

UNIVERSITY OF BAHRAIN

College of Science



Department of Mathematics

Generating Relations of Two Index and Two Variable Hermite Polynomials

A Thesis Submitted in Partial Fulfillment of the Requirements for the
Master of Science Degree in Mathematics

Submitted by
Shaikha Khaled Alshomeli
University Number: 20120125

Supervised by
Dr. Mohannad Shahwan
Associate Professor
University of Bahrain

Kingdom of Bahrain

September 2023



Thesis Discussion Committee Decision

Student Name: SHAIKHA KHALED AHMED ALSHOMELI	Student ID: 20120125
Department: Mathematics	College: Science
Program: Masters in Mathematics	
Thesis Title: Generating relations of two-index and two-variable Hermite polynomials.	
Supervisor Name: Dr.Mohannad Shahwan	Student GPA: 3.37

The Discussion Committee met on 24-Oct-23 In S41-2000. Upon discussing the student whose details are stated above in public, the discussion committee recommends the following:

The student is awarded a grade A- in the thesis subject.

- ☐ Without making any changes to the thesis.
- ☒ After making the modifications attached with this decision.

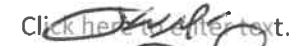
Discussion Committee Members Approval:

Name

1. Dr.Mohannad Shahwan (Advisor)
2. Dr.Nasser Metwally (Internal Examiner)
3. Prof.Zeyad Al-Zhour (External Examiner)

Signature

 Click here to enter text.

 Click here to enter text.

 Click here to enter text.

Table of Contents

Abstract	iv
Dedication	v
Acknowledgment	vi

Chapter 1: Introduction

1.1 Statement of Problem	1
1.2 Justification of Problem	1
1.3 Literature Review	2
1.4 Theoretical Framework	3
1.5 Methods and Procedures	3
1.6 Background and History	4
1.6.1 Hermite Function	4
1.6.2 Lie Algebra	5
1.6.3 Weisner's Group Theoretical Method	6

Chapter 2: Generating Relations of Two Index Generalized Hermite Polynomials

2.1 Introduction	10
2.2 Finding Generating Relations Using Weisner's Method	11
2.2.1 Recurrence Relations	11
2.2.2 First Order Differential Operators	16
2.2.3 Commutative Relations of Operators	18
2.2.4 Extended Form of Group Generated by Operators	29
2.2.5 Generating Functions	36
2.3 Some Applications	40

Chapter 3: Generating relations of Two index and Two variable Hermite polynomials	43
3.1 Introduction	43
3.2 Finding Generating Relations using Weisner's Method	46
3.2.1 Recurrence Relations	46
3.2.2 First Order Differential Operators	53
3.2.3 Commutative Relations of The Operators	54
3.2.4 Extended Form of The Group Generated by Operators	64
3.2.5 Generating Functions	69
3.3 Some Applications	74
 Chapter 4: Summary, Conclusion, and Directions for Future Research	
4.1 Summary	76
4.2 Conclusion	77
4.3 Future studies	78
 Bibliography and References	79

Abstract

In this thesis we establish four generating relations for the extended Hermite polynomials.

Among the cases discussed in this thesis are: Hermite polynomials of two variables and Hermite polynomials with two index and two variables.

All the steps are written, starting with recurrence relations, commutative operator relations, and we gain some novel generalizations of well-known generating relations from the expanded form of the groups formed by the operators.

These polynomials are used in a variety of pure and applied mathematics domains. (Dattoli et al 2003,2004). They are instrumental in describing the quantum treatment (Kim and Noz 1991) of coupled harmonic oscillation. Through this thesis, we consider three generalizations: Hermite two index and four variables polynomial, Hermite polynomial of two index and two variables.

Dedication

To my father, my always supporter.

To my mother, the best teacher ever.

To my husband, grateful to have him.

Acknowledgment

Most importantly, “almighty Allah” for giving me the stamina and capacity to comprehend and finish this assignment.

My sincere appreciation goes out to Dr. Mohannad Shahwan, my supervisor, for his patient direction, support, and helpful criticism of this research endeavor. With him, my initiative was able to move further. I appreciate you and the additional instructors from the University of Bahrain, especially the mathematics department.

I want to thank my family. None of this would have been possible without their love and support. They have always supported me, and I am grateful for everything they have done to make my goals a reality.

Chapter1: Introduction

1.1 Statement of The Problem

This thesis work aims to study and derive generating relations with two index extended Hermite polynomials, and two index and two variable Hermite polynomials using Weisner's 1955 approach.

1.2 Justification of Problem

Weisner 1955 presented a method for obtaining generating functions for a set of functions that satisfy certain commutation relations (McBride 1971). The Hermite polynomials are among the functions that satisfy the required commutation relations (Weisner 1959).

Recently, Hermite polynomials show it is closed relation to Lie group and Lie algebra, especially Weisner's group theoretic method, come from the applications.

In this regard (Khan et al 2009, 2010) and (Shahwan 2004) derived many generating relations for the generalized Hermite polynomials. In this work, we study some of these works in detail and the same technique for discovering the producing relations for two index, and two variable Hermite polynomials. (Cesarano, C. 2014).

1.3 Literature Review

In (Dattoli et al. 1993-2000) the authors presented and explored the extended Hermite polynomial theory. Srivastava (H.S.P. 2001) defined the Hermite polynomials of multi-index single variable and obtained generating functions using the series manipulation method. (Pathan et al 2002-2003) and (Shahwan 2003) established original generalized Hermite polynomial generating functions $H_{m,n}(x, y)$ introduced by (Dattoli 1993).

Due to the importance of these functions and their applications, we use (Wesnier's 1955) method to get original generating relations of double index and double variable Hermite polynomials, $He_{m,n}(x, y)$ by correctly interpreting the indices m and n of the polynomials.

1.4 Theoretical Framework

The most significant definitions and statements of the Lie group and Lie algebra, special functions, and generating relations are presented in chapter 1 (Introduction). In Chapter 2, we go over the Weisner group theoretic method for creating relations with double indexes polynomial of Hermite $H_{m,n}(x, y, z, w, k)$ in detail.

We utilize the same strategy in Chapter 3 to obtain some additional generating relations of double index and double variable Hermite polynomials, $He_{m,n}(x, y)$ (Cesarano, C. 2014).

In Chapter 4, we summarize the main new results obtained for $He_{m,n}(x, y)$, and then show the importance of these results.

1.5 Methods and Procedures

Using the recurrence relations of generalized Hermite polynomials, we introduce the first order differential operators, called raising, lowering, and stationary operators, meeting specific commutation connections demonstrating that these operators form a Lie group (Chon 1961) and (Miller 1968). The extended form of the group formed by the provided operators is found. Then we employ a strategy proposed by (Weisner 1955), according to Taylor's series, to get generating relations see (McBride 1971).

1.6 Background and History

1.6.1 Hermite polynomials

Many special functions in mathematics play an important role in many physical and statistical applications. These special functions are the solutions of certain differential equations and integrals. Special functions are analytic, and most are considered functions of complex variables. Hermite polynomials are one of the leading and most important special functions used in many applications, such as probability, numerical analysis, and physics. The first Hermite polynomial $H_n(x)$ introduced by the French mathematician Charlike Hermite in 1864, which contains only one variable.

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \exp(2xt - t^2)$$

This function has been developed and modified by many mathematicians including (Dattoli 1992, 1993, 1995, 1999, 2000, 2005, 2007)

It is defined using the relation:

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n! (2x)^{n-2k}}{k! (n-2k)!}$$

Where $\left[\frac{n}{2}\right]$ = largest integer.

1.6.2 Lie Algebra

The theory of functions of mathematical physics is intimately related to Lie algebra and Lie group theory. Lie algebra is an algebra that is anti-commutative and satisfies Jacobi's identity. The commutator product of x and y is a Lie bracket defined by

$$[x, y] = xy - yx \quad \text{on associative algebra}$$

In other words: if g is an algebra over a field of real numbers F , with the product $[x, y]$, then g is a Lie algebra if:

1. $[x, x] = 0, \forall x \in g$.
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0,$
 $\forall x, y, z \in g$.

Anti-commutative may be written as $[x, y] = -[y, x]$.

An abelian algebra of Lie is a algebra of Lie with a commutative Lie bracket, i.e. $[x, y] = [y, x]$. Hence, the bracket for Lie in abelian Lie algebra is the identity that equals 0.

1.6.4 Weisner's Group Theoretical Method

There are many methods to derive generating relations of functions, including Rainville's, Truesdell's, Miller's, series manipulation, and Weisner's methods. Each of these methods suits some function that satisfies specific circumstances.

The Hermite function is one of those that meet Weisner's requirements, which we discuss in our work. Finding generating relations for Bessel polynomials, generalized Laguerre polynomials, and Gegenbauer polynomials can also be done using Weisner's method, as they satisfy Weisner's conditions.

Weisner's method obtains generating functions for special functions starting from their polynomials using Lie group representation theory and Lie algebras. This method was introduced by the Canadian mathematician Louis Weisner in 1955.

In this method, we use some partial differential equations besides Lie algebra.

To illustrate this method, we discuss its main steps through the following few lines and use it in more detail.

Weisner's group theoretic method steps:

1. Recurrence Relations:

Finding differential recurrence relations where the function satisfies.

2. First Order (Linear) Differential Operators:

Finding three types of operators which are called:

1. Raising operator (we represent it by J^+)
2. Lowering operator (we represent it by J^-)
3. Stationary operator (we represent it by J^3)

The one-parameter Lie group is produced by every first-order linear differential operator. We can identify the extended form of the group formed by the linear differential operator if we know it.

Another way to phrase it is, we will be able to determine the operator's effect. e^{cC} on $f(x, y)$ where,

$$e^{cC} \equiv 1 + cC + \frac{(cC)^2}{2!} + \frac{(cC)^3}{3!} + \dots \equiv \sum_{k=0}^{\infty} \frac{(cC)^k}{k!}$$

where C is supposed to be an operator.

3. Commutative Relations of the Operators

Using the commutator notation, $[x, y] = xy - yx$ to find the three relations:

1. $[J^3, J^-]$

2. $[J^3, J^+]$

3. $[J^+, J^-]$

The commutator formulas show that the operators J^3 , J^+ , and J^- generate a Lie group.

4. The Extended Form of Operator Generated Group

Here we need to find $e^{bJ^-} f(x, y)$ and $e^{cJ^+} f(x, y)$, where $f(x, y)$ is a random function with b and c are random constants,

$$e^{cC} f(x, y) = \varphi(x, y) g(x, y)$$

$e^{cC} f(x, y)$ is the transformed group's extended form generated by C .

5. Generating Functions

Weisner's method for discovering generating relations, which is based on Taylor's series in the form of

$$\exp \left[c \frac{d}{dX} F(X) \right] = F(X + C)$$

We consider the following three cases:

1. The effect of both J^+ and J^- together (i.e., $b \neq 0$ & $c \neq 0$)
2. The effect of J^- only ($c = 0$)
3. The effect of J^+ only ($b = 0$)

However, throughout our study, since we are dealing with two index Hermite polynomials, we have six operators where we consider their effect on the given polynomial that is raising, lowering and stationary operators will act on m and n , respectively.

Chapter 2: Generating Functions of Two Index

Generalized Hermite Polynomials

2.1 Introduction

The double index generalized Hermite polynomials are characterized by two different indices, four variables, and one parameter.

The following bilateral generating function defines these polynomials. (Dattoli, G, 2004):

$$\begin{aligned} & \exp(xt + yu + zt^2 + \omega u^2 + kut) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} H_{m,n}(x, y, z, \omega, k) \end{aligned} \quad (2.1)$$

in which the polynomials $H_{m,n}$ are expressly supplied by the series

$$H_{m,n}(x, y, z, \omega, k) = m! n! \sum_{l=0}^{\min(m,n)} \frac{k^l H_{m-1}(x, z) H_{n-1}(y, \omega)}{l! (m-l)! (n-l)!}.$$

Furthermore, these polynomials satisfy the simultaneous partial differential equations listed below.

$$2z \frac{\partial^2 v}{\partial x^2} + k \frac{\partial^2 v}{\partial x \partial y} + x \frac{\partial v}{\partial x} - mv + v = 0 \quad (2.2)$$

and

$$2w \frac{\partial^2 v}{\partial y^2} + k \frac{\partial^2 v}{\partial x \partial y} + y \frac{\partial v}{\partial y} - nv + v = 0 \quad (2.3)$$

2.2 Finding Generating Relations using Weisner's Method

2.2.1 Recurrence Relations

In this subsection, we derive the relations for pure and differential recurrence. We differentiate Hermite generating function of two index and two variables which are considered in (2.1) partially with respect to each of the variables x, y and t .

Case (1): Differentiate eq. (2.1) with respect to x , we obtain:

$$t \cdot \exp(xt + yu + zt^2 + \omega u^2 + kut)$$

$$= \frac{\partial}{\partial x} \left(\sum_{m,n=0}^{\infty} \frac{t^m u^n}{m! n!} H_{m,n}(x, y, z, \omega, k) \right)$$

$$t \cdot \sum_{m,n=0}^{\infty} \frac{t^m u^n}{m! n!} H_{m,n}(x, y, z, \omega, k) = \sum_{m,n=0}^{\infty} \frac{t^m u^n}{m! n!} \frac{\partial}{\partial x} H_{m,n}(x, y, z, \omega, k)$$

Replace m by $m - 1$ on the left side, and multiply and divide by m , we find that

$$\begin{aligned} \sum_{m=1,n=0}^{\infty} \frac{mt^m u^n}{m(m-1)! n!} H_{m-1,n}(x, y, z, \omega, k) \\ = \sum_{m,n=0}^{\infty} \frac{t^m u^n}{m! n!} \frac{\partial}{\partial x} H_{m,n}(x, y, z, \omega, k). \end{aligned}$$

Evaluating coefficients of $t^m u^n$, we obtain the following differential recurrence relation

$$\frac{\partial}{\partial x} H_{m,n}(x, y, z, w, k) = m H_{m-1,n}(x, y, z, w, k) \quad (2.4)$$

Case (2): Differentiate eq. (2.1) with respect to y , we get

$$u \cdot \exp(xt + yu + zt^2 + \omega u^2 + kut)$$

$$= \frac{\partial}{\partial y} \left(\sum_{m,n=0}^{\infty} \frac{u^n t^m}{n! m!} H_{m,n}(x, y, z, \omega, k) \right)$$

$$u \cdot \sum_{m,n=0}^{\infty} \frac{t^m u^n}{m! n!} H_{m,n}(x, y, z, \omega, k) = \sum_{m,n=0}^{\infty} \frac{t^m u^n}{m! n!} \frac{\partial}{\partial y} H_{m,n}(x, y, z, \omega, k)$$

Replace n by $n - 1$ on the left-hand side, and we get

$$\begin{aligned} \sum_{m,n=1}^{\infty} \frac{n t^m}{n! m!} \frac{u^n}{(n-1)!} H_{m,n-1}(x, y, z, \omega, k) \\ = \sum_{m,n=0}^{\infty} \frac{t^m u^n}{m! n!} \frac{\partial}{\partial y} H_{m,n}(x, y, z, \omega, k) \end{aligned}$$

Evaluating coefficients of $t^m u^n$, we get the following differential recurrence relation

$$\frac{\partial}{\partial y} H_{m,n}(x, y, z, \omega, k) = n H_{m,n-1}(x, y, z, \omega, k) \quad (2.5)$$

Case (3): Differentiate eq.(2.1) with respect to t , we get

$$(x + 2zt + ku) \cdot \exp(xt + yu + zt^2 + wu^2 + kut)$$

$$= \sum_{m,n=0}^{\infty} \frac{u^n t^{m-1}}{n! (m-1)!} H_{m,n}(x, y, z, \omega, k)$$

$$\Rightarrow x \sum_{m,n=0}^{\infty} \frac{u^n t^m}{n! m!} H_{m,n}(x, y, z, \omega, k)$$

$$+ 2z \sum_{m,n=0}^{\infty} \frac{t^{m+1} u^n}{m! n!} H_{m,n}(x, y, z, \omega, k)$$

$$+ k \sum_{m,n=0}^{\infty} \frac{t^m u^{n+1}}{m! n!} H_{m,n}(x, y, z, \omega, k)$$

$$= \sum_{m,n=0}^{\infty} \frac{t^{m-1} u^n}{(m-1)! n!} H_{m,n}(x, y, z, \omega, k)$$

$$\Rightarrow x \sum_{m,n=0}^{\infty} \frac{t^m u^n}{m! n!} H_{m,n}(x, y, z, \omega, k)$$

$$+ 2z \sum_{m=1,n=0}^{\infty} \frac{m}{m} \frac{t^m}{(m-1)!} \frac{u^n}{n!} H_{m-1,n}(x, y, z, \omega, k)$$

$$+ k \sum_{m=0,n=1}^{\infty} \frac{n}{n} \frac{t^m}{m!} \frac{u^n}{(n-1)!} H_{m,n-1}(x, y, z, w, k)$$

$$= \sum_{m=1,n=0}^{\infty} \frac{t^m u^n}{m! n!} H_{m+1,n}(x, y, z, \omega, k).$$

Now, comparing the coefficients of $t^m u^n$, we get

$$xH_{m,n}(x, y, z, w, k) + 2zmH_{m-1,n}(x, y, z, w, k) + knH_{m,n-1}(x, y, z, w, k) = H_{m+1,n}(x, y, z, \omega, k). \quad (2.6)$$

Substitute eq. (2.4) and eq. (2.5) in eq. (2.6), we get

$$\begin{aligned} & \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) H_{m,n}(x, y, z, \omega, k) \\ & = H_{m+1,n}(x, y, z, \omega, k) \end{aligned} \quad (2.7)$$

Case (4): Differentiate eq.(2.1) with respect to u , we get

$$(y + 2wu + kt) \cdot \exp(xt + yu + zt^2 + wu^2 + kut)$$

$$= \sum_{m,n=0}^{\infty} \frac{t^m}{m!} \frac{u^{n-1}}{(n-1)!} H_{m,n}(x, y, z, \omega, k)$$

$$\Rightarrow y \sum_{m,n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} H_{m,n}(x, y, z, \omega, k)$$

$$+ 2w \sum_{m,n=0}^{\infty} \frac{t^m}{m!} \frac{u^{n+1}}{n!} H_{m,n}(x, y, z, \omega, k)$$

$$+ k \sum_{m,n=0}^{\infty} \frac{t^{m+1}}{m!} \frac{u^n}{n!} H_{m,n}(x, y, z, \omega, k)$$

$$= \sum_{m,n=0}^{\infty} \frac{t^m}{m!} \frac{u^{n-1}}{(n-1)!} H_{m,n}(x, y, z, \omega, k)$$

$$\begin{aligned}
& y \sum_{m,n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} H_{m,n}(x, y, z, \omega, k) \\
& \quad + 2w \sum_{m=0, n=1}^{\infty} \frac{n}{n} \frac{t^m}{m!} \frac{u^n}{(n-1)!} H_{m,n-1}(x, y, z, \omega, k) \\
& \quad + k \sum_{m=1, n=0}^{\infty} \frac{m}{m} \frac{t^m}{(m-1)!} \frac{u^n}{n!} H_{m-1,n}(x, y, z, w, k) \\
& \quad = \sum_{m=0, n=-1}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} H_{m,n+1}(x, y, z, \omega, k)
\end{aligned}$$

Comparing the coefficients of $t^m u^n$, we get

$$\begin{aligned}
& yH_{m,n}(x, y, z, w, k) + 2wnH_{m,n-1}(x, y, z, w, k) + \\
& kmH_{m-1,n}(x, y, z, w, k) = H_{m,n+1}(x, y, z, \omega, k)
\end{aligned} \tag{2.8}$$

Substitute eq. (2.4) and eq. (2.5) in eq. (2.8), we get

$$\left(y + 2w \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right) H_{m,n}(x, y, z, w, k) = H_{m,n+1}(x, y, z, w, k) \tag{2.9}$$

2.2.2 First Order Differential Operators

Let $m = p \frac{\partial}{\partial p}$, $n = s \frac{\partial}{\partial s}$ and replace them in eq. (2.2) and eq. (2.3),

respectively, we get

$$2z \frac{\partial^2 u}{\partial x^2} + k \frac{\partial^2 u}{\partial x \partial y} + x \frac{\partial u}{\partial x} - p \frac{\partial u}{\partial p} = 0 \quad (2.10)$$

and

$$2w \frac{\partial^2 u}{\partial y^2} + k \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial u}{\partial y} - s \frac{\partial u}{\partial s} = 0 \quad (2.11)$$

We find that: $f(x, y, z, w, k, p, s) = H_{m,n}(x, y, z, w, k)p^m s^n$ is a solution of eq. (2.10) and eq. (2.11), since $H_{m,n}(x, y, z, w, k)$ is a solution of eq. (2.2) and eq. (2.3). Now we use eqs. (2.4), (2.5), (2.7), and (2.9) to find the first order differential operators $\{A_i: i = 1, 2, 3, 4, 5, 6\}$

$$f(x, y, z, w, k, p, s) = H_{m,n}(x, y, z, w, t)p^m s^n.$$

For $A_1 = p \frac{\partial}{\partial p}$, we get:

$$A_1[H_{m,n}(x, y, z, \omega, k)p^m s^n] = mH_{m,n}(x, y, z, \omega, k)p^m s^n.$$

For $A_2 = s \frac{\partial}{\partial s}$, we get

$$A_2[H_{m,n}(x, y, z, \omega, k)p^m s^n] = nH_{m,n}(x, y, z, \omega, k)p^m s^n.$$

For $A_3 = p^{-1} \frac{\partial}{\partial x}$, we get:

$$A_3[H_{m,n}(x, y, z, w, k)p^m s^n] = mH_{m-1,n}(x, y, z, w, k)p^{m-1}s^n.$$

For $A_4 = p \left[x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right]$, we get

$$A_4[H_{m,n}(x, y, z, \omega, k)p^m s^n] = H_{m+1,n}(x, y, z, \omega, k)p^{m+1}s^m$$

For $A_5 = s^{-1} \frac{\partial}{\partial y}$, we get:

$$A_5[H_{m,n}(x, y, z, w, k)p^m s^n] = nH_{m,n-1}(x, y, z, w, k)p^m s^{n-1}$$

Finally, for $A_6 = s \left[y + 2w \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right]$, we get:

$$A_6[H_{m,n}(x, y, z, \omega, k)p^m s^n] = H_{m,n+1}(x, y, z, \omega, k)p^m s^{m+1}.$$

We have used the recurrence relations to introduce the three types of operators: Stationary, lowering, and raising operators.

2.2.3 Commutative Relations of the Operators

In this subsection, employ the commutator notation $[A,B]$ with $[A,B]f = (AB - BA)f$ where

$$f = f(x, y, z, w, k, p, s) = H_{m,n}(x, y, z, w, k) p^m s^n .$$

To find all possible commutation relations between the operators

Case (1): We have:

$$\begin{aligned} [A_1, A_2]f &= A_1 A_2 f - A_2 A_1 f \\ &= \left[\left(p \frac{\partial}{\partial p} \right) \left(s \frac{\partial}{\partial s} \right) \right] f - \left[\left(s \frac{\partial}{\partial s} \right) \left(r \frac{\partial}{\partial p} \right) \right] f = 0 \\ \Rightarrow [A_1, A_2]f &= 0 \\ \Rightarrow [A_1, A_2] &= 0. \end{aligned}$$

Case (2): We have:

$$\begin{aligned} [A_1, A_3]f &= A_1 A_3 f - A_3 A_1 f \\ &= \left[\left(p \frac{\partial}{\partial p} \right) \left(p^{-1} \frac{\partial}{\partial x} \right) \right] f - \left[\left(p^{-1} \frac{\partial}{\partial x} \right) \left(p \frac{\partial}{\partial p} \right) \right] f \\ &= p \cdot \left[p^{-1} \cdot \frac{\partial^2 f}{\partial p \partial x} - \frac{1}{p^2} \frac{\partial f}{\partial x} \right] - \left[p^{-1} p \cdot \frac{\partial^2 f}{\partial x \partial p} \right] \\ &= p p^{-1} \frac{\partial^2 f}{\partial p \partial x} - \frac{1}{p} \frac{\partial f}{\partial x} - p^{-1} p \frac{\partial^2 f}{\partial x \partial p} = -p^{-1} \frac{\partial f}{\partial x} = -A_3 f \\ \Rightarrow [A_1, A_3]f &= -A_3 f \\ \Rightarrow [A_1, A_3] &= -A_3. \end{aligned}$$

Case (3): We have:

$$\begin{aligned}
[A_1, A_4]f &= A_1 A_4 f - A_4 A_1 f \\
&= \left[\left(p \frac{\partial}{\partial p} \right) \left(p \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \right) \right] f \\
&\quad - \left[p \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \left(p \frac{\partial}{\partial p} \right) \right] f \\
&= \left[p \left[p \left(x \frac{\partial}{\partial p} + 2z \frac{\partial^2}{\partial p \partial x} + k \frac{\partial^2}{\partial p \partial y} \right) + \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \right] \right] f \\
&\quad - p^2 \left[x \frac{\partial}{\partial p} + 2z \frac{\partial^2}{\partial p \partial x} + k \frac{\partial^2}{\partial y \partial p} \right] f \\
&= p \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f \\
&\Rightarrow [A_1, A_4]f = A_4 f \\
&\Rightarrow [A_1, A_4] = A_4.
\end{aligned}$$

Case (4): We have:

$$\begin{aligned}
[A_1, A_5]f &= A_1 A_5 f - A_5 A_1 f \\
&= \left[\left(p \frac{\partial}{\partial p} \right) \left(s^{-1} \frac{\partial}{\partial y} \right) \right] f - \left[\left(s^{-1} \frac{\partial}{\partial y} \right) \left(p \frac{\partial}{\partial p} \right) \right] f = 0 \\
&\therefore [A_1, A_5]f = 0.
\end{aligned}$$

Case (5): We have:

$$[A_1, A_6]f = A_1 A_6 f - A_6 A_1 f$$

$$\begin{aligned}
&= \left(p \frac{\partial}{\partial p} \right) \left(s \left(y + 2\omega \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right) \right) f \\
&\quad - \left(s \left(y + 2\omega \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right) \right) \left(p \frac{\partial}{\partial p} \right) f = 0
\end{aligned}$$

$$\Rightarrow [A_1, A_6]f = 0$$

$$\Rightarrow [A_1, A_6] = 0.$$

Case (6): We have:

$$\begin{aligned}
[A_2, A_3]f &= A_2 A_3 f - A_3 A_2 f \\
&= \left[\left(s \frac{\partial}{\partial s} \right) \left(p^{-1} \frac{\partial}{\partial p} \right) \right] f - \left[\left(p^{-1} \frac{\partial}{\partial p} \right) \left(s \frac{\partial}{\partial s} \right) \right] f = 0
\end{aligned}$$

$$\Rightarrow [A_2, A_3]f = 0$$

$$\Rightarrow [A_2, A_3] = 0.$$

Case (7): We have:

$$\begin{aligned}
[A_2, A_4]f &= A_2 A_4 f - A_4 A_2 f \\
&= \left[\left(s \frac{\partial}{\partial s} \right) \left(P \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \right) \right] f \\
&\quad - \left[P \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \left(s \frac{\partial}{\partial s} \right) \right] f = 0
\end{aligned}$$

$$\Rightarrow [A_2, A_4]f = 0$$

$$\Rightarrow [A_2, A_4] = 0.$$

Case (8): We have:

$$\begin{aligned}
[A_2, A_5]f &= A_2 A_5 f - A_5 A_2 f \\
&= \left[\left(s \frac{\partial}{\partial s} \right) \left(s^{-1} \frac{\partial}{\partial y} \right) \right] f - \left[\left(s^{-1} \frac{\partial}{\partial y} \right) \left(s \frac{\partial}{\partial s} \right) \right] f \\
&= s \cdot \frac{-1}{s^2} \frac{\partial f}{\partial y} + s \cdot s^{-1} \frac{\partial^2 f}{\partial s \partial y} - s^{-1} \cdot s \frac{\partial^2 f}{\partial y \partial s} \\
&= \frac{-1}{s} \frac{\partial f}{\partial y} = -A_5 f \\
&\Rightarrow [A_2, A_5]f = -A_5 f \\
&\Rightarrow [A_2, A_5] = -A_5.
\end{aligned}$$

Case (9): We have:

$$\begin{aligned}
[A_2, A_6]f &= A_2 A_6 f - A_6 A_2 f \\
&= \left[\left(s \frac{\partial}{\partial s} \right) \left(s \left(y + 2w \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right) \right) \right] f - \left[\left(s \left(y + 2w \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right) \right) \left(s \frac{\partial}{\partial s} \right) \right] f \\
&= s \left[\left(y + 2w \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right) + s \left(y \frac{\partial}{\partial s} + 2w \frac{\partial^2}{\partial s \partial y} + k \frac{\partial^2}{\partial s \partial x} \right) \right] f \\
&\quad - s \left[s \left(y \frac{\partial}{\partial s} + 2w \frac{\partial^2}{\partial s \partial y} + k \frac{\partial^2}{\partial s \partial x} \right) \right] f \\
&= \left[s \left(y + 2w \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right) \right] f = A_6 f
\end{aligned}$$

$$\Rightarrow [A_2, A_6]f = A_6f$$

$$\Rightarrow [A_2, A_6] = A_6.$$

Case (10): We have:

$$[A_3, A_4]f = A_3A_4f - A_4A_3f$$

$$\begin{aligned} &= \left[\left(p^{-1} \frac{\partial}{\partial x} \right) \left(p \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \right) \right] f \\ &\quad - \left[p \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \left(p^{-1} \frac{\partial}{\partial x} \right) \right] f \\ &= \left[p^{-1} p \left(x \frac{\partial}{\partial x} + 2z \frac{\partial^2}{\partial x^2} + k \frac{\partial^2}{\partial y \partial x} + 1 \right) \right] f \\ &\quad - \left[p^{-1} p \left(x \frac{\partial}{\partial x} + 2z \frac{\partial^2}{\partial x^2} + k \frac{\partial^2}{\partial y \partial x} \right) \right] f = f \end{aligned}$$

$$\Rightarrow [A_3, A_4]f = f$$

$$\Rightarrow [A_3, A_4] = 1.$$

Case (11): We have:

$$[A_3, A_5]f = A_3A_5f - A_5A_3f$$

$$= \left[\left(p^{-1} \frac{\partial}{\partial x} \right) \left(s^{-1} \frac{\partial}{\partial y} \right) \right] f - \left[\left(s^{-1} \frac{\partial}{\partial y} \right) \left(p^{-1} \frac{\partial}{\partial x} \right) \right] f = 0$$

$$\Rightarrow [A_3, A_5]f = 0$$

$$\Rightarrow [A_3, A_5] = 0.$$

Case (12): We have:

$$\begin{aligned}
[A_3, A_6]f &= A_3 A_6 f - A_6 A_3 f \\
&= \left[\left(p^{-1} \frac{\partial}{\partial x} \right) \left(s \left(y + 2\omega \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right) \right) \right] f \\
&\quad - \left[\left(s \left(y + 2\omega \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right) \right) \left(p^{-1} \frac{\partial}{\partial x} \right) \right] f = 0
\end{aligned}$$

$$\Rightarrow [A_3, A_6]f = 0$$

$$\Rightarrow [A_3, A_6] = 0.$$

Case (13): We have:

$$\begin{aligned}
[A_4, A_5]f &= A_4 A_5 f - A_5 A_4 f \\
&= \left[p \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \left(s^{-1} \frac{\partial}{\partial y} \right) \right] f \\
&\quad - \left[\left(s^{-1} \frac{\partial}{\partial y} \right) \left(p \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \right) \right] f = 0
\end{aligned}$$

$$\Rightarrow [A_4, A_5]f = 0$$

$$\Rightarrow [A_4, A_5] = 0.$$

Case (14): We have:

$$\begin{aligned}
[A_4, A_6]f &= A_4 A_6 f - A_6 A_4 f \\
&= \left[\left(p \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \right) \left(s \left(y + 2\omega \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right) \right) \right] f \\
&\quad - \left[\left(s \left(y + 2w \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right) \right) \left(p \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \right) \right] f \\
&= ps \left[x \left(y + 2\omega \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right) + 2z \left(y \frac{\partial}{\partial x} + 2w \frac{\partial^2}{\partial x \partial y} + k \frac{\partial^2}{\partial x^2} \right) \right. \\
&\quad \left. + k \left(y \frac{\partial}{\partial y} + 2w \frac{\partial^2}{\partial y^2} + k \frac{\partial^2}{\partial y \partial x} + 1 \right) \right] f \\
&\quad - sp \left[y \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) + 2w \left(x \frac{\partial}{\partial y} + 2z \frac{\partial^2}{\partial x \partial y} + k \frac{\partial^2}{\partial y^2} \right) \right. \\
&\quad \left. + k \left(x \frac{\partial}{\partial x} + 2z \frac{\partial^2}{\partial x^2} + k \frac{\partial^2}{\partial y \partial x} + 1 \right) \right] f = 0 \\
&\Rightarrow [A_4, A_6]f = 0 \\
&\Rightarrow [A_4, A_6] = 0.
\end{aligned}$$

Case (15): We have:

$$\begin{aligned}
[A_5, A_6]f &= A_5 A_6 f - A_6 A_5 f \\
&= \left[\left(s^{-1} \frac{\partial}{\partial y} \right) \left(s \left(y + 2w \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right) \right) \right] f - \left[\left(s \left(y + 2w \frac{\partial}{\partial y} + \right. \right. \right. \\
&\quad \left. \left. k \frac{\partial}{\partial x} \right) \right) \left(s^{-1} \frac{\partial}{\partial y} \right) \right] f
\end{aligned}$$

$$\begin{aligned}
&= ss^{-1} \left[\left(y \frac{\partial}{\partial y} + 2w \frac{\partial^2}{\partial y^2} + k \frac{\partial^2}{\partial y \partial x} + 1 \right) \right] f \\
&\quad - ss^{-1} \left[y \frac{\partial}{\partial y} + 2w \frac{\partial^2}{\partial y^2} + k \frac{\partial^2}{\partial y \partial x} \right] f = f \\
&\Rightarrow [A_5, A_6]f = f \\
&\Rightarrow [A_5, A_6] = 1.
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
&[A_1, A_2] = 0, [A_2, A_3] = 0, [A_3, A_4] = 1, [A_4, A_5] = 0, [A_5, A_6] = 1, \\
&[A_1, A_3] = -A_3, [A_2, A_4] = 0, [A_3, A_5] = 0, [A_4, A_6] = 0, \\
&[A_1, A_4] = A_4, [A_2, A_5] = -A_5, [A_3, A_6] = 0, \\
&[A_1, A_5] = 0, [A_2, A_6] = A_6, \\
&[A_1, A_6] = 0.
\end{aligned}$$

Since the foregoing connections are not isomorphic to any commutation relations obeyed by a well-known Lie algebra of a local Lie group, according to Weisner's group theoretic technique, as a result, we are unable to connect the set of differential operators. $\{J^3, J^-, J^+\}$ with a particular Lie algebra, only that these differential operators generate three-dimensional Lie algebra can be said.

Therefore, the preceding commutation relations show that the set of operators.

$\{A_i: i = 1, 2, 3, 4, 5, 6\}$ make some Lie algebra λ .

Next, we can form the differential operators from eqs. (2.10) and (2.11).

$$L_1 = 2z \frac{\partial^2}{\partial x^2} + k \frac{\partial^2}{\partial x \partial y} + x \frac{\partial}{\partial x} - p \frac{\partial}{\partial p}$$

and

$$L_2 = 2w \frac{\partial^2}{\partial y^2} + k \frac{\partial^2}{\partial x \partial y} + y \frac{\partial}{\partial y} - s \frac{\partial}{\partial s}$$

Which can be expressed as

$$L_1 = A_3 A_4 - m - 1$$

$$L_2 = A_5 A_6 - n - 1$$

and this can be verified very easily.

We have:

$$\begin{aligned} A_3 A_4 (H_{m,n}(x, y, z, w, k) p^m s^n) &= A_3 (H_{m+1,n}(x, y, z, w, k) p^{m+1} s^n) \\ &= (m+1) H_{m,n}(x, y, z, w, k) p^m s^n. \end{aligned}$$

But

$$\begin{aligned} A_3 A_4 (H_{m,n}(x, y, z, w, k) p^m s^n) &= \left[\left(p^{-1} \frac{\partial}{\partial x} \right) \left(p \left(x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \right) \right] (H_{m,n}(x, y, z, w, k) p^m s^n) \\ &= \left[x \frac{\partial}{\partial x} + 2z \frac{\partial^2}{\partial x^2} + k \frac{\partial^2}{\partial x \partial y} + 1 \right] (H_{m,n}(x, y, z, w, k) p^m s^n) \end{aligned}$$

$$\Rightarrow L_1 = A_3 A_4 - m - 1$$

where m is replaced by $p \frac{\partial}{\partial p}$

$$\therefore L_1 = 2z \frac{\partial^2}{\partial x^2} + k \frac{\partial^2}{\partial x \partial y} + x \frac{\partial}{\partial x} - p \frac{\partial}{\partial p}$$

We have

$$\begin{aligned} A_5 A_6 (H_{n,m}(x, y, z, w, k) p^m s^n) &= A_5 (H_{m,n+1}(x, y, z, w, k) p^m s^{n+1}) \\ &= (n+1) H_{m,n}(x, y, z, w, k) p^m s^n \end{aligned}$$

$$\Rightarrow L_2 = A_5 A_6 - n - 1.$$

But

$$\begin{aligned}
& A_5 A_6 (H_{m,n}(x, y, z, w, k) p^m s^n) \\
&= \left[\left(s^{-1} \frac{\partial}{\partial y} \right) \left(s \left(y + 2\omega \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right) \right) \right] (H_{m,n}(x, y, z, w, k) p^m s^n) \\
&= s^{-1} \cdot s \left[y \frac{\partial}{\partial y} + 2\omega \frac{\partial^2}{\partial y^2} + k \frac{\partial^2}{\partial y \partial x} + 1 \right] f \\
&= \left[y \frac{\partial}{\partial y} + 2\omega \frac{\partial^2}{\partial y^2} + k \frac{\partial^2}{\partial y \partial x} + 1 \right] f
\end{aligned}$$

$$\Rightarrow L_2 = A_5 A_6 - n - 1$$

where n is replaced by $s \frac{\partial}{\partial s}$.

$$\therefore L_2 = 2\omega \frac{\partial^2}{\partial y^2} + k \frac{\partial^2}{\partial y \partial x} + y \frac{\partial}{\partial y} - s \frac{\partial}{\partial s},$$

which commutes with $\{A_i: i = 1, 2, 3, 4, 5, 6\}$.

That is: $[L_1, A_i] = 0$ and $[L_2, A_i] = 0$.

2.2.4 Extended Forms of The Group Generated by Operators

The extended forms of the group formed by will be found here

$$\{A_i: i = 1, 2, 3, 4, 5, 6\}.$$

Case(1): For $A_1 = p \frac{\partial}{\partial p}$.

Let a_1 be any variable, then

$$\frac{dp}{da_1} = p$$

$$\int \frac{dp}{p} = \int da_1$$

$$\ln p = a_1 + \ln A$$

Applying the initial condition when $a_1 = 0, p(0) = p_0$

$$\Rightarrow \ln p_0 = \ln A \Rightarrow p_0 = A$$

$$\Rightarrow \ln p = a_1 + \ln p_0$$

$$\therefore p = p_0 e^{a_1}$$

$$\frac{dx}{da_1} = 0$$

$$\Rightarrow dx = 0$$

$$\Rightarrow x = k.$$

Applying the initial condition when $a_1 = 0, x(0) = x_0$

$$\Rightarrow x_0 = k$$

$$\Rightarrow x = x_0$$

Also,

$$\frac{dy}{da_1} = 0, \quad \frac{dz}{da_1} = 0, \quad \frac{dw}{da_1} = 0, \quad \frac{dk}{da_1} = 0, \quad \frac{ds}{da_1} = 0$$

$$\Rightarrow y = y_0, \quad z = z_0, \quad w = w_0, \quad k = k_0, \quad s = s_0$$

$$\frac{dv}{da_1} = 0 \Rightarrow dv = 0 \Rightarrow v = k.$$

Applying the initial condition when $a_1 = 0, v(0) = 1 \Rightarrow k = 1, v = 1$

$$\therefore e^{a_1 A_1} f(x, y, z, w, k, p, s) = f(x_0, y_0, z_0, w_0, k_0, p_0 e^{a_1}, s_0) \quad (2.12)$$

Case(2): For $A_2 = s \frac{\partial}{\partial s}$.

By using the same calculation as in case(1), we get

$$\therefore e^{a_2 A_2} f(x, y, z, w, k, p, s) = f(x_0, y_0, z_0, w_0, k_0, p_0, s_0 e^{a_2}) \quad (2.13)$$

Case(3): For $A_3 = p^{-1} \frac{\partial}{\partial x}$.

Let a_3 be any variable

$$\frac{dx}{da_3} = p^{-1}$$

$$\int dx = \int p_0^{-1} da_3$$

$$x = \frac{a_3}{p_0} + k.$$

Applying the initial condition when $a_3 = 0, x(0) = x_0 \Rightarrow k = x_0$

$$x = \frac{a_3}{p_0} + x_0$$

Also,

$$\frac{dy}{da_3} = 0, \quad \frac{dz}{da_3} = 0, \quad \frac{dw}{da_3} = 0, \quad \frac{dk}{da_3} = 0,$$

$$\frac{dp}{da_3} = 0, \quad \frac{ds}{da_3} = 0$$

$$\Rightarrow y = y_0, \quad z = z_0, \quad w = w_0, \quad k = k_0, \quad p = p_0, \quad S = S_0$$

$$\text{and } \frac{dv}{da_3} = 0 \Rightarrow v = 1, \text{ then}$$

$$e^{a_3 A_3} f(x, y, z, w, k, p, s) = f\left(x_0 + \frac{a_3}{p_0}, y_0, z_0, w_0, k_0, p_0, s_0\right) \quad (2.14)$$

$$\textbf{Case (4):} \text{ For } A_4 = p \left[x + 2z \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right]$$

$$\Rightarrow A_4 = px + 2pz \frac{\partial}{\partial x} + pk \frac{\partial}{\partial y}.$$

Let a_4 be any variable

$$\frac{dx}{da_4} = 2pz, \quad \frac{dy}{da_4} = pk$$

$$\int dx = \int 2p_0 z_0 da_4$$

$$x = 2p_0 z_0 a_4 + k.$$

When using the initial condition $a_4 = 0, \quad x(0) = x_0$

$$x = 2p_0 z_0 a_4 + x_0$$

$$\int dy = \int p_0 k_0 da_4.$$

While the initial condition is being applied $a_4 = 0, y(0) = y_0$

$$y = p_0 k_0 a_4 + y_0$$

$$\Rightarrow \frac{dz}{da_4} = 0, \quad \frac{dw}{da_4} = 0, \quad \frac{dk}{da_4} = 0, \quad \frac{dp}{da_4} = 0, \quad \frac{ds}{da_4} = 0$$

$$z = z_0 \quad w = w_0, \quad k = k_0, \quad p = p_0, \quad s = s_0$$

$$\frac{dv}{da_4} = pxv$$

$$\frac{dv}{v} = p_0(x_0 + 2p_0 z_0 a_4) da_4$$

$$\int \frac{dv}{v} = \int (p_0 x_0 + 2p_0^2 z_0 a_4) da_4$$

$$\ln v = p_0 x_0 a_4 + 2z_0 p_0^2 \frac{a_4^2}{2} + k.$$

Applying the initial condition when $a_4 = 0, v(0) = 1, \ln(1) = 0$

$$\ln v = p_0 x_0 a_4 + z_0 p_0^2 a_4^2,$$

$$v = \exp(p_0 x_0 a_4 + z_0 p_0^2 a_4^2). \text{ then}$$

$$e^{a_4 A_4} f(x, y, z, \omega, k, p, s)$$

$$= \exp(p_0 x_0 a_4 + z_0 p_0^2 a_4^2)$$

$$\cdot f(x_0 + 2p_0 z_0 a_4, p_0 k_0 a_4 + y_0, z_0, \omega_0, k_0, p_0, s_0). \quad (2.15)$$

Case(5): For $A_5 = s^{-1} \frac{\partial}{\partial y}$

let a_5 be any variable, then

$$\frac{dy}{da_5} = s^{-1}$$

$$\int dy = \int \frac{1}{s_0} da_5$$

$$y = \frac{a_5}{s_0} + k.$$

Applying the initial condition when $a_5 = 0$, $y(0) = y_0$

$$\Rightarrow y_0 = k$$

$$y = y_0 + \frac{a_5}{s_0}$$

$x = x_0, z = z_0, k = k_0, s = s_0, p = p_0, w = w_0$, then

$$e^{a_5 A_5} f(x, y, z, w, k, p, s) = f\left(x_0, y_0 + \frac{a_5}{s_0}, z_0, w_0, k_0, p_0, s_0\right) \quad (2.16)$$

Case(6): For $A_6 = s \left[y + 2\omega \frac{\partial}{\partial y} + k \frac{\partial}{\partial x} \right] = sy + 2sw \frac{\partial}{\partial y} + sk \frac{\partial}{\partial x}$

Let a_6 be any variable, then

$$\frac{dx}{da_6} = sk$$

$$\Rightarrow \int dx = \int s_0 k_0 da_6$$

$$\Rightarrow x = s_0 k_0 a_6 + k$$

$$\frac{dy}{da_6} = 2sw$$

$$\int dy = \int 2s_0\omega_0 da_6$$

$$y = 2s_0\omega_0 a_6 + k$$

The first condition is being applied when $a_6 = 0 \Rightarrow y(0) = y_0$

$$y = 2s_0\omega_0 a_6 + y_0$$

$$\frac{dv}{da_6} = sy$$

Substitute $y = 2s_0\omega_0 a_6 + y_0$

$$\frac{dv}{da_6} = s_0(2s_0\omega_0 a_6 + y_0)v$$

$$\frac{dv}{v} = (2s_0^2\omega_0 a_6 + s_0y_0)da_6$$

$$\ln v = 2s_0^2\omega_0 \frac{a_6^2}{2} + s_0y_0 a_6 + k$$

Applying the initial condition when:

$$a_6 = 0, v(0) = 1; \ln(1) = 0 \Rightarrow k = 0$$

$$\Rightarrow \ln v = \omega_0 s_0^2 a_6^2 + s_0 y_0 a_6$$

$$e^{\ln v} = v = \exp(s_0 y_0 a_6 + \omega_0 s_0^2 a_6^2).$$

Also

$$\Rightarrow \frac{dz}{da_6} = 0, \quad \frac{dw}{da_6} = 0, \quad \frac{dk}{da_4} = 0, \quad \frac{dp}{da_6} = 0, \quad \frac{ds}{da_6} = 0$$

$$z = z_0, \quad w = w_0, \quad k = k_0, \quad p = p_0, \quad s = s_0, \quad \text{then}$$

$$\begin{aligned}
& e^{a_6 A_6} f(x, y, z, w, k, p, s) \\
& = \exp(s_0 y_0 a_6 + w_0 s_0^2 a_6^2) \\
& \quad \cdot f(x_0 + s_0 k_0 a_6, y_0 + 2s_0 w_0 a_6, z_0, w_0, k_0, p_0, s_0). \quad (2.17)
\end{aligned}$$

Therefore, generally speaking, we have the extended forms of the groups produced by

$$A_i (i = 1, 2, 3, 4, 5, 6)$$

following is:

$$\begin{aligned}
& e^{a_1 A_1} f(x, y, z, \omega, k, p, s) = f(x, y, z, \omega, k, p e^{a_1}, s), \\
& e^{a_2 A_2} f(x, y, z, \omega, k, p, s) = f(x, y, z, \omega, k, p, s e^{a_2}), \\
& e^{a_3 A_3} f(x, y, z, w, k, p, s) = f\left(x + \frac{a_3}{p}, y, z, w, k, p, s\right), \\
& e^{a_4 A_4} f(x, y, z, \omega, k, p, s) \\
& = \exp(p x a_4 + z p^2 a_4^2) \cdot f(x + 2p z a_4, p k a_4 + y, z, w, k, p, s), \\
& e^{a_5 A_5} f(x, y, z, w, k, p, s) = f\left(x, y + \frac{a_5}{s}, z, w, k, p, s\right), \\
& e^{a_6 A_6} f(x, y, z, w, k, p, s) \\
& = \exp(s y a_6 + w s^2 a_6^2) \cdot f(x + s k a_6, y + 2s w a_6, z, w, k, p, s).
\end{aligned}$$

2.2.5 Generating Functions

From the previous section, we have the extended forms of the groups produced by $A_i (i = 1, 2, 3, 4, 5, 6)$ from the eqs.(2.13)-(2.17)

We find the action of these operators as follow:

$$e^{a_6 A_6 + \dots + a_2 A_2 + a_1 A_1} f(x, y, z, w, k, p, s) = \exp(sy a_6 + ws^2 a_6^2 p x a_4 + zp^2 a_4^2) \cdot f\left(x + ska_6 + 2pza_4 + \frac{a_3}{p}, y + 2swa_6 + \frac{a_5}{s} + pka_4, z, w, k, pe^{a_1}, se^{a_2}\right). \quad (2.18)$$

By putting $a_i = 0 (i = 1, 2, 3, 4)$; $a_5 = b$, $a_6 = c$ in eq. (2.18) and letting

$$f(x, y, z, w, k, p, s) = H_{m,n}(x, y, z, w, k) p^m s^n,$$

we get:

$$\begin{aligned} e^{cA_6} e^{bA_5} H_{m,n}(x, y, z, w, k) p^m s^n \\ = \exp(ay_0 c + \omega_0 s_0^2 c^2) \\ \cdot H_{m,n}\left(x_0 + s_0 k_0 c, y_0 + 2s_0 \omega_0 c \right. \\ \left. + \frac{b}{s_0}, z_0, w_0, y_0, k_0\right) P_0^m s_0^n. \end{aligned} \quad (2.19)$$

But

$$\begin{aligned} e^{cA_6} e^{bA_5} (H_{m,n}(x, y, z, w, k) p^m s^n) \\ = e^{bA_5} \sum_{k=0}^{\infty} \frac{c^k A_6^k}{k!} H_{m,n}(x, y, z, w, k) p^m s^n \\ = e^{bA_5} \sum_{k=0}^{\infty} \frac{c^k}{k!} H_{m,n+k}(x, y, z, w, t) p^m s^{n+k} \end{aligned}$$

$$\begin{aligned}
& \therefore e^{cA_6} e^{bA_5} (H_{m,n}(x, y, z, w, k) p^m s^n) \\
&= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{b^l}{l!} \frac{c^k}{k!} H_{m,n+k-l}(x, y, z, \omega, k) p^m s^{n+k-l}.
\end{aligned} \tag{2.20}$$

We get the desired generating function by combining eqs. (2.19) and (2.20).

$$\begin{aligned}
& \exp(ay_0 c + \omega_0 s_0^2 c^2) \\
& \cdot H_{m,n} \left(x_0 + s_0 k_0 c, y_0 + 2s_0 \omega_0 c + \frac{b}{s_0}, z_0, w_0, y_0, k_0 \right) \\
&= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{b^l}{l!} \frac{c^k}{k!} H_{m,n+k-l}(x, y, z, \omega, k) s^{k-l}.
\end{aligned} \tag{2.21}$$

Next, some special cases of the generating relation eq. (2.21) are interesting.

By putting $b = 0$, $s = p = 1$ in eq. (2.21), we get:

$$\begin{aligned}
& \exp(ay_0 c + \omega_0 c^2) \cdot H_{m,n}(x_0 + k_0 c, y_0 + 2\omega_0 c, z_0, w_0, y_0, k_0) \\
&= \sum_{k=0}^{\infty} \frac{c^k}{k!} H_{m,n+k}(x, y, z, \omega, k).
\end{aligned}$$

If we let $c = 0$, $s = p = 1$ in eq. (2.21), we get the particular case:

$$H_{m,n}(x_0, y_0 + b, z_0, w_0, y_0, k_0) = \sum_{l=0}^{\infty} \frac{b^l}{l!} H_{m,n-l}(x, y, z, \omega, k).$$

In general, we have

$$\begin{aligned}
& H_{m,n}(x, y + b, z, w, y, k) \\
&= \sum_{l=0}^{\infty} \frac{b^l}{l!} H_{m,n-l}(x, y, z, \omega, k).
\end{aligned} \tag{2.22}$$

Further, by putting $a_i = 0 (i = 1, 2, 5, 6)$; $a_3 = a$, $a_4 = d$ in eq. (2.18) and letting

$$f(x, y, z, \omega, k, p, s) = H_{m,n}(x, y, z, \omega, k) p^m s^n,$$

we get:

$$\begin{aligned} & e^{dA_4} e^{aA_3} (H_{m,n}(x, y, z, w, k) p^m s^n) \\ &= \exp(p_0 x_0 d + z_0 p_0^2 d^2) \\ & \quad \cdot H_{m,n} \left(x_0 + 2p_0 z_0 d + \frac{a}{p_0}, y_0 \right. \\ & \quad \left. + p_0 k_0 d, z_0, w_0, k_0 \right) p_0^m s_0^n \end{aligned} \quad (2.23)$$

But

$$\begin{aligned} & e^{dA_4} e^{aA_3} (H_{m,n}(x, y, z, w, k) p^m s^n) \\ &= e^{aA_3} \sum_{k=0}^{\infty} \frac{d^k A_4^k}{k!} (H_{m,n}(x, y, z, \omega, k) p^m s^n) \\ &= e^{aA_3} \sum_{k=0}^{\infty} \frac{d^k}{k!} H_{m+k,n}(x, y, z, \omega, k) p^{m+k} s^n \end{aligned}$$

$$\begin{aligned} & e^{dA_4} e^{aA_3} (H_{m,n}(x, y, z, w, k) p^m s^n) \\ &= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{a^l}{l!} \frac{d^k}{k!} H_{m+k-l,n}(x, y, z, \omega, k) p^{m+k-l} s^n. \end{aligned} \quad (2.24)$$

Combining eqs. (2.23) and (2.24), we get another generating relation in the form:

$$\begin{aligned} & \exp(p_0 x_0 d + z_0 p_0^2 d^2) \\ & \quad \cdot H_{m,n} \left(x_0 + 2p_0 z_0 d + \frac{a}{p_0}, y_0 + p_0 k_0 d, z_0, w_0, k_0 \right) \\ &= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{a^l}{l!} \frac{d^k}{k!} H_{m+k-l,n}(x, y, z, \omega, k) p^{k-l}. \end{aligned} \quad (2.25)$$

By putting $a = 0$, $s = p = 1$ in eq. (2.25), we get the particular case:

$$\begin{aligned} & \exp(x_0 d + z_0 d^2) \cdot H_{m,n}(x_0 + 2z_0 d, y_0 + k_0 d, z_0, w_0, k_0) \\ &= \sum_{k=0}^{\infty} \frac{d^k}{k!} H_{m+k,n}(x, y, z, \omega, k). \end{aligned}$$

In general, we have,

$$\begin{aligned} & \exp(xd + zd^2) \cdot H_{m,n}(x + 2zd, y + kd, z, w, k) \\ &= \sum_{k=0}^{\infty} \frac{d^k}{k!} H_{m+k,n}(x, y, z, \omega, k). \end{aligned} \tag{2.26}$$

Also, by putting $d = 0$, $s = p = 1$ in (2.25), we get:

$$H_{m,n}(x_0 + a, y_0, z_0, w_0, k_0) = \sum_{l=0}^{\infty} \frac{a^l}{l!} H_{m-l,n}(x, y, z, \omega, k)$$

In general, we have,

$$\begin{aligned} & H_{m,n}(x + a, y, z, w, k) \\ &= \sum_{l=0}^{\infty} \frac{a^l}{l!} H_{m-l,n}(x, y, z, \omega, k). \end{aligned} \tag{2.27}$$

2.3 Some Applications

Numerous applications in mathematical physics, quantum mechanics, and related fields can be made of the relations discovered for the two-index Hermite polynomials $H_{m,n}(x, y, z, \omega, k)$ using Weisner's group theoretic approach. Some applications that might be used include:

1. Quantum mechanics: Two-index Hermite polynomials $H_{m,n}(x, y, z, \omega, k)$ are used in the study of quantum systems, such as multi-particle systems, harmonic oscillators with two degrees of freedom, and quantum field theory. The relations obtained through Weisner's group theoretic method can provide insights into the properties and behavior of these quantum systems, such as energy spectra, wave functions, and quantum states.
2. Mathematical physics: The obtained relations can be used to understand physical systems mathematical properties and symmetries involving two-index Hermite polynomials. For example, they can be used to derive conservation laws, study the behavior of physical observables, and analyze the symmetry-breaking patterns in physical systems.

3. Applied mathematics: The obtained relations can be utilized in applied mathematics for solving problems involving two-index Hermite $H_{m,n}(x, y, z, \omega, k)$ polynomials comprise eigenvalue issues, boundary value problems, and partial differential equations. They can also be used in the mathematical modeling of physical phenomena that involve two-variable Hermite polynomials, such as in signal processing, image analysis, and statistical mechanics.
4. Quantum information theory: Two-index Hermite polynomials $H_{m,n}(x, y, z, \omega, k)$ are used in quantum information theory for studying quantum entanglement, quantum computation, and quantum communication. The relations obtained through Weisner's group theoretic method can provide insights into the properties and manipulation of quantum information encoded in systems involving two-variable Hermite polynomials.
5. Symmetry analysis: The obtained relations can be used in the symmetry analysis of mathematical and physical systems to understand the underlying symmetries and to classify different solutions based on their symmetry properties. This can have applications in crystallography, materials science, and mathematical modeling of physical phenomena.

Overall, the relations obtained through Weisner's group theoretic method can be used in a wide variety of mathematical physics applications, quantum mechanics, applied mathematics, and related fields, providing insights into the properties, behavior, and symmetries of systems involving two-index Hermite polynomials.

Chapter 3: Generating Relations of Two Index and Two Variables Hermite Polynomials

3.1 Introduction

This chapter focuses on generating relations for Hermite polynomials with two indexes and two variables $He_{m,n}$, utilizing Weisner's method. Some various properties will be discussed, including the recurrence relations, first-order differential operators, and generating functions.

Let $\hat{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ be the associated matrix where a, b, and c are real numbers, and let $\underline{z} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\underline{h} = \begin{pmatrix} t \\ u \end{pmatrix}$ so transpose of $\underline{h}$ is $\underline{h}^t = (t \quad u)$

be the vectors in the space $\mathbb{R}^2$. It follows:

$$\begin{aligned} (1) \quad \underline{h}^t \hat{M} \underline{z} &= (t \quad u) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (t \quad u) \begin{pmatrix} ax + by \\ bx + cy \end{pmatrix} \\ &= atx + bty + bxu + cuy. \end{aligned} \tag{3.1}$$

$$\begin{aligned} (2) \quad \frac{1}{2} \underline{h}^t \hat{M} \underline{h} &= \frac{1}{2} (t \quad u) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} t \\ u \end{pmatrix} = \frac{1}{2} (t \quad u) \begin{pmatrix} at + bu \\ bt + cu \end{pmatrix} \\ &= \frac{1}{2} [at^2 + 2btu + cu^2]. \end{aligned} \tag{3.2}$$

Hermite polynomials with two indexes and two variables $He_{m,n}(x,y)$ presented by the generating function listed below (Cesarano, 2014):

$$e^{\frac{h^t}{2}\hat{M}_z - \frac{1}{2}h^t\hat{M}_h} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} He_{m,n}(x,y) \quad (3.3)$$

These polynomials are used in a variety of pure and applied mathematics domains (Dattoli et al 2003,2004); they are instrumental in describing the quantum treatment (Kim and Noz 1991) of coupled harmonic oscillation.

It is worth noting that the Hermite polynomials have double indexes and double variables $He_{m,n}(x,y)$ satisfies the following recurrence relation.

$$\begin{aligned} (ax + by - at - bu) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} He_{m,n}(x,y) \\ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m \frac{t^{m-1}}{m!} \frac{u^n}{n!} He_{m,n}(x,y). \end{aligned} \quad (3.4)$$

Moreover, if we substitute eqs. (3.1) and (3.2) in eq. (3.3), we obtain the following useful generating relation:

$$e^{atx+bty+bxu+cuy-\frac{1}{2}[at^2+2btu+cu^2]} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} He_{m,n}(x,y) \quad (3.5)$$

Eq. (3.5) is the main generating relation which we use in this Chapter.

3.2 Finding Generating Relations Using Weisner's Method

3.2.1 Recurrence Relation

We derive the pure and differential recurrence relations in this subsection. We differentiate Hermite generating function with two index and two variables defined in eq. (3.5) partially with respect to each of the variables x, y, u and t as seen below:

Case (1): Taking the partly derivative with respect to x in eq. (3.5) gives us the following.

$$(at + bu)e^{atx+bty+cuy-\frac{1}{2}[at^2+2btu+cu^2]+bxu} \\ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} \frac{\partial}{\partial x} He_{m,n}(x, y).$$

Next, using eq. (3.5) in the above equation, we get:

$$a \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^{m+1}}{m!} \frac{u^n}{n!} He_{m,n}(x, y) + b \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^{n+1}}{n!} He_{m,n}(x, y) \\ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} \frac{\partial}{\partial x} He_{m,n}(x, y).$$

Replace m by $m - 1$ in the left-hand side's first term and divide by m and replace n by $n - 1$ in the left-hand side's second term and multiply and divide by n in the above eq. , we get:

$$\begin{aligned}
& a \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{mt^m}{m!} \frac{u^n}{n!} H_{e_{m-1,n}}(x, y) + b \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{nu^n}{n!} H_{e_{m,n-1}}(x, y) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} \frac{\partial}{\partial x} H_{e_{m,n}}(x, y).
\end{aligned}$$

Comparing the coefficients of $t^m u^n$, we get

$$\frac{\partial}{\partial x} H_{e_{mm}}(x, y) = amH_{e_{m-1,n}(n,n)} + bnH_{e_{m,n-1}}(x, y) \quad (3.6)$$

Case(2): Differentiate (3.5) partially with respect to y , we get

$$\begin{aligned}
& (bt + cu)e^{atx+bty+cuy-\frac{1}{2}[at^2+2btu+cu^2]+bxu} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} \frac{\partial}{\partial y} H_{e_{m,n}}(x, y).
\end{aligned}$$

Using (3.5), we get

$$\begin{aligned}
& b \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^{m+1}}{m!} \frac{u^n}{n!} H_{e_{m,n}}(x, y) + c \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^{n+1}}{n!} H_{e_{m,n}}(x, y) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} \frac{\partial}{\partial y} H_{e_{m,n}}(x, y).
\end{aligned}$$

Replace m by $m - 1$ in the left-hand side's first term and divide by m and replace n by $n - 1$ in the left-hand side's second term and multiply and divide by n in the above eq., we get:

$$\begin{aligned}
& b \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{mt^m}{m!} \frac{u^n}{n!} H_{e_{m-1,n}}(x, y) + c \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{nu^n}{n!} H_{e_{m,n-1}}(x, y) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} \frac{\partial}{\partial y} H_{e_{m,n}}(x, y).
\end{aligned}$$

Comparing coefficients of $t^m u^n$, we get

$$\frac{\partial}{\partial y} H_{e_{m,n}}(x, y) = bm H_{e_{m-1,n}}(x, y) + cn H_{e_{m,n-1}}(x, y) \quad (3.7)$$

Case(3): Differentiate (3.5) partially with respect to u , we get

$$\begin{aligned}
& ((bx + cy) - (bt + cu)) e^{atx+bty+cuy-\frac{1}{2}[at^2+2btu+cu^2]+bxu} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{nu^{n-1}}{n!} H_{e_{m,n}}(x, y). \\
& (bx + cy) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} H_{e_{m,n}}(x, y) - b \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^{m+1}}{m!} \frac{u^n}{n!} H_{e_{m,n}}(x, y) \\
& - c \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^{n+1}}{n!} H_{e_{m,n}}(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{nu^{n-1}}{n!} H_{e_{m,n}}(x, y)
\end{aligned}$$

Replace m with $m-1$ in the second term on the left-hand side of the following equation and multiply and divide by m , we get by replacing n by $n-1$ in the third term of the left-hand side and n by $n-1$ on the right-hand side.

$$\begin{aligned} & (bx + cy) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m u^n}{m! n!} H_{e_{m,n}}(x, y) - \\ & b \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{mt^m u^n}{m! n!} H_{e_{m-1,n}}(x, y) \\ & - c \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{t^m nu^n}{m! n!} H_{e_{m,n-1}}(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m u^n}{m! n!} H_{e_{m,n+1}}(x, y). \end{aligned}$$

Now, contrasting the coefficients of $t^m u^n$, we get

$$\begin{aligned} H_{e_{m,n+1}}(x, y) &= (bx + cy)H_{e_{m,n}}(x, y) - bmH_{e_{m-1,n}}(x, y) - \\ & cnH_{e_{m,n-1}}(x, y) \end{aligned} \quad (3.8)$$

Case(4): Differentiate eq. (3.5) partially with respect to t , we get

$$\begin{aligned} & ((ax + by) - (at + bu))e^{atx+bty+cuy-\frac{1}{2}[at^2+2btu+cu^2]+bxu} \\ & = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{mt^{m-1} u^n}{m! n!} H_{e_{m,n}}(x, y). \\ & (ax + by) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m u^n}{m! n!} H_{e_{m,n}}(x, y) - a \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^{m+1} u^n}{m! n!} H_{e_{m,n}}(x, y) \end{aligned}$$

$$-b \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^{n+1}}{n!} H_{e_{m,n}}(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{mt^{m-1}}{m!} \frac{u^n}{n!} H_{e_{m,n}}(x, y)$$

Now, replace m with m-1 in the second term on the left-hand side of the following equation and multiply and divide by m, we infer by replacing n by n-1 in the third term of the left-hand side and n by n-1 on the right-hand side that

$$\begin{aligned} (bx + cy) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} H_{e_{m,n}}(x, y) - a \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{mt^m}{m!} \frac{u^n}{n!} H_{e_{m-1,n}}(x, y) \\ - b \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{t^m}{m!} \frac{nu^n}{n!} H_{e_{m,n-1}}(x, y) \\ = \sum_{m=-1}^{\infty} \sum_{n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} H_{e_{m+1,n}}(x, y). \end{aligned}$$

Now, Comparing the coefficients of $t^m u^n$, we get

$$\begin{aligned} He_{m+1,n}(x,y) \\ = (ax + by)He_{m,n}(x,y) - amHe_{m-1,n}(x,y) \\ - bnHe_{m,n-1}(x,y). \end{aligned} \tag{3.9}$$

If we put eq. (3.9) in eq. (3.6), we get:

$$\frac{\partial}{\partial x} H_{e_{m,n}}(x, y) = (ax + by)He_{m,n}(x, y) - He_{m+1,n}(x, y)$$

$$\Rightarrow \left[(ax + by) - \frac{\partial}{\partial x} \right] He_{m,n}(x, y) = He_{m+1,n}(x, y). \quad (3.10)$$

Similarly, if we put eq. (3.8) in eq. (3.7) and simplify, we get:

$$\begin{aligned} \frac{\partial}{\partial y} He_{m,n}(x, y) &= He_{m,n+1}(x, y) - (bx + cy) He_{m,n}(x, y) \\ \Rightarrow \left[(bx + cy) - \frac{\partial}{\partial y} \right] He_{m,n}(x, y) &= He_{m,n+1}(x, y). \end{aligned} \quad (3.11)$$

Next, multiply eq. (3.6) by b and eq. (3.7) by a , and we get:

$$\begin{aligned} b \frac{\partial}{\partial x} He_{m,n}(x, y) \\ = abm He_{m-1,n}(x, y) + b^2 n He_{m,n-1}(x, y) \end{aligned} \quad (3.12)$$

$$a \frac{\partial}{\partial y} He_{m,n}(x, y) = abm He_{m-1,n}(x, y) + acn He_{m,n-1}(x, y). \quad (3.13)$$

Subtract eq. (3.13) from eq. (3.12); we find that

$$\begin{aligned} \left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right] He_{m,n}(x, y) &= (b^2 - ac) n He_{m,n-1}(x, y) \\ &= -(ac - b^2) n He_{m,n-1}(x, y) \end{aligned}$$

$$\left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right] He_{m,n}(x, y) = -\Delta n He_{m,n-1}(x, y)$$

$$-\frac{1}{n\Delta} \left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right] He_{m,n}(x, y) = He_{m,n-1}(x, y). \quad (3.14)$$

Similarly, if we multiply eq. (3.6) by c and eq. (3.7) by b , we get:

$$c \frac{\partial}{\partial x} He_{m,n}(x, y) = acmHe_{m-1,n}(x, y) + bcnHe_{m,n-1}(x, y). \quad (3.15)$$

$$b \frac{\partial}{\partial y} He_{m,n}(x, y) = b^2 mHe_{m-1,n}(x, y) + bcnHe_{m,n-1}(x, y). \quad (3.16)$$

Subtract eq. (3.15) from eq. (3.16), yields

$$\begin{aligned} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] He_{m,n}(x, y) &= (ac - b^2)mHe_{m-1,n}(x, y) \\ \Rightarrow \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] He_{m,n}(x, y) &= \Delta mHe_{m-1,n}(x, y) \\ \Rightarrow \frac{1}{m\Delta} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] He_{m,n}(x, y) &= He_{m-1,n}(x, y). \end{aligned} \quad (3.17)$$

where $\Delta = ac - b^2$ and the associative matrix is $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$,

where a, b , and c are real numbers such that: $a, b, c > 0$,

$$\Delta = ac > b^2, t \neq u, |t|, |u| < \infty.$$

3.2.2 First Order Differential Operator

Using the recurrence relations obtained in the previous section, we can derive the following operators.

Let $f(x, y, p, s) = He_{m,n}(x, y)p^m s^n$.

1) For $A_1 = p \frac{\partial}{\partial p}$, we get:

$$A_1[He_{m,n}(x, y)p^m s^n] = mHe_{m,n}(x, y)p^m s^n.$$

2) For $A_2 = s \frac{\partial}{\partial s}$, we get:

$$A_2[He_{m,n}(x, y)p^m s^n] = nHe_{m,n}(x, y)p^m s^n.$$

3) For $A_3 = \frac{1}{p\Delta} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right]$, we get:

$$A_3[He_{m,n}(x, y)p^m s^n] = mHe_{m-1,n}(x, y)p^{m-1} s^n.$$

4) For $A_4 = p \left[(ax + by) - \frac{\partial}{\partial x} \right]$, we get:

$$A_4[He_{m,n}(x, y)p^m s^n] = He_{m+1,n}(x, y)p^{m+1} s^n.$$

5) For $A_5 = -\frac{1}{s\Delta} \left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right]$, we get:

$$A_5[He_{m,n}(x, y)p^m s^n] = nHe_{m,n-1}(x, y)p^m s^{n-1}.$$

6) For $A_6 = s \left[(bx + cy) - \frac{\partial}{\partial y} \right]$, we get:

$$A_6[He_{m,n}(x, y)p^m s^n] = He_{m,n+1}(x, y)p^m s^{n+1}.$$

3.2.3 Commutative Relations of the Operators

We employ the commutator notation $[A, B]$ with $[A, B]f =$

$(AB - BA)f$ where $f(x, y, s, p) = He_{m,n}(x, y)p^m s^n$ in order to find all possible commutation relations between the first order differential operators

$A_i, \{i = 1, 2, \dots, 6\}$ mentioned in the previous subsection.

Case (1): $[A_1, A_2]$

$$\begin{aligned} [A_1, A_2]f &= A_1 A_2 f - A_2 A_1 f \\ &= \left[\left(p \frac{\partial}{\partial p} \right) \left(s \frac{\partial}{\partial s} \right) \right] f - \left[\left(s \frac{\partial}{\partial s} \right) \left(r \frac{\partial}{\partial p} \right) \right] f = 0 \\ \Rightarrow [A_1, A_2]f &= 0 \\ \Rightarrow [A_1, A_2] &= 0. \end{aligned}$$

Case (2): $[A_1, A_3]$

$$\begin{aligned} [A_1, A_3]f &= A_1 A_3 f - A_3 A_1 f \\ &= \left[\left(p \frac{\partial}{\partial p} \right) \left(\frac{1}{p\Delta} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] \right) \right] f - \left[\left(\frac{1}{p\Delta} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] \right) \left(p \frac{\partial}{\partial p} \right) \right] f \\ &= \left(\frac{1}{\Delta} \left[c \frac{\partial^2 f}{\partial p \partial x} - b \frac{\partial^2 f}{\partial p \partial y} \right] \right) - \left(\frac{1}{\Delta} \left[c \frac{\partial^2 f}{\partial p \partial x} - b \frac{\partial^2 f}{\partial p \partial y} \right] \right) = 0 \\ \Rightarrow [A_1, A_3]f &= 0 \\ \Rightarrow [A_1, A_3] &= 0. \end{aligned}$$

Case (3): $[A_1, A_4]$

$$[A_1, A_4] = A_1 A_4 f - A_4 A_1 f$$

$$\left[\left(p \frac{\partial}{\partial p} \right) p \left((ax + by) - \frac{\partial}{\partial x} \right) \right] f - \left[p \left((ax + by) - \frac{\partial}{\partial x} \right) \left(p \frac{\partial}{\partial p} \right) \right] f$$

$$= (ax + by) p^2 \frac{\partial f}{\partial p} - p^2 \frac{\partial^2 f}{\partial p \partial x} - (ax + by) p^2 \frac{\partial f}{\partial p} + p^2 \frac{\partial^2 f}{\partial p \partial x} = 0$$

$$\Rightarrow [A_1, A_4] f = 0$$

$$\Rightarrow [A_1, A_4] = 0.$$

Case (4): $[A_1, A_5]$

$$[A_1, A_5] f = A_1 A_5 f - A_5 A_1 f$$

$$= \left[\left(p \frac{\partial}{\partial p} \right) \left(-\frac{1}{s\Delta} \left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right] \right) \right] f - \left[\left(-\frac{1}{s\Delta} \left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right] \right) \left(p \frac{\partial}{\partial p} \right) \right] f$$

$$= -\frac{p}{s\Delta} \left[b \frac{\partial^2 f}{\partial p \partial x} - a \frac{\partial^2 f}{\partial p \partial y} \right] + \frac{p}{s\Delta} \left[b \frac{\partial^2 f}{\partial p \partial x} - a \frac{\partial^2 f}{\partial p \partial y} \right] = 0$$

$$\Rightarrow [A_1, A_5] f = 0$$

$$\Rightarrow [A_1, A_5] = 0.$$

Case (5): $[A_1, A_6]$

$$\begin{aligned}
[A_1, A_6]f &= A_1 A_6 f - A_6 A_1 f \\
&= \left[\left(p \frac{\partial}{\partial p} \right) s \left((bx + cy) - \frac{\partial}{\partial y} \right) \right] f - \left[s \left((bx + by) - \frac{\partial}{\partial y} \right) \left(p \frac{\partial}{\partial p} \right) \right] f \\
&= (bx + cy) sp \frac{\partial f}{\partial p} - sp \frac{\partial^2 f}{\partial p \partial y} - (bx + cy) sp \frac{\partial f}{\partial p} + sp \frac{\partial^2 f}{\partial p \partial y} = 0 \\
&\Rightarrow [A_1, A_6]f = 0 \\
&\Rightarrow [A_1, A_6] = 0.
\end{aligned}$$

Case (6): $[A_2, A_3]$

$$\begin{aligned}
[A_2, A_3]f &= A_2 A_3 f - A_3 A_2 f \\
&= \left[\left(s \frac{\partial}{\partial s} \right) \left(\frac{1}{p\Delta} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] \right) \right] p - \left[\left(\frac{1}{p\Delta} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] \right) \left(s \frac{\partial}{\partial s} \right) \right] f \\
&= \left(\frac{s}{p\Delta} \left[c \frac{\partial^2 f}{\partial s \partial x} - b \frac{\partial^2 f}{\partial s \partial y} \right] \right) - \left(\frac{s}{p\Delta} \left[c \frac{\partial^2 f}{\partial s \partial x} - b \frac{\partial^2 f}{\partial s \partial y} \right] \right) = 0 \\
&\Rightarrow [A_2, A_3]f = 0 \\
&\Rightarrow [A_2, A_3] = 0.
\end{aligned}$$

Case (7): $[A_2, A_4]$

$$\begin{aligned}
[A_2, A_4]f &= A_2 A_4 f - A_4 A_2 f \\
&= \left[\left(s \frac{\partial}{\partial s} \right) p \left((ax + by) - \frac{\partial}{\partial x} \right) \right] f - \left[p \left((ax + by) - \frac{\partial}{\partial x} \right) \left(s \frac{\partial}{\partial s} \right) \right] f
\end{aligned}$$

$$= (ax + by)ps \frac{\partial f}{\partial s} - ps \frac{\partial^2 f}{\partial s \partial x} - (ax + by)ps \frac{\partial f}{\partial s} + ps \frac{\partial^2 f}{\partial s \partial x} = 0$$

$$\Rightarrow [A_2, A_4]f = 0$$

$$\Rightarrow [A_2, A_4] = 0.$$

Case (8): $[A_2, A_5]$

$$[A_2, A_5]f = A_2 A_5 f - A_5 A_2 f$$

$$= \left[\left(s \frac{\partial}{\partial s} \right) \left(-\frac{1}{s\Delta} \left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right] \right) \right] f - \left[\left(-\frac{1}{s\Delta} \left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right] \right) \left(s \frac{\partial}{\partial s} \right) \right] f$$

$$= -\frac{1}{\Delta} \left[b \frac{\partial^2 f}{\partial s \partial x} - a \frac{\partial^2 f}{\partial s \partial y} \right] + \frac{1}{\Delta} \left[b \frac{\partial^2 f}{\partial s \partial x} - a \frac{\partial^2 f}{\partial s \partial y} \right] = 0$$

$$\Rightarrow [A_2, A_5]f = 0$$

$$\Rightarrow [A_2, A_5] = 0.$$

Case (9): $[A_2, A_6]$

$$[A_2, A_6]f = A_2 A_6 f - A_6 A_2 f$$

$$= \left[\left(s \frac{\partial}{\partial s} \right) s \left((bx + cy) - \frac{\partial}{\partial y} \right) \right] f - \left[s \left((bx + by) - \frac{\partial}{\partial y} \right) \left(s \frac{\partial}{\partial s} \right) \right] f$$

$$= (bx + cy)s^2 \frac{\partial f}{\partial s} - s^2 \frac{\partial^2 f}{\partial s \partial y} - (bx + cy)s^2 \frac{\partial f}{\partial s} + s^2 \frac{\partial^2 f}{\partial s \partial y} = 0$$

$$\Rightarrow [A_2, A_6]f = 0$$

$$\Rightarrow [A_2, A_6] = 0.$$

Case (10): $[A_3, A_4]$

$$[A_3, A_4]f = A_3A_4f - A_4A_3f$$

$$\begin{aligned}
&= \left[\left(\frac{1}{p\Delta} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] \right) p \left((ax + by) - \frac{\partial}{\partial x} \right) \right] f \\
&\quad - \left[p \left((ax + by) - \frac{\partial}{\partial x} \right) \left(\frac{1}{p\Delta} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] \right) \right] f \\
&= \frac{1}{\Delta} \left[c \frac{\partial}{\partial x} (ax + by) - b \frac{\partial}{\partial y} (ax + by) \right] f - \frac{1}{\Delta} \left[c \frac{\partial^2}{\partial x^2} - b \frac{\partial^2}{\partial y \partial x} \right] f - \\
&\quad \frac{1}{\Delta} \left[c \frac{\partial}{\partial x} (ax + by) - b \frac{\partial}{\partial y} (ax + by) \right] f + \frac{1}{\Delta} \left[c \frac{\partial^2}{\partial x^2} - b \frac{\partial^2}{\partial y \partial x} \right] f = 0 \\
&\Rightarrow [A_3, A_4]f = 0 \\
&\Rightarrow [A_3, A_4] = 0.
\end{aligned}$$

Case (11): $[A_3, A_5]$

$$[A_3, A_5]f = A_3A_5f - A_5A_3f$$

$$\begin{aligned}
&= \left[\left(\frac{1}{p\Delta} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] \right) \left(-\frac{1}{s\Delta} \left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right] \right) \right] f \\
&\quad - \left[\left(-\frac{1}{s\Delta} \left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right] \right) \left(\frac{1}{p\Delta} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] \right) \right] f = 0 \\
&\Rightarrow [A_3, A_5]f = 0 \\
&\Rightarrow [A_3, A_5] = 0.
\end{aligned}$$

Case (12): $[A_3, A_5]$

$$[A_3, A_6]f = A_3A_6f - A_6A_3f$$

$$\begin{aligned} & \left[\left(\frac{1}{p\Delta} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] \right) \left(s \left((bx + cy) - \frac{\partial}{\partial y} \right) \right) \right] f \\ & - \left[\left(s \left((bx + cy) - \frac{\partial}{\partial y} \right) \right) \left(\frac{1}{p\Delta} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] \right) \right] f = 0 \end{aligned}$$

$$\Rightarrow [A_3, A_6]f = 0$$

$$\Rightarrow [A_3, A_6] = 0.$$

Case (13): $[A_4, A_5]$

$$[A_4, A_5]f = A_4A_5f - A_5A_4f$$

$$\begin{aligned} & = \left[p \left((ax + by) - \frac{\partial}{\partial x} \right) \left(-\frac{1}{s\Delta} \left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right] \right) \right] f \\ & - \left[\left(-\frac{1}{s\Delta} \left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right] \right) p \left((ax + by) - \frac{\partial}{\partial x} \right) \right] f = 0 \end{aligned}$$

$$\Rightarrow [A_4, A_5] = 0.$$

Case (14): $[A_4, A_6]$

$$[A_4, A_6]f = A_4A_6f - A_6A_4f$$

$$\begin{aligned} & = p \left((ax + by) - \frac{\partial}{\partial x} \right) s \left((bx + cy) - \frac{\partial}{\partial y} \right) f \\ & - \left[s \left((bx + cy) - \frac{\partial}{\partial y} \right) p \left((ax + by) - \frac{\partial}{\partial x} \right) \right] f \end{aligned}$$

$$\Rightarrow [A_4, A_6]f = 0$$

$$\Rightarrow [A_4, A_6] = 0.$$

Case (15): $[A_5, A_6]$

$$\begin{aligned} [A_5, A_6]f &= A_5 A_6 f - A_6 A_5 f \\ &= \left[\left(-\frac{1}{s\Delta} \left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right] \right) \left(s(bx + cy) - \frac{\partial}{\partial y} \right) \right] f - \left[s \left((bx + cy) - \frac{\partial}{\partial y} \right) \left(-\frac{1}{s\Delta} \left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right] \right) \right] f \end{aligned}$$

$$\Rightarrow [A_5, A_6]f = 0$$

$$\Rightarrow [A_5, A_6] = 0.$$

Thus from the previous commutator relations, we find that

$$[A_1, A_2] = 0, [A_2, A_3] = 0, [A_3, A_4] = 0, [A_4, A_5] = 0, [A_5, A_6] = 0,$$

$$[A_1, A_3] = 0, [A_2, A_4] = 0, [A_3, A_5] = 0, [A_4, A_6] = 0,,$$

$$[A_1, A_4] = 0, [A_2, A_5] = 0, [A_3, A_6] = 0,,$$

$$[A_1, A_5] = 0, [A_2, A_6] = 0,$$

$$[A_1, A_6] = 0.$$

According to Weisner's group theoretic method, we cannot connect the set of differential operators because the above relations are not isomorphic to any commutation relations obeyed by a well-known Lie algebra of a local Lie group. $\{A_i: i = 1, 2, 3, 4, 5, 6\}$ with a certain Lie algebra, we can only conclude that these differential operators generate three-dimensional Lie algebra.

Therefore, The commutation relations shown above demonstrate that the set of operators $\{A_i: i = 1, 2, 3, 4, 5, 6\}$ make some Lie algebra λ .

And from these operators, we can introduce the partial differential equations satisfied by two index and two variables $He_{m,n}(x, y)$

$$L_1 = A_3 A_4 - m - 1,$$

and this can be verified very easily

$$\begin{aligned} A_3 A_4 (He_{m,n}(x, y) p^m s^n) &= A_3 (He_{m+1,n}(x, y) p^{m+1} s^n) \\ &= (m + 1) He_{m,n}(x, y) p^m s^n. \end{aligned}$$

But

$$\begin{aligned} A_3 A_4 (He_{m,n}(x, y) p^m s^n) \\ = \left[\left(\frac{1}{p\Delta} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] \right) \left(p \left[(ax + by) - \frac{\partial}{\partial x} \right] \right) \right] (He_{m,n}(x, y) p^m s^n) \end{aligned}$$

$$= \frac{1}{\Delta} \left[c(ax + by) \frac{\partial}{\partial x} - b(ax + by) \frac{\partial}{\partial y} - c \frac{\partial^2}{\partial x^2} + b \frac{\partial^2}{\partial x \partial y} \right. \\ \left. + \Delta \right] (He_{m,n}(x, y) p^m s^n)$$

$$\Rightarrow L_1 = A_3 A_4 - m - 1,$$

where m is replaced by $p \frac{\partial}{\partial p}$

$$\therefore L_1 = \frac{1}{\Delta} \left[c(ax + by) \frac{\partial}{\partial x} - b(ax + by) \frac{\partial}{\partial y} - c \frac{\partial^2}{\partial x^2} + b \frac{\partial^2}{\partial x \partial y} + \Delta \right] \\ - p \frac{\partial}{\partial p} - 1$$

$$L_2 = A_5 A_6 - n - 1$$

$$A_5 A_6 (He_{n,m}(x, y) p^m s^n) = A_5 (He_{m,n+1}(x, y) p^m s^{n+1}) \\ = (n + 1) He_{m,n}(x, y) p^m s^n.$$

But

$$A_5 A_6 (He_{m,n}(x, y) p^m s^n) \\ = \left[\left(\frac{1}{s\Delta} \left[-b \frac{\partial}{\partial x} + a \frac{\partial}{\partial y} \right] \right) \left(s \left[(bx + cy) - \frac{\partial}{\partial y} \right] \right) \right] (He_{m,n}(x, y) p^m s^n) \\ = \frac{1}{\Delta} \left[-b(bx + cy) \frac{\partial}{\partial x} + a(bx + cy) \frac{\partial}{\partial y} - a \frac{\partial^2}{\partial y^2} + b \frac{\partial^2}{\partial x \partial y} \right. \\ \left. + \Delta \right] (He_{m,n}(x, y) p^m s^n) \\ = \frac{1}{\Delta} \left[-b(bx + cy) \frac{\partial}{\partial x} + a(bx + cy) \frac{\partial}{\partial y} - a \frac{\partial^2}{\partial y^2} + b \frac{\partial^2}{\partial x \partial y} \right. \\ \left. + \Delta \right] (He_{m,n}(x, y) p^m s^n),$$

$$\Rightarrow L_2 = A_5 A_6 - n - 1, \text{ where } n \text{ is replaced by } s \frac{\partial}{\partial s},$$

$$\therefore L_2 = \frac{1}{\Delta} \left[-b(bx + cy) \frac{\partial}{\partial x} + a(bx + cy) \frac{\partial}{\partial y} - a \frac{\partial^2}{\partial y^2} + b \frac{\partial^2}{\partial x \partial y} + \Delta \right] \\ - s \frac{\partial}{\partial s} - 1$$

3.2.4 Extended Form of the Group Generated by Operators

The extended form of the group created by $\{A_i: i = 1,2,3,4,5,6\}$ is found in this subsection.

Case (1): For $A_1 = p \frac{\partial}{\partial p}$

let a_1 be any variable, we get:

$$\frac{dp}{da_1} = p$$

$$\int \frac{dp}{p} = \int da_1$$

$$\ln p = a_1 + \ln A.$$

Applying the initial condition when $a_1 = 0, p(0) = p_0$

$$\ln p_0 = 0 + \ln A$$

$$A = p_0$$

$$\ln p = a_1 + \ln p_0$$

$$e^{\ln p} = e^{a_1} \cdot e^{\ln p_0} = p_0 e^{a_1}$$

$$\Rightarrow p = p_0 e^{a_1}.$$

$$\frac{dx}{da_1} = 0, \quad \frac{dy}{da_1} = 0, \quad \frac{ds}{da_1} = 0$$

$$\Rightarrow x = x_0, \quad y = y_0, \quad s = s_0$$

$$\frac{dv}{da_1} = 0 \Rightarrow dv = 0 \Rightarrow v = k.$$

Applying the initial condition when $a_1 = 0$, $v(0) = 1 \Rightarrow k = 1$, $v = 1$

$$\Rightarrow e^{a_1 A_1} f(x, y, p, s) = f(x_0, y_0, p_0 e^{a_1}, s_0) \quad (3.18)$$

Similarly, proceeding in the same manner as in case (1), we can derive the following cases:

Case (2): For $A_2 = s \frac{\partial}{\partial s}$

let a_2 be any variable, we get:

$$\Rightarrow e^{a_2 A_2} f(x, y, p, s) = f(x_0 y_0, p_0, s_0 e^{a_2}). \quad (3.19)$$

Case (3): For $A_3 = \frac{1}{p\Delta} \left[c \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right] = \frac{c}{p\Delta} \frac{\partial}{\partial x} - \frac{b}{p\Delta} \frac{\partial}{\partial y}$.

Let a_3 be any variable, then

$$\frac{dp}{da_3} = 0, \quad \frac{ds}{da_3} = 0,$$

$$p = p_0, s = s_0$$

$$\frac{dx}{da_3} = \frac{c}{p_0\Delta}$$

$$dx = \frac{c}{p_0\Delta} da_3$$

$$x = \frac{c}{p_0\Delta} a_3 + k.$$

$$\text{When } a_3 = 0 \Rightarrow x(0) = x_0 \Rightarrow x_0 = k$$

$$\Rightarrow x = \frac{c}{p_0\Delta} a_3 + x_0$$

$$\text{Similarly, for } \frac{dy}{da_3} = \frac{-b}{p_0\Delta}$$

$$\Rightarrow y = y_0 - \frac{b}{p_0\Delta} a_3$$

$$\Rightarrow e^{a_3 A_3} f(x, y, p, s)$$

$$= f\left(x_0 + \frac{c}{p_0\Delta} a_3, y_0 - \frac{b}{p_0\Delta} a_3, p_0, s_0\right). \quad (3.20)$$

$$\textbf{Case (4):} \text{ For } A_5 = -\frac{1}{s\Delta} \left[b \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} \right] = -\frac{b}{s\Delta} \frac{\partial}{\partial x} + \frac{a}{s\Delta} \frac{\partial}{\partial y}$$

let a_5 be any variable, we get:

$$e^{a_5 A_5} f(x, y, p, s)$$

$$= f\left(x_0 - \frac{b}{s_0\Delta} a_5, y_0 + \frac{a}{s_0\Delta} a_5, p_0, s_0\right). \quad (3.21)$$

Case (5): For $A_4 = p \left[(ax + by) - \frac{\partial}{\partial x} \right]$

let a_4 be any variable, we get:

$$\frac{dp}{da_4} = 0 \quad \frac{ds}{da_4} = 0$$

$$p = p_0, \quad s = s_0$$

$$\frac{dx}{da_4} = -1$$

$$dx = -p_0 da_4$$

$$x = -p_0 a_4 + k$$

$$\text{When } a_4 = 0 \Rightarrow x(0) = x_0$$

$$x = -p_0 a_4 + x_0$$

$$\Rightarrow x = x_0 - p_0 a_4$$

$$\frac{dy}{da_4} = 0 \Rightarrow y = y_0$$

$$\frac{dv}{da_4} = p_0(ax + by)v$$

$$\frac{dv}{v} = p_0(ax + by)da_4$$

$$\ln v = p_0(ax_0 + by_0)a_4 + k$$

when $a_4 = 0, v(0) = 1, \ln 1 = 0, k = 0$

$$\therefore v = \exp(ap_0x_0a_4 + bp_0y_0a_4)$$

$$e^{a_4A_4}f(x, y, p, s)$$

$$= \exp(ap_0x_0a_4 + by_0a_4) \cdot f(x_0 - p_0a_4, y_0, p_0, s_0) \quad (3.22)$$

Case (6): For $A_6 = s \left[(bx + cy) - \frac{\partial}{\partial y} \right]$

let a_4 be any variable we get:

$$e^{a_6A_6}f(x, y, p, s)$$

$$= \exp(bs_0x_0a_6 + cs_0y_0a_6) \cdot f(x_0, y_0 - s_0a_6, p_0, s_0) \quad (3.23)$$

3.2.5 Generating Functions

From the eqs. (3.18) -(3.23) we find the action of these operators as follows:

$$\begin{aligned}
& e^{a_6 A_6 + \dots + a_2 A_2 + a_1 A_1} f(x, y, p, s) \\
&= e^{a_5 A_5 + \dots + a_2 A_2 + a_1 A_1} \exp(b s_0 x_0 a_6 + c s_0 y_0 a_6) \cdot f(x_0, y_0 - s_0 a_6, p_0, s_0) \\
&= e^{a_4 A_4 + a_3 A_3 + a_2 A_2 + a_1 A_1} \exp(b s_0 x_0 a_6 + c s_0 y_0 a_6) \cdot f\left(x_0 - \frac{b}{s_0 \Delta} a_5, y_0 - \right. \\
&\quad \left. s_0 a_6 + \frac{a}{s_0 \Delta} a_5, p_0, s_0\right) \\
&= e^{a_3 A_3 + a_2 A_2 + a_1 A_1} \exp\left(a\left(x - \frac{b}{s_0 \Delta} a_5\right) a_4 + b(y_0 - s_0 a_6) a_4\right) \\
&\quad \cdot f\left(x_0 - \frac{b}{s_0 \Delta} a_5 - a_4, y_0 - s_0 a_6 + \frac{a}{s_0 \Delta} a_5, p_0, s_0\right) \\
&= e^{a_2 A_2 + a_1 A_1} \exp\left(a\left(x_0 - \frac{b}{s_0 \Delta} a_5\right) a_4 + b(y_0 - s_0 a_6) a_4\right) \\
&\quad \cdot f\left(x_0 - \frac{b}{s_0 \Delta} a_5 - a_4 + \frac{c}{p_0 \Delta} a_3, y_0 - s_0 a_6 + \frac{a}{s_0 \Delta} a_5 \right. \\
&\quad \left. - \frac{b}{p_0 \Delta} a_3, p_0, s_0\right) \\
&= \exp\left(a\left(x - \frac{b}{s_0 \Delta} a_5\right) a_4 + b(y_0 - s_0 a_6) a_4\right) \\
&\quad \cdot f\left(x_0 - \frac{b}{s_0 \Delta} a_5 - a_4 + \frac{c}{p_0 \Delta} a_3, y_0 - s_0 a_6 + \frac{a}{s_0 \Delta} a_5 \right. \\
&\quad \left. - \frac{b}{p_0 \Delta} a_3, p_0 e^{a_1}, s_0 e^{a_2}\right). \tag{3.24}
\end{aligned}$$

By putting $a_i = 0, i(1 = 1, 2, 3, 4), a_5 = g, a_6 = h,$

and writing $f(x, y, s, p) = He_{m,n}(x, y)p^m s^n$ in eq. (3.24), we get:

$$\begin{aligned}
& e^{hA_6} e^{gA_5} (He_{m,n}(x, y)p^m s^n) \\
&= e^{gA_5} (\exp(bsxh + csyh)) \cdot f(x, y - sh, p, s) \\
&= \exp(bsxh + csyh) \cdot f\left(x - \frac{b}{s\Delta}g, y - sh + \frac{a}{s\Delta}g, p, s\right) \\
&= \exp(bxh + cyh) \cdot He_{m,n}\left(x - \frac{b}{s\Delta}g, y - sh + \frac{a}{s\Delta}g\right)p^m s^n. \quad (3.25)
\end{aligned}$$

But

$$\begin{aligned}
e^{hA_6} e^{gA_5} (He_{m,n}(x, y)p^m s^n) &= e^{gA_5} \sum_{k=0}^{\infty} \frac{h^k A_6^k}{k!} (He_{m,n}(x, y)p^m s^n) \\
&= e^g \sum_{k=0}^{\infty} \frac{h^k}{k!} (He_{m,n+k}(x, y)p^m s^{n+k}) \\
&= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{g^l h^k}{l! k!} (He_{m,n+k-l}(x, y)p^m s^{n+k-l}). \quad (3.26)
\end{aligned}$$

By combining eq. (3.25) with eq. (3.26), we get the following required

relation:

$$\begin{aligned}
& \exp(bxh + cyh) \cdot He_{m,n} \left(x - \frac{b}{s\Delta} g, y - sh + \frac{a}{s\Delta} g \right) p^m s^n \\
&= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{g^l h^k}{l! k!} He_{m,n+k-l}(x, y) p^m s^{n+k-l}.
\end{aligned} \tag{3.27}$$

Now, some special cases of eq. (3.27) are of interest:

1) Putting $g = 0, p = s = 1$ in eq. (3.27), we find that

$$\begin{aligned}
& \exp(bxh + cyh) \cdot He_{m,n}(x, y - h) \\
&= \sum_{k=0}^{\infty} \frac{h^k}{k!} He_{m,n+k}(x, y).
\end{aligned} \tag{3.28}$$

2) Putting $h = 0, p = s = 1$ in eq. (3.27), we obtain

$$He_{m,n} \left(x - \frac{b}{s_0\Delta} g, y + \frac{a}{s_0\Delta} g \right) = \sum_{l=0}^{\infty} \frac{g^l}{l!} He_{m,n-l}(x, y). \tag{3.29}$$

Next, we derive another generating relation.

By putting $a_i = 0, i(1 = 1, 2, 5, 6), a_3 = r, a_4 = t$

and writing $f(x, y, s, p) = He_{m,n}(x, y) p^m s^n$ in (3.24), we get:

$$\begin{aligned}
& e^{tA_4} e^{rA_3} (He_{m,n}(x, y) p^m s^n) \\
&= \exp(axt + byt) \cdot f \left(x + \frac{c}{p\Delta} r - tp, y - \frac{b}{p\Delta} r, p, s \right)
\end{aligned}$$

$$= \exp(axy + byt) \cdot He_{m,n} \left(x + \frac{c}{p\Delta} r - tp, y - \frac{b}{p\Delta} r \right) p^m s^n$$

But

$$e^{tA_4} e^{rA_3} (He_{m,n}(x, y) p^m s^n) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{r^l t^k}{l! k!} He_{m+k-l,n}(x, y) p^{m+k-l} s^n.$$

Hence, we have

$$\begin{aligned} & \exp(axy + byt) \cdot He_{m,n} \left(x + \frac{c}{p\Delta} r - t, y - \frac{b}{p\Delta} r \right) p^m s^n \\ &= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{r^l t^k}{l! k!} He_{m+k-l,n}(x, y) p^{m+k-l} s^n. \end{aligned} \quad (3.30)$$

Now, some special cases of eq. (3.30) are of interest:

1) Putting $r = 0, p = s = 1$ in eq. (3.30), we get:

$$\begin{aligned} & \exp(axy + byt) \cdot He_{m,n}(x - t, y) \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} He_{m+k,n}(x, y). \end{aligned} \quad (3.31)$$

2) putting $t = 0, p = s = 1$ in eq. (3.30), we get:

$$He_{m,n} \left(x + \frac{c}{\Delta} r, y - \frac{b}{\Delta} r \right) = \sum_{l=0}^{\infty} \frac{r^l}{l!} He_{m-l,n}(x, y). \quad (3.32)$$

where $\Delta = ac - b^2$ and a , b , and c are positive real numbers.

It is important to note the following interesting special case.

3) letting $a = 2$, $b = -t$, $y = 1$, and $c = m = n = 0$, in eq. (3.31), we get the known result (Miller, 1968):

$$\exp(2xt - t^2) = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x) \quad (3.33)$$

3.3 Some Applications

The Hermite polynomials $He_{m+k,n}(x, y)$ have several applications, some of them listed below.

1. Quantum mechanics: In quantum mechanics, Hermite polynomials are used to describe the energy states of simple harmonic oscillators, which are fundamental to understanding the behavior of atoms and molecules. To calculate the wave functions of the oscillator in the presence of a magnetic field, $He_{m+k,n}(x, y)$ are required.
2. Statistical mechanics: $He_{m+k,n}(x, y)$ are also frequently employed in statistical mechanics to characterize particle distribution in a system. $He_{m+k,n}(x, y)$ can be used to derive the probability distribution function such that the normal distribution $N(0, \sigma^2)$.

$w_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}}$ can be described as

$$w_\sigma(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! 2^m \sqrt{2\pi(\sigma^2 + 1)^{2m+1}}} H_{2n}(x)$$

3. Signal processing: $He_{m+k,n}(x, y)$ can approximate functions and generate smooth curves or surfaces. Two index and two variables Hermite polynomials can create basis functions for polynomial approximations of surfaces with cylindrical symmetry.

4. Optics: $He_{m+k,n}(x, y)$ can also be used in optics to describe light propagation through an optical system, such as a lens or a mirror. Further, $He_{m+k,n}(x, y)$ can be used to calculate the intensity distribution of a laser beam as it propagates through a lens, which has critical applications in laser physics and optical engineering.

Overall, Hermite polynomials are an extremely versatile and powerful tool in mathematics and their many applications, such as physics, statistics, signal processing, and optics.

Overall, Hermite polynomials are essential mathematical tools with various implications in different fields of science and engineering.

Hermite polynomials, denoted as $H_n(x, y)$, are used in statistics and probability theory to describe the Gaussian distribution, also known as the normal distribution. These polynomials have applications in physics, engineering, and finance, where Gaussian distributions are commonly encountered.

Chapter 4: Summary, Conclusion, and Directions for Future Research

4.1 Summary

In Chapter 1, we reviewed the birth of Weisner's group theoretic method and its various steps in obtaining generating relations. Also, we reviewed the development of Hermite polynomials and their extensions (among numerous ones), together with their diverse applications to various research areas, particularly quantum mechanics, mathematical physics, and other branches.

Furthermore, in Chapter 1, we included a historical background, definitions, and some relations that greatly interest this study.

In Chapter 2, by Weisner's method, recurrence relations for four different cases, differential operators, commutative relation of differential operators, and there have been discovered extended forms of the group produced by operators, from which several generating relations for the two indexes and four variables Hermite polynomials $H_{m,n}(x, y, z, k)$ have been established.

Similarly, in Chapter 3, we used the same procedure as in Chapter 2, and we obtained several generating relations for two index and two variable

Hermite polynomials $He_{m,n}(x, y)$. It is noted that the obtained results can be considered as generalizations of some known results.

4.2 Conclusion

The process of obtaining generating functions takes place by many methods. Lie group is considered the current method for obtaining such generating relations. Weisner (1955) is the first to make considerable progress in this area, who demonstrates the group-theoretic relevance of hypergeometric generating functions, Hermite, and Bessel functions. Weisner's technique is methodically presented by Miller (1980) and McBride (1971), giving its solid foundation. Miller (1968) takes Weisner's theory further by linking it to Schrodinger's factorization method, and due in its definitive form to Infeld and Hull.

Indeed, generating relations has significant implications in several fields, including mathematics, physics, engineering, and computer science. These relations serve as a valuable instrument to analyze and describe intricate systems and processes by breaking them down into simpler components, exactly the partial differential equation that the equation defined.

On the other hand, $H_n(x)$ are a set of orthogonal polynomials named after Charles Hermite (1864). They are commonly used in physics and engineering, particularly when examining quantum mechanics and additionally, when solving partial differential equations, where they describe a system's energy levels and wave functions. They are also used to solve partial differential equations, particularly in studying heat transfer and fluid dynamics. Additionally, Hermite polynomials are used in the field of finance, specifically in the pricing of options and other financial derivatives. They are also used in statistics, particularly in the study of probability distributions.

4.3 Future studies

As directions for further research, that the results in this thesis could lead naturally to further progress in the theory of Hermite polynomials.

Indeed, a further natural extension of the present investigation would consist of establishing new applications in physics, engineering, harmonic oscillators in quantum mechanics,

Furthermore, computer science and discussing the connections with known families of Hermite polynomials.

Bibliography and References

1. Cesarano, C.(2014).A note on generalized Hermite polynomials, Inter.J.Appl.Math.Inf.,(8),1-6.
2. Cohn, P.M. (1961). Lie groups, Cambridge University Press.
3. Dattoli, G., Khan, S. and Yasmin, G. (2007). Generating relations of Hermite-Tricomi functions using a representation of Lie algebra τ_3 , Georgain Mathematical Journal, (14)(1), 99-107.
4. Dattoli, G., Srivastava, H.M and Khan, S. (2005). Operational versus Lie- algebraic methods and the theory of multi-variable Hermite polynomials, Integral Transforms, and Special Functions, (16)(1), 81-91.
5. Dattoli, G., Lorenzutta, S.,Ricci,P.E., Cesarano, C.,(2004). "On a family of hybrid polynomials," *Integral Transforms and Special Functions*, 15 (6), pp. 485-490.
6. Dattoli, G., Ricci, P.E., Cesarano, C., Vazquez, L., (2003)."Special polynomials and fractional calculus," *Mathematical and Computer Modeling*, 37 (7-8), pp. 729-733.
7. Dattoli, G. (2000). Differential Identities, Generalized Hermite Polynomials and applications in physics and mathematics, Centre de Recherches Mathematiques, CRM Proceedings and Lecture Notes, (25), 83-92.
8. Dattoli, G., Lorenzutta, S., Cesarano, C., and Sacchetti, D., (2000). Higher order derivatives of exponential functions and generalized forms of Kampe de Fériét-Bell polynomials, Int. Math. Stat. Sci. 9(2) 111-123.
9. Dattoli, G. and Torre, A., (2000). Exponential operators, quasi-monomials and generalized polynomials, Radiation Physics and Chemistry (57), 21-26.

- 10.Dattoli, G., Torre, A. and Lorenzutta, S. (1999). Operational Identities and properties of ordinary and generalized special functions, J. Math. Anal. Appl., (236), 399-414.
- 11.Dattoli, G., Lorenzutta, S., Svrivastava, H.M and Torre, A. (1999). Some families of mixed generating functions and Generalized Polynomials, Le Matematiche, LIV-Fasc, (I), 147-157.
- 12.Dattoli, G., Torre, A. and Mazzacurati, S., (1998). Monomiality and integrals involving Laguerre polynomials, Rend. Mat. Appl. VII 18(3), 565-574.
- 13.Dattoli, G. and Torre, A. (1995). Phase-space formalism, Generalized Hermite Polynomials, Orthogonal functions and certain- annihilation operators, J. Math. Phys., (36)(4), 1636-1644.
- 14.Dattoli, G., Chiccoli, C., Lorenzutta, S., Maino, G. and Torre, A. (1993). Generalized Bassel functions and generalized Hermite Polynomials, J. Math. Anal. Appl., (178), 509-516.
15. Kim, Y.S., Noz, M.E. (1991)."Phase-space picture of quantum mechanics," *World Scientific, Singapore*.
- 16.Khan, S. and Raza, N. (2010). 2-variable generalized Hermite matrix polynomials and Lie algebra representation, Reports on Math. Phys., (66),2.
- 17.Khan, S. and Khan, R. (2009). Lie-theoretic generating relations involving multi-variable Hermite- Tricomi functions, Integral Transforms and Special functions, (20)(5), 365-375.
- 18.McBride, E. (1971). Obtaining generating functions, Springer-Verlag Berlin Heidelberg New York.
- 19.Miller, W., Jr. (1968). Lie theory and special functions, Academic Press, New York, and London.

- 20.Pathan, M.A., Khan, S. and Yasmin, G. (2003). Lie theory and two variable generalized Hermite polynomials, Ital. J. Pure and Applied Math., (14), 129-136.
- 21.Pathan, M.A., Subuhi Khan and Ghazala Yasmin, (2002). Representation of a Lie algebra $G(0, 1)$ and three variable generalized Hermite polynomials $H(x, y, z)$, Integral Transforms and Special functions (13), 59-64,
- 22.Shahwan, M.(2004).Lie theoretic generating functions of two index generalized Hermite polynomials. South East Asian J.Math.&Math.Sc.(2)(2),1-7.
- 23.Shahwan, M. (2003). Lie theoretic generating functions of generalized Hermite Polynomials, Mutah Lil-Buhuth wad-Dirasat, (18), 21-27.
- 24.Srivastava, H.S.P., (2001). On multi-index Hermite polynomials, Proceeding c:.,he Second International Conference of the Society for Special Functions and their Applications (SSFA) (1), 177-184.
- 25.Weisner, L. (1959). Generating functions for Hermite functions, Canad. J. Math., (11), 141-147.
- 26.Weisner, L. (1955). Group-theoretic origin of certain generating functions, Pacific J. Math., (5), 1033-1039.
- 27.Dattoli, G., Khan, S., & Ricci, P. E. (2008). On Crofton–Glaisher type relations and derivation of generating functions for Hermite polynomials, including the multi-index case. Integral Transforms and Special Functions, 19(1), 1-9.
- 28.Cesarano, C., Fornaro, C., & Vazquez, L. (2015). A note on a special class of Hermite polynomials. Int. J. Pure Appl. Math, 98(2), 261-273.

29. Cesarano, C. L. E. M. E. N. T. E., Cennamo, G. M., & Placidi, L. U. C. A. (2014). Humbert polynomials and functions in terms of Hermite polynomials towards applications to wave propagation. *WSEAS Transactions on Mathematics*, 13, 595-602.
30. Scerrato, D., Giorgio, I., & Rizzi, N. L. (2016). Three-dimensional instabilities of pantographic sheets with parabolic lattices: numerical investigations. *Zeitschrift für angewandte Mathematik und Physik*, 67, 1-19.
31. Dattoli, G. (2003). Incomplete 2D Hermite polynomials: properties and applications. *Journal of mathematical analysis and Applications*, 284(2), 447-454.
32. Wünsche, A. (2000). General Hermite and Laguerre two-dimensional polynomials. *Journal of Physics A: Mathematical and General*, 33(8), 1603.
33. Dattoli, G., Lorenzutta, S., Maino, G., & Torre, A. (1997). Theory of multi-index multi-variable Bessel functions and Hermite polynomials. *Le Matematiche*, 52(1), 179-197.
34. Shrivastava, H. S. P. (2001). Some generating function relations of multi-index Hermite polynomials. *Mathematical and Computational Applications*, 6(3), 189-194.
35. Wünsche, A. (2005). Generalized Zernike or disc polynomials. *Journal of Computational and applied mathematics*, 174(1), 135-163.
36. Nai'ma Ai't Jemma and Allal Ghanm (2014) ON A CLASS OF TWO-INDEX REAL HERMITE POLYNOMIALS. *Palestine Journal of Mathematics* Vol. 3(2), 185- 190.
37. Ahmed Ali Al-Gonaha, *Malaya J. Mat.* (2014) Generating relations involving 2-variable Hermite matrix polynomials 2(3) 236-242.

ملخص

في هذه الرسالة تمت دراسة كيفية توليد العلاقات ثنائية المؤشر وثنائية المتغيرات متعددة الحدود المرتبطة بدالة هيرمايت من خلال نهج ويزنر.

تشمل الحالات التي ناقشناها في هذه الرسالة؛ توليد العلاقات بين مؤشرين متعددي الحدود هيرمايت المعممة، توليد علاقات بين مؤشرين واثنين من متعددات حدود هيرمايت المتغيرة.

تم توضيح كافة الخطوات بتفاصيلها المرتبطة بنهج ويزنر بدءًا من إيجاد العلاقات التكرارية ثم الحصول على العلاقات المؤثرة والعلاقات التبادلية بينهم، إضافة إلى الصيغة الممتدة للمجموعات الناتجة من هذه العلاقات المؤثرة، لننتهي بالحصول على علاقات مولدة جديدة والتي هي تعميم لعلاقات أساسية معروفة سابقاً.

جامعة البحرين

قسم الرياضيات



كلية العلوم

الحصول على علاقات مولدة من مؤشرين واثنين من متغيرات هيرمايت المطور

أطروحة مقدمة كجزء من متطلبات الحصول على درجة الماجستير في الرياضيات

إعداد

شيخة خالد الشوملي
الرقم الجامعي: ٢٠١٢٠١٢٥

إشراف

د. مهند جمال شهوان
أستاذ مشارك
جامعة البحرين

مملكة البحرين

سبتمبر ٢٠٢٣ م