



University of Bahrain
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Pseudo – Valuation Domains

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requirements for the degree of**

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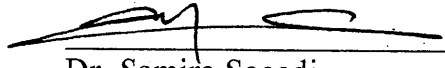
"Pseudo-Valuation Domains"

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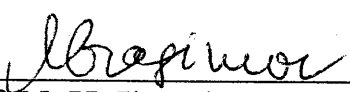
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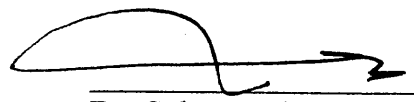
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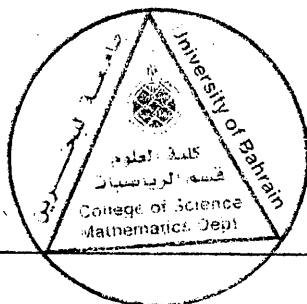
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ABSTRACT الخلاصة

جميع الحلقات الواردة في هذه الرسالة هي حلقات تكاملية وإبدالية ذات عنصر محايد. في المرجع [HH1] ، قدم هيدستروم وهسون مفهوم مجال النصف (valuation) . هذا المفهوم له ارتباط وثيق بحلقات (valuation) حيث أنها تشترك معها في الكثير من الخصائص خصوصاً المتعلقة منها بمجموعة المثاليات والعمليات عليها. لقد تم تعميم هذه المفاهيم بشكل طبيعي ومنطقي ويمكن الرجوع للتفاصيل في المرجع المشار إليه بالرمز [DF3].

تنقسم هذه الرسالة التي بين أيدينا إلى ثلاثة فصول رئيسية. يتحدث الفصل الأول في قسمه الأول عن التعاريف الأساسية للموضوع إلى جانب العديد من الخواص المكافئة لمفهوم مجال نصف valuation . أما القسم الثاني والثالث فهما يدرسان العلاقة بين مجال نصف valuation (PVD) و مجال valuation . أما القسم الرابع والأخير فهو يتناول مجال نصف valuation في قالب جديد من منظور نظرية الفئات وهو يقدم مفهوم فئة القسمة ويمكن الرجوع للمراجع المشار إليها بالرموز [O], [AN1] للاستزادة والنظر في التفاصيل.

أما الفصل الثاني في قسمه الأول فهو يقدم طريقة فعالة لإنشاء مجالات نصف valuation . أما القسم الثاني فهو يدرس الحلقات الفوقية للمجال نصف valuation .

أما القسم الثالث فهو يتطرق إلى العلاقة بين المجالات نصف valuation وبين المجالات النيوثيرية. أما القسم الأخير من هذا الفصل فهو يقدم مفهوم الهومومورفيزم القوي المحلي وعلاقته بمفهوم المجالات نصف valuation .

الفصل الثالث والأخير يعمم مفهوم المجالات نصف valuation . وبهذا التعميم ينتج نوعان من حلقات نصف valuation يتم دراستهما كل على حدة مما يساهم في تعميم بعض المفاهيم الواردة في الفصلين الأول والثاني.

NOMENCLATURE

$\in, \notin$	Belong to, does not belong to
$\subseteq$	Is contained in
$\subset$	Properly contained in
$\not\subset$	Is not contained in
$\cap$	The Intersection
$\sum$	Summation
$\cong$	Isomorphic to
$=, \neq$	Equal to, not equal to
$\geq$	Greater than or equal
$\emptyset$	Empty set
$\Rightarrow$	Implies
$(M: M)_S$	The set $\{x \in S : xM \subseteq M\}$
R	The integral domain R
K	The fraction field of R
M^{-1}_S	The set $\{x \in S : xM \subseteq R\}$
$A \setminus B$	Belong to A but not B
a / b	a over b
$k[y]$	The polynomial ring with indeterminate y over the field k
$k(y)$	The fraction field of the ring $k[y]$
T_Q	The localization of T at Q

ABBREVIATIONS

PVD	Pseudo-valuation domain
VD	Valuation domain
(R, M)	Integral domain with maximal ideal M
$\text{Frac}(R)$	The fraction field of the integral domain R
$\text{Spec}(R)$	The spectra of R , the set of all prime ideals of R
$N(R)$	The set of non-unit elements of R

LIST OF CONTENT

Chapters & sections	page
ACKNOWLEDGMENT	i
ABSTRACT	ii
NOMENCLATURE & ABBREVIATIONS	iii
INTRODUCTION	vi

CHAPTER I

1. Pseudo – valuation domains	2
2. Valuation domains	6
3. Associated valuation domains	9
4. Group of divisibility.....	12

CHAPTER II

1. Pullback constructions	15
2. Overring of pseudo – valuation domains.....	21
3. Noetherian domains	27
4. Strong local homomorphism	29

CHAPTER III

1. Valuation subring of an integral domain	37
2. Pseudo – valuation subring of an integral domain	39
3. First type of pseudo – valuation subrings	42
4. Second type of pseudo – valuation subrings	47

LIST OF REFERENCES	vii
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ABSTRACT

Throughout this thesis, all rings are commutative with identity and the letter R denotes an integral domain with quotient field K . Hedstrom and Houston [HH1] introduced the concept of *Pseudo – valuation domain* (**PVD** for short). The name "*Pseudo – valuation*" domain is justified by the fact that these rings share many ideal – theoretic properties with the valuation rings to which they are closely related. The motivations for studying such a generalization of a valuation domain are various and meaningful [DF3]. **PVD** have been studied extensively in [HH2], [D2], [AN], [AD1], [B1], [DF2], [OK], [B5], [B2] and [B4].

This thesis is divided into three chapters. In chapter I, several characterizations for **PVD** are given in section 1. Sections 2 and 3 deal with the relationship between **PVD** and valuation domain, and that between **PVD** and associated valuation domain respectively. Section 4 provides a new characterization of **PVD** by using the notion of group of divisibility [AN1], [O].

Chapter II, produces a beautiful way of constructing **PVD**'s in section 1. Section 2 studies the overrings of a **PVD**, while section 3 investigates the relation between **PVD** and Noetherian domains. Finally, section 4 introduces the notion of *strong local homomorphism* and how to relate **PVD**'s to valuation domains through this notion.

Chapter III, generalizes the study of **PVD** to the context of arbitrary integral domains by defining the notion of pseudo–valuation subring R in an integral domain S . This generalization provides two types of **PVD**'s. Most of the concepts in chapters I and II are generalized in this chapter by a natural way.

CHAPTER I

1. PSEUDO-VALUATION DOMAINS.

Definition 1.1 Let R be a domain with quotient field K . A prime ideal P of R is called *strongly prime* if $x, y \in K$ and $xy \in P$ imply that $x \in P$ or $y \in P$.

We start by a characterization of such prime ideal.

Proposition 1.2 Let P be a prime ideal of a domain R with quotient field K . Then the following are equivalent:

- (1) P is strongly prime.
- (2) $x^{-1}P \subseteq P$ whenever $x \in K \setminus R$.

Proof. Assume that P is a strongly prime ideal of R . If $x \in K \setminus R$ and $p \in P$, then $p = (p x^{-1}) x \in P$, whence $p x^{-1} \in P$. Thus $x^{-1}P \subseteq P$. Conversely, assume that $x^{-1}P \subseteq P$ whenever $x \in K \setminus R$, and let $ab \in P$. If $a, b \in R$, there is nothing to prove. Hence we may assume that $a \notin R$, so that $a^{-1}P \subset P$ and $b = a^{-1}(a b) \in P$.

Definition 1.3 A domain R is called *pseudo-valuation domain* (*PVD*, for short) if each prime ideal of R is strongly prime.

We will provide several characterizations of pseudo-valuation domains. The first characterization uses the set $N(R)$ of all non-units of R .

Theorem 1.4 *Let R be a domain with quotient field K . The following statements are equivalent.*

- (1) R is a PVD.
- (2) For each $x \in K \setminus R$ and $a \in N(R)$, we have $(a + x)R = xR$.
- (3) For each $x \in K \setminus R$ and $a \in N(R)$, we have $x^{-1}a \in R$.

Proof. (1) $\Rightarrow$ (2): Let $x \in K \setminus R$ and a be an element of $N(R)$. Then $a \in P$ for some prime ideal P , so that $x^{-1}a \in x^{-1}P \subseteq P \subset R$. Hence $\frac{x+a}{x} = 1 + ax^{-1} \in R$ and

$(x+a)R \subseteq xR$; on the other hand, $x+a \notin R$ so that $(x+a)^{-1}P \subset P$. Thus $\frac{a}{x+a} \in$

$P \subseteq R$. Since $\frac{a}{x+a} = 1 - \frac{x}{x+a}$, we have $\frac{x}{x+a} \in R$ and $xR \subseteq (x+a)R$.

(2) $\Rightarrow$ (3): Let $x \in K - R$ and a be an element of $N(R)$. In view of (2), we have $\frac{x+a}{x} = 1 + ax^{-1} \in R$, whence $ax^{-1} \in R$ also.

(3) $\Rightarrow$ (1): Let P be prime ideal and take $ab \in P$ with $a, b \in K$. We may assume that $b \notin R$. By hypothesis since $ab \in N(R)$, then $a = b^{-1}(ab) \in R$. We claim that a is a non-unit; otherwise $b = a^{-1}(ab) \in P$, a contradiction. We apply the hypothesis again to get $b^{-1}a \in R$. Thus $a^2 = (b^{-1}a)(ab) \in P$ and $a \in P$, as desired.

Before going to the next characterization, we prove the following proposition.

Proposition 1.5 *In a PVD R , the prime ideals are linearly ordered. In particular R is local.*

Proof. Let P and Q be prime ideals with $P \not\subseteq Q$, and suppose that $a \in P \setminus Q$. Then for each $b \neq 0$, $b \in Q$, we have $\frac{a}{b} \notin R$. Hence $(\frac{b}{a})P \subseteq P$ by Proposition 1.2. Thus $b = (\frac{b}{a})a \in P$ and we have $Q \subseteq P$.

To simplify the notations, we will use frequently (R, M) to denote a local ring R with maximal ideal M .

The next characterization of *PVD* uses the fact that a *PVD* is local with maximal ideal M .

Theorem 1.6 *Let (R, M) be a local ring with maximal ideal M . The following assertions are equivalent.*

- (1) R is a *PVD*.
- (2) For each ideals I, J of R , then $I \subseteq J$ or $MJ \subseteq MI$.
- (3) For I, J two ideals of R , then $I \subseteq J$ or $MJ \subseteq I$.
- (4) M is strongly prime.

Proof. (1) $\Rightarrow$ (2): Assume that $I \not\subseteq J$, and pick $a \in I \setminus J$. For each $b \neq 0$, $b \in J$ we have $\frac{a}{b} \notin R$, so that $(\frac{b}{a})M \subseteq M$ and $Mb \subseteq Ma \subseteq MI$. It follows that $MJ \subseteq MI$.

(2) $\Rightarrow$ (3): For I, J two ideals of R , if $I \not\subseteq J$, then $MJ \subseteq MI \subseteq I$.

(3) $\Rightarrow$ (4): Let $a, b \in R$, $b \neq 0$ with $\frac{a}{b} \notin R$. We shall show that $(\frac{b}{a})M \subseteq M$. By

Proposition 1.2, this will suffice. Since $\frac{a}{b} \notin R$, we have $(a) \not\subseteq (b)$, whence $M(b) \subseteq$

(a) and $M(\frac{b}{a}) \subseteq R$. If $M(\frac{b}{a}) = R$, then $M = R(\frac{a}{b})$ and

$\frac{a}{b} \in R$, a contradiction. Hence $(\frac{b}{a})M \subseteq M$, as desired.

(4) $\Rightarrow$ (1): Let $x \in K$, $x \notin R$ and P be a prime ideal of R . Let $p \in P$ and note that since $P \subset M$, we have $x^{-1}p \in M$. Hence $x^{-1}(p x^{-1}) \in M$, whence $(x^{-1}p)^2 = (x^{-1}p x^{-1}) p \in P$. Since P is a prime ideal and $x^{-1}p \in R$, we therefore have $x^{-1}p \in P$.

The last characterization uses the comparability conditions.

Theorem 1.7 Let (R, M) be a local ring. The following assertions are equivalent.

- (1) R is *PVD*.
- (2) For $x \in K$, and P a prime ideal of R , then xR and P are comparable.
- (3) For $a, b \in R$, and P a prime ideal of R , then aP and bR are comparable.
- (4) For $a, b \in R$, aM and bR are comparable.
- (5) For $x \in K$, xR and M are comparable.

Proof. (1) $\Rightarrow$ (2): Let $x \in K$ and P a prime ideal. If $x \in P$, then $xR \subseteq P$. If $x \notin P$, we will show that $P \subseteq xR$. Let $p \in P$, then $p = (x^{-1}p) x$. As P is strongly prime, then $x^{-1}p \in P$. Thus $x^{-1}P \subseteq P$, so $P \subseteq xP \subseteq xR$.

(2) $\Rightarrow$ (3): Let $a, b \in R$ and P be a prime ideal, then the assertion is clear if $a = 0$ or $b = 0$. We may suppose that $a \neq 0$, $b \neq 0$, and consider $x = \frac{b}{a} \in K$. By assertion, xR and P are comparable. Either $xR \subseteq P$, so $\frac{b}{a}R \subseteq P$ and $bR \subseteq aP$, or $P \subseteq xR$, so $P \subseteq \frac{b}{a}R$ and $aP \subseteq bR$.

(3) $\Rightarrow$ (4): Clear.

(4) $\Rightarrow$ (5): Let $x \in K$, then $x = \frac{b}{a}$, for some $a \in R \setminus \{0\}$ and $b \in R$. We have either $aM \subseteq bR$ or $bR \subseteq aM$, so $M \subseteq \frac{b}{a}R$ or $\frac{b}{a}R \subseteq M$. It follows that $M \subseteq xR$ or $xR \subseteq M$.

(5) $\Rightarrow$ (1): We will show that M is strongly prime. Let $xy \in M$, $x, y \in K$ and $x \notin M$. This implies that $xR \not\subset M$. So $M \subseteq xR$. Thus $xy \in xR$ implies $xy = xr$, so $y \in R$. If $y \notin M$, then $y^{-1} \in R$. So $x = (xy) y^{-1} \in M$, since $xy \in M$ and $y^{-1} \in R$, a contradiction. Hence $y \in M$.

2. VALUATION DOMAINS

We start by recalling the definition of a valuation domain.

Definition 2.1. Let V be an integral domain with quotient field K . If one of the following equivalent conditions is satisfied, then V is called a *valuation domain*.

- (1) If A and B are two ideals of V , then $A \subseteq B$ or $B \subseteq A$.
- (2) If $a, b \in V$, then $(a) \subseteq (b)$ or $(b) \subseteq (a)$.
- (3) For every $x \in K$, $x \in V$ or $x^{-1} \in V$.

Now we will investigate the relationship between valuation domains and *PVDs*.

Proposition 2.2 Every valuation domain is a *PVD*.

Proof. Let V be a valuation domain, and let P be a prime ideal in V . Suppose that $xy \in P$ where $x, y \in K$, the quotient field of V . If both x and y are in V , then we are done. Suppose that $x \notin V$. Since V is a valuation domain, we have $x^{-1} \in V$. Hence $y = (xy) x^{-1} \in P$, as desired.

The converse of the previous proposition is not true. The following example produces the construction of a *PVD* that is not a valuation domain.

Example 2.3 Let $K \subset L$ be an extension of fields and x be an indeterminate over L . Let $V = L[x]_{(x)}$. Then V is a valuation domain with maximal ideal $M = (x)L[x]_{(x)}$ and quotient field $L(x)$.

Now, consider $R = K + M$.

(i) R is a subring of V : Let $x, y \in R$. Then $x = a + m$ and $y = b + m'$, where $a, b \in K, m, m' \in M$. We have

$$x - y = (a - b) + (m - m') \in K + M = R$$

and
$$xy = ab + (am' + bm + mm') \in K + M = R.$$

(ii) $\text{Frac}(R) = L(x)$: We have $R \subset V \subset L(x)$, so $\text{Frac}(R) \subseteq L(x)$. Now, let $t = p/q \in L(x)$, with $p \in L[x]$ and $q \in L[x] \setminus \{0\}$. Pick $m \in M, m \neq 0$. Then $t = (pm)/(qm)$, with $pm \in M \subseteq R$ and $qm \neq 0, qm \in M \subseteq R$. Thus, $t \in \text{Frac}(R)$.

(iii) M is a maximal ideal of R : Note first the following isomorphic relations $R/M = (K + M)/M \cong K/(K \cap M) = K/(0) \cong K$. So R/M is a field and M is a maximal ideal of R .

(iv) R is local: Let $x \in R \setminus M$. Then $x = a + m$ with $a \in K \setminus \{0\}$ and $m \in M$. Since $x \in V \setminus M$, then x is a unit of V . But $x^{-1} = a' + m'$ with $a' \in L \setminus \{0\}$ and $m' \in M$. Whence,

$$1 = xx^{-1} = aa' + (am' + a'm + mm').$$

Hence, we have $1 - aa' = am' + a'm + mm' \in M \cap L = (0)$. So $aa' = 1$ and $a' = a^{-1} \in K$. Therefore, $x^{-1} = a^{-1} + m' \in R$, and x is a unit of R .

(v) R is a *PVD*: Let $x, y \in \text{Frac}(R)$ such that $xy \in M$. Since V is a valuation domain, then V is a *PVD*. It results, since $x, y \in \text{Frac}(V) = \text{Frac}(R)$ and M is strongly prime in V , that $x \in M$ or $y \in M$. Thus M is a strongly prime in R .

(vi) R is not a valuation domain: Let $z \in L \setminus K$. We have $z \in L(x) = \text{Frac}(R) = \text{Frac}(V)$. If $z \in R$, then $z = a + m$, where $a \in K$ and $m \in M$. Since $z - a \in M \cap L = (0)$, then $z = a \in K$, a contradiction. Hence $z \notin R$. Also $z^{-1} \in L \setminus K$. By a similar proof, we can show that $z^{-1} \notin R$. Hence R is not a valuation domain.

However, we have a partial converse in the sense of the following result.

Proposition 2.4 *Let R be a *PVD* with maximal ideal M . If P is a non-maximal prime ideal of R , then R_P is a valuation domain.*

Proof. Let K denote the quotient field of R , and let $x \in K$. If $x \in R$, then $x \in R_P$. If $x \notin R$, then $x^{-1}M \subset M$, since R is a *PVD*. Choose $m \in M \setminus P$. Then $x^{-1} = (mx^{-1})/m \in R_P$. Hence R_P is a valuation domain.

3. ASSOCIATED VALUATION DOMAINS

Theorem 3.1 *Let (R, M) be a local domain. Then The following assertions are equivalent:*

- (1) R is PVD.
- (2) R has a unique valuation overring V with maximal ideal M .
- (3) There exists a valuation overring V in which every prime ideal of R is also a prime ideal of V .

Proof. (1) $\Rightarrow$ (2): Let R be a PVD. Then there is a valuation domain (W, N) such that $N \cap R = M$ [G, Theorem 19.6]. Let $V = W_M$. Then V is a valuation overring of R with maximal ideal MW_M . We claim that $M = MW_M$. It is clear that $M \subseteq MW_M$. For the reverse containment, let $x \in MW_M$. Then $x = m/a$ for some $m \in M$ and $a \in W \setminus M$. If $a \in R$, then $a \in R \setminus M$, so a is a unit of R and $x = a^{-1}m \in MR \subseteq M$. If $a \notin R$, then $a^{-1}M \subseteq M$ since M is a strongly prime ideal. Thus $x = a^{-1}m \in M$. Therefore, $M = MW_M$.

Now we show that V is unique: Let V' be a valuation overring of R different from V with maximal ideal M . If $V \not\subseteq V'$, then pick $v \in V \setminus V'$. Then $v^{-1} \in V'$. As v^{-1} is not a unit of V' , then $v^{-1} \in M$. Thus $1 = v v^{-1} \in VM \subseteq M$ a contradiction. Therefore, $V \subseteq V'$ and, likewise we show that $V' \subseteq V$. Thus $V = V'$.

(2) $\Rightarrow$ (3): Let V be a valuation overring of R with maximal ideal M . We have to prove that every prime ideal P of R is a prime ideal of V . If $p \in P$ and $v \in V$, then $p \in M$, so that $vp \in M$. Thus $v^2 p \in M$, whence $(vp)^2 \in P$. Hence $vp \in P$ and P is an ideal of V . Now let $xy \in P$ and $x, y \in V$. If both x and y are in R , then $x \in P$ or $y \in P$. Thus assume that $x \notin R$, so that $x \notin M$ and $x^{-1} \in V$. Thus, since P is an ideal of V , we have $y = x^{-1}(xy) \in P$. Hence P is a prime ideal of V .

(3) $\Rightarrow$ (1): Let V be the given valuation overring and let P be a prime ideal of R . Then since every prime ideal of R is also prime in V , and since V is a *PVD*, then P is a strongly prime of R . Thus R is a *PVD*.

Definition 3.2 The unique valuation domain V corresponding to a *PVD* R will be called the *associated valuation domain* of R .

Note that a valuation domain V has itself as the associated valuation domain.

We shall give more informations about the associated valuation overring of a *PVD*. We need the following notations:

Notations: Let (R, M) be a local ring with quotient field K . Denote

- (i) $(M : M) = \{x \in K : xM \subseteq M\}$
- (ii) $M^{-1} = (R : M) = \{x \in K : xM \subseteq R\}$

It is clear that $(M : M) \subseteq M^{-1}$ and that $(M : M)$ is an overring of R that shares the ideal M with R . However, M^{-1} is not in general a ring [AN1].

Proposition 3.3 *If R is a PVD, then $(M : M)$ is the associated valuation overring of R .*

Proof. Suppose that V is the associated valuation of R . Since M is an ideal of V , then $MV \subseteq M$. So $V \subseteq (M : M)$. To prove the reverse containment, let $x \in (M : M)$. If $x \notin V$, then $x^{-1} \in V$, since V is a valuation domain. But $xM \subseteq M$, then $1 = x x^{-1} \in M$, a contradiction. So $x \in V$.

Note that if (R, M) is a local domain such that the overring $(M : M)$ is a valuation overring, then R is not in general a *PVD*.

Example 3.4: Let k be a field, x an indeterminate over k and $V = k[[x]]$. The ring of all power series in one indeterminate x over k . Consider $R = k[[x^2, x^3]]$. Then R is a local ring with maximal ideal $M = (x^2, x^3)$.

(i) R is not a PVD since M is not strongly prime ($x^2 \in M$, but $x \notin M$).

(ii) $(M : M)$ is a valuation domain. In fact, $(M : M) = k[[x]]$. For $f \in M$, $f = x^2g + x^3h$ with $g, h \in R$. As $xf = x^3g + (x^2)^2h \in M$, then $x \in (M : M)$. It follows that $V = k[[x]] \subseteq (M : M)$. In other way, $(M : M)$ is a valuation domain since it is an overring of the valuation ring V . If P is the maximal ideal of $(M : M)$, then P is a prime ideal of V , so $P = (x^n)V$ for some positive integer $n \geq 1$. But if $n > 1$, then P is not a prime ideal of V , since in this case, $x^n \in P$ and $x \notin P$. Thus $n = 1$, $P = (x)V$ and $(M : M) = V$.

Proposition 3.5 *Let (R, M) be a PVD which is not a valuation domain and (V, M) a valuation overring of R . If I is a non-zero principal ideal of R , then I is not an ideal of V .*

Proof. Suppose that $I = Ra$ is a non-zero ideal of V . Then $I = VI = V(Ra) = Va$. Choose an element $v \in V \setminus R$. Then $va \in I$, so that $va = ra$ with $r \in R$ and $v = r \in R$, a contradiction.

Corollary 3.6 *If R is a PVD and has a non-zero principal prime ideal, then R is a valuation domain.*

Proof. Assume that R is not a valuation domain and $V \neq R$ be its associated valuation overring. If P is a non-zero principal prime ideal of R , then P is not an ideal of V , by Proposition 3.5. This contradicts Theorem 3.1.

We now show that the associated valuation overring of a *PVD* (R, M) is exactly M^{-1} in the case where R is not a valuation domain.

Theorem 3.7 *let (R, M) be a local domain which is not a valuation domain. Then R is a *PVD* if and only if $V = M^{-1}$ is a valuation overring of R with maximal ideal M .*

Proof. Assume that R is a *PVD* and let $V = (M : M)$ be its associated valuation overring. We will prove that $M^{-1} = V$. We have already noted that $V = (M : M) \subseteq M^{-1}$. Let us prove the inclusion in the other side. Let $x \in M^{-1}$, then $xM \subseteq R$, so xM is an ideal of R . If $xM = R$, then $M = Rx^{-1}$ is a non-zero principal ideal of R . According to Corollary 3.6, R is a valuation domain, a contradiction. It follows that xM is a proper ideal of R , and so $xM \subseteq M$, hence $x \in (M : M) = V$. Therefore, M^{-1} is a valuation overring of R with maximal ideal M .

The converse is a direct consequence of Theorem 3.1, since if $V = M^{-1}$ is a valuation overring with maximal ideal M , then R is a *PVD*.

4. GROUP OF DIVISIBILITY.

In this section, we will provide a new characterization of a *PVD* by using the notion of group of divisibility [O]. Let us recall some definitions and terminologies about the group of divisibility of an integral domain.

Let R be an integral domain and K be its quotient field. If $(K^*, \cdot)$ is the group of units of K , then $(K^*, \cdot)$ is an Abelian group. The set of units $U(R)$ of R is a normal subgroup of K^* .

1) Set $G = K^* / U(R)$, the factor group of K^* modulo $U(R)$. Every element of G is a coset and has the form $xU(R)$, for some $x \in K^*$. Moreover, since G is an Abelian group, the group operation of G will be denoted additively as follows: $xU(R) + yU(R) = (xy)U(R)$, for all elements $xU(R)$ and $yU(R)$ of G .

2) We define a relation $\leq$ on G as follows:

For two elements $xU(R)$ and $yU(R)$ of G , $xU(R) \leq yU(R) \Leftrightarrow yR \subseteq xR$.

$\leq$ is a partial order. Indeed,

- $\leq$ is reflexive: Let $xU(R) \in G$. As $xR \subseteq xR$, then $xU(R) \leq xU(R)$.
- $\leq$ is anti-symmetric: Let $xU(R), yU(R) \in G$ such that $xU(R) \leq yU(R)$ and $yU(R) \leq xU(R)$. We have $yR \subseteq xR$ and $xR \subseteq yR$, so $xR = yR$. Thus $xU(R) = yU(R)$.
- $\leq$ is transitive: Let $xU(R), yU(R), zU(R) \in G$ such that $xU(R) \leq yU(R)$ and $yU(R) \leq zU(R)$. We have $yR \subseteq xR$ and $zR \subseteq yR$, so $zR \subseteq xR$. Thus $xU(R) \leq zU(R)$.

3) The partial order relation $\leq$ is compatible with the group operation $+$:

Suppose that $xU(R) \leq yU(R)$ and let $zU(R) \in G$. Then $yR \subseteq xR$ implies that $y = xr$ for some $r \in R$. So $yz = (zx)r \in (zx)R$. It follows that $(yz)R \subseteq (zx)R$, so $(zx)U(R) \leq (yz)U(R)$. Hence $zU(R) + xU(R) \leq yU(R) + zU(R)$.

Consequently, $(G, \leq)$ is an ordered Abelian group called the **group of divisibility of K with respect to R** .

4) The additive identity $1U(R) = U(R)$ of G will be denoted by the symbol 0 . An element $g = xU(R)$ of G is called **positive** if $0 \leq g$. Since we have $1U(R) \leq xU(R)$ if and only if $xR \subseteq R$, or also if and only if $x \in R$, then the set of all positive elements of G is $G_+ = \{xU(R) : x \in R\}$.

Theorem 5.2 *Let R be a domain with quotient field K and group of divisibility G . Then the following assertions are equivalent.*

- (i) R is a PVD.
- (ii) For each $g \in G$, either $0 \leq g$ or $g < h$ for all $h \in G$ with $0 < h$.

Proof. (i) $\Rightarrow$ (ii): R is local with maximal ideal M . Let g be a non-positive element of G . Then $g = xU(R)$ for some element $x \in K \setminus R$ (in fact, $x \in R$ implies that $0 \leq g = xU(R)$). It results that $x^{-1}M \subseteq M$, since R is PVD. Let $h \in G$, $0 < h$, then $h = yU(R)$ for some $y \in R$. But y is not a unit, if it is so, then $h = U(R) = 0$, a contradiction. So $y \in M$. It follows that $m = x^{-1}y \in M$. Therefore, $0 \leq mU(R) = x^{-1}U(R) + yU(R) = -g + h$. If $-g + h = 0$, then $g = h$, a contradiction since $0 < h$ while g is a non-positive element. Hence $0 < -g + h$ and $g < h$.

(ii) $\Rightarrow$ (i): We first show that R is local. If R has two distinct maximal ideals M and N , then we can choose $x \in M \setminus N$ and $y \in N \setminus M$. Let $g = (xy^{-1})U(R)$ and $h = yU(R)$. We have $0 < h$ since $y \notin U(R)$, $y \in R$. By hypotheses, we have either $0 \leq g$ or $g < h$. If $0 \leq g$, then $(xy^{-1})U(R) \geq 0$, which implies that $xy^{-1} \in R$. Hence $x = yr \in N$ for some $r \in R$, a contradiction. But, if $g < h$, then $yR \subseteq (xy^{-1})R$, so $y = rx y^{-1}$. It follows that $y^2 = rx \in M$. Thus $y \in M$, a contradiction. Hence R is local, we call M its maximal ideal. We will show that R is PVD. In view of Theorem 1.6, it suffices to prove that M is strongly prime. Let $x \in K \setminus R$. We need to prove that $x^{-1}M \subseteq M$. Let $m \in M$. Since $xU(R)$ is a non-positive element of G and $0 < mU(R)$, then $xU(R) < mU(R)$. This implies that $mR \subseteq xR$ and $(x^{-1}m)R \subseteq R$. Hence, $x^{-1}m \in R$. If $x^{-1}m \notin M$, then $x^{-1}m = u$ is a unit of R . So $m = ux$ and $mU(R) = uU(R) + xU(R) = xU(R)$, a contradiction.

CHAPTER II

1. PULLBACK CONSTRUCTION.

We start by recalling some results concerning pullback rings.

Let (T, M) be a local ring, $K = T / M$, its residue field and $\varphi : T \rightarrow T/M$ be the canonical epimorphism. Let D be a subring of K , $i: D \rightarrow T/M = K$ the inclusion homomorphism and R the inverse image of D by φ illustrated by the following square diagram:

$$\begin{array}{ccc}
 R = \varphi^{-1}(D) & \xrightarrow{\quad} & D \\
 \downarrow j & & \downarrow i \\
 T & \xrightarrow{\quad \varphi \quad} & T/M = K
 \end{array}$$

In this case,

- (i) $R = \varphi^{-1}(D)$ is a subring of T called *the pullback ring* (T, M, D) .
- (ii) R and T share the same ideal M . Indeed, $M + M \subseteq M$ since M is an ideal of T .

Moreover, $RM \subseteq TM \subseteq M$.

- (iii) $R / M \cong D$. Indeed, $R = \varphi^{-1}(D)$. So $D = \varphi(R)$, since φ is onto. Thus

$$R / M \cong \varphi(R) / \varphi(M) = D / (0) \cong D.$$

(iv) $\text{Frac}(R) = \text{Frac}(T)$:

It is clear that $\text{Frac}(R) \subseteq \text{Frac}(T)$. For the reverse containment, let $z \in \text{Frac}(T)$, then $z = t / t'$, for some $t \in T$ and $t' \in T \setminus \{0\}$. Pick $m \in M$, $m \neq 0$, we have $z = t m / t' m$. As $t m \in R$ and $t' m \in R \setminus \{0\}$, then $z \in \text{Frac}(R)$.

The following theorem provides a new characterization of a *PVD* in terms of pullback rings.

Theorem 1.1 *The following assertions are equivalent:*

- (i) R is *PVD*.
- (ii) R is the pullback ring (V, M, k) where V is a valuation overring of R with maximal ideal M and k is a subfield of V/M . In this case, V is the associated valuation overring of R .

Proof. (i) $\Rightarrow$ (ii): Let V be the associated valuation overring of R , $k = R / M$ and R_1 the pullback ring (V, M, k) .

$$\begin{array}{ccc}
 R_1 = \varphi^{-1}(k) & \xrightarrow{\quad} & k \\
 \downarrow & & \downarrow \\
 V & \xrightarrow{\quad \varphi \quad} & V/M = K
 \end{array}$$

We want to show that $R = R_1$. Since $k = R / M = \varphi(R)$, then $R_1 = \varphi^{-1}(k) = \varphi^{-1}(\varphi(R)) \supseteq R$. Now, we will prove the reverse containment. Let $z \in R_1$, then $\varphi(z) \in k = \varphi(R)$. So $\varphi(z) = \varphi(r)$ for some $r \in R$. Thus $z - r \in \text{Ker } \varphi = M$. Thus $z \in r + M \subseteq R$.

(ii) $\Rightarrow$ (i): Suppose that R is the pullback ring (V, M, k) , where V is a valuation overring of R with maximal ideal M and k is a subfield of V/M .

(1) Since φ is an epimorphism, then $\varphi(R) = \varphi(\varphi^{-1}(k)) = k$. Now from the second isomorphism theorem, we have, $R/M \cong \varphi(R) / \varphi(M) = k / (0) \cong k$.

Thus R/M is a field and M is a maximal ideal of R .

(2) If R has another maximal ideal, say $N \neq M$, then let $m \in M \setminus N$, and consider the multiplicative set $S = \{m^t : t \geq 0\}$ of R and V . We have $R_S = V_S$. Indeed, $R_S \subseteq V_S$, and if $x \in V_S$, then $x = v/m^t = vm/m^{t+1}$ for some $v \in V$. Since $m \in M$, then $vm \in M \subseteq R$, and $m^{t+1} \in S$. So $x \in R_S$. Now $N \cap S = \emptyset$, then NR_S is a prime ideal of $R_S = V_S$. So $NR_S = QV_S$ for some prime ideal Q of V . Therefore, we have $N = NR_S \cap R = QV_S \cap R = QV_S \cap V \cap R = Q \cap R$. But $Q \subseteq M$ the maximal ideal of V . So $N \subseteq Q \subseteq M$, a contradiction.

(3) R has a valuation overring V with the same maximal ideal M , so R is a *PVD* with the associated valuation V , by the application of Theorem 3.1 Chap I.

This theorem enables us to provide several examples of a *PVD*.

Example 1.2 Let K be a proper subfield of a field L and x be an indeterminate over L . Let

$$V = L[[x]] = \{a_0 + a_1x + a_2x^2 + \dots : a_0, a_1, a_2, \dots \in L\}$$

the ring of power series over L in one indeterminate x .

It is known that V is a valuation ring [G]. We can also notice that V can be written as $V = L[[x]] = L + M$ where $M = xL[[x]]$ is the maximal ideal of V . Now, let $R = K + M = K + xL[[x]]$, then R is the pullback ring (V, M, K) . By application of Theorem 1.1, we deduce that R is a *PVD* with V as its associated valuation overring.

$$\begin{array}{ccc}
R & \longrightarrow & K \\
\downarrow & & \downarrow \\
V & \xrightarrow{\varphi} & V/M = L
\end{array}$$

Lemma 1.3 *If (R, M) is a PVD and P is a prime ideal of R , then R/P is a PVD.*

Proof. If $P = M$, then R/P is a field, so R/P is a PVD. Let us suppose that $P \subset M$ and consider the associated valuation domain V of R . According to Theorem 3.1(iii), R and V have the same prime ideals. In particular P and M are both prime ideals of R and V , so R/P and V/P share the same maximal ideal M/P . As, in addition, V/P is a valuation overring of R/P , then R/P is a PVD and V/P is its associated valuation domain, by Theorem 3.1(ii).

Theorem 1.4 *Let (W, N) be a valuation ring with residue ring $W/N = K$, D a subring of K and R the pullback ring (W, N, D) . If $K = \text{Frac}(D)$, then the following assertions are equivalent.*

- (i) R is PVD.
- (ii) D is PVD.

Proof. Consider the following square diagram:

$$\begin{array}{ccc}
R = \varphi^{-1}(D) & \longrightarrow & D \\
\downarrow & & \downarrow \\
W & \xrightarrow{\varphi} & K = W/N
\end{array}$$

(i) $\Rightarrow$ (ii): Suppose that R is a PVD . Then, since φ is an epimorphism, we have $\varphi(R) = \varphi(\varphi^{-1}(R)) = D$. By the second isomorphism theorem, we have

$$R/N \cong \varphi(R) / \varphi(N) = D / (0) \cong D.$$

Since D is an integral domain, then R/N is an integral domain and hence N is a prime ideal of R . Now, as R is a PVD , then R/N is a PVD by lemma 1.3, and so is D .

(ii) $\Rightarrow$ (i): Suppose now that D is a PVD . To show that R is a PVD , we proceed by steps:

(1) We will prove that $N \subset M$ for each maximal ideal M of R : Let $n \in N$, for every $r \in R$, $1 + rn$ is a unit element of W (since $1 + rn \in 1 + N \subseteq W$, but $1 + rn \notin N$). Then there is $z \in W$ such that $(1 + rn)z = 1$, so $z = 1 - rzn \in 1 + N \subseteq R$, so z is a unit of R . Thus $1 + rn$ is a unit of R . If $N \not\subset M$, then take $n \in N \setminus M$, then $M + (n) = R$, so $m + rn = 1$ for some $m \in M$, $r \in R$. It follows that $m = 1 + (-r)n$ is a unit of R , a contradiction.

(2) R is local: if R has two maximal ideals M, M' , then M/N and M'/N are maximal ideals of $R/N = D$. But this is false since D is local.

(3) Now, we will prove that R is a PVD by showing that M is a strongly prime ideal. Let $xy \in M$, where $x, y \in \text{Frac}(R) = \text{Frac}(W)$, the quotient field of R .

- If $x \notin R$, then $x \notin N$. So either $x \notin W$, then $x^{-1} \in N \subseteq W$ since W is a valuation domain, or $x \in W$, then x is a unit of W , and so x^{-1} is a unit of W . In both cases, we find that $x^{-1} \in W$. It results that $x^{-1}(xy) = y \in M$. Similarly, from $y \notin W$, we obtain that $x \in M$.

- If $x, y \in W$, then $x + N$ and $y + N$ belong to $W/N = K = \text{Frac}(D)$. Now, $(xy) + N \in M/N$. Since M/N is the maximal ideal of $R/N = D$, then M/N is strongly prime, since D is supposed to be a PVD . It follows that $x + N \in M/N$ or $y + N \in M/N$.

Thus $x \in M + N \subseteq M$ or $y \in M + N \subseteq M$. Thus M is strongly prime ideal, and R is a *PVD*.

By application of this theorem, we can build more examples of *PVD*.

Example 1.5 Let $\mathbf{C}$ be the field of all complex numbers and $x_1, x_2, \dots, x_n$ be n indeterminates over $\mathbf{C}$. Then R_1 the pullback $(V_1, M_1, \mathbf{R})$ defined by the following square diagram:

$$\begin{array}{ccc} R_1 & \longrightarrow & \mathbf{R} \\ \downarrow & & \downarrow \\ V_1 = \mathbf{C}[x_1]_{(x_1)} & \xrightarrow{\varphi} & V_1 / M_1 = \mathbf{C} \end{array}$$

where $\mathbf{R}$ is the field of real numbers. Then $R_1 = \mathbf{R} + x_1 \mathbf{C}[x_1]_{(x_1)}$ is a *PVD* with $\text{Frac}(R_1) = \mathbf{C}(x_1)$, by Theorem 1.1.

Consider R_2 the pullback ring (V_2, M_2, R_1) defined by the following square diagram:

$$\begin{array}{ccc} R_2 & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ V_2 = \mathbf{C}(x_1)[x_2]_{(x_2)} & \xrightarrow{\varphi} & \mathbf{C}(x_1) = V_2 / M_2 \end{array}$$

$R_2 = R_1 + x_2 \mathbf{C}(x_1)[x_2]_{(x_2)} = \mathbf{R} + x_1 \mathbf{C}[x_1]_{(x_1)} + x_2 \mathbf{C}(x_1)[x_2]_{(x_2)}$ is a PVD with quotient field $\mathbf{C}(x_1, x_2)$, by Theorem 1.4.

Continuing this procedure of construction, we get a new PVD R_n with quotient field $\mathbf{C}(x_1, x_2, \dots, x_n)$ defined by

$$R_n = \mathbf{R} + x_1 \mathbf{C}[x_1]_{(x_1)} + x_2 \mathbf{C}(x_1)[x_2]_{(x_2)} + \dots + x_n \mathbf{C}(x_1, \dots, x_{n-1})[x_n]_{(x_n)}.$$

2. OVERRINGS OF A PSEUDO-VALUATION DOMAIN

Definition 2.1 *An extension of rings $R \subset T$ is said to be incomparable if two prime ideals $Q_1 \subseteq Q_2$ of T such that $Q_1 \cap R = Q_2 \cap R$, implies that $Q_1 = Q_2$.*

Lemma 2.2 *Let R be a PVD and let T be an overring. If Q is proper prime ideal in T , then every prime ideal of R contained in $Q \cap R$ is also a prime ideal of T .*

Proof. Let P be prime ideal of R , with $P \subseteq Q \cap R$. To show that P is an ideal of T , it suffices to show that $t p \in P$ for all $t \in T$ and $p \in P$. Now, $p = (t p) t^{-1} \in P$ implies that $t p \in P$ or $t^{-1} \in P$. However, if $t^{-1} \in P$, then we necessarily have $t^{-1} \in Q$. But, this implies that $t^{-1}t = 1 \in Q$, a contradiction. Thus $t p \in P$ and P is indeed an ideal of T . That P is a prime ideal of T follows easily from the fact that P is strongly prime in R .

Theorem 2.3 *Let R be PVD. If T is an overring of R such that the extension $R \subset T$ satisfies the incomparability, then T is PVD and every prime ideal of T is a prime ideal of R .*

Proof. Let Q be a prime ideal of T . We claim that Q is also a prime ideal of R . Clearly $Q \cap R$ is a prime ideal of R , whence $Q \cap R$ is a prime ideal in T , by Lemma 2.2. Thus $Q \cap R \subseteq Q$ are two prime ideals of T lying over $Q \cap R$ in R . Since the incomparability holds, we must have $Q = Q \cap R$, so that Q is a prime ideal of R . Since R and T have the same quotient field and Q is strongly prime in R , it follows easily that Q is a strongly prime ideal in T . Thus T is a PVD.

If T is an overring of R such that the extension $R \subset T$ is integral, then $R \subset T$ satisfies the incomparability and we can apply Theorem 2.3.

Theorem 2.4 *Let R be PVD. If T is an overring of R such that the extension $R \subset T$ is an integral extension, then*

- (i) T is PVD.
- (ii) R and T have the same prime ideals.

Proof. According to Theorem 2.3, T is PVD. Moreover, every prime ideal of T is a prime ideal of R . It remains to prove that every prime ideal P of R is also a prime ideal of T . By integrality, there is a prime ideal Q of T such that $Q \cap R = P$. We claim that $Q = P$. If $Q \neq P$, then let $v \in Q \setminus P$ and v is integral over R . So there is a dependence relation

$$v^n + r_1 v^{n-1} + \dots + r_{n-1} v + r_n = 0$$

with minimal degree n . We have $r_n \in Tv \cap R \subseteq Q \cap R = P$. If $n = 1$, then we have $v + r_1 = 0$ implies that $v \in P$ a contradiction. Suppose that $n \geq 2$ and let $w = v^{n-1} + r_1 v^{n-2} + \dots + r_{n-1}$. This implies that $v w + r_n = 0$, so $vw \in P$. Since P is strongly prime and $v \notin P$, this implies that $w \in P$. But

$$v^{n-1} + r_1 v^{n-2} + \dots + r_{n-1} - w = 0$$

is another dependence relation with degree $n-1$, contradicts the minimality of n .

Definition 2.5 Let $R \subset S$ be an extension of integral domain. Then the set of all element $s \in S$ that are integral over R , denoted by R' is called the integral closure of R in S . If $S = \text{Frac}(R)$, then R' is called the integral closure of R .

Corollary 2.6 If R is a PVD with maximal ideal M , then R' is a PVD with maximal ideal M and associated valuation domain $(M : M)$.

Lemma 2.7 Let R be a PVD with maximal ideal M . Let T be an overring of R . If $s^{-1} \in T$ for some $s \in M \setminus \{0\}$, then T is a valuation domain.

Proof. Let $y \in T \setminus R$. Then $y = \frac{a}{t}$, where $a \in R$ and $t \in R \setminus \{0\}$. Since aR and tM are comparable, it is necessary that $tM \subset aR$ since $\frac{a}{t} \notin R$. Thus $st = ar$ for some $r \in R$. Hence $a \neq 0$. Thus $y^{-1} = \frac{t}{a} = \frac{r}{s} \in T$ since $s^{-1} \in T$. Hence $T \setminus R \subset U(T)$ the set of units of T . Now, to show that T is a valuation domain, we need only prove that aT and bT are comparable for $a, b \in R$. Since R is a PVD, then aM and bR are comparable. If $aM \subset bR$, then $as = bx$ for some $x \in R$, and hence $a = b(\frac{x}{s}) \in bT$ gives $aT \subset bT$. Otherwise, $bR \subseteq aM$, then $bT \subseteq aT$. Hence T is a valuation domain.

Lemma 2.8 Let R be a PVD with maximal ideal M and quotient field K . If $y \in K \setminus R$, then $y^{-1} \in (M : M)$.

Proof. Since R is a PVD, then $(M : M) = V$ is a valuation domain. Therefore, for $y \in K \setminus R$, either $y \in K \setminus V$, so $y^{-1} \in V$, or $y \in V \setminus R$, so $y \notin M$ and $y^{-1} \in V$. Hence, in both cases, $y^{-1} \in V$.

Theorem 2.8 Let R be a PVD with maximal ideal M . Then the following are equivalent:

- (i) $R' = (M : M)$.
- (ii) Every overring of R is a PVD.

Proof. (i) $\Rightarrow$ (ii) : Suppose that $R' = (M : M)$. Let T be an overring of R . If there is $s \in M \setminus \{0\}$ such that $s^{-1} \in T$, then T is a valuation domain, by Lemma 2.6. Hence T is a PVD. Suppose that $s^{-1} \notin T$ for all $s \in M \setminus \{0\}$. Then let $y \in T \setminus R$, where $y = r / s$, $r \in R$ and $s \in R \setminus \{0\}$. If $s \notin M$, then s is a unit of R , which implies that $y \in R$, a contradiction. So $s \in M \setminus \{0\}$ and $s^{-1} \notin T$. By Lemma 2.7, $y^{-1} \in (M : M) = R'$. Hence y^{-1} is integral over R , so it satisfies a dependence equation:

$$\left(\frac{1}{y}\right)^n + a_{n-1}\left(\frac{1}{y}\right)^{n-1} + \dots + a_1\left(\frac{1}{y}\right) + a_0 = 0,$$

where $a_i \in R$ for $i = 0, 1, 2, \dots, n-1$. Multiply by y^{n-1} , we get

$$\frac{1}{y} + a_{n-1} + \dots + a_1 y^{n-2} + a_0 y^{n-1} = 0.$$

This implies that $\frac{1}{y} = -a_{n-1} - a_{n-2} - \dots - a_1 y^{n-2} - a_0 y^{n-1} \in R[y] \subseteq T$. So y is a unit of T .

Also $y^{-1} \notin R$, otherwise, if $y^{-1} \in R$, then $t = y^{-1} \in M$ and $t \in M \setminus \{0\}$ with $t^{-1} = y \in T$, a contradiction. Consequently, if $y \in T \setminus R$, then $y^{-1} \in T \setminus R$. So $y = (y^{-1})^{-1} \in (M : M) = R'$, by Lemma 2.7.

(1) M is an ideal of T : Indeed, let $x \in T$.

- If $x \in T \setminus R$, then $x \in (M : M)$, so $xM \subseteq M$.
- If $x \in R$, then $xM \subseteq M$.

(2) M is a maximal ideal of T : Let $z \in T \setminus M$. If $z \in T \setminus R$, then z is a unit of T by our previous proof. If $z \in R$, then z is a unit of R , and hence z is a unit of T .

(3) M is the unique maximal ideal of T : Suppose that M' is a maximal ideal of T such that $M' \neq M$. If $a \in M' \setminus M$, then a is a unit of T , a contradiction.

(4) M is a strongly prime ideal of R . This implies that M is a strongly prime ideal of T . Hence T is a *PVD*.

To prove the converse, we need the following two lemmas concerning the pullback rings.

Lemma 2.9 *Let (T, M) be a valuation ring with residue field $T / M = K$, D a subring of K and R the pullback ring (T, M, D) . Then T is integral over R if and only if K is algebraic over D .*

Proof. We have the following square diagram:

$$\begin{array}{ccc} \varphi^{-1}(D) = R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & T/M = K \end{array}$$

It is clear that if T is integral over R , then $T/M = K$ is integral over $R/M = D$ [G].

Suppose now that K is algebraic over D , and let $t \in T$. Then $z = \varphi(t)$ is algebraic over D , so $z^n + d_{n-1}z^{n-1} + \dots + d_0 = 0$ for some $d_0, d_1, \dots, d_{n-1} \in D$. But each $d_i = \varphi(r_i)$ for some $r_i \in R$. Hence $\varphi[t^n + r_{n-1}t^{n-1} + \dots + r_0] = 0$.

It follows that $t^n + r_{n-1}t^{n-1} + \dots + r_0 \in \ker(\varphi) = M$. So $t^n + r_{n-1}t^{n-1} + \dots + r_0 = m \in M$. Therefore, we have $t^n + r_{n-1}t^{n-1} + \dots + (r_0 - m) = 0$ and t is integral over R .

In the case where R is a *PVD*, we have the following:

Lemma 2.10 *Let (R, M) be a PVD. Then $R' = (M : M)$ if and only if $(M : M)/M$ is algebraic over R/M .*

Proof. It suffices to notice that R is the pullback ring

$$\begin{array}{ccc} \varphi^{-1}(k) = R & \longrightarrow & k = R/M \\ \downarrow & & \downarrow \\ V = (M : M) & \xrightarrow{\varphi} & K = V/M \end{array}$$

and apply Lemma 2.9. Indeed, if $R' = (M : M)$, then $V = (M : M)$ is integral over R , so $(M : M)/M$ is algebraic over R/M . Conversely, if $(M : M)/M$ is algebraic over R/M , then this implies that V is integral over R , which implies that $V \subseteq R'$. But by Corollary 2.5, $R' \subseteq V$, so $V = R' = (M : M)$.

Let us complete the proof of Theorem 2.8:

Suppose that $R' \neq (M : M)$. By Corollary 2.5, we have $R' \subset (M : M)$. From Lemma 2.10, $(M : M)/M$ is not algebraic over $R/M = K$. Let $t \in K$, a transcendental element over K . Then $K[t]$ is not a local ring since it has an infinite maximal ideals (for instance $N_a = (t - a)$, where $a \in K$).

Now consider the following diagram:

$$\begin{array}{ccc}
 R' & \longrightarrow & k \\
 \downarrow & & \downarrow \\
 R_1 = \varphi^{-1}(K[t]) & \longrightarrow & K[t] \\
 \downarrow & \varphi & \downarrow \\
 V & \longrightarrow & K
 \end{array}$$

$R_1 = \varphi^{-1}(K[t])$ is an intermediate ring between R' and V with an infinitely many maximal ideals $M_a = \varphi^{-1}((t - a))$, since

$$R_1 / M_a = \varphi^{-1}(K[t]) / \varphi^{-1}((t - a)) \cong K[t] / (t - a) \cong K.$$

So we found a ring R_1 between R' and $(M : M)$ that is not local, so R_1 is not a *PVD* overring of R .

3. NOETHERIAN DOMAINS

Lemma 3.1 *If R Noetherian domain with integral closure R' and M is a maximal ideal of R , then $(M : M) \subseteq R'$.*

Proof. Let $x \in (M : M)$, then $xM \subseteq M$. Since R is Noetherian, then M is finitely generated, say $M = Rz_1 + Rz_2 + \dots + Rz_n$ for non-zero elements $z_1, z_2, \dots, z_n$ of R . Therefore we have

$$\begin{aligned}
x z_1 &= a_{11} z_1 + a_{12} z_2 + \dots + a_{1n} z_n \\
x z_2 &= a_{21} z_1 + a_{22} z_2 + \dots + a_{2n} z_n \\
&\vdots \\
x^{-1} z_n &= a_{n1} z_1 + a_{n2} z_2 + \dots + a_{nn} z_n
\end{aligned}$$

That is

$$\begin{aligned}
(a_{11} - x) z_1 + a_{12} z_2 + \dots + a_{1n} z_n &= 0 \\
a_{21} z_1 + (a_{22} - x) z_2 + \dots + a_{2n} z_n &= 0 \\
&\vdots \\
a_{n1} z_1 + a_{n2} z_2 + \dots + (a_{nn} - x) z_n &= 0
\end{aligned}$$

Since the elements $z_1, z_2, \dots, z_n$ are different from zero, we necessarily have

$$d = \begin{vmatrix} a_{11} - x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - x & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - x \end{vmatrix} = 0$$

which is an equation of integral dependence of x over R . Thus x is integral over R and $x \in R'$.

Theorem 3.2: *Let R be a Noetherian domain with quotient field K and integral closure R' . The following assertions are equivalent:*

- (i) R is a pseudo-valuation domain.
- (ii) For each $x \in K \setminus R$, we have $x^{-1} \in R'$.

Proof. Assume that R is a *PVD* with maximal ideal M . If $x \in K \setminus R$, then $x^{-1}M \subseteq M$, so $x^{-1} \in (M : M)$. Since R is Noetherian, then $(M : M) \subseteq R'$, by Lemma 3.1. Thus $x^{-1} \in R'$. Conversely, assume that $x \in K \setminus R$ and let P be a prime ideal of R . By Lying over theorem [G], there is a prime ideal P' of R' such that $P' \cap R = P$. Since $x^{-1} \in R'$, then $x^{-1}P \subseteq x^{-1}P' \subseteq P'$. We claim that $x^{-1}P \subseteq R$, in which case $x^{-1}P \subseteq P' \cap R = P$, and we are done. To prove the claim, suppose that there exists $p \in P$, with $x^{-1}p \notin R$. Then $x p^{-1} \in R'$ by hypothesis, whence $1 = (x p^{-1})(x^{-1}p) \in P'$, a contradiction.

Corollary 3.3 *If R is a *PVD* and Noetherian, then every overring of R is a *PVD*.*

Proof. In view of Theorem 2.8, it suffices to show that $R' = (M : M)$. But this follows easily by combining Lemma 3.1 and Corollary 2.5. Indeed, since R is *PVD*, we have that inclusion $R' \subseteq (M : M)$, and since R is Noetherian, we have the reverse inclusion $(M : M) \subseteq R'$.

4. STRONG LOCAL HOMOMRPHISMS

We start by the following definition [ADE]:

Definition 4.1 If (R, M) and (S, N) are each local domain, then a ring homomorphism $f: R \rightarrow S$ is called **strong local homomorphisim** (sl-homomorphism, for short) if $f(M) = N$.

In this section, if (R, M) is a local domain with quotient field K , we will denote $T(R) = (M : M) = \{x \in K : xM \subseteq M\}$ and i_R the inclusion map $R \rightarrow T(R)$.

Remarks 4.2 It is easy to verify the following statements:

- (1) The composite of any two sl- homomorphisms is also a sl- homomorphism.
- (2) If (R, M) is a local domain, then the inclusion map $i_R : R \rightarrow T(R)$ is a sl- homomorphism if and only if $T(R)$ is local with maximal ideal M . In particular, if R is a PVD with associated valuation domain V , then the inclusion $i_R : R \rightarrow V$ is a sl- homomorphism .

Theorem 4.3 Let (R, M) and (S, N) be local domains such that S is not a field . If $f : (R, M) \rightarrow (S, N)$ is a sl-homomorphism , then there exists a unique ring homomorphism $T(f) : T(R) \rightarrow T(S)$ such that the following diagram commutes, i.e. $T(f) \circ i_R = i_S \circ f$.

$$\begin{array}{ccc}
 R & \xrightarrow{f} & S \\
 i_R \downarrow & & \downarrow i_S \\
 T(R) & \xrightarrow{F = T(f)} & T(S)
 \end{array}$$

Proof. Since f is a sl- homomorphism and S is not a field, we have $f(M) = N \neq 0$. So we can pick $m_o \in M$ such that $f(m_o) \neq 0$, m_o fixed. Define

$$\begin{aligned}
 F : T(R) = (M : M) & \rightarrow T(S) = (N : N) \\
 z & \rightarrow f(zm_o) / f(m_o)
 \end{aligned}$$

Note that $z \in (M : M)$ implies that $zm_o \in M$. It results that $z = m/m_o$ for some element $m \in M$. Thus $F(z) = f(m) / f(m_o) = f(zm_o) / f(m_o)$

- (1) F is well – defined: We want to show that $F(z)N \subseteq N$. Let $n \in N$, then $n = f(m)$ for some $m \in M$. We observe the following

$$\begin{aligned}
F(z) \cdot n &= F(z) f(m) \\
&= [f(zm_o) / f(m_o)] \cdot f(m) \\
&= f(zm_o m) / f(m_o) \\
&= [f(zm)f(m_o)] / f(m_o) \\
&= f(zm) \quad , \text{ since } z m \in M.
\end{aligned}$$

But $f(z m) \in f(M) = N$. This implies that $F(z) \in (N : N) = T(S)$.

(2) F is independent of the choice of m_o . To see this, let $m_1 \in M$ such that $f(m_1) \neq 0$. For $z \in T(R)$, we have

$$\begin{aligned}
f(zm_o)f(m_1) &= f(zm_o m_1) \\
&= f(zm_1 m_o) \\
&= f(zm_1)f(m_o)
\end{aligned}$$

This implies that $f(z m_o) / f(m_o) = f(zm_1) / f(m_1) = F(z)$.

(3) F is a homomorphism. Let $z_1, z_2 \in T(R)$, then for the multiplication we have:

$$\begin{aligned}
F(z_1 z_2) &= f(z_1 z_2 m_o) / f(m_o) \\
&= [f(z_1 z_2 m_o) \cdot f(m_o)] / [f(m_o)]^2 \\
&= f(z_1 m_o z_2 m_o) / [f(m_o)]^2 \\
&= [f(z_1 m_o) / f(m_o)] \cdot [f(z_2 m_o) / f(m_o)] \\
&= F(z_1) \cdot F(z_2).
\end{aligned}$$

Now for the addition we have:

$$\begin{aligned}
F(z_1 + z_2) &= f[(z_1 + z_2)m_o] / f(m_o) \\
&= f(z_1 m_o + z_2 m_o) / f(m_o) \\
&= [f(z_1 m_o) / f(m_o)] + [f(z_2 m_o) / f(m_o)] \\
&= F(z_1) + F(z_2)
\end{aligned}$$

(4) We have $(F \circ i_R)(r) = (i_S \circ f)(r)$. Indeed,

$$\begin{aligned}
 (F \circ i_R)(r) &= F(r) \\
 &= f(r m_0) / f(m_0) \\
 &= [f(r) \cdot f(m_0)] / f(m_0) \\
 &= f(r) \\
 &= (i_S \circ f)(r)
 \end{aligned}$$

(5) Uniqueness of F : Assume that there exists a homomorphism G from $T(R)$ to $T(S)$ such that $G \circ i_R = i_S \circ f$. Then

$$\begin{aligned}
 G(z) f(m_0) &= G(z) [i_S \circ f(m_0)] \\
 &= G(z) [G \circ i_R(m_0)] \\
 &= G(z) G(m_0) \\
 &= G(z m_0) \\
 &= G[i_R(z m_0)] \\
 &= [G \circ i_R](z m_0) \\
 &= [i_S \circ f](z m_0) \\
 &= i_S[f(z m_0)] \\
 &= f(z m_0)
 \end{aligned}$$

Therefore $G(z) = f(z m_0) / f(m_0) = F(z)$. This implies that F is unique and is denoted by $T(f)$.

Corollary 4.4: *Let (R, M) and (S, N) be PVD with S not a field and V and W are respectively the associated valuation domain of R and S . Then every strong-homomorphism f from R to S can be extended uniquely to a strong homomorphism $T(f)$ from V to W .*

Proof. This result is a direct application of previous theorem. It remains to show that $T(f)$ is a strong local homomorphism.

Note that V has M as a maximal ideal and W has N as a maximal ideal. Now

$$\begin{aligned}
 T(f)(M) &= (T(f) \circ i_R)(M) \\
 &= (i_S \circ f)(M) \\
 &= i_S(f(M)) \\
 &= i_S(N) \\
 &= N
 \end{aligned}$$

The following example shows that the hypothesis " S is not a field" cannot be deleted from Theorem 4.3.

Example 4.5 There exists a sl- homomorphism $F : R \rightarrow S$ such that S is a field and there are infinitely many ring homomorphisms $g : T(R) \rightarrow T(S)$ such that $i_S \circ f = g \circ i_R$.

Let K be a field and let x, y two indeterminates over K . Then

- $V = K(x)[y]_{(y)} = K(x) + M$ is a valuation domain with maximal ideal $M = (y)K(x)[y]_{(y)}$.

- $R = K + M$ is a PVD whose canonically associated valuation domain is $T(R) = (M : M) = V$.

- Consider the sl- homomorphism $f : R \rightarrow S = K(x)$ defined by $f(k + m) = k$ for all $k \in K$ and $m \in M$.

- For each positive integer n , we define $g_n : T(R) = V \rightarrow T(S) = S = K(x)$ by $g_n(F + m) = F(x^n)$ for all $F \in K(x)$ and $m \in M$.

It is easy to check that g_n is a sl- homomorphism and $i_S \circ f = g_n \circ i_R$ for each n . Moreover, we have constructed an infinitely many extensions of f , for if i, j are distinct positive integers, then $g_i(x) = x^i \neq x^j = g_j(x)$.

In the following proposition, we study the decomposition of a sl-homomorphism into two sl-homomorphisms.

Proposition 4.5 *Let $u : R \rightarrow V$ be sl-homomorphism from a local ring (R, M) to a valuation ring (V, N) , then u can be decomposed into two sl-homomorphisms*

$$R \xrightarrow[u_1]{} R_1 \xrightarrow[i_R]{} V$$

such that u_1 is onto and R_1 is PVD with associated valuation V .

Proof. Let $R_1 = u(R)$ and $u_1 : R \rightarrow R_1$ defined by $u_1(x) = u(x)$. Clearly u_1 is onto. Since $\text{Ker}(u)$ is an ideal of R , then $\text{Ker}(u) \subseteq M$. Then $u_1(M) = u(M) = N$ is a prime ideal of R_1 such that $R_1 / N = u_1(R) / u_1(M) \cong R / M$. So N is a maximal ideal of R_1 . Let $z \in R_1 \setminus N$, $z = u(t)$ for $t \in R \setminus M$. As t is a unit in R , then $z = u(t)$ is a unit of $u(R) = R_1$. Therefore R_1 is a local ring with maximal ideal N . By [ChapI, Theorem 3.1], R_1 is a PVD with associated valuation V .

Corollary 4.6 *The following are equivalent.*

- (i) R is PVD
- (ii) *There is an injective sl-homomorphism from R to a valuation domain.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial since if V is the associated valuation domain of R , then take the inclusion sl-homomorphism $u : R \rightarrow V$.

(ii) $\Rightarrow$ (i): Suppose that there is a sl-homomorphism $u : R \rightarrow V$ from R to a valuation domain V . Then by Proposition 4.5, u can be decomposed into two sl-homomorphisms

$$R \xrightarrow{u_1} R_1 \xrightarrow{i_R} V$$

where $R_1 = u_1(R)$ and u_1 is onto. Since u_1 is also injective, then u_1 is an isomorphism. Hence $R_1 \cong R$. Since R_1 is *PVD*, this implies that R is a *PVD*.

It is known that if $f: R \rightarrow S$ is a homomorphism and $u: R \rightarrow T$ is an epimorphism such that $\text{Ker}(u) \subseteq \text{Ker}(f)$, then there is a unique homomorphism $g: T \rightarrow S$ such that $g \circ u = f$. We produce here in the following similar result concerning strong local homomorphism, but without supposing u is onto.

Corollary 4.7 *Let $f: (R, M) \rightarrow (W, M')$ be a sl-homomorphism where W is valuation domain. If $u: (R, M) \rightarrow (V, N)$ is a sl-homomorphism from R to a valuation domain, then there is a unique sl-homomorphism $g: V \rightarrow W$ such that $g \circ u = f$ if and only if $\text{Ker}(u) \subseteq \text{Ker}(f)$.*

Proof. Consider the following diagram:

$$\begin{array}{ccc}
 (R, M) & \xrightarrow{f} & (W, M') \\
 \downarrow u_1 & \nearrow f_1 & \uparrow \\
 R_1 = u_1(R) & & \\
 \downarrow i_R & \nearrow g & \\
 (V, N) & &
 \end{array}$$

It is clear that if g exists, then for every $x \in \text{Ker}(u)$, we have $f(x) = g(u(x)) = g(0) = 0$, i.e. $x \in \text{Ker}(f)$. For the converse, let $R_1 = u_1(R)$. Then the sl-homomorphism $u_1: R \rightarrow R_1$ defined by $u_1(x) = u(x)$ is onto, and satisfies $\text{Ker}(u_1) \subseteq \text{Ker}(f)$. It follows that there is a unique homomorphism $f_1: R_1 \rightarrow W$ such that $f_1 \circ u_1 = f$. Moreover, f_1 is a sl-homomorphism since $f_1(N) = f_1(u_1(M)) = f(M) = M'$. By the Corollary 4.4, there is a unique sl-homomorphism $g: V \rightarrow W$ such that $g \circ i_R = f_1$. Thus $g \circ u = g \circ i_R \circ u_1 = f_1 \circ u_1 = f$.

CHAPTER III

1. VALUATION SUBRING OF AN INTEGRAL DOMAIN

An integral domain R with quotient field K is said to be a valuation domain if it satisfies the following property: whenever $x \in K$, we have $x \in R$ or $x^{-1} \in R$. It is natural to think about domains R satisfying the above property with replacing the field K by a given domain S containing R ; thus the following definition is legitimated [AE].

Definition 1.1 Let $R \subset S$ be an extension of integral domains. We say that R is a *valuation subring of S* (R is a *VD in S* , for short) if R is a subring of S and whenever $x \in S$, we have $x \in R$ or $x^{-1} \in R$.

Recall that a prime ideal P of a ring R is said to be *divided* if $PR_P = P$ [D1]. For instance, it is known that every prime ideal of a valuation domain is divided [G].

Lemma 1.2 Let P be a prime ideal of R . The following assertions are equivalent:

- (i) P is divided.
- (ii) Every ideal I of R is comparable to P .
- (iii) Every principal ideal rR of R is comparable to P .

Proof. (i) $\Rightarrow$ (ii): Let I be an ideal of R . If $I \not\subseteq P$, then $I \cap (R \setminus P) \neq \emptyset$, so $IR_P = R_P$.

There are $a_j \in I$ and $\lambda_j \in R_P$ such that $\sum_{j=1}^n a_j \lambda_j = 1$. For each element p of P , we

have $p = p(1) = \sum_{j=1}^n a_j (\lambda_j p) \in I(PR_P) \subseteq IP \subseteq IR \subseteq I$. Thus $P \subseteq I$.

(ii) $\Rightarrow$ (iii): Is trivial.

(iii) $\Rightarrow$ (i): Let $x \in PR_P$. Suppose by way of contradiction that $x \notin P$. Then $x = p / r$ with $p \in P$ and $r \in R \setminus P$. The principal ideal rR is comparable to P . Since $rR \not\subseteq P$, then $P \subseteq rR$, so $p = ra$ for some $a \in R$. It follows that $p / r = a = x \in R \cap PR_P = P$, a contradiction.

The following result links VD subrings to divided prime ideals and valuation rings. To show that, we introduce the conductor of S in R for an extension $R \subset S$ of integral domains [G]. Recall that

$$[R : S] = \{x \in R : xS \subseteq R\}$$

is called the conductor of S in R . It is easy to show that $[R : S]$ is an ideal shared by R and S . Moreover, $[R : S]$ is the largest ideal shared by R and S , since if I is another ideal shared by R and S , then $IS \subseteq I \subseteq R$, and so $I \subseteq [R : S]$.

Theorem 1.3 *Let $R \subseteq S$ be an extension of rings. Then the following statements are equivalent:*

- (i) R is a VD in S ;
- (ii) There is a divided prime ideal P of R such that $S = R_P$ and the quotient R/P is a valuation domain.

Proof. (i) $\Rightarrow$ (ii): Let P be a maximal ideal of S . We claim that $P \subset R$. To show that, we will prove that $P \cap (S \setminus R) = \emptyset$. In fact, if there exists $x \in P \cap (S \setminus R)$, then $x \in P$ and $x^{-1} \in R \subseteq S$, so $1 = x x^{-1} \in P$, a contradiction. It follows that $PS \subseteq P \subset R$

R , and whence P is contained in the conductor $[R: S]$ of S in R . Consequently, $P = [R: S]$ since P is a maximal ideal of S . It follows that S is a quasilocal domain with maximal ideal $P = [R: S]$.

Clearly, the integral domain R/P may be considered as a subring of the field $K = S/P$. let $u = s + P \in K$, with $s \in S$. Since $s \in R$ or $s^{-1} \in R$, we get $u = s + P \in R/P$ or $u^{-1} = s^{-1} + P \in R/P$. Therefore, R/P is a valuation domain with quotient field K . Since S is a local domain with maximal ideal P , it is easy to see that $R_P \subseteq S_{R-P} = S$. Now, if $s \in S \setminus R$, then $s^{-1} \in R$, so that $s^{-1} \notin P$. It follows that $s = 1 / s^{-1} \in R_P$. Thus $S = R_P$ and $P = PS = PR_P$.

(ii) $\Rightarrow$ (i): Of course, we have $\text{Frac}(R/P) = R_P/PR_P = S/P$. Therefore, if $x \in S$, then $x + P \in S/P = \text{Frac}(R/P)$. As R/P is a valuation domain, we get $x + P \in R/P$ or $(x + P)^{-1} \in R/P$. If we suppose that $x + P \in R/P$, then $x \in R$. Suppose now that $(x + P)^{-1} \in R/P$, then there is $y \in R$ such that $x^{-1} - y \in P$. Thus $x^{-1} \in y + P \subseteq R$ is a VD in S .

Corollary 1.4 *If $R \subset S$ is an extension of integral domains such that R is a VD in S , then R and S are local.*

Proof. The proof of the previous theorem shows that S is local. Furthermore, there is a divided prime ideal P of R such that the quotient R/P is a valuation domain, by Theorem 1.3. Let M/P be the maximal ideal of the local R/P , where M is a maximal ideal of R containing P . Let Q be a prime ideal of R . Since each ideal of R is comparable to P (Lemma 1.2), then either $Q \subseteq P \subseteq M$, or $P \subseteq Q$. In this last case, we find that Q/P is an ideal of R/P , so $Q/P \subseteq M/P$ and $Q \subseteq M$. Hence R is local with maximal ideal M .

2. PSEUDO-VALUATION SUBRING OF AN INTEGRAL DOMAIN

Now, we introduce a definition of pseudo-valuation subring R of an integral domain S [AA]. We denote by $N(R)$ the set of all non unit elements of R .

Definition 2.1 Let $R \subset S$ be an extension of integral domains. We say that R is a *pseudo-valuation subring* of S (R is a **PVD** in S , for short) if R is a subring of S and $x^{-1}a \in R$, for each $x \in S \setminus R$, $a \in N(R)$.

Remarks 2.2 Let R be an integral domain.

- (1) R is a *PVD* in its quotient field $S = \text{Frac}(R)$ means that R is a *PVD*.
- (2) If R is a *PVD* in S , then $\text{Frac}(R) = \text{Frac}(S)$. Indeed, if $x \in S \setminus R$, $a \in N(R)$, then $ax^{-1} = r \in R \setminus \{0\}$, so $x = a/r \in \text{Frac}(R)$.
- (3) If R is a *VD* in S , then R is *PVD* in S . Indeed, if $x \in S \setminus R$, then $x^{-1} \in R$, whence $x^{-1}a \in R$, for each $a \in N(R)$.

Proposition 2.3. *If R is PVD in S , then R is local.*

Proof. Suppose that R is not local. Let M and N be two distinct maximal ideals of R . Then $M + N = R$. Hence there exist $m \in M$, $n \in N$ such that $m + n = 1$. For $x \in S \setminus R$, we have $x^{-1}m$, $x^{-1}n \in R$. Thus $x^{-1} \in R$. It follows that R is a valuation subring of S , against our assumption, by corollary 1.4.

In the following, we generalize the notion of strongly prime ideal, to be convenient with this global study [DF4], [B6].

Definition 2.4 Let $R \subset S$ be an extension of integral domains. A prime ideal P of R is said to be ***S-strong*** if whenever $x, y \in S$ satisfy $xy \in P$, then either $x \in P$ or $y \in P$. If each prime ideal of R is S -strong, we say that S is a ***strong extension*** of R .

Proposition 2.5. *If R is a PVD in S , then S is a strong extension of R .*

Proof. Let P be a prime ideal of R and $a, b \in S$ such that $ab \in P$. If $a, b \in R$, then there is nothing to prove. Suppose that one of the two elements a, b is not in R ; say for instance $b \notin R$. As $ab \in N(R)$, we get $b^{-1}(ab) = a \in R$. If a is a unit of R , then $b = a^{-1}(ab) \in P$, a contradiction, so that $a \in N(R)$. Thus $b^{-1}a \in R$. Therefore, $a^2 = (b^{-1}a)(ab) \in P$ and consequently, $a \in P$.

Remark 2.6 The converse of the above Proposition does not hold. It suffices to take a non-local domain R and $S = R[X]$. Then R is not PVD in S since R is assumed to be not local. However, S is a strong extension of R . Indeed, let P be a prime ideal of R and $fg \in P$, for some $f, g \in S = R[x]$. So $\deg(fg) = 0$. This implies that $\deg(f) + \deg(g) = 0$. So $\deg(f) = \deg(g) = 0$, and f and g are two elements of R . Since P is a prime ideal of R , then $f \in P$ or $g \in P$.

The following Theorem provides a characterization of a PVD subring R of an integral domain S . One can notice the similarity of this result with [Chap I, Theorem 1.4].

Theorem 2.7 *Let R be a subring of a domain S . Then the following statements are equivalent:*

- (i) R is a PVD in S .
- (ii) For each $x \in S \setminus R$ and each $P \in \text{spec}(R)$, we have $x^{-1}P \subseteq P$.
- (iii) For each $x \in S \setminus R$ and each $a \in N(R)$, we have $(x + a)R = xR$.

Proof. (i) $\Rightarrow$ (ii): Let $p \in P$ and $x \in S \setminus R$. Then, $x^{-1}p \in R$, so $p = ax$ for some $a \in R$. As $p = ax \in P$ and $x \notin P$, then $a \in P$, by Proposition 2.5. Thus $x^{-1}p \in P$.

(ii) $\Rightarrow$ (iii): Let $a \in N(R)$, and $x \in S \setminus R$. Then there exists a prime ideal P of R such that $aR \subseteq P$. Hence $\frac{a}{a+x} \in P$, since $(x+a) \in S \setminus R$. But

$$\frac{x+a}{x} = 1 + x^{-1}a \in 1 + P \subseteq R,$$

so that $(x+a)R \subseteq xR$. On the other hand, $\frac{x}{a+x} = 1 - \frac{a}{a+x} \in 1 + P \subseteq R$, and consequently, $xR \subseteq (x+a)R$.

(iii) $\Rightarrow$ (i): Let $x \in S \setminus R$ and $a \in N(R)$. Then $(x+a)R = xR$. Hence $\frac{x+a}{x} \in R$, i.e., $1 + ax^{-1} \in R$. Thus $ax^{-1} \in R$.

We will produce another characterization of a *PVD* subring R of an integral domain S . We need the following definition [OK]. Let P be a prime ideal of a ring R and let S be an overring of R . Then (P, S) is called a **comparable pair** if $P \subseteq xR$ or $xR \subseteq P$ for each element $x \in S$.

Theorem 2.8 *Let $R \subseteq S$ be an extension of rings such that R is local with maximal ideal M . Then the following statements are equivalent:*

- (i) R is a *PVD* in S .
- (ii) For each $x \in S \setminus R$, $x^{-1}M \subseteq M$.
- (iii) (M, S) is a comparable pair.

Proof. (i) $\Rightarrow$ (ii): From Theorem 2.7.

(ii) $\Rightarrow$ (iii): Let $x \in S$. If $x \in M$, then $xR \subseteq M$. Let $x \in S \setminus M$.

- If $x \in R$, then x is a unit of R and $M \subseteq xR = R$.
 - If $x \notin R$, then $x^{-1}M \subseteq M$, so $M \subseteq Mx \subseteq xR$.
- (iii) $\Rightarrow$ (i): Let $x \in S \setminus R$. Then $M \subseteq xR$. Hence $x^{-1}M \subseteq R$, and $x^{-1}a \in x^{-1}M \subseteq R$, for every $a \in N(R)$. Therefore, R is a *PVD* in S .

Corollary 2.9 *Let R be a *PVD* in S and M the maximal ideal of R . Then for each prime ideal $P \neq M$ of R , the localization R_P is a *VD* in S_P .*

Proof. Let $z \in S_P$. Then $z = s/\alpha$ with $s \in S$ and $\alpha \in R \setminus P$. This implies that $z\alpha \in S$.

- If $z\alpha \in R$, then $z = z\alpha/\alpha \in R_P$.
- If $z\alpha \notin R$, then $(z\alpha)^{-1}M \subseteq M$, by Theorem 2.8. Pick an element $m \in M \setminus P$, we find that $z^{-1} = (z\alpha)^{-1}m\alpha m^{-1} \in Rm^{-1} \subseteq R_P$.

If R is a *PVD* in S and M is the maximal ideal of R , then two cases have to be considered:

- If $SM \neq S$, we say that we have a ***first type*** of *PVD* subrings.
- If $SM = S$, we say that we have a ***second type*** of *PVD* subrings.

The next two sections deal with these two types.

3. FIRST TYPE OF PSEUDO-VALUATION SUBRINGS

This section is devoted to the study of pseudo-valuation subrings R of a ring S such that $MS \neq S$, where M is the maximal ideal of R . Before embarking in the study of such *PVD* subrings, we need a new characterization of divided prime ideal in the following sense.

Lemma 3.1 *Let P be a prime ideal of an integral domain. The following assertions are equivalent:*

- (i) P is divided.
- (ii) P is R_P -strong.
- (iii) R is the pullback ring $(R_P, PR_P, R/P)$.

Proof. (i) $\Rightarrow$ (ii): Let $x, y \in R_P$. If $xy \in P$, then $xy \in PR_P = P$. As PR_P is a prime ideal of R_P , then $x \in PR_P = P$ or $y \in PR_P = P$.

(ii) $\Rightarrow$ (i): If P is not divided, then there is $z \in PR_P$ and $z \notin P$. As $z = p/\alpha$ with $p \in P$ and $\alpha \in R \setminus P$, then $z\alpha = p \in P$ with $z \notin P$ and $\alpha \notin P$. It results that P is not R_P -strong.

(i) $\Rightarrow$ (iii): Since $PR_P = P$, then P is shared by R and R_P . If R_1 is the pullback ring $(R_P, PR_P, R/P)$ then $R_1 = \varphi^{-1}(R/P) = R + PR_P = R + P = R$ (See the square diagram below).

$$\begin{array}{ccc}
 R = R_1 & \xrightarrow{\quad} & R/P \\
 \downarrow j & & \downarrow i \\
 R_P & \xrightarrow{\quad \varphi \quad} & R_P/PR_P \cong \text{Frac}(R/P)
 \end{array}$$

(iii) $\Rightarrow$ (i): If R is the pullback ring $(R_P, PR_P, R/P)$, then $R = \varphi^{-1}(R/P) = R + PR_P$, so $PR_P \subseteq R$. By contraction we get $PR_P \cap R = PR_P = P$. Thus P is divided.

The set of all prime ideal of an integral domain R is called **spectra** of R and denoted $\text{Spec}(R)$.

Lemma 3.2 *Let $R \subseteq S$ be an extension of integral domains such that R is local with maximal ideal M . Then the following statements are equivalent:*

- (i) $\text{Spec}(R) = \text{Spec}(S)$.
- (ii) S is local with maximal M .

Proof. The implication (i) $\Rightarrow$ (ii): is trivial. Let us prove (ii) $\Rightarrow$ (i). R and S share the same maximal ideal M . So if $Q \in \text{Spec}(S)$, then $Q \subseteq M \subseteq R$, then $Q \cap R = Q$ is a prime ideal of R . Thus $\text{Spec}(S) \subseteq \text{Spec}(R)$. Let $P \in \text{Spec}(R)$. If $P = M$, then $P \in \text{Spec}(S)$. Suppose that $P \subset M$. Pick $m \in M \setminus P$ and consider the multiplicative set $N = \{ m^t : t \geq 0 \}$. Then $R_N = S_N$. So $PR_N = PS_N$ is a prime ideal of S_N since $P \cap N = \emptyset$. Set $Q = PS_N \cap S$, then $Q \in \text{Spec}(S)$ and $Q \cap R = PS_N \cap S \cap R = PR_N \cap R = P$. But $Q \subseteq M$, then $Q \cap R = Q = P \in \text{Spec}(S)$.

Theorem 3.3 *Let $R \subseteq S$ be an extension of integral domains such that R is local with maximal ideal M . Suppose that $MS \neq S$. Then the following statements are equivalent:*

- (i) R is a PVD in S .
- (ii) M is a divided prime ideal of S .
- (iii) M is a prime ideal of S and $\text{Spec}(R) = \text{Spec}(S_M)$.
- (iv) M is a prime ideal of S and for each $z \in S_M \setminus R$, $z^{-1} \in S_M$.
- (v) M is a prime ideal of S and R is a PV_d in S_M .
- (vi) M is a prime ideal of S and for each $z \in S \setminus R$, $M = zM$.
- (vii) M is a prime ideal of S and R is the pullback ring $(S_M, MS_M, R/M)$.

Proof: (i) $\Rightarrow$ (ii): Let Q be a maximal ideal of S containing MS . Then $M \subseteq MS \cap R \subseteq Q \cap R$, and so $Q \cap R = M$ since M is the maximal of R . Let $s \in S$. If $Ms \not\subseteq R$, then $ms \notin R$ for some $m \in M$. As $ms \in S$, we get $(ms)^{-1}m \in M$, by Theorem 2.8.

Hence $s^{-1} \in M$, so that $1 = s (s^{-1}) \in Q$, against the maximality of Q . It follows that $Ms \subseteq R$. Now, we will prove that $Ms \subseteq M$. Indeed, if Ms contains a unit of R , then $Ms = R$. Consequently, $M = Rs^{-1}$, and so $s^{-1} \in M$, this gives, as previously, $1 \in Q$. Thus, $Ms \subseteq M$ and M is an ideal of S . As M is S -strong, by Proposition 2.5, then M is a prime ideal of S . It remains to show that $MS_M = M$. Let $x \in MS_M$, then $x = m/q$ for some $m \in M$ and $q \in S \setminus M$.

- If $q \in R$, then q is a unit of R , so that $x = mq^{-1} \in M$.
- If $q \notin R$, then $q^{-1}M \subseteq M$, i.e., $M \subseteq qM$. In particular, there is an element $m' \in M$ such that $m = qm'$. Thus $x = m' \in M$.

(ii) $\Rightarrow$ (iii): Now, $R \subseteq S_M$ is a pair of rings with the same maximal ideal $MS_M = M$. Thus $\text{Spec}(R) = \text{Spec}(S_M)$, by Lemma 3.2.

(iii) $\Rightarrow$ (iv): Let $z \in S_M \setminus R$. Since $MS_M = M \subset R$, we have $z \notin MS_M$. But S_M is local with maximal ideal MS_M . Thus z is a unit of S_M .

(iv) $\Rightarrow$ (ii): Let $z \in MS_M$. Then $z^{-1} \notin S_M$. It follows that $z \in R$, so that $z \in MS_M \cap R = M$.

(ii) $\Rightarrow$ (i): Let $z \in S \setminus R$. Then $z \notin M$. Hence $(1/z)M \subseteq MS_M = M$. Hence R is PVD in S by Theorem 2.8.

Thus we have proved the following equivalences:

$$(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv).$$

(v) $\Rightarrow$ (i): If an element $x \in S \setminus R$, then $x \in S_M \setminus R$, and this implies that $x^{-1}M \subseteq M$ since R is supposed to be PVD in S_M . Hence R is PVD in S by Theorem 2.8.

(iv) $\Rightarrow$ (v): Let $z \in S_M \setminus R$. Then $z^{-1} \in S_M$ and consequently, $z^{-1}MS_M \subseteq MS_M$. But M is a divided prime ideal of S , thus $z^{-1}M \subseteq M$.

(vi) $\Rightarrow$ (i): Let $z \in S \setminus R$, then $M = zM$, so $z^{-1}M = M$. Hence R is PVD in S .

(i) $\Rightarrow$ (vi): Let $z \in S \setminus R$, we have $z^{-1}M \subseteq M$, so $M \subseteq zM$. But M is an ideal of S , so $zM \subseteq M$. Therefore, $M = zM$.

(ii) $\Leftrightarrow$ (vii): Straightforward by Lemma 3.1.

Corollary 3.4 Let $R \subset S$ be an extension of integral domains such that R is a local with maximal ideal M . If $MS \neq S$ and $R \subset S$ incomparable (in particular, if S is integral over R), then the following statements are equivalent:

- (i) R is a PVD in S .
- (ii) $\text{Spec}(R) = \text{Spec}(S)$.
- (iii) S is local with maximal ideal M .

Proof. (ii) $\Leftrightarrow$ (iii): From Lemma 3.2.

(iii) $\Rightarrow$ (i): This implication comes directly from Theorem 3.3(ii) since M , as a maximal ideal of S , is a divided prime ideal of S .

(i) $\Rightarrow$ (iii): Since M is a divided prime ideal of S , it is comparable to each prime ideal of S . If Q is a maximal ideal of S , then $M \subseteq Q$. As $M \cap R = Q \cap R = M$, then $M = Q$, by the incomparability property.

Remark 3.5

- 1) In Corollary 3.4, (ii) $\Leftrightarrow$ (iii) is always satisfied in view of Lemma 3.2.
- 2) The implication (iii) $\Rightarrow$ (i) always holds according to Theorem 3.3(ii) since M , as a maximal ideal of S , is divided prime ideal. Therefore, if $R \subset S$ is an extension of integral domains such that $\text{Spec}(R) = \text{Spec}(S)$, then R is PVD in S (indeed, in this case the extension $R \subset S$ satisfies the incomparability property and $MS = M \neq S$).

Example 3.6 Let R be local domain with maximal ideal M . If R is PVD in S and $MS \neq S$, then S is not necessarily local:

Let k be a field and $L = k(Y)$ be the quotient field of the polynomial ring $k[Y]$. Set $T = L[[X]]$ and $M = XT$. Let R and S be respectively the two pullbacks rings $S = (T, M, k[Y])$ and $R = (S, M, k)$ illustrated by the following two square diagrams:

$$\begin{array}{ccc}
R & \longrightarrow & R/M \\
\downarrow & & \downarrow \\
S & \longrightarrow & k[Y] \\
\downarrow & & \downarrow \\
T = L[[X]] & \longrightarrow & L = k(Y)
\end{array}$$

Clearly, S is not quasilocal since $S/M \cong k[Y]$, and M is a prime ideal of S . We claim that M is divided in S . In fact, to show that $MS_M = M$, it suffices to prove that $S_M = T$. Since $S \subseteq T$, then $S_M \subseteq T_M = T$. For the reverse containment, let $t \in T$. It is clear that if $t \in M$, then $t \in S \subseteq S_M$. Suppose that $t \notin M$. Set $t' = \varphi(t)$, then $t' \in L = \text{Frac}(k[Y])$ and $t' \neq 0$. There are $\alpha' \in k[Y] \setminus \{0\}$ and $s' \in k[Y]$ such that $t' = \frac{s'}{\alpha'}$. As φ is onto, there are s and $\alpha \in S$ such that $s' = \varphi(s)$ and $\alpha' = \varphi(\alpha)$. Moreover, $s \notin M$ since $s' \neq 0$. It follows that $\varphi(t) = \frac{\varphi(s)}{\varphi(\alpha)}$, i.e., $\varphi(t\alpha) = \varphi(s)$. Hence $t\alpha - s = m \in M = \text{Ker}(\varphi)$. Therefore, $t = \frac{s+m}{\alpha} \in S_M$. Consequently, $S_M = T$, $MS_M = MT = M$ and so R is PVD in S by Theorem 3.3(ii).

4. SECOND TYPE OF PSEUDO-VALUATION SUBRINGS

This section is devoted to the study of pseudo-valuation subrings R of an integral domain S such that $MS = S$, where M is the maximal ideal of R .

Lemma 4.1. *Let R be a PVD in S of the second type. Then for each $x \in S \setminus R$, we have $x^{-1} \in S$.*

Proof. Let M be the maximal ideal of R . As $MS = S$, we get $\sum_{i=1}^n m_i s_i = 1$ for some elements $m_i \in M$ and $s_i \in S$. If $x \in S \setminus R$, then $x^{-1}m_i \in M$ for each $1 \leq i \leq n$. Thus $x^{-1} = \sum_{i=1}^n (x^{-1}m_i)s_i \in S$.

Notations 4.2 Let (R, M) be a quasilocal domain and S be a domain containing R . Define

$$(M : M)_S = \{x \in S \mid xM \subseteq M\}$$

and

$$M_S^{-1} = \{x \in S \mid xM \subseteq R\}.$$

Note that $(M : M)_S$ is strictly contained in S , when $MS = S$ while $M(M : M)_S = M$.

Theorem 4.3 *Let $R \subset S$ be an extension of integral domain such that R is local with maximal ideal M . Suppose that $MS = S$. Then the following statements are equivalent:*

- (i) R is PVD in S .
- (ii) $(M : M)_S$ is a VD in S with maximal ideal M .

If one of the two statements is satisfied, then $(M : M)_S$ is the unique intermediate ring between R and S that is a valuation subring of S with maximal ideal M .

Proof. (i) $\Rightarrow$ (ii): As R is PVD in S , then for each $x \in S \setminus R$, $x^{-1}M \subseteq M$, so that $x^{-1} \in (M : M)_S$. Thus $(M : M)_S$ is VD in S . To show that M is the maximal ideal of $(M : M)_S$, let $x \in (M : M)_S$ be not unit. If $x \notin M$, then $x \notin R$, so $x^{-1}M \subseteq M$; thus $x^{-1} \in (M : M)_S$, a contradiction. Therefore $x \in M$.

(ii) $\Rightarrow$ (i): If $(M : M)_S$ is VD in S with maximal ideal M , then $\text{Spec}(R) = \text{Spec}((M : M)_S)$, by Lemma 3.2. Let $x \in S \setminus R$.

- If $x \in (M : M)_S$, then $x \in (M : M)_S \setminus M$. Hence x is a unit of $(M : M)_S$. Hence $x^{-1} \in (M : M)_S$ and $x^{-1}M \subseteq M$.

- If $x \notin (M : M)_S$, then $x^{-1} \in (M : M)_S$ since $(M : M)_S$ is a valuation subring of S . Thus $x^{-1}M \subseteq M$. It follows that R is *PVD* in S , by application of Theorem 3.8.

For the uniqueness, let T be a subring of S containing R such that T is *VD* in S with maximal ideal M . Since $T \subseteq (M : M)_S$, then T is *VD* in $(M : M)_S$ (if $x \in (M : M)_S \setminus T$, then $x \in (M : M)_S \setminus R$, so $x^{-1} \in R \subseteq T$). Now, by Theorem 1.3, there is a divided prime ideal Q of T such that $T_Q = (M : M)_S$; hence $QT_Q = Q = M$ and $(M : M)_S = T_Q = T$.

Corollary 4.4 *Let $R \subseteq S$ be an extension of domains. Then the following statements are equivalent:*

- (i) R is *VD* in S with maximal ideal M .
- (ii) R is *PVD* in S with maximal ideal M and $R = (M : M)_S$.

Proof. (ii) $\Rightarrow$ (i) results directly from Theorem 4.3. Let us prove (i) $\Rightarrow$ (ii). We know in view of Remark 2.2 that each *VD* in S is *PVD* in S . Furthermore, it is clear that $R \subseteq (M : M)_S$. For the reverse containment, let $x \in S \setminus R$, then $x^{-1} \in R$, so $x^{-1} \in M$. As $x x^{-1} = 1 \notin M$, we necessarily have $x \notin M$. Therefore, $x \notin (M : M)_S$.

Lemma 4.5 *Let (R, M) be a quasilocal domain which is *PVD* in a domain S . If R has a non zero principal prime ideal, then $R = (M : M)_S$.*

Proof. Set $T = (M : M)_S$. According to Theorem 4.3, T is quasilocal with maximal ideal M . Hence $\text{Spec}(R) = \text{Spec}(T)$, by Lemma 3.2. Suppose that R has a nonzero principal prime ideal $P = rR$. Then P is a nonzero prime ideal of T . If $t \in T$, then, as

$P = TP = rT$, we get $tr \in P$, so $tr = \alpha r$ for some $\alpha \in R$. Hence, $t = \alpha \in R$. Therefore, $R = T$.

Proposition 4.6 Let $R \subset S$ be an extension of integral domains such that R is local with maximal ideal M and $R \neq (M : M)_S$. Then the following statements are equivalent:

- (i) R is PVD in S ;
- (ii) M_S^{-1} is a valuation subring of S with maximal ideal M .

Proof. It is sufficient to show that $M_S^{-1} = (M : M)_S$ and then apply Theorem 4.3. It is clear that $(M : M)_S \subseteq (R : M)_S$. For the reverse containment, let $x \in (R : M)_S$, then $xM \subseteq R$. Hence xM is an ideal of R . If we suppose that $xM \not\subseteq M$, then $xM = R$, so that $M = Rx^{-1}$ is a principal prime ideal of R and consequently, $R = (M : M)_S$ by Lemma 4.5, a contradiction. Hence $xM \subseteq M$, i.e., $x \in (M : M)_S$.

Now, we are in a position to give a characterization of subring PVD of an integral domain of the second type.

Theorem 4.7 Let $R \subset S$ be an extension of domains such that R is local with maximal ideal M and $MS = S$. Then the following statements are equivalent:

- (i) R is PVD in S .
- (ii) There is a divided prime ideal P of R such that $S = R_P$ and R/P is a PVD.
- (iii) $S = R_P$ for a prime ideal P of R and $R \subset S$ is strong.
- (iv) $S = R_P$ for a divided prime ideal P of R and M is S -strong.
- (v) For $z \in S \setminus R$, $z^{-1} \in S$, and M is S -strong.

Proof. (i) $\Rightarrow$ (ii): Set $T = (M : M)_S$, then we have $R \subseteq T \subset S$ and T is a VD in S with maximal ideal M (The inclusion $T \subset S$ is strict since $MT \subseteq M$ while $MS = S$). By

Theorem 1.3, there exists a divided prime ideal P of T such that $S = T_P$ and T/P is a valuation domain. As $\text{Spec}(R) = \text{Spec}(T)$ by Lemma 3.2, then P is a prime ideal of R . Moreover $P \subset M$ ($P \neq M$ since $PS = PT_P = P$ while $MS = M$). We claim that $R_P = T_P = S$. Indeed, it is clear that $R_P \subseteq T_P$. Let $z \in T_P$, then $z = t/a$ with $t \in T$ and $a \in T \setminus P$. Pick $m \in M \setminus P$, we have $z = (tm)/(am)$ with $tm \in R$ and $am \in R \setminus P$. Hence $z \in R_P$. Thus $T_P \subseteq R_P$ and $R_P = T_P = S$. It follows that $PR_P = PT_P = P$, i.e., P is a divided prime ideal of R . It remains to show that R/P is a pseudo-valuation domain. The quotient field of R/P is $\text{Frac}(R/P) = R_P/PR_P = S/P$. Let $\bar{z} = z + P$ an element of $\text{Frac}(R/P)$ such that $\bar{z} = z + P \notin R/P$, where $z \in S$. Then $z \in S \setminus R$, so that $z^{-1}M \subseteq M$. Since S is local with maximal ideal P , we have $z^{-1} \in S$. Let $\varphi : S \rightarrow S/P$ be the canonical epimorphism. Then $\varphi(z^{-1}M) = \varphi(z^{-1})\varphi(M) \subseteq \varphi(M)$. Hence, $\overline{z^{-1}}(M/P) \subseteq M/P$. Thus R/P is *PVD* in its quotient field, that is R/P is a *PVD*.

(ii) $\Rightarrow$ (iii): Suppose that $S = R_P$ for some prime ideal P of R . To show that $R \subset R_P$ is a strong extension, we will prove that each prime ideal Q of R is R_P -strong. Note first that Q is comparable to P , by Lemma 1.2, and that P is R_P -strong since P is divided, by Lemma 3.1. Let $x, y \in R_P$ such that $xy \in Q$. If $P \subset Q$, then $(x + P)(y + P) \in Q/P$. As R/P is a *PVD* with quotient field $R_P/P = R_P/PR_P = \text{Frac}(R/P)$ by hypothesis, then Q/P is R_P/P -strong. It follows that either $x + P \in Q/P$ or $y + P \in Q/P$. Thus Q contains either x or y , as required. Suppose now that $P \subset Q$. If both x and y are in R , the assertion follows since Q is prime. However, at least one of x, y is in P because xy belongs to P and $P = PR_P$ is a prime ideal of R_P . Accordingly, we may suppose that $x \in P$ and $y \in R_P \setminus P$. In this case $x = (xy)y^{-1} \in QR_P \cap R = Q$, as desired.

(iii) $\Rightarrow$ (iv): As $R \subset R_P = S$ is a strong extension, then M is S -strong. Furthermore, since P is R_P -strong, then P is divided, by Lemma 3.1.

(iv) $\Rightarrow$ (v): Let $z \in S \setminus R$. Then $z \notin P = PR_P$. Hence z is a unit of $R_P = S$.

(v) $\Rightarrow$ (i): Let $x \in S \setminus R$ and $m \in M$. We have $(x^{-1}m)x \in M$. Now, as $x^{-1} \in S$, $x \notin M$ and M is S -strong, we get $x^{-1}m \in M$. Therefore, R is PVD in S , by Theorem 2.8.

Proposition 4.9. *Let R be an integral domain. Then the following statements are equivalent:*

- (i) R is a PVD .
- (ii) R is PVD in a valuation overring.

Proof. (i) $\Rightarrow$ (ii): Let V be the valuation ring associated to R . As R is PVD in its quotient field, then R is PVD in V .

(ii) $\Rightarrow$ (i): Suppose that R is PVD in a valuation overring V and let M be the maximal ideal of R . Let $x \in K \setminus R$. If $x \in V$, then $x \in V \setminus R$, so $x^{-1}M \subseteq M$. Suppose now that $x \notin V$. Since V is a valuation domain, then $x^{-1} \in V$. To show that $x^{-1}M \subseteq M$, we consider two cases:

- If $MV = M$, then R is a PVD in V of first type. By application of Theorem 3.3, M is a prime ideal of V , so that $x^{-1}M \subseteq M$.
- If $MV \neq V$, then R is a PVD in V of second type. By application of Theorem 4.7, there exists a divided prime ideal P of R such that $V = R_P$. Whence, $x^{-1} \in P \subset R$, and $x^{-1}M \subseteq M$.

It follows that $x^{-1}M \subseteq M$ for every $x \in K \setminus R$. Hence, R is a PVD , by Theorem 2.8.

5. PULLBACK CONSTRUCTIONS.

It is always of interest to study pullback constructions in order to provide examples and counter examples.

Theorem 5.1 *Let S be an integral domain, N a maximal ideal of S , $K = S / N$, D a subring of K and R be the pullback ring of (S, N, D) . Then R is *PVD* in S if and only if either*

- (i) $\text{Spec}(R) = \text{Spec}(S)$, or
- (ii) $S = R_N$, $NR_N = N$ and D is *PVD*.

Proof. (i) $\Rightarrow$ (ii): Suppose that R is *PVD* in S . Then R is local with maximal ideal M . As N is a common ideal of R and S , we have $N \subseteq M$.

- If $MS \neq S$, then $M = N$, by Theorem 3.3. Thus $\text{Spec}(R) = \text{Spec}(S)$, by Lemma 3.2.
 - If $MS = S$, then $N \subset M$, so $S = R_N$, $N = NR_N$ and $R/N \cong D$ is *PVD*, by Theorem 4.7.
- (ii) $\Rightarrow$ (i): If $\text{Spec}(R) = \text{Spec}(S)$, it is clear that R is *PVD* in S , by Remark 3.5. Now, if $S = R_N$, $NR_N = N$ and D is *PVD*, then $R/N \cong D$ is *PVD*. This shows, in particular, that R is local with a maximal ideal M strictly containing N ($M \neq N$ since if $M = N$, then $\text{Spec}(R) = \text{Spec}(S)$ and R is local since if R has two maximal ideals M and M' , then these maximal ideals are comparable to N , so M and M' contain N . It follows that M / N and M' / N are two maximal ideals of $R / N \cong D$, a contradiction). Moreover, we have $MS = S$, then by application of Theorem 4.7, we can say that R is *PVD* in S .

The following result gives condition making *PVD* a transitive property.

Proposition 5.2 *Let R be *PVD* in T and M the maximal ideal of R . If $MT \neq T$, and if T is *PVD* in S , then R is *PVD* in S .*

Proof. In view of Theorem 3.3, M is a prime ideal of T . Let $z \in S \setminus R$. Two cases have to be considered:

- If $z \in T$, then $z \in T \setminus R$, and so $z^{-1}M \subseteq M$, since R is *PVD* in T .

- If $z \notin T$, then $z \in S \setminus T$, so $z^{-1}M \subseteq M$ since T is *PVD* in S .

The following example shows that, without the condition $MT \neq T$ in Proposition 5.2 does not hold.

Example 5.3

Let K be a field, V_1 the valuation ring $V_1 = K(X, Y, Z)[t]_{(t)} = K(X, Y, Z) + N$, where $N = tV_1$, and let $T = K(X, Y) + N$. Then T is a *PVD* with quotient field $K(X, Y, Z, t)$, which is not a valuation ring (Chapter II, Theorem 1.1) (in fact $Z \in V_1$, but $Z \notin T$ and $Z^{-1} \notin T$). Let $V_2 = K(X)[Y]_{(Y)}$; then V_2 is a valuation ring with maximal ideal $N' = YV_2$. The domain $D = K + N'$ is a *PVD*, again by (Chapter II, Theorem 1.1).

Now, let us consider $R = D + N = K + M$ with $M = N + N'$. Then R is a pullback ring illustrated by the following square diagram

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/N \cong K(X, Y) \end{array}$$

As $\text{Frac}(D) = K(X, Y)$, then $NR_N = N$, $T = R_N$ by a similar argument to Example 3.6), and $R/N \cong D$ is a *PVD*.

Hence, by Theorem 5.1, R is *PVD* in T with $MT = T$. But $N \subset M$ and $R_N = T$ is not a valuation ring. Thus R is not a *PVD* (Chapter I, Proposition 2.4). Therefore, R is *PVD* in T and T is a *PVD* with $MT = T$, but R is not *PVD*.

Corollary 5.4 Let R be a local ring with maximal ideal M . Suppose that the integral closure R' of R in S is strictly contained in S . Then R is *PVD* in S if and only if R' is *PVD* in S with maximal ideal M .

Proof. If R is PVD in S , then R is PVD in R' . Hence R' is local with maximal ideal M , by Corollary 3.4. Moreover $R' \subseteq (M : M)_S$. As $(M : M)_S$ is a VD subring of S with maximal ideal M , then R' is PVD in S , by Theorem 4.3. Conversely, suppose that R' is PVD in S with maximal ideal M ; then $\text{Spec}(R) = \text{Spec}(R')$, by Lemma 3.2 and consequently, R is PVD in R' , by remark 3.5. Finally, as $MR' \neq R'$, we conclude that R is PVD in S , by Proposition 5.2.

Proposition 5.5 *Let R be a local ring with maximal ideal M . Suppose that R is PVD in S , then each ring T between R and S is comparable to the ring $(M : M)_S$.*

Proof. The result is clear if $(M : M)_S$ is equal to R or S . Let us suppose that we have the strict inclusions $R \subset (M : M)_S \subset S$. Two cases arise:

- If $MT \neq T$ and R is PVD in T , then $M \in \text{Spec}(T)$ by Theorem 3.3, and consequently $T \subseteq (M : M)_S$.

- If $MT = T$, then there exists $t_1, t_2, \dots, t_n \in T$ and $m_1, m_2, \dots, m_n \in M$ such that

$\sum_{i=1}^n m_i t_i = 1$. For each $z \in (M : M)_S$, we have $zM \subseteq M$. Then $z = 1 \cdot z =$

$\sum_{i=1}^n (m_i z) t_i \in MT = T$. Thus $(M : M)_S \subseteq T$.

Next, we investigate extension of rings $R \subset S$ such that each ring between R and S is PVD in S .

Theorem 5.6 *Let R be a local ring with maximal ideal M and R' its integral closure in S . Then the following statements are equivalent:*

- (i) *Every ring between R and S is a PVD in S .*
- (ii) *$R' = (M : M)_S$ and R is PVD in S .*

(iii) S is integral over R and local with maximal ideal M or R' is a VD subring of S with maximal ideal M ,

Proof. (i) $\Rightarrow$ (ii): Clearly R is PVD in S . It remains to show that $R' = (M : M)_S$. We know, from Corollary 3.4, that $R' \subseteq (M : M)_S$ and R' is local with maximal ideal M . Set $k = R'/M$, we get the pullback:

$$\begin{array}{ccc} R' & \longrightarrow & k \\ \downarrow & & \downarrow \\ (M : M)_S & \longrightarrow & (M : M)_S / M \end{array}$$

Suppose that $k \subset (M : M)_S / M$ is not algebraic. Let $x \in (M : M)_S$ such that $t = x + M \in (M : M)_S / M$ is transcendental over k . We have $k \subset k[t^2] \subset k[t] \subseteq (M : M)_S$. Hence $\varphi^{-1}(k[t^2]) = T$ is an intermediate ring strictly between R and $(M : M)_S$. As $T/M \cong k[t^2]$ is not local, then T is not PVD in S , a contradiction. It follows that $k \subset (M : M)_S / M$ is algebraic. Therefore, $(M : M)_S$ is integral over R' (Chap II, Lemma 2.9). But R' is integrally closed in S , this yields $R' = (M : M)_S$.

(ii) $\Rightarrow$ (iii): Either $S = (M : M)_S = R'$, then S is integral over R with $\text{Spec}(R) = \text{Spec}(S)$, or $R' = (M : M)_S \subset S$, then R' is a VD of S with maximal ideal M , by Theorem 4.3.

(iii) $\Rightarrow$ (i): Suppose that S is integral over R , and local with maximal ideal M . Then $\text{Spec}(R) = \text{Spec}(S)$, by Lemma 3.2, and so R is PVD in S , by Remark 3.5. Let T be a ring between R and S . Since S is integral over T , then T is local with maximal ideal M . It follows that $\text{Spec}(T) = \text{Spec}(S)$ and so T is PVD in S .

Suppose now that R' is a VD subring of S with maximal ideal M , then R' is PVD in S and $MR' \neq R'$. As $\text{Spec}(R) = \text{Spec}(R')$ from lemma 3.2, then R is PVD in R' . By Corollary 5.2, R is PVD in S . Now, using Theorem 4.3, $(M : M)_S$ is the unique

intermediate ring between R and S , that is a VD subring of S . Thus $R' = (M : M)_S$. Let T be an intermediate ring between R and S . Then by Proposition 5.5, we have either $T \subseteq R'$, so T is PVD in S , by Theorem 4.3; or $R' \subseteq T$. In this last case, it is clear that T is a VD subring of S , thus T is PVD in S .

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