

**A STUDY OF DEGENERATE ELLIPTIC
PARTIAL DIFFERENTIAL EQUATIONS**

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DISSERTATION

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ABSTRACT

In this thesis, two types of second order elliptic partial differential equations will be studied. The first type is the following equation

$$A(x, y) \frac{\partial^2 u}{\partial x^2} + 2B(x, y) \frac{\partial^2 u}{\partial x \partial y} + C(x, y) \frac{\partial^2 u}{\partial y^2} = 0$$

for a function u of Sobolev class $\mathcal{W}_{loc}^{2,2}(\Omega)$. Here A, B and C are measurable functions on Ω with $A > 0, C > 0$ and $AC - B^2 > 0$ a.e.

Our main result will be that u is of class $\mathcal{C}^1(\Omega)$ provided that $\frac{(A + C)^2}{AC - B^2}$ is locally integrable on Ω .

The second equation we will study is the non-homogeneous p -harmonic equation

$$\operatorname{div} |\nabla u|^{p-2} \nabla u = \operatorname{div} f$$

for a function $u \in \mathcal{W}^{1,p}(\mathbb{R}^n)$, where $f \in \mathcal{W}^{1,q}(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Our main result is the following:

THEOREM. *Let u be a non-homogeneous p -harmonic function on $\mathbb{R}^n$ of class $\mathcal{W}^{1,p}(\mathbb{R}^n)$, where $1 < p \leq 2$. If $g(x) = |\nabla u(x)|^{\frac{p-2}{2}} \nabla u(x)$, then*

$$g \in \mathcal{W}^{1,2}(\mathbb{R}^n, \mathbb{R}^n)$$

and the following uniform estimate holds

$$\|Dg\|_{\mathbf{L}^2} \leq C(p, n) \|f\|_{\mathbf{L}^q}^{\frac{2-p}{2p-2}} \|Df\|_{\mathbf{L}^q}.$$

Among other applications, this theorem will be used to establish higher integrability of ∇u .

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CHAPTER I
INTRODUCTION

1.1 Notation.

We begin by reviewing the notation we shall employ. In this dissertation, Ω will stand for an open set in Euclidean space $\mathbb{R}^n$, for some n . The standard inner product on $\mathbb{R}^n$ will be denoted by $\langle \cdot, \cdot \rangle$. If $u : \Omega \rightarrow \mathbb{R}^m$ is a function on Ω , the *support* of u is defined by

$$\text{spt } u = \Omega \cap \overline{\{x : u(x) \neq 0\}},$$

where the closure of a set $S \subset \mathbb{R}^n$ is denoted by $\overline{S}$. If $S \subset \Omega$, $\overline{S}$ is compact and also $\overline{S} \subset \Omega$, we shall say that S is *compactly contained* in Ω and write $S \subset\subset \Omega$. $C(\cdots)$ will denote various constants, depending upon the listed parameters, which are not necessarily the same at the various occurrences.

The symbol

$$B(x, r) = \{y : |x - y| < r\}$$

denotes the open ball with center x and radius r . A closed ball will be denoted by

$$\overline{B}(x, r) = \{y : |x - y| \leq r\}.$$

Here $|\cdot|$ is the standard norm on $\mathbb{R}^n$. Also, we shall employ the same notation $|\cdot|$ for the Lebesgue measure, since no confusion will arise. When clarity requires it, we shall write $|\cdot|_k$ to indicate the dimension of the measure. If A and B are two measurable sets with $A \subset B$, we say that A has *full measure* (in B) if $|A| = |B|$.

The partial derivatives of a function f with respect to a variable x will be denoted by $\frac{\partial f}{\partial x}$ or f_x . Also, we will make use of the familiar complex differentiation operators

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

and

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).$$

If $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -tuple of non-negative integers, α is called a *multi-index* and we define $|\alpha| = \sum_{i=1}^n \alpha_i$. Higher order derivatives are denoted by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

1.2 Function Spaces.

Throughout this work, standard function spaces will be employed. In this section, we will discuss those spaces and list some facts about them for later use.

Let m be a non-negative integer, the space of m -times continuously differentiable functions is

$$\mathcal{C}^m(\Omega) = \{u : u : \Omega \rightarrow \mathfrak{R}, D^\alpha u \text{ is continuous for all } \alpha, |\alpha| \leq m\}.$$

We also define

$$\mathcal{C}^\infty(\Omega) = \bigcap_{m=1}^{\infty} \mathcal{C}^m(\Omega).$$

Now let $k = 0, 1, \dots, \infty$ and set

$$\mathcal{C}_o^k(\Omega) = \mathcal{C}^k(\Omega) \cap \{u : \text{spt } u \subset\subset \Omega\}.$$

Let $0 < \beta \leq 1$. A function $u : \Omega \rightarrow \mathfrak{R}^m$ is said to be *Hölder continuous* with exponent β if

$$|u(x) - u(y)| \leq C |x - y|^\beta$$

for some constant C and for all $x, y \in \Omega$. The space of all Hölder continuous functions with exponent β will be denoted by $\mathcal{C}^\beta(\Omega)$. For functions that possess some differentiability we define

$$\mathcal{C}_{loc}^{1,\beta}(\Omega) = \{u : u \in \mathcal{C}^1(\Omega) \text{ and } \nabla u \in \mathcal{C}^\beta(D) \text{ for all compact } D \subset \Omega\}.$$

The familiar Lebesgue spaces will be denoted, as usual, by $L^p(\Omega)$. The norm on $L^p(\Omega)$ will be denoted by $\|\cdot\|_{p,\Omega}$. If Ω is the entire space $\mathfrak{R}^n$, we shall suppress Ω and write $\|\cdot\|_p$. The local Lebesgue spaces are

$$L_{loc}^p(\Omega) = \{u : u \in L^p(\Omega') \text{ for all } \Omega' \subset\subset \Omega\}.$$

We denote by $\mathbf{L}^p(\Omega, \mathfrak{R}^n)$ the space of vector-valued functions whose coordinate functions belong to $\mathbf{L}^p(\Omega)$. The norm on $\mathbf{L}^p(\Omega, \mathfrak{R}^n)$ is

$$\|u\|_{\mathbf{L}^p(\Omega, \mathfrak{R}^n)} = \sum_{i=1}^n \|u_i\|_{p, \Omega}.$$

If Ω is a bounded set with positive measure and $u \in \mathbf{L}^1(\Omega)$, we shall denote the average integral of u over Ω by

$$\oint_{\Omega} u = \frac{1}{|\Omega|} \int_{\Omega} u.$$

Now we want to define Sobolev spaces. Before we do that, we need to discuss the notion of weak derivatives. Let $u \in \mathbf{L}_{loc}^1(\Omega)$. For a given multi-index α , a function $v \in \mathbf{L}_{loc}^1(\Omega)$ is called the α^{th} *weak derivative* of u if

$$\int_{\Omega} \phi v = (-1)^{|\alpha|} \int_{\Omega} u D^{\alpha} \phi$$

for all test functions $\phi \in \mathcal{C}_0^{\infty}(\Omega)$. We write $v = D^{\alpha}u$. Clearly, $D^{\alpha}u$ is uniquely determined up to a set of Lebesgue measure zero. For notational convenience we write $D^0u = u$.

DEFINITION 1.1. For $p \geq 1$ and k a non-negative integer, we define the Sobolev space

$$\mathcal{W}^{k,p}(\Omega) = \{u : D^{\alpha}u \in \mathbf{L}^p(\Omega), |\alpha| \leq k\}.$$

The space $\mathcal{W}^{k,p}(\Omega)$ is equipped with a norm

$$\|u\|_{p,k,\Omega} = \left(\int_{\Omega} \sum_{|\alpha| \leq k} |D^{\alpha} u(x)|^p dx \right)^{\frac{1}{p}}$$

which is clearly equivalent to

$$\sum_{|\alpha| \leq k} \|D^{\alpha} u\|_{p,\Omega}.$$

Furthermore, we set

$$\mathcal{W}^{k,p}(\Omega, \mathbb{R}^m) = \{f = (f_1, f_2, \dots, f_m) : f_i \in \mathcal{W}^{k,p}(\Omega), i = 1, \dots, m\}.$$

It is well-known that $\mathcal{W}^{k,p}(\Omega)$ is a Banach space. The space $\mathcal{W}_0^{k,p}(\Omega)$ is defined to be the closure of $\mathcal{C}_0^{\infty}(\Omega)$ relative to the above norm. Local Sobolev spaces are defined as

$$\mathcal{W}_{loc}^{k,p}(\Omega) = \{u : u \in \mathcal{W}^{k,p}(\Omega'), \text{ for all } \Omega' \subset\subset \Omega\}.$$

We record the following theorems on Sobolev spaces for future use.

THEOREM 1.2 (MEYERS-SERRIN). *The space*

$$\mathcal{C}^{\infty}(\Omega) \cap \mathcal{W}^{k,p}(\Omega)$$

is dense in $\mathcal{W}^{k,p}(\Omega)$.

THEOREM 1.3 (SOBOLEV'S IMBEDDING THEOREM). *The space $\mathcal{W}^{m,p}(\Omega)$ is continuously imbedded in $\mathbf{L}^q(\Omega)$ for $1 \leq p \leq q \leq \frac{np}{n-mp}$. When $p < q$ this imbedding is actually a compact operator.*

THEOREM 1.4 (SOBOLEV-POINCARÉ'S INEQUALITY). *For each $1 \leq s < n$ there exists a constant $A(n, s)$ such that*

$$\left(\int_{B(x,r)} |f(y) - f_{B(x,r)}|^{\frac{n-s}{n-s}} dy \right)^{\frac{n-s}{n-s}} \leq r A(n, s) \left(\int_{B(x,r)} |\nabla f(y)|^s dy \right)^{\frac{1}{s}}$$

for all $B(x, r) \subset \mathbb{R}^n$, $f \in \mathcal{W}^{1,s}(B(x, r))$. Here $f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy$.

1.3 Mappings with Integrable Distortion.

In this section we assume that $\Omega \subset \mathbb{R}^n$ is a domain, that is, open and connected.

We shall consider mappings $f = (f_1, f_2, \dots, f_n) : \Omega \rightarrow \mathbb{R}^n$ of Sobolev class $\mathcal{W}_{loc}^{1,n}(\Omega, \mathbb{R}^n)$. Its total differential $Df(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is identified with the

$n \times n$ -matrix $Df(x) = \left[\frac{\partial f_i}{\partial x_j} \right]$. The Jacobian $J(x, f) = \det Df(x)$ is assumed

to be nonnegative a.e., in other words f preserves orientation.

DEFINITION 1.5. A mapping $f \in \mathcal{W}_{loc}^{1,n}(\Omega, \mathbb{R}^n)$ is said to have *finite distortion* if

$$|Df(x)|^n \leq n^{\frac{n}{2}} K(x) J(x, f) \text{ a.e.}$$

where the function K satisfies $1 \leq K(x) < \infty$ for almost every $x \in \Omega$.

Here $|Df(x)|$ denotes the Hilbert-Schmidt norm of the total differential $Df(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$. It follows from the assumption that $Df(x)$ vanishes whenever $J(x, f)$ does. The *distortion quotient* at the point $x \in \Omega$ is defined by

$$K(x, f) = \frac{|Df(x)|^n}{n^{\frac{n}{2}} J(x, f)} \geq 1.$$

If $J(x, f) = 0$ we put $K(x, f) = 1$. Therefore, $K(\cdot, f)$ is defined almost everywhere in Ω .

A mapping $f \in \mathcal{W}_{loc}^{1,n}(\Omega, \mathbb{R}^n)$ is said to be *M-quasiregular*, $1 \leq M < \infty$, if $\|K(\cdot, f)\|_{\infty, \Omega} \leq M$. In addition, if f is a homeomorphism, f is said to be *M-quasiconformal*. Further, if $K(\cdot, f) \in \mathbf{L}_{loc}^1(\Omega)$, then f is said to be of *integrable distortion*.

A well-known result in the theory of quasiregular mappings states that a non-constant quasiregular mapping is open and discrete. For the proofs of these results, we refer the reader to [19]. In two dimensions this fact has been established by Bojarski [6]. He proved Stoilow's type factorization

$$f = \phi \circ h^{-1}$$

with $h : \Omega' \rightarrow \Omega$ a homeomorphism and $\phi : \Omega' \rightarrow \mathbb{R}^2$ a holomorphic function.

Recently, Iwaniec and Šverák [15] proved that this factorization remains valid for 2-dimensional mappings with integrable distortion. Therefore, such mappings are also open and discrete. Here is a precise formulation of their result.

THEOREM 1.6 (IWANIEC-ŠVERÁK). *Let Ω be a bounded domain in the complex plane and let $f \in \mathcal{W}^{1,2}(\Omega, \mathbb{C})$ be a mapping whose distortion quotient $K = K(\cdot, f)$ is integrable on Ω . Then there exists a homeomorphism $h \in \mathcal{W}^{1,2}(\Omega', \Omega)$ and a holomorphic function $\phi \in \mathcal{W}^{1,2}(\Omega', \mathbb{C})$ such that*

$$f = \phi \circ h^{-1}.$$

Moreover,

$$\int_{\Omega'} |Dh(\omega)|^2 d\sigma(\omega) \leq \int_{\Omega} K(z) d\sigma(z)$$

and

$$\int_{\Omega'} |\phi'(\omega)|^2 d\sigma(\omega) \leq \int_{\Omega} \left| \frac{\partial}{\partial z} f(z) \right| d\sigma(z).$$

1.4 What is Ahead.

In Chapter II we will study solutions to the equation

$$A(x, y) \frac{\partial^2 u}{\partial x^2} + 2B(x, y) \frac{\partial^2 u}{\partial x \partial y} + C(x, y) \frac{\partial^2 u}{\partial y^2} = 0,$$

where $u \in \mathcal{W}_{loc}^{2,2}(\Omega)$. To do so, we will introduce the so called complex gradient of u , $f = u_x - iu_y \in \mathcal{W}_{loc}^{1,2}(\Omega, \mathbb{C})$. By utilizing the theory of mappings with integrable distortion, we will show that f is continuous, open, and discrete. The continuity of f immediately implies that u is of class $\mathcal{C}^1(\Omega)$. The discreteness of f will be used to prove the following unique continuation result

THEOREM. If $E \subset \Omega$ with Hausdorff dimension greater than 1 and if $u|_E = 0$. Then $u \equiv 0$ on Ω .

Our next goal is to find an estimate of the modulus of continuity of f . The result obtained has nothing to do with the solution u on the part of f . Here is a precise statement

THEOREM. Let $f = (f_1, \dots, f_m) : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a continuous open map of class $\mathcal{W}^{1,n}(\Omega, \mathbb{R}^m)$. Then

$$|f(a) - f(b)| \leq \frac{C(n)}{\sqrt[n]{\log \frac{2R}{|a-b|}}} \sum_{i=1}^m \|\nabla f_i\|_{\mathbf{L}^n(B(x, 2R))}$$

for all $a, b \in B(x, R)$, where $B(x, 2R) \subset \Omega$.

The discussion of the above result and its prerequisites is carried out in Chapter III.

The study of the non-homogeneous p -harmonic equation

$$\operatorname{div} |\nabla u|^{p-2} \nabla u = \operatorname{div} f$$

is the content of Chapter IV. We will prove that for $1 < p \leq 2$, $|\nabla u|^{\frac{p-2}{2}} \nabla u$ is of class $\mathcal{W}^{1,2}(\mathbb{R}^n, \mathbb{R}^n)$ and its total differential satisfies a uniform estimate by means of f . These results show immediately that the only p -harmonic function (that is, $f \equiv 0$) of class $\mathcal{W}^{1,p}(\mathbb{R}^n)$ is the zero function. Another application yields higher integrability of ∇u . A third application shows that in three dimensions, u is Hölder continuous.

CHAPTER II

NON-UNIFORMLY ELLIPTIC EQUATIONS

2.1 Introduction.

We are concerned with equations of the following type

$$A(x, y) \frac{\partial^2 u}{\partial x^2} + 2B(x, y) \frac{\partial^2 u}{\partial x \partial y} + C(x, y) \frac{\partial^2 u}{\partial y^2} = 0 \quad (2.1)$$

where u is a function of Sobolev class $\mathcal{W}_{loc}^{2,2}(\Omega)$. Throughout this chapter, we shall assume that $\Omega \subset \mathbb{R}^2$ is a domain, that is, open and connected. Here A, B , and C are measurable functions on Ω . The *non-uniform ellipticity* of equation (2.1) means that the quadratic form $A(x, y) \xi^2 + 2B(x, y) \xi \zeta + C(x, y) \zeta^2$ is positive definite at almost every point $(x, y) \in \Omega$, that is,

- (1) $A(x, y) > 0$ and $C(x, y) > 0$
- (2) $A(x, y) C(x, y) - B^2(x, y) > 0$.

Let us begin with a brief historical account. The study of the second order equations with smooth coefficients goes back to Poincaré, Fredholm, Hilbert and many others. At the beginning of these developments the solutions were found by using integral representations of fundamental solutions via Green's function. Also, the so-called parametrix method was applied by Hilbert and

his school to solve some boundary value problems. The equations with Hölder continuous coefficients were systematically investigated by Schauder in the 1930's by using his fixed point method. We will not enter into this extremely rich literature. In this chapter we emphasize the case when the coefficients are only measurable. The first significant result in this direction is due to Alexandrov who studied such equations in connection with the Monge-Ampère Equation. In two dimensions the methods of complex functions proved most efficient, see the work of Bers and Nirenberg [4] and [5]. They assumed that the equation is *uniformly elliptic* which means that there are positive constants M and ϵ such that

$$(1) \quad \epsilon < A(x, y) \leq M \text{ and } \epsilon < C(x, y) \leq M$$

$$(2) \quad A(x, y)C(x, y) - B^2(x, y) > \epsilon .$$

Our goal is to examine equation (2.1) under weaker assumptions. In order to formulate these assumptions, we introduce the so called *distortion function*

$$\mathcal{K}(x, y) = \frac{[A(x, y) + C(x, y)]^2}{2[A(x, y)C(x, y) - B^2(x, y)]} - 1. \quad (2.2)$$

In terms of the coefficient matrix

$$\mathcal{M} = \begin{bmatrix} A & B \\ B & C \end{bmatrix},$$

we may express $\mathcal{K}$ by

$$\mathcal{K} = \frac{(\text{Tr } \mathcal{M})^2}{2 \det \mathcal{M}} - 1 = \frac{1}{2} \left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} \right),$$

where $\lambda_1 = \lambda_1(x, y)$ and $\lambda_2 = \lambda_2(x, y)$ are the eigenvalues of $\mathcal{M}$. Note that $\mathcal{K} = \frac{1}{2} \left(\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_2}} - \frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1}} \right)^2 + 1$. Thus $\mathcal{K}(x, y) \geq 1$ *a.e.* and equality occurs if and only if $\lambda_1(x, y) = \lambda_2(x, y)$ which means that $\mathcal{M}$ is a multiple of the identity matrix, that is, $A(x, y) = C(x, y)$ and $B(x, y) = 0$. At such points our equation reduces to the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

It is in that sense that $\mathcal{K}$ measures how far our equation is from the Laplacian.

Our basic assumption will be that

$$\mathcal{K} \in \mathbf{L}_{loc}^1(\Omega). \quad (2.3)$$

2.2 The Complex Gradient.

In our study of PDEs in two variables, it will be convenient to use complex numbers. We shall write $z = x + iy$ and recall the familiar Cauchy-Riemann operators

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).$$

Let $u \in \mathcal{W}_{loc}^{2,2}(\Omega)$ be a solution of equation (2.1). The *complex gradient* of u is

defined by

$$f(z) = u_x - i u_y = 2 \frac{\partial u}{\partial z}.$$

Obviously, this is a complex function of Sobolev class $\mathcal{W}_{loc}^{1,2}(\Omega, \mathbb{C})$. In accordance with Section 1.3 we can now state the following:

DEFINITION 2.1. A mapping $F = (\phi, \psi)$ of Sobolev class $\mathcal{W}_{loc}^{1,2}(\Omega, \mathbb{R}^2)$ is said to be of *finite distortion* if

$$|DF(z)|^2 \leq 2K(z) \det DF(z) \quad a.e. \quad (2.4)$$

where $1 \leq K(z) < \infty \quad a.e.$ If, moreover, there exists such a K with $K \in \mathbf{L}_{loc}^1(\Omega)$, then F is said to be of *integrable distortion*.

The smallest $K(z)$ for which (2.4) holds will be called the *distortion* of F at the point z . Here DF stands for the differential of F

$$DF = \begin{bmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{bmatrix}.$$

Its Hilbert-Schmidt norm is defined by

$$|DF|^2 = \phi_x^2 + \phi_y^2 + \psi_x^2 + \psi_y^2.$$

By Hadamard's inequality we always have

$$2 \det DF(z) \leq |DF(z)|^2$$

and equality occurs if and only if $DF(z)$ is a matrix of similarity transformation, that is, a multiple of an orthogonal matrix. Therefore, the case $K(z) \equiv 1$ is very special. Complex functions whose distortion equals one everywhere are holomorphic.

2.3 Study of Equation (2.1) via Mappings with Integrable Distortion.

Let u be a solution of equation (2.1) and let $f = u_x - i u_y$ be its complex gradient. We have $|Df|^2 = u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2$ and $\det Df = u_{xy}^2 - u_{xx}u_{yy}$. We will prove that f has integrable distortion. This will be achieved by showing that $K(\cdot, f)$ is dominated by the function $\mathcal{K}$ at (2.2).

THEOREM 2.2. *Let u be a solution of equation (2.1) with complex gradient $f = u_x - i u_y$. Then f is of integrable distortion and*

$$K(z, f) \leq \mathcal{K}(z) = \frac{A^2(z) + C^2(z) + 2B^2(z)}{2[A(z)C(z) - B^2(z)]}. \quad (2.5)$$

Moreover, inequality (2.5) is sharp.

PROOF. For the purpose of this proof, we omit the dependence of functions on the variable z . Consider the following two cases:

Case I: $B = 0$. Therefore, $u_{yy} = -\frac{A}{C}u_{xx}$. The inequality $2AC \leq A^2 + C^2$

implies that

$$\begin{aligned}
AC(A^2 + C^2)u_{xx}^2 + 2AC^3u_{xy}^2 &\leq AC(A^2 + C^2)u_{xx}^2 + (A^2 + C^2)C^2u_{xy}^2, \\
AC[(A^2 + C^2)u_{xx}^2 + 2C^2u_{xy}^2] &\leq (A^2 + C^2)(ACu_{xx}^2 + C^2u_{xy}^2), \\
\frac{A^2}{C^2}u_{xx}^2 + u_{xx}^2 + 2u_{xy}^2 &\leq \frac{A^2 + C^2}{AC} \left(\frac{A}{C}u_{xx}^2 + u_{xy}^2 \right).
\end{aligned}$$

Hence

$$u_{yy}^2 + u_{xx}^2 + 2u_{xy}^2 \leq \frac{A^2 + C^2}{AC} (u_{xy}^2 - u_{xx}u_{yy}).$$

Case II: $B \neq 0$. We have

$$\begin{aligned}
0 &\leq \frac{1}{4B^2} [(A^2 - AC + 2B^2)u_{xx} - (C^2 - AC + 2B^2)u_{yy}]^2 \\
&= (A^2 + C^2 + 2B^2) \left[\left(\frac{Au_{xx} + Cu_{yy}}{2B} \right)^2 - u_{xx}u_{yy} \right] \\
&\quad - (AC - B^2) \left[u_{xx}^2 + u_{yy}^2 + 2 \left(\frac{Au_{xx} + Cu_{yy}}{2B} \right)^2 \right] \\
&= (A^2 + C^2 + 2B^2) (u_{xy}^2 - u_{xx}u_{yy}) - (AC - B^2) (u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2).
\end{aligned}$$

Therefore,

$$u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 \leq \frac{A^2 + C^2 + 2B^2}{AC - B^2} (u_{xy}^2 - u_{xx}u_{yy}).$$

In either case we conclude with (2.5). The sharpness of this estimate will be established through the subsequent example. ■

Let us remark that these elegant arguments were found as a result of a far reaching refinements of our first lengthy calculation based on minimization of some nonlinear expressions.

EXAMPLE 2.3.

(1) Let Ω be the first quadrant of $\mathbb{R}^2$, $\Omega = \{(x, y) : x > 0, y > 0\}$. Consider the following scalar functions on Ω

$$A(x, y) = y, B \equiv 0, C(x, y) = x \text{ and } u(x, y) = x^3 - y^3.$$

Clearly, $Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0$.

In this case, $\mathcal{K}(x, y) = \frac{x^2 + y^2}{2xy}$ and $\frac{|Df(x, y)|^2}{2 \det Df(x, y)} = \frac{x^2 + y^2}{2xy}$, where

$f = u_x - i u_y$. Therefore,

$$\frac{x^2 + y^2}{2xy} \leq K(z, f) \leq \mathcal{K}(x, y) = \frac{x^2 + y^2}{2xy},$$

and $K(z, f) = \mathcal{K}(x, y)$ in this case.

The next example deals with the case $B \neq 0$.

(2) Let $\Omega = \mathbb{R}^2$, $A(x, y) \equiv 2$, $B(x, y) \equiv C(x, y) \equiv 1$ and $u(x, y) = x^2 + 4y^2 - 6xy$. As above, one easily finds that $K(z, f) = \mathcal{K}(x, y)$. The above examples show that inequality (2.5) is sharp. The following example shows that a strict inequality may occur.

(3) Let $\Omega = \mathbb{R}^2$, $A \equiv 2$, $B \equiv 0$, $C \equiv 3$ and $u(x, y) = 3x^2 + 4xy - 2y^2$. One easily sees that $K(z, f) = \frac{21}{20}$ whereas $\mathcal{K}(x, y) = \frac{13}{12}$.

COROLLARY 2.4. *The complex gradient $f = u_x - iu_y$ satisfies the Beltrami equation*

$$f_{\bar{z}} = \mu(z)f_z.$$

Ellipticity of this equation is expressed by the inequality

$$|\mu| \leq \frac{\sqrt{(A-C)^2 + 4B^2}}{A+C} < 1 \text{ a.e.}$$

PROOF. It is a simple matter to see that $u_{xx} = f_{\bar{z}} + \frac{1}{2}(f_z + \overline{f_z})$, that $u_{yy} = f_{\bar{z}} - \frac{1}{2}(f_z + \overline{f_z})$ and that $u_{xy} = \frac{1}{2i}(\overline{f_z} - f_z)$. Thus

$$u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 = 2(|f_z|^2 + |f_{\bar{z}}|^2),$$

and

$$u_{xy}^2 - u_{xx}u_{yy} = |f_z|^2 - |f_{\bar{z}}|^2.$$

Theorem (2.1) implies

$$2(|f_z|^2 + |f_{\bar{z}}|^2) \leq 2\mathcal{K}(|f_z|^2 - |f_{\bar{z}}|^2).$$

Therefore,

$$|f_{\bar{z}}|^2 \leq \frac{\mathcal{K}-1}{\mathcal{K}+1}|f_z|^2 = \frac{(A-C)^2 + 4B^2}{(A+C)^2}|f_z|^2.$$

So we have

$$|\mu| = \frac{|f_{\bar{z}}|}{|f_z|} \leq \frac{\sqrt{(A-C)^2 + 4B^2}}{A+C}.$$

This completes the proof. ■

We now recall Theorem 1.6 which establishes Stoilow's type factorization of mappings with integrable distortion. It asserts among other things:

COROLLARY 2.5. *The complex gradient of a solution of equation (2.1) is open and discrete.*

It is worth mentioning that in Theorem 1.6 Ω was assumed to be bounded. This assumption is not needed in Corollary 2.5 since openness and discreteness are local properties.

We see from the Stoilow factorization that $f(z) = F(h^{-1}(z))$, where F is holomorphic and h is a homeomorphism. In particular, f is continuous. Consequently, we have the following regularity result

COROLLARY 2.6. *Every solution of equation (2.1) is of class $C^1(\Omega)$ and its gradient is differentiable a.e.*

The second assertion follows from Theorem 2.7 which is due to Gehring and Lehto [2]. Before stating this theorem, let us note that the gradient of the solution of equation (2.1) is a mapping of class $\mathcal{W}_{loc}^{1,2}(\Omega, \mathbb{R}^2)$ and, therefore, has finite partial derivatives a.e.

THEOREM 2.7(GEHRING-LEHTO). *Let $O \subset \mathbb{C}$ be open and let $g : O \rightarrow \mathbb{C}$ be an open mapping with finite partial derivatives a.e. Then g is differentiable a.e. in O .*

2.4 Unique Continuation Property.

One important feature of the complex gradient of a solution to equation (2.1) is that it has isolated zeros. It is well-known that if two Sobolev functions coincide on a set of positive measure, so do their partial derivatives. Therefore, if two solutions of equation (2.1) agree on a set $E \subset \Omega$ with $|E| > 0$, they must be equal on Ω . This holds because their complex gradients would agree on a set with a limit point.

The assumption $|E| > 0$ can be relaxed substantially. It suffices to assume that E has Hausdorff dimension greater than one. Before we go any further, we need some preparation. We will be interested in sequences in $\mathbb{R}^2$ that approach their limits in a certain direction.

A half-line issuing from a point $a \in \mathbb{R}^2$ and passing through another point b will be denoted by $\overrightarrow{ab}$. The *direction* $\overrightarrow{ab}$, denoted by $\text{dir}(\overrightarrow{ab})$, is defined to be the point $\frac{b-a}{|b-a|}$ in the unit sphere S^1 .

DEFINITION 2.8. Let $a \in \mathbb{R}^2$ and $s \in S^1$. A sequence $\{a_n\}$ in $\mathbb{R}^2$ with $a_n \neq a$ is said to *s-converge* to a or to *approximate a in the s -direction* if and only if the following two conditions hold:

- (1) $\lim_{n \rightarrow \infty} a_n = a$
- (2) $\lim_{n \rightarrow \infty} \text{dir}(\overrightarrow{aa_n}) = s$.

Now we recall the definition of Hausdorff measure and dimension.

DEFINITION 2.9. Let $A \subset \mathbb{R}^n$ and $0 \leq s < \infty$. Define

$$\mathcal{H}^s(A) = \lim_{\delta \downarrow 0} \inf \left\{ \sum_{j=1}^{\infty} (\text{diam } C_j)^s : A \subset \bigcup_{j=1}^{\infty} C_j, \text{diam } C_j \leq \delta \right\}.$$

We call $\mathcal{H}^s$ the *s-dimensional Hausdorff measure* on $\mathbb{R}^n$. The *Hausdorff dimension* of a set $A \subset \mathbb{R}^n$ is defined to be

$$\dim A = \inf \{s : \mathcal{H}^s(A) = 0\}.$$

Note the following elementary but useful fact. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}^2$ is Lipschitz with Lipschitz constant M . Then for any set $A \subset \mathbb{R}$

$$\mathcal{H}^1(f(A)) \leq M\mathcal{H}^1(A). \quad (2.6)$$

The following theorem is adapted from Saks [20], Lemma 3.1. The proof provided there is rather short and many details are missing. Here we will give a detailed argument.

THEOREM 2.10. *Let R be a planar set and $\theta \in S^1$ a fixed direction, and let P be the set of points of R that cannot be approximated in θ -direction by points in R . Then P is a countable union of sets each of which has finite 1-dimensional Hausdorff measure. In particular $\dim P \leq 1$.*

PROOF. By changing the coordinate system, if necessary, we may assume that $\theta = (0, 1)$. Define for $n \in \mathbb{N}$

$$P_n = \{(x, y) \in P : (a, b) \in R, |x - a| \leq \frac{1}{n}, |y - b| \leq \frac{1}{n} \text{ imply } b - y \leq n|a - x|\}.$$

We claim that $P = \bigcup_{n=1}^{\infty} P_n$. Let $p = (x, y) \in P$. Since the isolated points of R obviously belong to P_n for sufficiently large n , we may assume that p is a limit point of R . Suppose that $p \notin P_n$ for all n . Choose a sequence $p_m = (x_m, y_m)$ in R of distinct terms converging to p . Then for every $n \in \mathbb{N}$, $|p_m - p| < \frac{1}{n}$ provided m is sufficiently large. This implies $|x_m - x| < \frac{1}{n}$ and $|y_m - y| < \frac{1}{n}$.

Since $p \notin P_n$ we must have $n|x_m - x| < y_m - y$. Consequently,

$$n^2(x_m - x)^2 + (x_m - x)^2 < (x_m - x)^2 + (y_m - y)^2$$

or

$$\frac{(x_m - x)^2}{(x_m - x)^2 + (y_m - y)^2} < \frac{1}{n^2 + 1}.$$

This simply means that $\frac{(x_m - x)^2}{(x_m - x)^2 + (y_m - y)^2} \rightarrow 0$. Now we observe that

$\frac{(y_m - y)^2}{(x_m - x)^2 + (y_m - y)^2} = 1 - \frac{(x_m - x)^2}{(x_m - x)^2 + (y_m - y)^2} \rightarrow 1$. Therefore,

$\frac{x_m - x}{\sqrt{(x_m - x)^2 + (y_m - y)^2}} \rightarrow 0$ and $\frac{y_m - y}{\sqrt{(x_m - x)^2 + (y_m - y)^2}} \rightarrow 1$ since

$y_m - y > 0$. In other words, p can be approximated in $(0,1)$ -direction by points in R . This is a contradiction. Therefore, $P \subset \bigcup_{n=1}^{\infty} P_n$, the other inclusion being evident.

In what follows it will be convenient to break each P_n into subsets of small diameter. Precisely we write $P_n = \bigcup_{k=1}^{\infty} P_{n,k}$ with $\text{diam } P_{n,k} < \frac{1}{n}$. Now if

$(x_1, y_1), (x_2, y_2) \in P_{n,k}$, then $|(x_1, y_1) - (x_2, y_2)| < \frac{1}{n}$ which implies that

$|x_1 - x_2| < \frac{1}{n}$ and $|y_1 - y_2| < \frac{1}{n}$. Thus

$$|y_1 - y_2| \leq n |x_1 - x_2| \quad (2.7)$$

by the definition of P_n .

Next, let $Q_{n,k}$ be the orthogonal projection of $P_{n,k}$ onto the x -axis. The latter inequality shows that under this projection each point in $Q_{n,k}$ corresponds to a unique point in $P_{n,k}$. Therefore, we may define the inverse map $f_{n,k} : Q_{n,k} \rightarrow \mathbb{R}^2$. Another application of (2.7) shows that $f_{n,k}$ is Lipschitz. Indeed, given $x_1, x_2 \in Q_{n,k}$ there exist unique y_1 and y_2 so that $f_{n,k}(x_1) = (x_1, y_1)$ while $f_{n,k}(x_2) = (x_2, y_2)$. Thus $|f_{n,k}(x_1) - f_{n,k}(x_2)| \leq |x_1 - x_2| + |y_1 - y_2| \leq (n+1)|x_1 - x_2|$ as desired. Consequently, each set $P_{n,k}$

has finite 1-dimensional Hausdorff measure by (2.6). As a matter of fact we have

$$\mathcal{H}^1(P_{n,k}) = \mathcal{H}^1(f_{n,k}(Q_{n,k})) \leq (n+1)\mathcal{H}^1(Q_{n,k}) \leq \frac{n+1}{n}.$$

Observing that $P = \bigcup_{n,k \in \mathbb{N}} P_{n,k}$ completes the proof. ■

COROLLARY 2.11. *Let $E \subset \mathbb{R}^2$ be a set of Hausdorff dimension greater than 1. Then we can find a subset $F \subset E$ of dimension also greater than 1 with the following property. For every point $a \in F$ there are two sequences $\{a_n\}$ and $\{a'_n\}$ in E each of distinct terms such that $\{a_n\}$ approximates a in $(1,0)$ -direction while $\{a'_n\}$ approximates a in $(0,1)$ -direction.*

PROOF. First apply the above theorem for $\theta = (0,1)$, to obtain a set P with $\dim P \leq 1$ such that every point in $E - P$ can be approximated in $(0,1)$ -direction by points in E . Next apply the above theorem once more with $\theta = (1,0)$ to obtain a set Q with $\dim Q \leq 1$ such that every point in $E - P - Q$ can be approximated in $(1,0)$ -direction by points in E . The assertion now follows by letting $F = E - (P \cup Q)$. ■

We are now in a position to state and prove the main result of this section.

THEOREM 2.12 (UNIQUE CONTINUATION THEOREM). *If $E \subset \Omega$ with $\dim E > 1$ and if u is a solution of equation (2.1) with $u|_E = 0$. Then $u \equiv 0$ on Ω .*

REMARK 2.13. Recall that solutions of (2.1) are continuous. Precisely, this means that u has a continuous representative. In the following proof we will be dealing with this continuous function. The heart of the matter in Theorem 2.12 is the case when $|E| = 0$. This makes the continuity of u essential because it cannot be redefined on E .

PROOF OF THEOREM 2.12. Let F be as in Corollary 2.11. Fix $a \in F$ and let T_1 and T_2 be the unit vectors issuing from a with directions $(1,0)$ and $(0,1)$, respectively. Furthermore, let $\{x_n\}$ be a sequence of distinct terms in E that $(1,0)$ -converges to a . Note that $u(x_n) = u(a) = 0$. By using Taylor's expansion

$$u(x) - u(a) = \nabla u(a) \cdot (x - a) + o(|x - a|),$$

we have

$$0 = \frac{u(x_n) - u(a)}{|x_n - a|} = \nabla u(a) \cdot \frac{(x_n - a)}{|x_n - a|} + \frac{o(|x_n - a|)}{|x_n - a|}.$$

Letting n go to infinity, we obtain

$$\nabla u(a) \cdot T_1 = 0.$$

In much the same way, we find that $\nabla u(a) \cdot T_2 = 0$. Thus $\nabla u(a) = 0$. Therefore, $\nabla u|_F = 0$ and, of course, the same is true for its complex gradient. F is uncountable and thus has a limit point in Ω . So $\nabla u \equiv 0$ and u is constant in Ω . But we know that $u|_E = 0$, consequently $u \equiv 0$, as desired. ■

An immediate consequence of Theorem 2.12 is the following

COROLLARY 2.14. *If two solutions of equation (2.1) agree on a set of Hausdorff dimension greater than one, then they are equal.*

Let us remark that the assumption that $\dim E > 1$ is by no means necessary. For example, E could be the topological comb space.

CHAPTER III

MODULUS OF CONTINUITY

3.1 Introduction.

In this chapter we want to estimate the modulus of continuity of the complex gradient of a solution of equation (2.1). It turns out that for our results we only need to exploit the continuity and the openness of the complex gradient of the solution of equation (2.1). We also need the fact that the complex gradient belongs to the Sobolev class $\mathcal{W}^{1,2}(\Omega)$.

It is perhaps more interesting to note that our arguments remain valid in arbitrary dimension, say n , once the mappings in question are continuous, open, and belong to the Sobolev class $\mathcal{W}^{1,n}(\Omega)$. Therefore, we present these results in higher dimensions.

3.2 Measure Theoretic Results.

In this section we will prove some measure theoretic results. They will be used to give an estimate of the modulus of continuity of continuous open mappings of class $\mathcal{W}^{1,n}(\Omega)$, where Ω is an open set in $\mathbb{R}^n$. We will make use of well-known facts from integration theory such as the Fubini Theorem and results on modes of convergence.

Throughout this section we fix $x_0 \in \mathbb{R}^n$ and $R > 0$. In addition, we set $B = B(x_0, R)$ and $S_r = S(x_0, r)$, where r is some positive number. In what follows we use the notation $d\sigma(x)$ for the $(n - 1)$ -dimensional area element on S_r .

Our first result is the following

PROPOSITION 3.1. *If $P \subset B$ has full measure, that is $|P| = |B|$. Then for almost every r , $0 < r < R$, we have $|P \cap S_r|_{n-1} = |S_r|_{n-1}$.*

PROOF. Let $E_r = S_r - P$, $0 < r < R$ and $E = \cup E_r = B - P$. Hence $|E| = 0$. By Fubini's theorem, we have

$$0 = \int_B \chi_E(x) dx = \int_0^R \int_{S_r} \chi_E(x) d\sigma(x) dr.$$

Therefore, $\int_{S_r} \chi_E(x) d\sigma(x) = 0$ for almost every $r \in (0, R)$. On the other hand $\int_{S_r} \chi_E(x) d\sigma(x) = |E \cap S_r|_{n-1} = |E_r|_{n-1}$. Consequently, $|E_r|_{n-1} = 0$ for almost every r . This is precisely what the proposition asserts. ■

PROPOSITION 3.2. *Let $\{\phi_j\}$ be a sequence converging in $L^p(B)$ to $\phi \in L^p(B)$, for some $1 \leq p < \infty$. Then $\{\phi_j\}$ has a subsequence that converges to ϕ in $L^p(S_r)$ for almost every $r \in (0, R)$.*

PROOF. We have

$$\lim_{j \rightarrow \infty} \int_B |\phi_j(x) - \phi(x)|^p dx = 0.$$

This is the same as

$$\lim_{j \rightarrow \infty} \int_0^R \int_{S_r} |\phi_j(x) - \phi(x)|^p d\sigma(x) dr = 0. \quad (3.1)$$

Letting $\psi_j(r) = \int_{S_r} |\phi_j(x) - \phi(x)|^p d\sigma(x)$, equation (3.1) states that

$$\psi_j(r) \rightarrow 0 \text{ in } \mathbf{L}^1(0, R) \text{ as } j \rightarrow \infty.$$

Therefore, $\{\psi_j\}$ has a subsequence $\{\psi_{j_k}\}$ that converges to 0 *a.e.* on $(0, R)$.

This means that for almost every $r \in (0, R)$ we have

$$\lim_{k \rightarrow \infty} \int_{S_r} |\phi_{j_k}(x) - \phi(x)|^p d\sigma(x) = 0,$$

completing the proof. ■

An immediate inference from Propositions 3.1 and 3.2 is the following

COROLLARY 3.3. *Suppose that the sequence $\{h_j\}$ converges to h in $\mathcal{W}^{1,p}(B)$. Then there is a subsequence $\{h_{j_{k_s}}\}$ such that for almost every $r \in (0, R)$, each of the following holds as $s \rightarrow \infty$:*

- (1) $h_{j_{k_s}} \rightarrow h$ in $\mathbf{L}^p(S_r)$
- (2) $\nabla h_{j_{k_s}} \rightarrow \nabla h$ in $\mathbf{L}^p(S_r, \mathbb{R}^n)$
- (3) $h_{j_{k_s}} \rightarrow h$ *a.e.* in S_r
- (4) $\nabla h_{j_{k_s}} \rightarrow \nabla h$ *a.e.* in S_r .

PROOF. By definition of the norm in $\mathcal{W}^{1,p}(B)$ we know that h_j converges to h in $L^p(B)$ and that ∇h_j converges to ∇h in $L^p(B, \mathbb{R}^n)$. Applying Proposition 3.2, we obtain a subsequence h_{j_k} and a set $A_1 \subset (0, R)$ with full measure such that h_{j_k} converges to h in $L^p(S_r)$ for every $r \in A_1$. Applying Proposition 3.2 once more to the sequence ∇h_{j_k} , we obtain a subsequence $\nabla h_{j_{k_s}}$ and a set $A_2 \subset (0, R)$ with full measure such that $\nabla h_{j_{k_s}}$ converges to ∇h in $L^p(S_r, \mathbb{R}^n)$ for all $r \in A_2$. Evidently $A_1 \cap A_2$ has full measure in $(0, R)$. Thus the subsequence $\{h_{j_{k_s}}\}$ satisfies (1) and (2). For simplicity of notation, we denote the last subsequence obtained by h_j . Now h_j converges to h in $L^p(B)$ and thus has a subsequence h_{j_k} that converges pointwise to h on a set $P \subset B$ with full measure. Proposition 3.1 shows that there exists a set $A_3 \subset (0, R)$ with full measure such that $|P \cap S_r|_{n-1} = |S_r|_{n-1}$ for every $r \in A_3$. This means that h_{j_k} converges to h a.e. on S_r for every $r \in A_3$. Now repeat the above argument for the sequence ∇h_{j_k} to obtain a set $A_4 \subset (0, R)$ with full measure and a subsequence $\nabla h_{j_{k_s}}$ that converges to ∇h a.e. on S_r for every $r \in A_4$. It is clear that $h_{j_{k_s}}$ satisfies (1)-(4). ■

Our next prerequisite is concerned with Sobolev's inequality on spheres, also referred to as the Gehring Oscillation Lemma. The interested reader is referred to [13].

LEMMA 3.4. *Let $h \in C^1(B)$ and $p \in (n-1, \infty)$. Then there exists a constant*

$A(n, p)$ such that for every $r \in (0, R)$ and for every $a, b \in S_r$ we have

$$|h(a) - h(b)| \leq rA(n, p) \left(\int_{S_r} |\nabla h(y)|^p d\sigma(y) \right)^{\frac{1}{p}}.$$

In the following theorem, we will prove that certain Sobolev functions on a ball are actually continuous when restricted to almost every sphere within the ball.

THEOREM 3.5. *Suppose that $h \in \mathcal{W}^{1,p}(B)$, where $n - 1 < p < \infty$. Then $h \in \mathcal{C}(S_r)$ for almost every $r \in (0, R)$.*

PROOF. Choose a sequence $\{h_j\}$ in $\mathcal{W}^{1,p}(B) \cap \mathcal{C}^\infty(B)$ with $h_j \rightarrow h$ in $\mathcal{W}^{1,p}(B)$ as $j \rightarrow \infty$. By Corollary 3.3, there is a subsequence $\{h_{j_k}\}$ such that, for almost every $r \in (0, R)$, the following are true as $k \rightarrow \infty$:

- (1) $h_{j_k} \rightarrow h$ in $\mathbf{L}^p(S_r)$
- (2) $\nabla h_{j_k} \rightarrow \nabla h$ in $\mathbf{L}^p(S_r, \mathbb{R}^n)$
- (3) $h_{j_k} \rightarrow h$ a.e. in S_r .

Now fix such $r \in (0, R)$ and a point $b \in S_r$ for which $\lim_{k \rightarrow \infty} h_{j_k}(b) = h(b)$, and let $\epsilon > 0$ be given. Choose N large enough so that

$$|h_{j_k}(b) - h_{j_s}(b)| < \frac{\epsilon}{2}$$

and

$$\|\nabla h_{j_k} - \nabla h_{j_s}\|_{\mathbf{L}^p(S_r, \mathbb{R}^n)} \leq \frac{\epsilon |S_r|^{\frac{1}{p}}}{2 r A(n, p)}$$

for all $k, s \geq N$.

By Lemma 3.4, we have

$$|(h_{j_k} - h_{j_s})(a) - (h_{j_k} - h_{j_s})(b)| \leq \frac{rA(n, p)}{|S_r|^{\frac{1}{p}}} \|\nabla h_{j_k} - \nabla h_{j_s}\|_{\mathbf{L}^p(S_r, \mathbb{R}^n)}$$

for all $a \in S_r$, whenever $k, s \geq N$.

Thus

$$|h_{j_k}(a) - h_{j_s}(a)| \leq |(h_{j_k} - h_{j_s})(b)| + \frac{rA(n, p)}{|S_r|^{\frac{1}{p}}} \|\nabla h_{j_k} - \nabla h_{j_s}\|_{\mathbf{L}^p(S_r, \mathbb{R}^n)} < \epsilon$$

for all $k, s \geq N$ and for every $a \in S_r$.

This simply means that $\{h_{j_k}\}$ is a Cauchy sequence in $\mathcal{C}(S_r)$ with respect to the sup norm. Thus $\{h_{j_k}\}$ converges uniformly on S_r to a function $g \in \mathcal{C}(S_r)$. Consequently, $h = g$ a.e. in S_r . This is precisely what is meant by writing $h \in \mathcal{C}(S_r)$. ■

Now we are in a position to considerably weaken the assumptions in Lemma 3.4. Our goal is to relax the assumption that h is $\mathcal{C}^1$.

THEOREM 3.6. *Let $h \in \mathcal{W}^{1,p}(B)$, $n - 1 < p < \infty$. Then, for almost every $r \in (0, R)$, we have*

$$h \in \mathcal{C}(S_r)$$

and

$$|h(a) - h(b)| \leq \frac{rA(n, p)}{|S_r|^{\frac{1}{p}}} \|\nabla h\|_{\mathbf{L}^p(S_r, \mathbb{R}^n)}$$

for every $a, b \in S_r$.

PROOF. The first assertion is the content of Theorem 3.5. To prove the second assertion, let $\{h_j\}$ be a sequence in $\mathcal{C}^\infty(B) \cap \mathcal{W}^{1,p}(B)$ converging to h in $\mathcal{W}^{1,p}(B)$. After passing to subsequences, if needed, Corollary 3.3, Lemma 3.4, and Theorem 3.5 yield the following :

$$(1) \quad |h_j(a) - h_j(b)| \leq \frac{rA(n,p)}{|S_r|^{\frac{1}{p}}} \|\nabla h_j\|_{\mathbf{L}^p(S_r, \mathbb{R}^n)} \text{ for all } j$$

$$(2) \quad h_j \rightarrow h \text{ uniformly on } S_r \text{ as } j \rightarrow \infty$$

$$(3) \quad \nabla h_j \text{ converges to } \nabla h \text{ in } \mathbf{L}^p(S_r, \mathbb{R}^n)$$

for almost every $r \in (0, R)$ and for every $a, b \in S_r$. The result now follows by letting j go to ∞ in (1) and using (2) and (3). ■

3.3 Modulus of Continuity.

In the next lemma we will prove a maximum principle for continuous open mappings.

LEMMA 3.7. *Let $g : \Omega \rightarrow \mathbb{R}$ be a continuous open mapping. If $\overline{B}(c, r) \subset \Omega$, then there exist $a, b \in \partial B(c, r)$ such that*

$$|g(x) - g(y)| \leq |g(a) - g(b)|$$

for all $x, y \in \overline{B}(c, r)$.

PROOF. Let $z \in B(c, r)$, so $g(z)$ belongs to the open set $g(B(c, r))$. Therefore there exists an $\epsilon > 0$ such that $(g(z) - \epsilon, g(z) + \epsilon) \subset g(B(c, r))$. This

means that z is neither a maximum nor a minimum of g on $B(c, r)$. Therefore, both the maximum and the minimum occur on $\partial B(c, r)$, say at the points b and a respectively. Thus $g(\overline{B}(c, r)) = [g(a), g(b)]$, and hence $|g(x) - g(y)| \leq |g(a) - g(b)|$ for all $x, y \in \overline{B}(c, r)$. ■

In the remaining part of this section, B_r stands for an open ball of radius r centered at an arbitrary point of $\mathbb{R}^n$.

LEMMA 3.8. *Let $\Omega \subset \mathbb{R}^n$ be open and let $h \in \mathcal{W}^{1,n}(\Omega) \cap \mathcal{C}(\Omega)$ be an open mapping. Then*

$$|h(a) - h(b)| \leq \frac{C(n)}{\sqrt[n]{\log \frac{2R}{|a-b|}}} \|\nabla h\|_{\mathbf{L}^n(B_{2R}, \mathbb{R}^n)} \quad (3.2)$$

for all $a, b \in B_R$ with $B_{2R} \subset \Omega$, where the constant $C(n)$ depends on the dimension only.

PROOF. Fix some positive number R with $B_{2R} \subset \Omega$ and let $a, b \in B_R$. Set $\delta = \frac{1}{2}|a - b|$ and $c = \frac{1}{2}(a + b)$. Consider the spheres $S(c, t)$, where $\delta \leq t \leq R$. It is evident that $S(c, t) \subset B_{2R}$ for all such t . The previous lemma implies that, for every $t \in [\delta, R]$, there are $x(t)$ and $y(t)$ in $S(c, t)$ with

$$|h(a) - h(b)| \leq |h(x(t)) - h(y(t))|.$$

By Theorem 3.6, we have

$$|h(x(t)) - h(y(t))| \leq \frac{tA(n)}{|S(c, t)|^{\frac{1}{n}}} \|\nabla h\|_{\mathbf{L}^n(S(c, t), \mathbb{R}^n)}$$

for almost every $t \in [\delta, R]$. Thus

$$|h(a) - h(b)|^n \leq \frac{t^n}{t^{n-1}} C'(n) \int_{S(c,t)} |\nabla h|^n$$

or

$$\frac{1}{t} |h(a) - h(b)|^n \leq C'(n) \int_{S(c,t)} |\nabla h|^n,$$

where $C'(n) = \frac{t^{n-1} A^n(n)}{|S(c,t)|}$. Note that $|S(c,t)| = \omega_{n-1} t^{n-1}$.

Integrating the above inequality with respect to t on $[\delta, R]$ and using Fubini's theorem we arrive at

$$|h(a) - h(b)|^n \log \frac{R}{\delta} \leq C'(n) \int_{\mathcal{A}(c,\delta,R)} |\nabla h|^n \leq C'(n) \int_{B_{2R}} |\nabla h|^n.$$

Here $\mathcal{A}(c, \delta, R)$ stands for the annulus $\{x \in \mathbb{R}^n : \delta \leq |x - c| \leq R\}$. Therefore,

$$|h(a) - h(b)| \leq \frac{C(n)}{\sqrt[n]{\log \frac{2R}{|a-b|}}} \|\nabla h\|_{L^n(B_{2R}, \mathbb{R}^n)},$$

where $C(n) = \sqrt[n]{C'(n)}$. ■

Note that mappings in $\mathcal{W}^{1,n}(\Omega)$ need not be continuous, for example $h(x) = \log \log \frac{1}{|x|}$ in the unit ball. On the other hand inequality (3.2) alone implies

continuity of h . It is therefore tempting to try to dispense with continuity in the assumptions of Lemma 3.8. Unfortunately this is not possible, at least by the method of our proof. We observe that continuity of h was used only to invoke Lemma 3.7. Part of the conclusion of this lemma is that the function g is bounded on closed disks. The following example demonstrates that mere openness of g is not sufficient for Lemma 3.7.

EXAMPLE 3.9. Let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be defined by

$$\phi(z) = \begin{cases} e^{\frac{1}{z}}, & \text{for } z \neq 0 \\ 1, & \text{for } z = 0. \end{cases}$$

Let $O \subset \mathbb{C}$ be open. If $0 \in O$ then $\phi(O) = \mathbb{C} - \{0\}$ is open, and if $0 \notin O$ then $\phi(O)$ is again open since ϕ is holomorphic on $\mathbb{C} - \{0\}$. Thus ϕ is an open mapping. Let $g(z) = \operatorname{Re}(\phi(z))$. Clearly g is open. However, it is not bounded on closed disks. In fact, $g(B(0, r)) = \mathbb{R}$ for every $r > 0$.

Finally, we are in a position to state and prove the main result of this chapter.

THEOREM 3.10. Let $f = (f_1, \dots, f_m) : \Omega \rightarrow \mathbb{R}^m$ be an open map of class $\mathcal{W}^{1,n}(\Omega, \mathbb{R}^m) \cap \mathcal{C}(\Omega, \mathbb{R}^m)$. Then

$$|f(a) - f(b)| \leq \frac{C(n)}{\sqrt[n]{\log \frac{2R}{|a-b|}}} \sum_{i=1}^m \|\nabla f_i\|_{\mathbf{L}^n(B_{2R}, \mathbb{R}^m)}$$

for all $a, b \in B_R$, with $B_{2R} \subset \Omega$.

PROOF. By using the previous lemma, we have

$$|f(a) - f(b)| = \sum_{i=1}^m |f_i(a) - f_i(b)| \leq \sum_{i=1}^m \frac{C(n)}{\sqrt[n]{\log \frac{2R}{|a-b|}}} \|\nabla f_i\|_{\mathbf{L}^n(B_{2R}, \mathfrak{R}^m)}.$$

Thus

$$|f(a) - f(b)| \leq \frac{C(n)}{\sqrt[n]{\log \frac{2R}{|a-b|}}} \sum_{i=1}^m \|\nabla f_i\|_{\mathbf{L}^n(B_{2R}, \mathfrak{R}^m)}. \quad \blacksquare$$

CHAPTER IV

THE p -HARMONIC EQUATION

4.1 Introduction and Statement of the Main Result.

Consider the homogeneous p -harmonic equation

$$\operatorname{div} |\nabla u|^{p-2} \nabla u = 0 \tag{4.1}$$

where $u \in \mathcal{W}^{1,p}(\Omega)$, $1 < p < \infty$. Equation (4.1) is understood in a weak sense (a precise definition follows). We say that u is a p -harmonic function on Ω .

Interest in the regularity problems for p -harmonic functions arises from several considerations. One particular compelling connection is with a problem of Gehring and Reich [10] concerning the degree of summability of the derivative of a quasiconformal mapping, recently solved by Astala [3].

Note that 2-harmonic functions are real analytic since they are harmonic in the usual sense. When $p \neq 2$, equation (4.1) is non-linear and it degenerates at the zeros of the gradient of u . Consequently, the smoothness of p -harmonic functions with $p \neq 2$ cannot be deduced from theorems on weak solutions of elliptic equations of divergence type.

However, Ural'tseva [22] and Uhlenback [21] proved independently that p -harmonic functions with $2 < p < \infty$ are in the class $\mathcal{C}_{loc}^{1,\alpha}$. In fact, they proved

this result for solutions of a class of non-linear degenerate elliptic systems, the p -harmonic equation being one of them. Evans [8] provided a simpler and shorter proof for the p -harmonic equation, but again only for $2 < p < \infty$. This result was extended by Lewis [17] to the full range $1 < p < \infty$. It is worth mentioning that the case $1 < p \leq 2$ can be reduced to $p \geq 2$ by Hodge duality argument, see [14].

In this chapter, we want to study non-homogeneous p -harmonic functions. They are weak solutions of the non-homogeneous p -harmonic equation

$$\operatorname{div} |\nabla u|^{p-2} \nabla u = \operatorname{div} f \quad (4.2)$$

where $f \in \mathcal{W}^{1,q}(\Omega, \mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{q} = 1$. Here is the precise formulation

DEFINITION 4.1. A function $u \in \mathcal{W}^{1,p}(\Omega)$, $1 < p < \infty$, is called *non-homogeneous p -harmonic* on Ω if and only if it solves (4.2) in the weak sense, that is,

$$\int_{\Omega} \langle |\nabla u(x)|^{p-2} \nabla u(x), \nabla \phi(x) \rangle dx = \int_{\Omega} \langle f(x), \nabla \phi(x) \rangle dx \quad (4.3)$$

for every test function $\phi \in \mathcal{C}_0^\infty(\Omega)$.

Note that by approximation, (4.3) also holds for every test function $\phi \in \mathcal{W}_0^{1,p}(\Omega)$. Since $\mathcal{W}_0^{1,p}(\mathbb{R}^n) = \mathcal{W}^{1,p}(\mathbb{R}^n)$, an interesting estimate can be obtained in the case $\Omega = \mathbb{R}^n$ by letting $\phi = u$. Namely,

$$\int_{\mathbb{R}^n} |\nabla u|^p = \int_{\mathbb{R}^n} \langle f, \nabla u \rangle \leq \int_{\mathbb{R}^n} |f| |\nabla u| \leq \|f\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^n)} \|\nabla u\|_{\mathbf{L}^p(\mathbb{R}^n, \mathbb{R}^n)}.$$

Thus

$$\|\nabla u\|_{\mathbf{L}^p(\mathbb{R}^n, \mathbb{R}^n)} \leq \|f\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^n)}^{q-1}. \quad (4.4)$$

Our goal is to prove the following theorem and use it to obtain higher integrability of ∇u .

THEOREM 4.2. *Let u be a non-homogeneous p -harmonic function on $\mathbb{R}^n$ of class $\mathcal{W}^{1,p}(\mathbb{R}^n)$, where $1 < p \leq 2$. If $g(x) = |\nabla u(x)|^{\frac{p-2}{2}} \nabla u(x)$, then*

$$g \in \mathcal{W}^{1,2}(\mathbb{R}^n, \mathbb{R}^n)$$

and the following uniform estimate holds

$$\|Dg\|_{\mathbf{L}^2(\mathbb{R}^n, \mathbb{R}^{n \times n})} \leq C(p, n) \|f\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{2-p}{2p-2}} \|Df\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}. \quad (4.5)$$

The proof of the above theorem will be given in Section 4.3. Observe that Theorem 4.2 implies a Liouville type theorem, that is, if u is a p -harmonic function of class $\mathcal{W}^{1,p}(\mathbb{R}^n)$ then $u = 0$ a.e. This holds because in such case we would have $Dg = 0$ a.e. and thus g is constant. But $g \in \mathbf{L}^2(\mathbb{R}^n)$. Hence $g = 0$ a.e. This shows that $|\nabla u| = 0$ a.e. which implies that u is constant. However, $u \in \mathbf{L}^p(\mathbb{R}^n)$. Therefore, $u = 0$ a.e.

Now we state a standard result that characterizes Sobolev functions in terms of boundedness of their difference quotients.

THEOREM 4.3. *If $F \in \mathcal{W}^{1,p}(\mathbb{R}^n)$, $1 \leq p < \infty$, then*

$$\int_{\mathbb{R}^n} |F(x+h) - F(x)|^p dx \leq |h|^p \int_{\mathbb{R}^n} |\nabla F(x)|^p dx \quad (4.6)$$

for every $h \in \mathbb{R}^n$. Conversely, if a function F in $L^p(\mathbb{R}^n)$ verifies the uniform bound

$$\int_{\mathbb{R}^n} |F(x+h) - F(x)|^p dx \leq C^p |h|^p$$

for every $h \in \mathbb{R}^n$, then $F \in \mathcal{W}^{1,p}(\mathbb{R}^n)$ and we have

$$\|\nabla F\|_{L^p(\mathbb{R}^n, \mathbb{R}^n)} \leq C.$$

In the sequel we shall use this theorem for $\mathbb{R}^n$ or $\mathbb{R}^{n \times n}$ valued functions, the results remain true but the constants involved in this theorem will change.

4.2 Auxiliary Results.

In this section, we prove two algebraic inequalities regarding vectors in $\mathbb{R}^n$. These inequalities are needed for the proof of Theorem 4.2. Also, we will prove an imbedding theorem of Sobolev functions.

Let $p > 1$. Consider the function $a(x)$ defined on $[0, 1]$ by

$$a(x) = \begin{cases} \frac{(x^{\frac{p}{2}} - 1)^2}{(x-1)(x^{p-1}-1)}, & \text{for } 0 \leq x < 1 \\ \frac{p^2}{4(p-1)}, & \text{for } x = 1. \end{cases}$$

By elementary considerations, one sees that $\lim_{x \rightarrow 1^-} a(x) = \frac{p^2}{4(p-1)}$. Therefore,

our function is continuous on $[0, 1]$ and hence bounded, say by $K_p \geq 1$.

Our first inequality is the following

LEMMA 4.4. *Let $A, B \in \mathbb{R}^n - \{0\}$. Then*

$$||A|^{\frac{p-2}{2}} A - |B|^{\frac{p-2}{2}} B|^2 \leq K_p \langle |A|^{p-2} A - |B|^{p-2} B, A - B \rangle.$$

PROOF. By the above discussion, we know that

$$(x^{\frac{p}{2}} - 1)^2 \leq K_p(x - 1)(x^{p-1} - 1), \quad \text{for } 0 \leq x < 1. \quad (4.7)$$

Note that the result is immediate whenever $|A| = |B|$. After interchanging $|A|$ and $|B|$ if necessary we may assume, without loss of generality, that $|B| >$

$|A| > 0$. Let $x = \frac{|A|}{|B|} < 1$.

Substituting x in (4.7) yields

$$\left(\frac{|A|^{\frac{p}{2}}}{|B|^{\frac{p}{2}}} - 1 \right)^2 \leq K_p \left(\frac{|A|}{|B|} - 1 \right) \left(\frac{|A|^{p-1}}{|B|^{p-1}} - 1 \right)$$

$$\left(|A|^{\frac{p}{2}} - |B|^{\frac{p}{2}} \right)^2 \leq K_p (|A| - |B|) (|A|^{p-1} - |B|^{p-1})$$

$$|A|^p + |B|^p - 2|A|^{\frac{p}{2}}|B|^{\frac{p}{2}} \leq K_p (|A|^p + |B|^p - |A||B|^{p-1} - |A|^{p-1}|B|).$$

Some elementary algebraic manipulations give

$$\begin{aligned} |A||B| \left(K_p |A|^{p-2} + K_p |B|^{p-2} - 2|A|^{\frac{p-2}{2}}|B|^{\frac{p-2}{2}} \right) \\ \leq (K_p - 1) (|A|^p + |B|^p). \end{aligned}$$

Observe that

$$\begin{aligned} K_p |A|^{p-2} + K_p |B|^{p-2} - 2|A|^{\frac{p-2}{2}}|B|^{\frac{p-2}{2}} \\ = \left(|A|^{\frac{p-2}{2}} - |B|^{\frac{p-2}{2}} \right)^2 + (K_p - 1) (|A|^{p-2} + |B|^{p-2}) \geq 0. \end{aligned}$$

The Cauchy-Schwarz Inequality implies that

$$\begin{aligned}
& \langle A, B \rangle \left(K_p |A|^{p-2} + K_p |B|^{p-2} - 2 |A|^{\frac{p-2}{2}} |B|^{\frac{p-2}{2}} \right) \\
& \leq |A| |B| \left(K_p |A|^{p-2} + K_p |B|^{p-2} - 2 |A|^{\frac{p-2}{2}} |B|^{\frac{p-2}{2}} \right) \\
& \leq (K_p - 1) (|A|^p + |B|^p).
\end{aligned}$$

By rearranging the last inequality, we arrive at

$$\begin{aligned}
& |A|^p + |B|^p - 2 \langle A, B \rangle |A|^{\frac{p-2}{2}} |B|^{\frac{p-2}{2}} \\
& \leq K_p (|A|^p + |B|^p - |A|^{p-2} \langle A, B \rangle - |B|^{p-2} \langle A, B \rangle).
\end{aligned}$$

This is the same as

$$\left| |A|^{\frac{p-2}{2}} A - |B|^{\frac{p-2}{2}} B \right|^2 \leq K_p \langle |A|^{p-2} A - |B|^{p-2} B, A - B \rangle,$$

completing the proof. ■

In the proof of our next inequality, we will make use of the following observation. Define $b(x)$ as follows

$$b(x) = \begin{cases} \frac{(x-1)^2(x+1)^{p-2}}{(x^{\frac{p}{2}}-1)^2}, & \text{for } 0 \leq x < 1 \\ \frac{2^p}{p^2}, & \text{for } x = 1. \end{cases}$$

Again, it is easy to see that $\lim_{x \rightarrow 1^-} b(x) = \frac{2^p}{p^2}$. Let $C_p = \max_{0 \leq x \leq 1} b(x)$. Note that

$$C_p \geq 1.$$

Our next inequality is

LEMMA 4.5. For all $A, B \in \mathbb{R}^n - \{0\}$ and $1 < p \leq 2$, the following inequality holds

$$|A - B|^2 (|A| + |B|)^{p-2} \leq C_p \left| |A|^{\frac{p-2}{2}} A - |B|^{\frac{p-2}{2}} B \right|^2.$$

PROOF. It is evident that the inequality holds whenever $|A| = |B|$. Therefore, we may assume that $|B| > |A| > 0$. The above discussion shows that, for $0 \leq x < 1$, we can write

$$(x^2 - 2x + 1)(x + 1)^{p-2} \leq C_p(x^p - 2x^{\frac{p}{2}} + 1)$$

or

$$(x^2 + 1)(x + 1)^{p-2} - C_p(x^p + 1) \leq 2x(x + 1)^{p-2} - 2C_p x^{\frac{p}{2}}.$$

Set $x = \frac{|A|}{|B|}$ in the above inequality to obtain

$$\begin{aligned} \left(\frac{|A|^2}{|B|^2} + 1 \right) \left(\frac{|A|}{|B|} + 1 \right)^{p-2} - C_p \left(\frac{|A|^p}{|B|^p} + 1 \right) \\ \leq 2 \frac{|A|}{|B|} \left(\frac{|A|}{|B|} + 1 \right)^{p-2} - 2C_p \frac{|A|^{\frac{p}{2}}}{|B|^{\frac{p}{2}}}. \end{aligned}$$

Multiplying by $-|B|^p$ yields

$$\begin{aligned} 2C_p |A|^{\frac{p}{2}} |B|^{\frac{p}{2}} - 2|A||B| (|A| + |B|)^{p-2} \\ \leq C_p (|A|^p + |B|^p) - (|A|^2 + |B|^2) (|A| + |B|)^{p-2} \end{aligned}$$

or

$$\begin{aligned} 2|A||B| \left(C_p |A|^{\frac{p-2}{2}} |B|^{\frac{p-2}{2}} - (|A| + |B|)^{p-2} \right) \\ \leq C_p (|A|^p + |B|^p) - (|A|^2 + |B|^2) (|A| + |B|)^{p-2}. \quad (4.8) \end{aligned}$$

As in the previous lemma, we want to replace $|A||B|$ by $\langle A, B \rangle$ via the Cauchy-Schwarz Inequality. To do so, we first need to ensure that

$$C_p |A|^{\frac{p-2}{2}} |B|^{\frac{p-2}{2}} - (|A| + |B|)^{p-2} \geq 0. \quad (4.9)$$

For this purpose we begin with an obvious inequality

$$|A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \leq \frac{1}{2} (|A| + |B|).$$

Hence

$$\begin{aligned} |A|^{\frac{2-p}{2}} |B|^{\frac{2-p}{2}} &\leq \frac{1}{2^{2-p}} (|A| + |B|)^{2-p} \\ (|A| + |B|)^{p-2} &\leq \frac{1}{2^{2-p}} |A|^{\frac{p-2}{2}} |B|^{\frac{p-2}{2}} \end{aligned}$$

As $C_p > \frac{1}{2^{2-p}}$, (4.9) follows.

Replacing $|A||B|$ by $\langle A, B \rangle$ in (4.8), we obtain

$$\begin{aligned} 2 \langle A, B \rangle \left(C_p |A|^{\frac{p-2}{2}} |B|^{\frac{p-2}{2}} - (|A| + |B|)^{p-2} \right) \\ \leq C_p (|A|^p + |B|^p) - (|A|^2 + |B|^2) (|A| + |B|)^{p-2}. \end{aligned}$$

After an obvious rearrangement this inequality takes the form

$$\begin{aligned} (|A|^2 + |B|^2) (|A| + |B|)^{p-2} - 2 \langle A, B \rangle (|A| + |B|)^{p-2} \\ \leq C_p \left(|A|^p + |B|^p - 2 \langle A, B \rangle |A|^{\frac{p-2}{2}} |B|^{\frac{p-2}{2}} \right) \end{aligned}$$

which is none other than

$$(|A| + |B|)^{p-2} |A - B|^2 \leq C_p \left| |A|^{\frac{p-2}{2}} A - |B|^{\frac{p-2}{2}} B \right|^2.$$

This completes the proof. ■

LEMMA 4.6. Suppose that $u \in \mathcal{W}_{loc}^{1,1}(\mathbb{R}^n)$ with $\nabla u \in \mathbf{L}^s(\mathbb{R}^n, \mathbb{R}^n)$, $1 \leq s < n$.

Then there exist constants c and $A(n, s)$ such that

$$u - c \in \mathbf{L}^{\frac{ns}{n-s}}(\mathbb{R}^n)$$

and

$$\|u - c\|_{\frac{ns}{n-s}} \leq A(n, s) \|\nabla u\|_{\mathbf{L}^s(\mathbb{R}^n, \mathbb{R}^n)}.$$

PROOF. Define $c_k = \oint_{B_k} u$, $k = 1, 2, \dots$. For $j < k$ we have

$$\begin{aligned} |B_j|^{\frac{n-s}{ns}} |c_k - c_j| &= \left(\int_{B_j} |c_k - c_j|^{\frac{ns}{n-s}} \right)^{\frac{n-s}{ns}} \\ &\leq \left(\int_{B_j} |c_k - u|^{\frac{ns}{n-s}} \right)^{\frac{n-s}{ns}} + \left(\int_{B_j} |u - c_j|^{\frac{ns}{n-s}} \right)^{\frac{n-s}{ns}} \\ &\leq \left(\int_{B_k} |c_k - u|^{\frac{ns}{n-s}} \right)^{\frac{n-s}{ns}} + \left(\int_{B_j} |u - c_j|^{\frac{ns}{n-s}} \right)^{\frac{n-s}{ns}}. \end{aligned}$$

Poincaré's inequality shows that the last expression is no larger than

$$A(n, s) \left(\int_{B_k} |\nabla u|^s \right)^{\frac{1}{s}} + A(n, s) \left(\int_{B_j} |\nabla u|^s \right)^{\frac{1}{s}}$$

which is in turn no larger than

$$A(n, s) \|\nabla u\|_{\mathbf{L}^s(\mathbb{R}^n, \mathbb{R}^n)}.$$

Therefore,

$$|c_k - c_j| \leq \frac{A(n, s) \|\nabla u\|_{\mathbf{L}^s(\mathbb{R}^n, \mathbb{R}^n)}}{j^{\frac{n-s}{s}}}.$$

Thus $\lim_{k \rightarrow \infty} c_k = c$ exists and

$$|c - c_j| \leq \frac{A(n, s) \|\nabla u\|_{\mathbf{L}^s(\mathbb{R}^n, \mathbb{R}^n)}}{j^{\frac{n-s}{s}}}.$$

Consequently,

$$\begin{aligned} \left(\int_{B_k} |u - c|^{\frac{ns}{n-s}} \right)^{\frac{n-s}{ns}} &\leq \left(\int_{B_k} |u - c_k|^{\frac{ns}{n-s}} \right)^{\frac{n-s}{ns}} + |c_k - c| |B_k|^{\frac{n-s}{ns}} \\ &\leq A(n, s) \left(\int_{B_k} |\nabla u|^s \right)^{\frac{1}{s}} + A(n, s) \|\nabla u\|_{\mathbf{L}^s(\mathbb{R}^n, \mathbb{R}^n)} \\ &\leq A(n, s) \|\nabla u\|_{\mathbf{L}^s(\mathbb{R}^n, \mathbb{R}^n)}. \end{aligned} \tag{4.10}$$

Now we let k go to ∞ and use the Monotone Convergence Theorem. This completes the proof. ■

Let us remark that, in general, the constant c cannot be dispensed with. One can see this by considering any non-zero constant function u . However, the constant c may be dispensed with under additional assumptions on the decay of u at ∞ . We shall encounter such a situation in the proof of Corollary 4.7.

4.3 The Proof of Theorem 4.2.

Having disposed of the above preliminaries we are now in a position to prove Theorem 4.2.

PROOF OF THEOREM 4.2. Recall the equation

$$\operatorname{div} |\nabla u|^{p-2} \nabla u = \operatorname{div} f$$

which simply means that

$$\int_{\mathbb{R}^n} \langle |\nabla u(x)|^{p-2} \nabla u(x), \nabla \phi(x) \rangle dx = \int_{\mathbb{R}^n} \langle f(x), \nabla \phi(x) \rangle dx \quad (4.11)$$

for every test function $\phi \in \mathcal{W}^{1,p}(\mathbb{R}^n)$. Fix an arbitrary $h \in \mathbb{R}^n$ and apply (4.11) to $\phi(x - h)$. By changing variable x into $x + h$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \langle |\nabla u(x + h)|^{p-2} \nabla u(x + h), \nabla \phi(x) \rangle dx \\ = \int_{\mathbb{R}^n} \langle f(x + h), \nabla \phi(x) \rangle dx. \end{aligned} \quad (4.12)$$

By subtracting (4.11) from (4.12), side by side, we arrive at

$$\begin{aligned} \int_{\mathbb{R}^n} \langle |\nabla u(x + h)|^{p-2} \nabla u(x + h) - |\nabla u(x)|^{p-2} \nabla u(x), \nabla \phi(x) \rangle dx \\ = \int_{\mathbb{R}^n} \langle f(x + h) - f(x), \nabla \phi(x) \rangle dx. \end{aligned}$$

It is natural to test this integral identity with $\phi(x) = u(x + h) - u(x)$.

$$\begin{aligned} \int_{\mathbb{R}^n} \langle |\nabla u(x + h)|^{p-2} \nabla u(x + h) - |\nabla u(x)|^{p-2} \nabla u(x), \nabla u(x + h) - \nabla u(x) \rangle dx \\ = \int_{\mathbb{R}^n} \langle f(x + h) - f(x), \nabla u(x + h) - \nabla u(x) \rangle dx. \end{aligned} \quad (4.13)$$

Now Lemma 4.4 yields

$$\begin{aligned}
& |g(x+h) - g(x)|^2 \\
&= \left| |\nabla u(x+h)|^{\frac{p-2}{2}} \nabla u(x+h) - |\nabla u(x)|^{\frac{p-2}{2}} \nabla u(x) \right|^2 \\
&\leq K_p \langle |\nabla u(x+h)|^{p-2} \nabla u(x+h) - |\nabla u(x)|^{p-2} \nabla u(x), \nabla u(x+h) - \nabla u(x) \rangle.
\end{aligned}$$

Upon integration, we use (4.13) and the Cauchy-Schwarz Inequality to obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} |g(x+h) - g(x)|^2 dx &\leq K_p \int_{\mathbb{R}^n} |f(x+h) - f(x)| |\nabla u(x+h) - \nabla u(x)| dx \\
&= K_p \int_{\mathbb{R}^n} |\nabla u(x+h) - \nabla u(x)| (|\nabla u(x+h)| + |\nabla u(x)|)^{\frac{p-2}{2}} \\
&\quad \times (|\nabla u(x+h)| + |\nabla u(x)|)^{\frac{2-p}{2}} |f(x+h) - f(x)| dx.
\end{aligned}$$

Hölder's inequality shows that the right hand side is no larger than

$$\begin{aligned}
& K_p \left[\int_{\mathbb{R}^n} \frac{|\nabla u(x+h) - \nabla u(x)|^2}{(|\nabla u(x+h)| + |\nabla u(x)|)^{2-p}} dx \right]^{\frac{1}{2}} \\
& \times \left[\int_{\mathbb{R}^n} (|\nabla u(x+h)| + |\nabla u(x)|)^p dx \right]^{\frac{2-p}{2p}} \\
& \times \left[\int_{\mathbb{R}^n} |f(x+h) - f(x)|^q dx \right]^{\frac{1}{q}}.
\end{aligned} \tag{4.14}$$

Note that the constraint $1 < p \leq 2$ is needed in (4.14). Here in the first integral, the indeterminate quotient $\frac{0}{0}$ may occur, and as a convention, we set it to 0 in this case.

The first factor in (4.14) is no larger than

$$\sqrt{C_p} K_p \left[\int_{\mathbb{R}^n} |g(x+h) - g(x)|^2 dx \right]^{\frac{1}{2}}$$

by Lemma 4.5. Consequently,

$$\begin{aligned} & \int_{\mathbb{R}^n} |g(x+h) - g(x)|^2 dx \\ & \leq \sqrt{C_p} K_p \left[\int_{\mathbb{R}^n} |g(x+h) - g(x)|^2 dx \right]^{\frac{1}{2}} \\ & \quad \times \left[\int_{\mathbb{R}^n} (|\nabla u(x+h)| + |\nabla u(x)|)^p dx \right]^{\frac{2-p}{2p}} \\ & \quad \times \left[\int_{\mathbb{R}^n} |f(x+h) - f(x)|^q dx \right]^{\frac{1}{q}}. \end{aligned}$$

Notice that

$$\begin{aligned} & \left[\int_{\mathbb{R}^n} (|\nabla u(x+h)| + |\nabla u(x)|)^p dx \right]^{\frac{2-p}{2p}} \\ & \leq \left[\left(\int_{\mathbb{R}^n} |\nabla u(x+h)|^p dx \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p dx \right)^{\frac{1}{p}} \right]^{\frac{2-p}{2}} \\ & = 2^{\frac{2-p}{2}} \|\nabla u\|_{\mathbf{L}^p(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{2-p}{2}}. \end{aligned}$$

Therefore, we have the following estimate of the $\mathbf{L}^2$ norm of $g(x+h) - g(x)$

$$\begin{aligned} \left[\int_{\mathbb{R}^n} |g(x+h) - g(x)|^2 dx \right]^{\frac{1}{2}} & \leq D(p) \left[\int_{\mathbb{R}^n} |\nabla u|^p dx \right]^{\frac{2-p}{2p}} \\ & \quad \times \left[\int_{\mathbb{R}^n} |f(x+h) - f(x)|^q dx \right]^{\frac{1}{q}}, \end{aligned}$$

where we have denoted by $D(p) = 2^{\frac{2-p}{2}} K_p \sqrt{C_p}$. The first integral above is finite, because $\nabla u \in \mathbf{L}^p(\mathbb{R}^n)$. Dividing both sides by $|h|$ yields

$$\begin{aligned} & |h|^{-1} \left(\int_{\mathbb{R}^n} |g(x+h) - g(x)|^2 dx \right)^{\frac{1}{2}} \\ & \leq D(p) \|\nabla u\|_{\mathbf{L}^p(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{2-p}{2}} |h|^{-1} \left(\int_{\mathbb{R}^n} |f(x+h) - f(x)|^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

Invoking Theorem 4.3, we have

$$|h|^{-1} \left(\int_{\mathbb{R}^n} |f(x+h) - f(x)|^q dx \right)^{\frac{1}{q}} \leq \left(\int_{\mathbb{R}^n} |Df(x)|^q dx \right)^{\frac{1}{q}}.$$

Therefore,

$$|h|^{-1} \left(\int_{\mathbb{R}^n} |g(x+h) - g(x)|^2 dx \right)^{\frac{1}{2}} \leq D(p) \|\nabla u\|_{\mathbf{L}^p(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{2-p}{2}} \|Df\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}$$

for every $h \in \mathbb{R}^n$. Now we invoke Theorem 4.3 again to conclude that $g \in \mathcal{W}^{1,2}(\mathbb{R}^n, \mathbb{R}^n)$ and satisfies the inequality

$$\|Dg\|_{\mathbf{L}^2(\mathbb{R}^n, \mathbb{R}^{n \times n})} \leq C(p, n) \|\nabla u\|_{\mathbf{L}^p(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{2-p}{2}} \|Df\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}.$$

Applying Estimate (4.4) we arrive at

$$\|Dg\|_{\mathbf{L}^2(\mathbb{R}^n, \mathbb{R}^{n \times n})} \leq C(p, n) \|f\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{2-p}{2}} \|Df\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}.$$

This completes the proof. $\blacksquare$

4.4 Regularity.

In this section we derive some implications of Theorem 4.2 by using Sobolev Imbedding Theorems.

COROLLARY 4.7. *Let $u \in \mathcal{W}^{1,p}(\mathbb{R}^n)$, $n > 2$, $1 < p \leq 2$, satisfy the equation*

$$\operatorname{div} |\nabla u|^{p-2} \nabla u = \operatorname{div} f,$$

where $f \in \mathcal{W}^{1,q}(\mathbb{R}^n, \mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $\nabla u \in \mathbf{L}^s(\mathbb{R}^n, \mathbb{R}^n)$ for all

$$p \leq s \leq \frac{np}{n-2} \text{ and}$$

$$\|\nabla u\|_{\mathbf{L}^s(\mathbb{R}^n, \mathbb{R}^n)} \leq C(p, n) \|f\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^n)}^{q-1-\theta} \|Df\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}^\theta \quad (4.15)$$

$$\text{where } \theta = \frac{n}{p} - \frac{n}{s}.$$

PROOF. Since $\mathcal{W}^{1,2}(\mathbb{R}^n, \mathbb{R}^n) \subset \mathbf{L}^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $2 \leq r \leq \frac{2n}{n-2}$, we find that $|g| = |\nabla u|^{\frac{p}{2}} \in \mathbf{L}^r(\mathbb{R}^n)$ for every such an r . Hence $|\nabla u| \in \mathbf{L}^s(\mathbb{R}^n)$, for every $s \in \left[p, \frac{np}{n-2}\right]$. In order to prove inequality (4.15) we need a precise estimate in Lemma 4.6. Accordingly, there exists a constant a such that

$$\|g - a\|_{\mathbf{L}^{\frac{2n}{n-2}}(\mathbb{R}^n, \mathbb{R}^n)} \leq C(n) \|Dg\|_{\mathbf{L}^2(\mathbb{R}^n, \mathbb{R}^{n \times n})}. \quad (4.16)$$

We claim that $a = 0$. To see this consider a ball B_R . Hölder's inequality implies

$$\left(\int_{B_R} |g - a|^2 \right)^{\frac{1}{2}} \leq |B_R|^{\frac{1}{n}} \left(\int_{B_R} |g - a|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \leq R C(n) \|Dg\|_{\mathbf{L}^2(\mathbb{R}^n, \mathbb{R}^{n \times n})}.$$

Therefore,

$$\begin{aligned}
|B_R|^{\frac{1}{2}}|a| &= \left(\int_{B_R} |a|^2 \right)^{\frac{1}{2}} \\
&\leq \left(\int_{B_R} |g-a|^2 \right)^{\frac{1}{2}} + \left(\int_{B_R} |g|^2 \right)^{\frac{1}{2}} \\
&\leq RC(n) \|Dg\|_{\mathbf{L}^2(\mathbb{R}^n, \mathbb{R}^{n \times n})} + \|g\|_{\mathbf{L}^2(\mathbb{R}^n, \mathbb{R}^n)}.
\end{aligned}$$

Thus

$$|a| \leq \frac{C(n) \|Dg\|_{\mathbf{L}^2(\mathbb{R}^n, \mathbb{R}^{n \times n})}}{R^{\frac{n-2}{2}}} + \frac{C(n) \|g\|_{\mathbf{L}^2(\mathbb{R}^n, \mathbb{R}^n)}}{R^{\frac{n}{2}}}.$$

Letting R go to ∞ yields $a = 0$.

Now inequality (4.16) reads as

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla u|^{\frac{np}{n-2}} &\leq C(n) \|Dg\|_{\mathbf{L}^2(\mathbb{R}^n, \mathbb{R}^{n \times n})}^{\frac{2n}{n-2}} \\
&\leq C(n, p) \|f\|_{\mathbf{L}^2(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{n(2-p)}{(p-1)(n-2)}} \|Df\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}^{\frac{2n}{n-2}}
\end{aligned}$$

by Theorem 4.2. This is the same as

$$\|\nabla u\|_{\mathbf{L}^{\frac{np}{n-2}}(\mathbb{R}^n, \mathbb{R}^n)} \leq C(n, p) \|f\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{2-p}{p(p-1)}} \|Df\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}^{\frac{2}{p}}.$$

On the other hand we have $\|\nabla u\|_{\mathbf{L}^p(\mathbb{R}^n, \mathbb{R}^n)} \leq \|f\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^n)}^{q-1}$ by (4.4). Now we

interpolate to obtain an L^s -estimate of the gradient of u , for $p \leq s \leq \frac{np}{n-2}$.

$$\begin{aligned}
\|\nabla u\|_{L^s(\mathbb{R}^n, \mathbb{R}^n)} &\leq \|\nabla u\|_{L^p(\mathbb{R}^n, \mathbb{R}^n)}^{1-\frac{n}{2}+\frac{np}{2s}} \|\nabla u\|_{L^{\frac{np}{n-2}}(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{n}{2}-\frac{np}{2s}} \\
&\leq C(n, p) \|f\|_{L^q(\mathbb{R}^n, \mathbb{R}^n)}^{(q-1)(1-\frac{n}{2}+\frac{np}{2s})} \|f\|_{L^q(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{2-p}{p(p-1)}(\frac{n}{2}-\frac{np}{2s})} \|Df\|_{L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}^{\frac{2}{p}(\frac{n}{2}-\frac{np}{2s})} \\
&= C(n, p) \|f\|_{L^q(\mathbb{R}^n, \mathbb{R}^n)}^{q-1-\theta} \|Df\|_{L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}^{\theta},
\end{aligned}$$

where $\theta = \frac{n}{p} - \frac{n}{s}$. This completes the proof. ■

An interesting C^α -regularity result follows from (4.15) when the dimension equals 3.

COROLLARY 4.8. *Under the assumptions of Corollary 4.7, if $n = 3$, then*

$u \in C^\alpha(\mathbb{R}^3)$ with $\alpha = \frac{1}{q}$. Moreover, we have the following uniform bound

$$|u(x) - u(y)| \leq C |x - y|^{\frac{1}{q}}, \text{ for all } x, y \in \mathbb{R}^3,$$

where $C = C(p) \|f\|_{L^q(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{2-p}{p(p-1)}} \|Df\|_{L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}^{\frac{2}{p}}$.

PROOF. Let us recall the well-known inequality, see [13], which holds in arbitrary dimension n ,

$$|u(x) - u(y)| \leq \frac{2sn}{s-n} |x - y|^{1-\frac{n}{s}} \|\nabla u\|_{L^s(\mathbb{R}^n, \mathbb{R}^n)}, \text{ for } s > n.$$

Substitute $s = 3p$ and $n = 3$ to obtain

$$\begin{aligned}
|u(x) - u(y)| &\leq \frac{18p}{3p-3} |x-y|^{\frac{1}{q}} \|\nabla u\|_{\mathbf{L}^{3p}(\mathbb{R}^n, \mathbb{R}^n)} \\
&= 6q |x-y|^{\frac{1}{q}} \|\nabla u\|_{\mathbf{L}^{3p}(\mathbb{R}^n, \mathbb{R}^n)} \\
&\leq C(p) \|f\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{2-p}{p(p-1)}} \|Df\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}^{\frac{2}{p}} |x-y|^{\frac{1}{q}},
\end{aligned}$$

completing the proof. $\blacksquare$

COROLLARY 4.9. *Under the assumptions of Corollary 4.7, if $1 < p < 2$ and $n \geq 4$, then*

$$\|u - c\|_{\frac{np}{n-2-p}} \leq C(p) \|f\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{2-p}{p(p-1)}} \|Df\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}^{\frac{2}{p}},$$

where c is a constant.

PROOF. By (4.15), with $s = \frac{np}{n-2}$, we have

$$\|\nabla u\|_{\mathbf{L}^{\frac{np}{n-2}}(\mathbb{R}^n, \mathbb{R}^n)} \leq C(n, p) \|f\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{2-p}{p(p-1)}} \|Df\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}^{\frac{2}{p}}.$$

Notice that $\frac{n}{n-2} < \frac{np}{n-2} < n$. Applying Lemma 4.6 with $\frac{np}{n-2}$ in place of s we obtain

$$\begin{aligned}
\|u - c\|_{\frac{np}{n-2-p}} &\leq C(n, p) \|\nabla u\|_{\mathbf{L}^{\frac{np}{n-2}}(\mathbb{R}^n, \mathbb{R}^n)} \\
&\leq C(p, n) \|f\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^n)}^{\frac{2-p}{p(p-1)}} \|Df\|_{\mathbf{L}^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}^{\frac{2}{p}},
\end{aligned}$$

as claimed. $\blacksquare$

REMARK 4.10. One would expect something interesting in the case $\frac{np}{n-2} = n$. However, because of the constraint $1 < p \leq 2$, this happens only when $n = 4$ and $p = 2$. In this case we are reduced to Laplace equation, the theory of which is well known and therefore uninteresting to us.

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