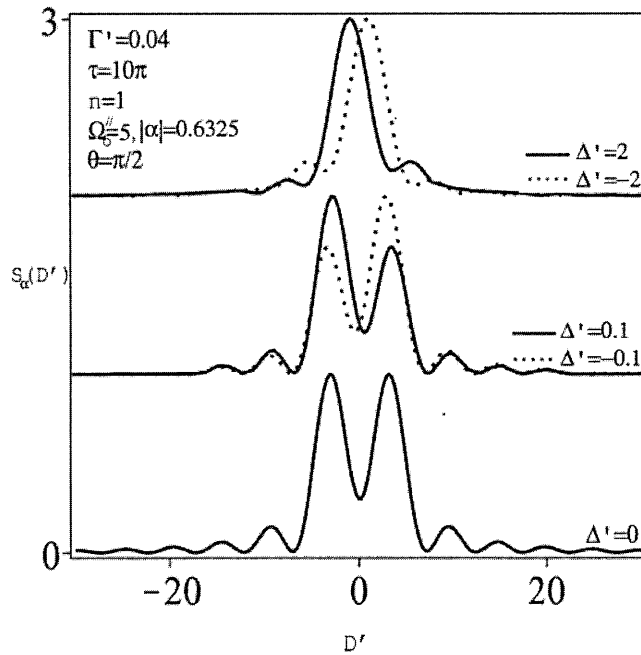


Figure 4.11

As Figure 4.9 but with $\Omega_0'' = 5, |\alpha| = 0.6325, \theta = \frac{\pi}{2}$
and versus $|\Delta'| = 0, 0.1$ and 2

4.4 Conclusion and Future Work

The main concern of the study in this thesis is the theoretical study of quantum mechanical model composed of a single harmonic Oscillator (HO) coupled to a sequential pulsed field of $\sin^2$ -shape.

Specifically, the transient emitted spectrum was investigated in the absence of any dissipation processes. Exact analytical expressions are derived for the mean photon number of the HO and the transient spectrum with arbitrary detuning for three initial conditions of the HO, namely, vacuum, number and coherent states. The main results are:

- (i) The mean photon number of the HO, with initial vacuum state (or number state) follow the same pattern of oscillatory behavior as the driving pulse for large detuning parameter. For the initial coherent state, the mean photon number of the HO shows increasing envelope oscillations with irregularity and asymmetry due to the phase of the initial coherent state.
- (ii) The transient spectrum with initial vacuum state at exact resonance is symmetric and develops from a single Lorentzian pulse to a single Lorentzian surrounded by weak oscillations in its tail with larger number of pulses. With initial number state $|n_0\rangle$, the balance between the initial excitation (n_0) and the strength of the driving pulse is to relatively amplify the oscillations in the wings of the single Lorentzian. For initial coherent state of phase θ , the spectrum at exact resonance is symmetric for both $\theta = 0, \frac{\pi}{2}$, with 'hole burning' structure in the Lorentzian peak and pronounced oscillations in the latter case of $\theta = \frac{\pi}{2}$. The 'hole burning' structure becomes asymmetric and washes away for larger detuning parameter.
- (iii) In principle, detection of the extra weak peak structure in the spectrum and its oscillatory pattern due to larger number of sequential pulses is possible in the light of experiments detecting ultra-weak signals (Vahlbruch

2008). Also, such ultra-weak spectrum signals are much used in optical signal processing (Mabuchi 2011).

Finally, future work of the studied model may include:

- (a) Other shapes of pulse-trains (e.g. Chirped Gaussian pulses, Train of Dirac δ -functions,...).
- (b) Time-dependent coupling of two quantized harmonic oscillators.

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APPENDIX A

EQUATIONS OF MOTION

There are two basic pictures where the quantum system can be described dynamically, namely, Schrödinger and Heisenberg pictures.

(1) Schrödinger's Picture:

In this picture, the quantum system state vectors are changing with time, while the operators are fixed.

Along the lines (and assumptions involved) of Dirac (Dirac 1981) (also see (Louisell 1973)), we outline the derivation of the time-dependent Schrodinger's equation of motion as follows.

(a) Let the state ket vector $|t\rangle$ is a particular state of several states of the system.

(b) Assume the state $|t_0\rangle$ determines $|t\rangle$; $t_0 < t$.

(c) The state $|t\rangle$ is generated from the state $|t_0\rangle$ via a linear operator $\hat{T}$ (that depends on (t) and (t_0)),

$$|t\rangle = \hat{T}|t_0\rangle \quad (\text{A.1})$$

(d) Assume the length of the states $|t\rangle$ and $|t_0\rangle$ are equal,

$$\langle t|t\rangle = \langle t_0|t_0\rangle \quad . \quad (\text{A.2})$$

Using (A.1) into (A.2), we have

$$\hat{T}^\dagger \hat{T} = \hat{T} \hat{T}^\dagger = 1 \quad (\text{A.3})$$

and so $\hat{T}$ is a unitary operator,

$$\hat{T} = (\hat{T}^\dagger)^{-1}, \hat{T}^\dagger = (\hat{T})^{-1},$$

i.e. $\hat{T}$ is invertible.

The main property of the unitary operator upon transforming from $|t_0\rangle \rightarrow |t\rangle$ is that it preserves the orthogonality and length of the state vectors.

(e) In the infinitesimal (continuum) limit case, assume the following limit exists,

$$\frac{d}{dt_0}|t\rangle = \lim_{t \rightarrow t_0} \frac{|t\rangle - |t_0\rangle}{t - t_0} \quad (\text{A.4})$$

$$= \lim_{t \rightarrow t_0} \left(\frac{\hat{T} - 1}{t - t_0} \right) |t_0\rangle$$

$$= \hat{T}_1 |t_0\rangle \quad (\text{A.5})$$

where we have used (A.1).

Denoting the limit operator,

$$\lim_{t \rightarrow t_0} \left(\frac{\hat{T} - 1}{t - t_0} \right) = \hat{T}_1$$

and use Taylor's expansion to first order in $(t - t_0)$ (linear approximation) we have ,

$$\hat{T} \simeq 1 + (t - t_0) \hat{T}_1 \quad (\text{A.6})$$

(f) Using (A.6) into (A.3), we get,

$$(1 + (t - t_0)\hat{T}_1^\dagger)(1 + (t - t_0)\hat{T}_1) = 1$$

$$(t - t_0)(\hat{T}_1^\dagger + \hat{T}_1) = 0, \quad \text{to } O(t - t_0).$$

$$\text{So, } \hat{T}_1^\dagger + \hat{T}_1 = 0 \quad . \quad (\text{A.7})$$

Hence $\hat{T}_1$ is a pure imaginary linear operator.

(g) For physical consideration, the limit operator is taken as,

$$\hat{T}_1 = (i\hbar)^{-1}\hat{H}(t_0) \quad (\text{A.8})$$

with $\hbar$ is the (reduced) Planck's constant and $\hat{H}$ is the total (real) energy operator of the quantum system.

Thus, (A.5) and (A.8) give,

$$i\hbar \frac{d}{dt_0} |t_0\rangle = \hat{H}(t_0) |t_0\rangle \quad .$$

Changing $t_0 \rightarrow t, |t\rangle \rightarrow |\psi(t)\rangle$ we have Schrödinger's equation of motion for the state $|\psi(t)\rangle$,

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \quad . \quad (\text{A.9})$$

(h) if $\hat{H}$ is time-independent (conservative system), the formal solution of (A.9) is,

$$|\psi(t)\rangle = e^{-i\hbar^{-1}\hat{H}t}|\psi(0)\rangle \quad (\text{A.10})$$

with the unitary evolution operator $e^{-i\hbar^{-1}\hat{H}t}$.

(2) Heisenberg's Equation:

In this picture, the system operators are changing with time while the state vectors are kept fixed. The quantum average of any operator $\hat{O}(t)$ in the state $|\psi(t)\rangle$ is,

$$\langle \hat{O}(t) \rangle = \langle \psi(t) | \hat{O}_s | \psi(t) \rangle \quad (\text{A.11})$$

where $\hat{O}_s$ represents the fixed operator in the Schrödinger picture. Using (A.10) and its bra equivalent, we get,

$$\langle \hat{O}(t) \rangle = \langle \psi(0) | e^{i\hbar^{-1}\hat{H}t} \hat{O}_s e^{-i\hbar^{-1}\hat{H}t} | \psi(0) \rangle \quad (\text{A.12})$$

$$= \langle \psi(0) | \hat{O}_H(t) | \psi(0) \rangle \quad (\text{A.13})$$

where,

$$\hat{O}_H(t) = e^{i\hbar^{-1}\hat{H}t} \hat{O}_s e^{-i\hbar^{-1}\hat{H}t} \quad (\text{A.14})$$

represents the time-dependent operator in Heisenberg picture. Differentiating (A.14) with respect to (t) (keeping operator ordering), we get,

$$i\hbar \frac{d}{dt} \hat{O}_H(t) = e^{i\hbar^{-1}\hat{H}t} \hat{O}_s e^{-i\hbar^{-1}\hat{H}t} \hat{H} - \hat{H} e^{i\hbar^{-1}\hat{H}t} \hat{O}_s e^{-i\hbar^{-1}\hat{H}t}$$

$$= \hat{O}_H \hat{H} - \hat{H} \hat{O}_H$$

$$= [\hat{O}_H(t), \hat{H}] \tag{A.15}$$

Note, if the commutator $[\hat{O}_H(t), \hat{H}] = 0$, then $\hat{O}_H$ is a constant of the motion.

APPENDIX (B)

THE COMMUTATOR $[\hat{q}, \hat{p}] = i\hbar$

The Heisenberg equation of motion, (A.15), can be alternatively achieved using Bohr's correspondence principle (Newing and Cunningham 1967). This principle, as one of the postulates of quantum mechanics, is spelled as follows.

Within classical mechanics, the equation of motion for a dynamical variable $a(t)$ (for simplicity of one degree of freedom) in terms of the total Hamiltonian $H = H(q, p)$ has the form ,

$$\frac{da}{dt} = \{a, H\} \quad (\text{B.1})$$

where,

$$\{a, H\} = \frac{\partial a}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial a}{\partial p} \frac{\partial H}{\partial q} \quad (\text{B.2})$$

is the classical Poisson bracket.

The transfer from classical mechanics, eq (B.1), to the corresponding (operator) quantum mechanics, the following replacement is used,

$$a \rightarrow \hat{A}, \quad H \rightarrow \hat{H}$$

and

$$\{a, H\} \rightarrow \frac{1}{i\hbar} [\hat{A}, \hat{H}] \quad (\text{B.3})$$

and hence Heisenberg equation of motion (A.15) is obtained.

In general, the classical Poisson bracket for any two variables, $u(q, p)$ and $v(q, p)$ (for one degree of freedom) is defined as,

$$\{u, v\} = \frac{\partial u}{\partial q} \frac{\partial v}{\partial p} - \frac{\partial u}{\partial p} \frac{\partial v}{\partial q} \quad . \quad (\text{B.4})$$

If $u = q$ and $v = p$ then (B.4) gives,

$$\{q, p\} = 1 \quad (\text{B.5})$$

and hence according to the correspondence principle, (B.3), we have the commutator relation,

$$[\hat{q}, \hat{p}] = i\hbar \quad . \quad (\text{B.6})$$

APPENDIX (C)

ANALYTICAL EXPRESSION OF EQUATION (4.8)

The explicit analytical expression for the spectrum $S_0(t, D, \Gamma)$ with initial vacuum state, eq (4.8), is given below.

$$S_0(t, D, \Gamma) = \frac{1}{2} \Gamma \Omega_0^2 (A_1^2 + A_2^2) \quad (\text{C.1})$$

with,

$$\begin{aligned} A_1 = & \frac{\Gamma t \cos Dt + Dt \sin Dt}{(\Gamma^2 + D^2)} \\ & + \frac{1}{(\Gamma^2 + D^2)^2} ((\Gamma^2 - D^2) e^{-\Gamma t} + (\Gamma^2 - D^2) \cos Dt \\ & \quad + 2\Gamma D \sin Dt) \\ & + \left[\frac{1}{4\omega_n(\Gamma^2 + D_-^2)} (D_- e^{-\Gamma t} + \Gamma \sin D_- t + D_- \cos D_- t) \right. \\ & \quad \left. + (\text{same}, \omega_n \rightarrow -\omega_n) \right], \quad (\text{C.2}) \\ A_2 = & \frac{Dt \cos Dt - \Gamma t \sin Dt}{(\Gamma^2 + D^2)} \\ & + \frac{1}{(\Gamma^2 + D^2)^2} (2\Gamma D e^{-\Gamma t} + 2\Gamma D \cos Dt \\ & \quad - (\Gamma^2 - D^2) \sin Dt) \end{aligned}$$

$$\begin{aligned}
& + [\frac{1}{4\omega_n(\Gamma^2 + D_-^2)}(-\Gamma e^{-\Gamma t} + \Gamma \cos D_- t + D_- \sin D_- t) \\
& + (same, \omega_n \rightarrow -\omega_n)], \tag{C.3}
\end{aligned}$$

where $D_- = D - 2\omega_n$, $\omega_n = n\omega_0$.

PUBLISHED ARTICLE

Transient spectrum of $\sin^2$ -pulsed driven harmonic oscillator

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Exact analytical results are derived for the coupled system of nondissipative single mode quantized harmonic oscillator (HO) and arbitrary pulse shape. Specifically, for a $\sin^2$ -pulse shape, the transient fluorescent spectrum is obtained in the general case of initial coherent state $|\alpha\rangle$ of the HO. The dominant central Lorentzian is surrounded by *weak* oscillations due to the larger number of sequential pulses which get amplified asymmetrically in the nonresonant case as a result of balancing and interference processes between the initial excitation ($|\alpha^2|$) and the strength of the exciting pulse. Further pronounced oscillations is noticed with “hole burning” structure in the central peak due to nonzero phase of the initial amplitude (α).

Keywords: Quantized harmonic oscillator; $\sin^2$ -pulse; transient spectrum.

1. Introduction

Quantal investigation of the scattered radiation of coupled systems of radiation–matter or radiation–radiation reveals information about its emitted frequencies, as well the energy change in the original (matter or radiation) due to such interaction (e.g. Refs. 1 and 2). Specifically, short (pico- or femto-second) laser pulses exciting nondissipative atomic systems has its crucial roles in ultrafast devices or spectroscopic applications³ and coherent population transfer^{4–6} and the recent trend to manipulate emission and absorption radiation processes to achieve superposition of atomic coherent state, which is a basic element in quantum information processing and quantum computation.^{7,8} Various shapes of laser pulses coupled to the simplest nondissipative 2-level atomic system have been theoretically examined in the literature.^{9–16} On the other hand, the coupling of nondissipative single mode quantised harmonic oscillator (HO) with multi-mode rectangular laser pulses have been

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investigated in Ref. 14 within and without rotating wave approximation (RWA), where fast oscillatory processes are concerned.

The pulsed-driven HO can be viewed in two ways:

- (i) A fundamental example of radiation–radiation interaction where the HO represents a single mode of the quantized em field in a Fock (number) or coherent state representation.
- (ii) A limiting case of many excited atoms contained in a very small volume $O(\lambda^3)$; $\lambda \sim 10^{-5}$ cm, is the optical radiation wavelength.¹⁷ Mathematically speaking, all atoms are assumed to be on the same single site, known as Dicke model.¹⁸ Further, recent study show that nano-biosensors detecting biomolecules are physically justified to be modeled as a perturbed HO (see e.g. Refs. 19, 20 and references therein). With this in mind, we study the pulsed-driven (quantized) HO where the exciting laser pulse is of $\sin^2$ -shape, i.e., sequential many-pulses. The main goal is to investigate the dynamics as well the transient scattered (fluorescent) spectrum.

The paper is presented as follows. In Sec. 2 to follow, we present the Hamiltonian model and the formal operator solutions of the corresponding Heisenberg equation for the HO variables, and hence the analytical expressions for the mean photon-number and the transient spectrum, for *arbitrary* pulse shape. The explicit calculations and computational results in the case of $\sin^2$ -pulse are presented with different initial states of the HO for the mean photon number and the transient spectrum in Secs. 3 and 4, respectively. A summary is given in Sec. 5.

2. The Model Hamiltonian and General Solutions

The Hamiltonian model representing the interaction of a single quantized (nondissipative) HO with a (classical) laser pulse within the RWA is of the following form¹⁴ (in units of $\hbar = 1$):

$$H = \omega_0 \hat{a}^\dagger \hat{a} + \Omega_0 f(t) (\hat{a}^\dagger e^{-i\omega_L t} + \hat{a} e^{i\omega_L t}), \quad (1)$$

where ω_0 is the HO frequency, ω_L is the oscillating frequency of the pulse envelope, $f(t)$ is the pulse shape (real) function and Ω_0 is the pulse strength (Rabi frequency).

The HO operators $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators, respectively, obeying the commutator algebra, $[\hat{a}, \hat{a}^\dagger] = 1$.

The first term of the Hamiltonian (1) is the unperturbed Hamiltonian of the HO — apart from the constant term $\frac{1}{2}\hbar\omega$ which does not affect the dynamics — while the rest of terms (in Ω_0) represent the interaction Hamiltonian between the HO and the pulse. Heisenberg equations of motion for the operators $\hat{a}$ and $\hat{a}^\dagger$ according to the Hamiltonian (1) are of the form,

$$\begin{aligned} \dot{\hat{a}} &= -i\omega_0 \hat{a} - i\Omega_0 f(t) e^{-i\omega_L t}, \\ \dot{\hat{a}}^\dagger &= i\omega_0 \hat{a}^\dagger + i\Omega_0 f(t) e^{i\omega_L t}. \end{aligned} \quad (2)$$

2.1. Arbitrary pulse shape

The formal operator solutions of (2) for arbitrary pulse shape $f(t)$ are

$$\begin{aligned}\hat{a}(t) &= \hat{a}(0)e^{-i\omega_0 t} - i\Omega_0 e^{-i\omega_0 t} \int_0^t f(t')e^{-i\Delta t'} dt' \\ &= e^{-i\omega_0 t} \hat{A}(t),\end{aligned}\tag{3a}$$

$$\hat{a}^\dagger(t) = e^{i\omega_0 t} \hat{A}^\dagger(t),\tag{3b}$$

where

$$\hat{A}(t) = \hat{a}(0) - i\Omega_0 \int_0^t f(t')e^{-i\Delta t'} dt',\tag{4}$$

and $\Delta = (\omega_L - \omega_0)$ is the frequency detuning parameter.

- (i) The mean photon number $n(t) = \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle$ of the HO, in an arbitrary state $|\rangle$ is then given by

$$n(t) = \langle \hat{a}^\dagger(0)\hat{a}(0) \rangle + \Omega_0^2 |I(t)|^2 - i\Omega_0 (\langle \hat{a}^\dagger(0) \rangle I(t) - c.c.),\tag{5}$$

where *c.c.* stands for complex conjugate, and

$$I(t) = \int_0^t f(t')e^{-i\Delta t'} dt'.\tag{6}$$

The first term in (5) represents the initial mean photon number of the HO, while the second term in Ω_0^2 represents the intensity of the (classical) pulse. The last term in (5) represents the change in the photon number $n(t)$ (i.e., energy) due to the exchange of initial amplitude $\langle \hat{a}^\dagger(0) \rangle$ of the HO with the driving pulse.

In the general case of initial coherent state $|\alpha\rangle$ for the HO, Eq. (5) takes the form

$$n_\alpha(t) = |\alpha|^2 + \Omega_0^2 |I(t)|^2 + 2\Omega_0 \text{Im}(\alpha^* I(t)).\tag{7}$$

For initial vacuum state $|0\rangle$, (7) is reduced to

$$n_0(t) = \Omega_0^2 |I(t)|^2,\tag{8}$$

while for initial number state $|n_o\rangle$, (7) reduced to

$$n_{n_o}(t) = n_o + \Omega_0^2 |I(t)|^2.\tag{9}$$

- (ii) The transient fluorescent spectrum in terms of the auto-correlation function $\langle \hat{a}^\dagger(t_1)\hat{a}(t_2) \rangle$ is given by^{14,21}

$$S(t, D, \Gamma) = 2\Gamma \int_0^t dt_1 \int_0^t dt_2 e^{(-\Gamma + iD)(t-t_1)} e^{(-\Gamma - iD)(t-t_2)} \langle \hat{A}^\dagger(t_1)\hat{A}(t_2) \rangle,\tag{10}$$

where $D = \omega - \omega_0$ is the frequency mismatch between the frequency (ω) of the instrument photon detector and the frequency (ω_0) of the HO and Γ is the detector's width (i.e., Γ^{-1} is the life-time or time-duration of the detected photon

in the instrument). The function $\langle \hat{A}^\dagger(t_1)\hat{A}(t_2) \rangle$, with $\hat{A}^\dagger$ (and its adjoint) given in (3), is the auto-correlation function for the shift (creation and annihilation) operators.

In the general case of initial coherent state $|\alpha\rangle$ for the HO and using (4) and its adjoint we get the following expression:

$$\langle \hat{A}^\dagger(t_1)\hat{A}(t_2) \rangle = |\alpha|^2 + \Omega_0^2 I^*(t_1)I(t_2) + i\Omega_0(\alpha I^*(t_1) - \alpha^* I(t_2)). \quad (11)$$

Inserting (11) into (10), we have the following form for the spectrum:

$$S_\alpha(t, D, \Gamma) = 2\Gamma e^{-2\Gamma t} [|\alpha|^2 |J_1(t)|^2 + \Omega_0^2 |J_2(t)|^2 - 2\Omega_0 \text{Im}(\alpha J_1^*(t)J_2(t))], \quad (12)$$

where

$$\begin{aligned} J_1 &= \int_0^t e^{(\Gamma-iD)t'} dt' \\ &= \frac{e^{(\Gamma-iD)t} - 1}{\Gamma - iD} \end{aligned} \quad (13a)$$

and

$$J_2 = \int_0^t e^{(\Gamma-iD)t'} I^*(t') dt'. \quad (13b)$$

For the initial vacuum state $|0\rangle$, (12) reduces to

$$S_0(t, D, \Gamma) = 2\Gamma \Omega_0^2 e^{-2\Gamma t} |J_2(t)|^2, \quad (14)$$

while for the initial number state $|n_o\rangle$, (12) reduces to

$$S_{n_o}(t, D, \Gamma) = 2\Gamma e^{-2\Gamma t} (n_o |J_1(t)|^2 + \Omega_0^2 |J_2(t)|^2). \quad (15)$$

Effect of the pulse shape is shown through the terms in $J_2(t)$ in the three cases of $S_\alpha(t)$, $S_o(t)$, $S_{n_o}(t)$, Eqs. (12), (14) and (15), respectively, while the terms in $J_1(t)$ are associated with the nonzero initial mean photon numbers $|\alpha|^2$, n_o in the two cases $S_\alpha(t)$, $S_{n_o}(t)$, Eqs. (12) and (15), respectively

2.2. $\sin^2$ -pulse shape

The case of the pulse field of envelope form

$$\begin{aligned} f(t) &= \sin^2(n\omega_0 t); \quad n = 1, 2, \dots \\ &= \sin^2(n\tau); \quad \tau = \omega_0 t \end{aligned} \quad (16)$$

represents a pulsed laser beam of beating frequency $n\omega_o$ in the period $[0, n\pi]$ with (π) is a one pulse duration (see Fig. 1). Equivalently, (16) describes a series of sequential pulses with one pulse lasts from $m\pi \rightarrow (m+1)\pi$; $m = 0, 1, 2, \dots$. In electrical engineering context, this is known as rectified full-wave or “hum”, which allows more electrical current to flow in one direction than the other (i.e., it converts alternating current, a.c., to direct current, d.c.).

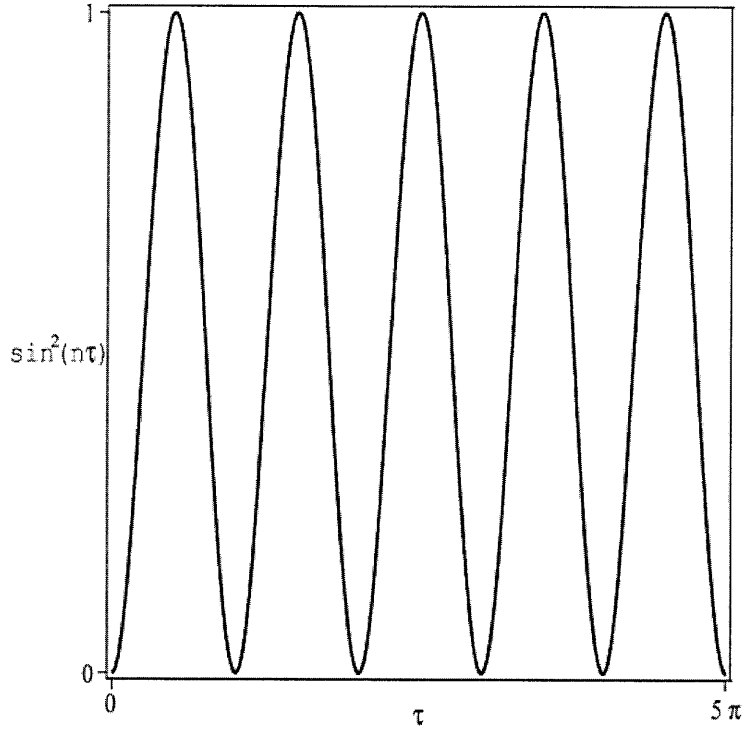


Fig. 1. Plot of $f(\tau) = \sin^2(n\tau)$ against $\tau = \omega_0 t$ in the period $[0, 5\pi]$.

With the pulse form (16), the integration (6) is now given as follows:

$$\begin{aligned}
 I(t) &= \int_0^t f(t') e^{-i\Delta t'} dt' \\
 &= \int_0^t \sin^2(\omega_n t') e^{-i\Delta t'} dt'; \quad \omega_n = n\omega_0 \\
 &= \begin{cases} \frac{1}{2} \left(t - \frac{\sin(2\omega_n t)}{2\omega_n} \right), & \Delta = 0 \\ I_r + iI_i, & \Delta \neq 0 \end{cases} \quad (17)
 \end{aligned}$$

where

$$I_r = \frac{\sin \Delta t}{2\Delta} - \frac{1}{4} \left(\frac{\sin(2\omega_n - \Delta)t}{2\omega_n - \Delta} + (\Delta \rightarrow -\Delta) \right), \quad (18a)$$

$$I_i = \frac{\cos \Delta t - 1}{2\Delta} + \frac{1}{4} \left(\frac{\cos(2\omega_n - \Delta)t - 1}{2\omega_n - \Delta} - (\Delta \rightarrow -\Delta) \right) \quad (18b)$$

and

$$|I(t)|^2 = I_r^2 + I_i^2. \quad (18c)$$

The expressions (7)–(9) for the mean photon number of the HO and for the corresponding transient spectra (12), (14), (15) with (17) and (18) are examined analytically and computationally in the following two sections in the special case of squared-sine pulse shape.

3. Mean Photon Number

There are two main cases to examine:

- (A) The two cases of mean photon numbers $n_o(t), n_{n_o}(t)$ with initial condition in the vacuum $|0\rangle$ and number $|n_o\rangle$ states of Eqs. (8) and (9), respectively, can be combined in one form, namely

$$n(t) = n_0 + \Omega_0^2 |I(t)|^2, \quad (19)$$

where $n_0 = 0$ in the $|0\rangle$ -state, $n_o = n_{n_o} \neq 0$ in the $|n_o\rangle$ -state. Define the scaled quantity for the mean photon number

$$\begin{aligned} N_1(t) &= \frac{n(t) - n_0}{\Omega_o'^2} \\ &= \frac{|I(\tau)|^2}{\max |I(\tau)|^2}, \end{aligned} \quad (20)$$

where $\Omega_o' = \Omega_0/\omega_0$ is the normalized Rabi frequency and the normalized quantity $|I(\tau)|^2 = \omega_0^2 |I(t)|^2$ is independent of the pulse strength Ω_o^2 .

In the resonant case ($\Delta = 0$), Fig. 2: full line, $N_1(\tau)$ shows a slowly monotonic growing behavior. This means that at exact resonance, having started initially the HO in a fixed $|n\rangle$ -state; $n \geq 0$, the interaction with the pulse causes the HO to shift to higher energy states. This is unlike the nonresonant case ($\Delta \neq 0$, Fig. 2) dashed and dotted lines, where the oscillatory behavior is a result of the dispersion, i.e., mismatch between the frequency ω_0 of the HO and that of the pulse envelope (ω_L). Note for small $|\Delta_1| = |\frac{\Delta}{\omega_0}| = 1$, each single pulse has duration of 2π , whereas with larger $|\Delta_1| = 10$ the oscillations increase with each pulse of duration π , as that of the original applied pulse (Fig. 1).

- (B) In the case of *initial coherent state* $|\alpha\rangle$, the mean photon number $n_\alpha(t)$ is given by (7). The scaled quantity

$$N_\alpha(t) = |\alpha|^2 + \Omega_o'^2 |I(\tau)|^2 + 2\Omega_o' \text{Im}(\alpha^* I(\tau)), \quad (21)$$

now depends on the normalized pulse strength parameter Ω_o' .

For *real* $\alpha = 1$, and resonant case ($\Delta = 0$), $N_\alpha(\tau)$ resembles that of $\Delta = 0$ in Fig. 2 in vacuum or number state. In the nonresonant case and small $\Delta_1 = \pm 1$, Fig. 3 $N_\alpha(\tau)$ shows the oscillatory behavior with however different phase that depends on the sign of Δ_1 due to the effect of the initial amplitude (α), (compare Fig. 3 with Fig. 2 for $|\Delta_1| = 1$).

Increasing $\Delta_1 = 10$ (Fig. 3) has the effect of increasing the envelope oscillations (each pattern of period π) as well the amplitudes of the beat oscillations. The case of $\Delta_1 = -10$, is the inverse of that of $\Delta_1 = 10$.

For complex $\alpha = |\alpha|e^{i\theta}$ with $\theta = \frac{\pi}{2}$, $|\alpha| = 1$, $N_\alpha(\tau)$ shows monotonic decrease in the resonant case ($\Delta_1 = 0$), while in the nonresonant case $|\Delta_1| = 10$, it shows oscillations of weak amplitudes and of period 2π (Fig. 4). With larger

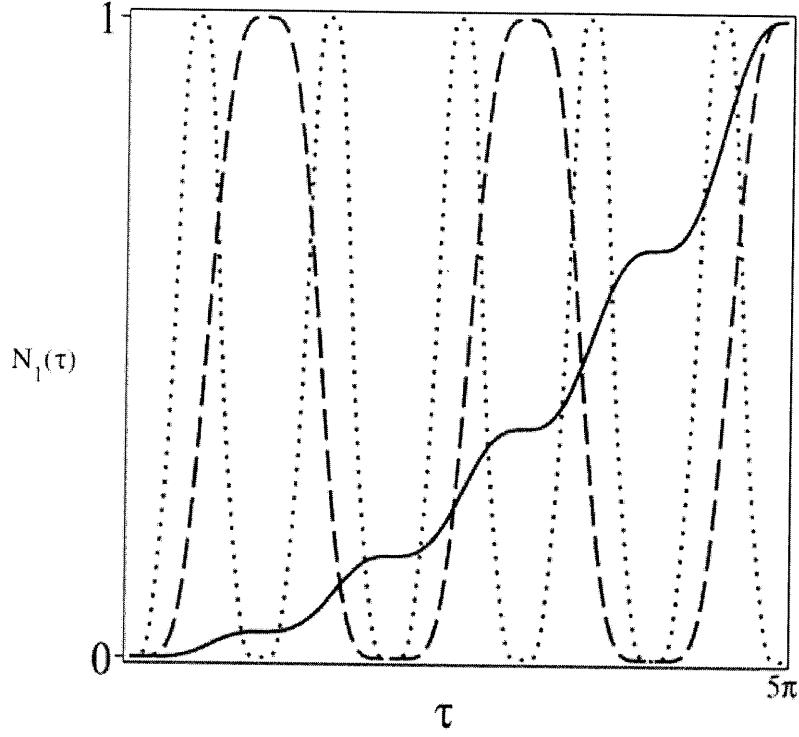


Fig. 2. Plot of $N_1(\tau)$ against τ for $\Delta_1 = 0$ (full line), $|\Delta_1| = 1$ (dashed line), 10 (dotted line).

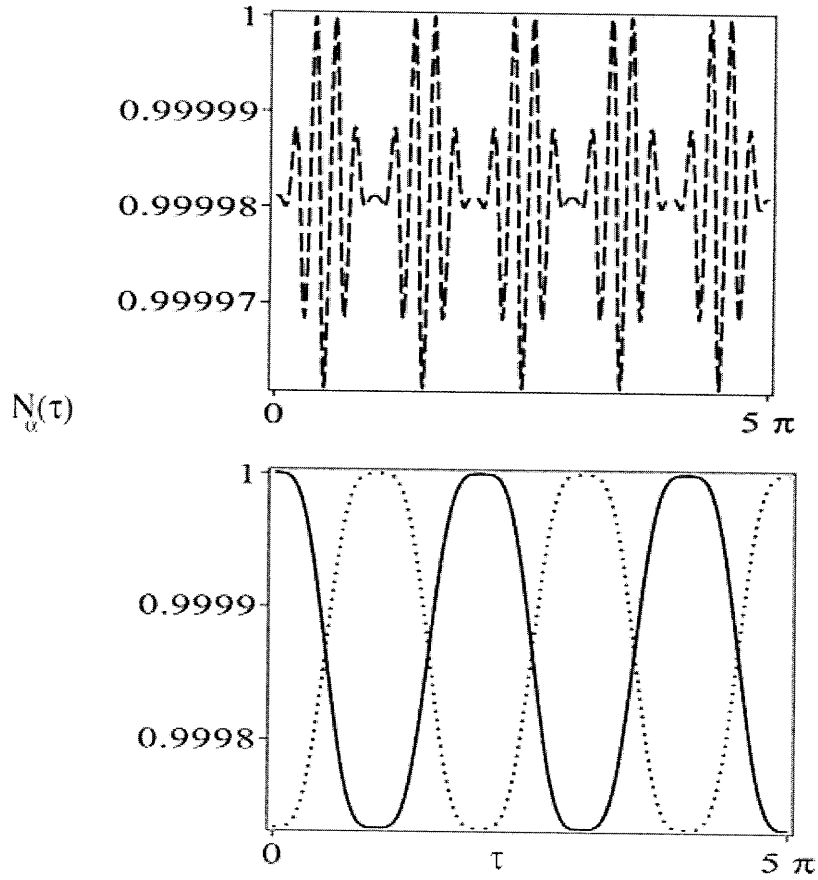


Fig. 3. Plot of $N_\alpha(\tau)$ with real $\alpha = 1$, and for $\Delta_1 = 1$ (full line), 1 (dotted line), 10 (dashed line) with $\Omega'_o = 10^{-3}$.

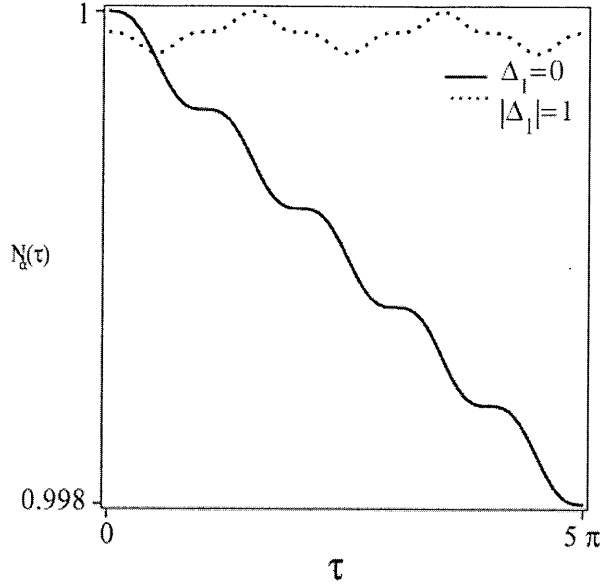


Fig. 4. Plot of $N_\alpha(\tau)$ for $|\alpha| = 1$, $\theta = \frac{\pi}{2}$, $\Delta'_1 = 0$ (full line), $|\Delta'_1| = 1$ (dotted line), with $\Omega'_o = 10^{-3}$.

$|\alpha| = 10$ and $\theta = \frac{\pi}{2}$, $|\Delta_1| = 10$, the weak oscillations in Fig. 4 turn to pronounced envelope and beating oscillations qualitatively similar to that of Fig. 3. Note that, the regularity of the oscillations occurs with phase $\theta = 0, \pi$ (i.e., real α).

4. Transient Spectrum

In the general case of the HO in the initial $|\alpha\rangle$ -state, the transient scattered spectrum $S_\alpha(t, D, \Gamma)$ is given by Eq. (12) with $J_{1,2}(t)$ given by 13(a) and 13(b), respectively. With the use of (17) into (13b) we have J_2 is explicitly given in the resonant and nonresonant cases as follows:

$$J_{2,\Delta=0}(t) = \frac{1}{2(\Gamma - iD)} \left[\left(t - \frac{1}{(\Gamma - iD)} \right) e^{(\Gamma - iD)t} + \frac{1}{(\Gamma - iD)} \right] + \frac{i}{8\omega_n} \left[\frac{(e^{(\Gamma - i(D - 2\omega_n))t} - 1)}{(\Gamma - i(D - 2\omega_n))} - \text{same}, (\omega_n \rightarrow -\omega_n) \right] \quad (22a)$$

and

$$J_{2,\Delta \neq 0}(t) = \frac{-i}{2\Delta} \left[\frac{e^{(\Gamma - i(D - \Delta))t} - 1}{\Gamma - i(D - \Delta)} - \frac{e^{(\Gamma - iD)t} - 1}{\Gamma - iD} \right] + \left[\frac{-i}{4(2\omega_n - \Delta)} \left(\frac{e^{(\Gamma - i(D + 2\omega_n - \Delta))t} - 1}{\Gamma - i(D + 2\omega_n - \Delta)} - \frac{e^{(\Gamma - iD)t} - 1}{\Gamma - iD} \right) + \text{same}, (\omega_n \rightarrow -\omega_n) \right]. \quad (22b)$$

Now, we consider the following three cases of initial state of the HO.

4.1. Vacuum state

In this case $\alpha, |\alpha|^2 \rightarrow 0$, and the spectrum is given by (14). In general, the analytical expression for the spectrum is too lengthy to write down. A general look at this expression shows that the spectrum structure comprises Lorentzians, dispersive-line shapes and their narrowed shapes centered at $D = 0, \pm 2\omega_n, n = 1, 2, \dots$ with interference terms and oscillating weight coefficients: $\cos Dt, \sin Dt, \cos D_{\mp}t, \sin D_{\mp}t$; with $D_{\mp} = D \mp 2\omega_n$.

The normalized spectrum $S_0(D) = S_0(t, D, \Gamma) / \max(S_0(t, D, \Gamma))$ which is independent of the pulse strength Ω_0 , in this case is plotted against the normalized detector's detuning $D' = D/\Gamma$ for fixed normalized detector's width $\Gamma' = \Gamma/\omega_0$ and detection time $\tau = \omega_0 t$. For $\tau = \pi, \Gamma' = 0.04, \Delta' = 0$ (resonance case) — Fig. 5 — the spectrum is symmetric and develops from a single Lorentzian at $D' = 0$ for $n = 1$ (single pulse) to a Lorentzian with surrounded two small side bands for $n = 2$ (two pulses), and to a Lorentzian surrounded by oscillations and very weak side bands at larger $|D'| \simeq 250$ for larger number of pulses ($n = 5$). These oscillations are due the larger number of sequential pulses over the period $[0, \pi]$.

In the off-resonance case $\Delta' = 5, -5$ with $\tau = 10\pi$, the main peak shifts to the right (left) with weak asymmetrical side bands and dispersive-line shapes (Fig. 6).

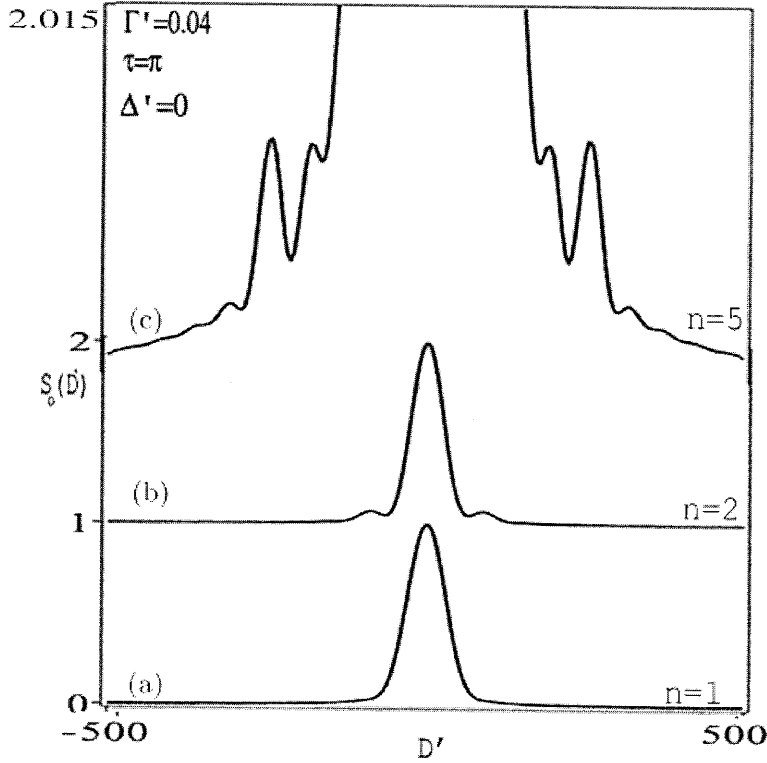


Fig. 5. The normalized spectrum with initial vacuum state $S_0(D')$ versus the normalized detector's detuning parameter D' at exact resonance ($\Delta' = 0$) and normalized detection time $\tau = \pi$ for different pulses $n = 1, 2, 5$.

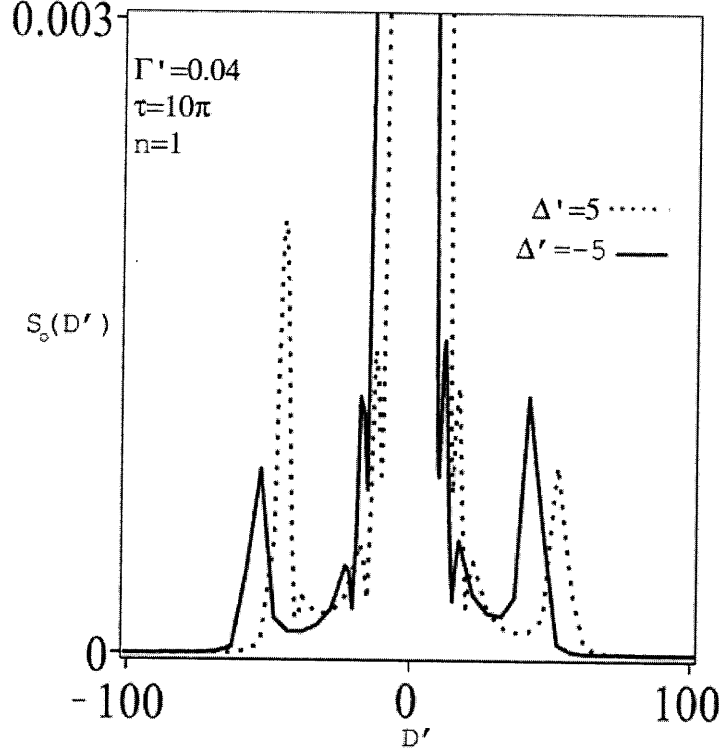


Fig. 6. The zooming of the spectrum $S_o(D')$ for the cases $\Delta' = 5, -5$ and $\tau = 10\pi$.

4.2. Number state

In this case $\alpha \rightarrow 0, |\alpha|^2 \rightarrow n_0$ and the spectrum is given by (15), which has an additive term in $n_0|J_1(t)|^2$ to the previous case of Eq. (14), namely, $\Omega_o^2|J_2(t)|^2$. The balance between the two terms in (15), namely in n_0 and Ω_0 , is best illustrated in Fig. 7 in the off-resonance case ($\Delta' \neq 0$) for equal weights $n_o = \Omega_o'' = 1$ ($\Omega_o'' = \frac{\Omega_o}{\Gamma}$) where the normalized spectrum $S_{n_0}(D')$ is essentially symmetric and the oscillations around the central peak are relatively prominent. For stronger pulse strength $\Omega_o'' > n_0$, the spectrum becomes asymmetric around the right side of the central peak.

4.3. Coherent state

In this case, the spectrum is given by the full expression in (12), where the last term represents an inference between the initial amplitude parameter (α) and the pulse strength (Ω_0).

For fixed parameters $\Gamma' = 0.04, \tau = 10\pi$ and $n = 1$, we note the following structures for the normalized spectrum $S_\alpha(D')$:

- (a) For real α , the balance between the pair of parameters (α, Ω_o'') is best shown in the nonresonant case ($\Delta' \neq 0$) — (Fig. 8) — where $S_\alpha(D')$ has asymmetry with additional smaller peak at $D' = \Delta' = 5$ for $\Omega_o'' > \alpha$. The opposite value of $\Delta' = -5$ only causes lesser asymmetry in the weak oscillations with $\Omega_o'' > \alpha$. For fixed pair values of (Ω_o'', α) with $\Omega_o'' > \alpha$, changing the detuning parameter $|\Delta'|$ we have essentially the same qualitative results with shifted symmetrical or

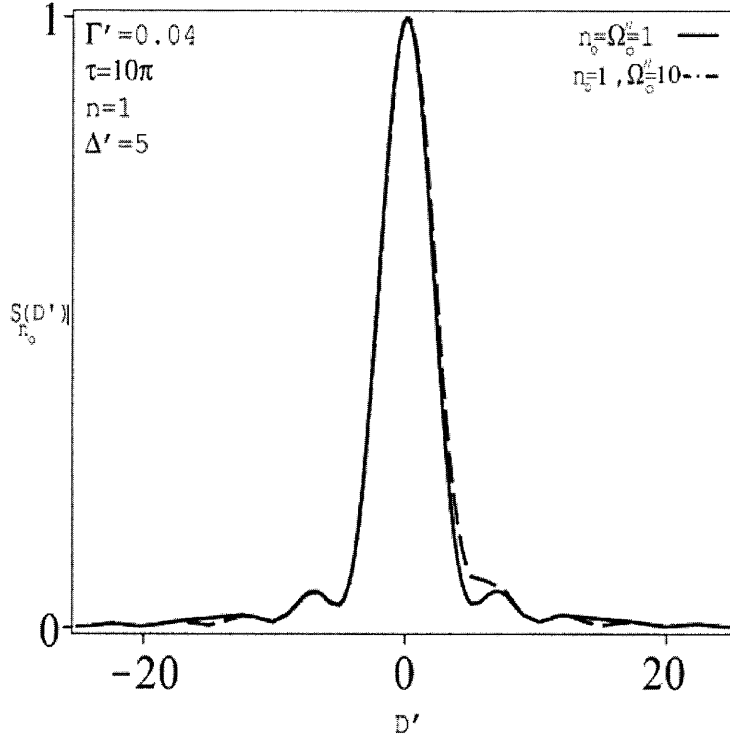


Fig. 7. The normalized spectrum $S_{n_o}(D')$ with initial number state for $\tau = 10\pi, n = 1, \Delta' = 5$ and for different pair values of (n_o, Ω_o'') .

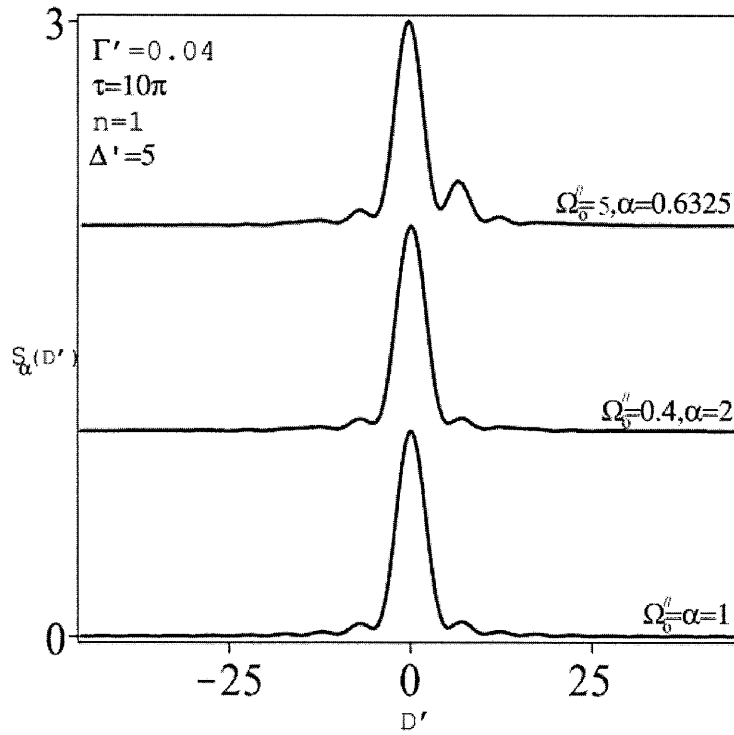


Fig. 8. The normalized spectrum $S_{\alpha}(D')$ with initial coherent state α ($\alpha = \text{real}$) and different values of (α, Ω_o'') .

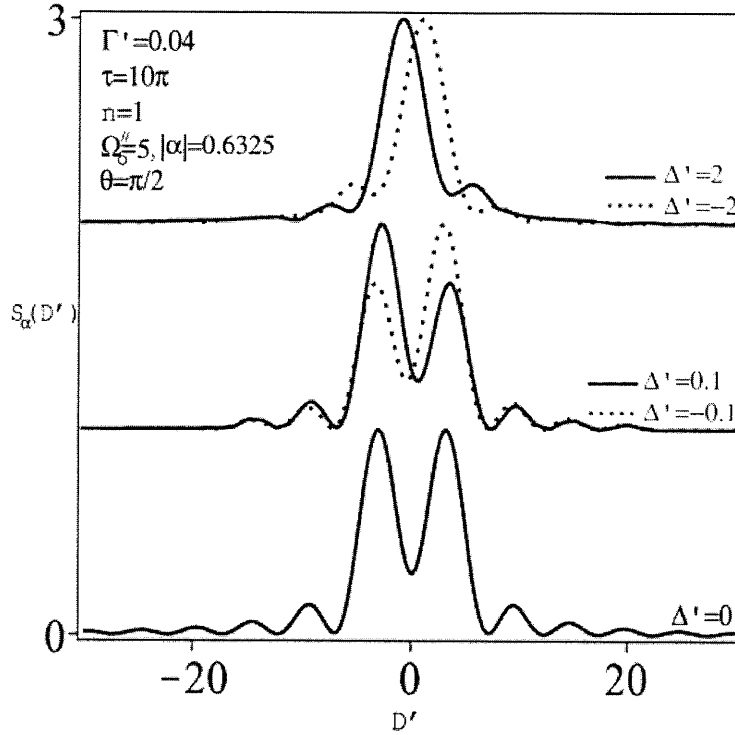


Fig. 9. As Fig. 8 but with $\Omega_o'' = 5$, $|\alpha| = 0.6325$ and versus $|\Delta'| = 0, 0.1$ and 2 .

asymmetrical splitting in the main peak that depends on how small the initial value of the parameter α .

- (b) For complex $\alpha = |\alpha|e^{i\theta}$, the phase parameter θ has a pronounced effect compared with the case of real α ($\theta = 0$) of Fig. 8. This is best shown in Fig. 9 for $\theta = \frac{\pi}{2}$ and $\Omega_o'' = 5$, $|\alpha| = 0.6325$ and varied values of $|\Delta'| = 0, 0.1, 2$.

At exact resonance ($\Delta' = 0$) the dominant single peak (for $\theta = 0$) splits into two peaks (hole burning) with pronounced oscillations in the spectrum wings. In the nonresonant case ($|\Delta'| \neq 0$), the hole burning structure becomes asymmetric with lesser oscillations and washes away with larger values of $|\Delta'| > 1$.

5. Summary

The transient scattered spectrum of a single (quantized) HO interacting with a pulsed laser beam of $\sin^2$ — shape have been investigated in the absence of any dissipation processes. Exact analytical expression are derived for the mean photon number of the HO and the transient spectrum in the resonant and nonresonant regimes for three initial conditions of the HO, namely, vacuum, number and coherent states. The main results are:

- (i) The mean photon number of the HO, with initial vacuum state (or number state) has oscillatory pattern as that of the applied pulse with large detuning parameter. With initial coherent state and increasing detuning, the mean photon number exhibits increase in the envelope oscillations as well in the

amplitude of the beat oscillations. The regularity of the oscillations occurs with initial coherent phase $\theta = 0$ or π , but skewness and asymmetry structure occur for $\theta \neq 0$.

- (ii) The transient spectrum with initial vacuum state at exact resonance is symmetric and starts with a single Lorentzian pulse ($n = 1$) and develops to a Lorentzian with extra two side bands for two pulses ($n = 2$) and tends to a single Lorentzian surrounded by weak oscillations in its tails for larger number of pulses. Oscillations are due to the larger number of sequential pulses. With initial number state $|n_o\rangle$, the balance between the initial excitation (n_o) and the strength of the applied pulse (Ω''_o) is to relatively amplify the oscillations in the wings of the single Lorentzian with asymmetry noted in the nonresonant case for $\Omega''_o > n_o$. For initial coherent state of phase θ , the spectrum at exact resonance is symmetric for both $\theta = 0, \frac{\pi}{2}$, with “hole burning” in the Lorentzian peak and pronounced oscillations in the latter case of $\theta = \frac{\pi}{2}$. With increased detuning the hole burning structure becomes asymmetric and washes away.

Finally, we note that detecting such weak extra side peaks and oscillations (with larger number of sequential pulses) in the scattered spectrum should not pose an experimental obstacle in the light of experiments detecting ultra-weak signal fields in Refs. 22–24. Ultra-weak signals are much desired with optical signal processing (e.g. Refs. 25 and 26 and references therein).

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الملخص

تختص الرسالة بدراسة نموذج رياضي " لتذبذب توافقي كمي "(Quantized Harmonic Oscillator) تحت تأثير مجال نبضي ضوئي ذات شكل (مربع دالة الجيب: $\sin^2$ - pulse). تم استنتاج حلول رياضية لمعادلات هيزنبرج المؤثرة (Heisenberg Operator Equations)، وكذلك حساب دالة الطيف الانتقالي (Transient Spectra) ثم عرض النتائج حسابيا (Computational) وكذلك مناقشتها لعدة حالات ابتدائية " للتذبذب التوافقي ".

جامعة البحرين

قسم الرياضيات

كلية العلوم



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أطروحة مقدمة كجزء من متطلبات الحصول على درجة الماجستير
في الرياضيات

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