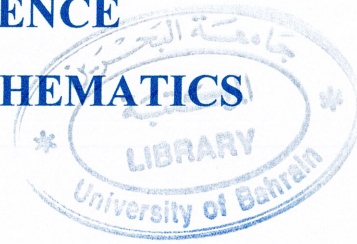


UNIVERSITY OF BAHRAIN
COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS



**Some mathematical aspects
of bubble dynamics**

BY
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To
My Family
And
My friends

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ABSTRACT

Because the effect of the small bubble pulsating which creates erosion of the materials , and the problems of the effect of strong pressure it caused , the study of bubbles collapse take place in the beginning of the last century

The first one who analysis the problem of bubbles was Lord Rayleigh (1917) whose derived the relation :

$$R \ddot{R} + \frac{3}{2} \left(\dot{R} \right)^2 = \frac{P(R) - P(\infty)}{\rho}$$

By some calculation we derived that the time t_{TC} required for total collapse of the bubbles by

$$t_{TC} = 0.915 \frac{R_0}{\sqrt{C}}$$

Where
$$C = \frac{P(\infty) - P(R)}{\rho}$$

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INTRODUCTION

Because the effect of the small bubble pulsating which creates erosion of the materials , and the problems of the effect of strong pressure it caused , the study of bubbles collapse take place in the beginning of the last century

In Chapter 1 :

The first one who analysis the problem of bubbles was Lord Rayleigh (1917) whose derived the relation :

$$R \ddot{R} + \frac{3}{2} \left(\dot{R} \right)^2 = \frac{P(R) - P(\infty)}{\rho}$$

By some calculation we derived that the time t_{TC} required for total collapse of the bubbles by

$$t_{TC} = 0.915 \frac{R_0}{\sqrt{C}}$$

Where $C = \frac{P(\infty) - P(R)}{\rho}$

In chapter 2:

We concerned with the collapse of bubble smaller than critical size which takes infinitely long time .

In chapter 3:

We study the gas bubbles in fluid by Euler and Reyleigh equations . It is shown that characteristic equation of the system has two complex conjugate roots with negative real part and an infinite number of real negative roots .

In Chapter 4 :

On the dynamics of gas bubble , there are two different types of motion depending on the initial radius of the bubble , oscillatory or monotonically .

CHAPTER 1

DYNAMICS OF THE BUBBLE IN THE IDEAL LIQUID

1.1 Introduction

The first analysis of a problem on cavitation and bubble dynamics was made by Lord Rayleigh (1917), who solved the problem of the collapse of an empty cavity in large mass of liquid.

Under the following assumption : that the cavity is of spherical symmetry with neglect of surface tension and liquid viscosity and with the assumption of liquid incompressibility.

Rayleigh showed from the momentum equation that the bubble boundary $R(t)$ obeyed the relation

$$R \ddot{R} + \frac{3}{2} \left(\dot{R} \right)^2 = \frac{P(R) - P(\infty)}{\rho} \quad (*)$$

Many papers devoted to bubble dynamics are reviewed in the books Leighton (1993) and Brennen (1995).

In this Chapter, Rayleigh equation is derived by using continuity equation and Euler's equation.

Rayleigh (1917) found that the time of collapsing of spherical cavity in an incompressible (non-viscous) fluid is finite time period. This result established by changing the variables in Rayleigh equation and solve resultant equation by separation of variables and the integration is evaluated by using Beta and Gamma functions.

1.2 Rayleigh Equation

Rayleigh Equation (*) which describes the dynamics of radial motion is derived here.

The equation defined by:

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \bar{V}) = 0 \quad (1.2.1)$$

is called the continuity equation.

A special case, which affords great simplification, is incompressible flow, where the density changes are negligible.

Then $\frac{\partial \rho}{\partial t} = 0$ regardless of whether the flow is steady or unsteady, and the density can be slipped out of the divergence in

Equation (1.2.1) and divided out, the result is $\text{div} \bar{V} = 0$

In spherical case

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 V) = 0 \quad , \quad r^2 V(r, t) = f(t)$$

$f(t)$ can be found from boundary condition

$$\Rightarrow f(t) = R^2 \dot{R} \quad \text{at the bubble surface.}$$

Incompressibility of the liquid means that the liquid velocity at a distance r from the bubble center is

$$V(r, t) = \frac{f(t)}{r^2} = \frac{R^2 \dot{R}}{r^2} \quad (1.2.2)$$

The rate of change of the velocity of a given fluid particle as it moves in space which is denoted by the total derivative $\frac{dV}{dt}$ has to be expressed in terms of quantities to points fixed in space :

$$\text{Thus } \frac{dV}{dt} = \frac{\partial V}{\partial t} \frac{\partial t}{\partial t} + \frac{\partial V}{\partial r} \frac{\partial r}{\partial t}$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + V \frac{\partial V}{\partial r} \quad (1.2.3)$$

$$\text{Since } V = \frac{R^2 \dot{R}}{r^2} \quad \text{by} \quad (1.2.2)$$

$$\Rightarrow \frac{\partial V}{\partial t} = \frac{R^2 \ddot{R}}{r^2} + \frac{2R\dot{R}^2}{r^2} \quad (1.2.4)$$

And

$$\frac{\partial V}{\partial r} = - \frac{2 R^2 \dot{R}}{r^3} \quad (1.2.5)$$

Substituting (1.2.2), (1.2.4) and (1.2.5) into (1.2.3) gives :

$$\begin{aligned} \frac{dV}{dt} &= \frac{R^2 \ddot{R}}{r^2} + \frac{2 R \dot{R}^2}{r^2} + \left(\frac{R^2 \dot{R}}{r^2} \right) \left(\frac{-2 R^2 \dot{R}}{r^3} \right) \\ \frac{dV}{dt} &= \frac{R^2 \ddot{R}}{r^2} + \frac{2 R \dot{R}^2}{r^2} - \frac{2 R^4 \dot{R}}{r^5} \end{aligned} \quad (1.2.6)$$

Where the equation defined by

$$\rho \frac{d \bar{V}}{dt} = - \frac{\partial P}{\partial r} \quad (1.2.7)$$

Is called the Euler's equation in one-dimensional motion and one of the fundamental equations of fluid dynamics .

Substituting (1.2.6) into (1.2.7) yields :

$$\rho \left(\frac{R^2 \ddot{R}}{r^2} + \frac{2 R \dot{R}^2}{r^2} - \frac{2 R^4 \dot{R}^2}{r^5} \right) = - \frac{\partial P}{\partial r} \quad (1.2.8)$$

Integrating both sides of (1.2.8) with respect to r

from $r = R$ to $r = \infty$ implies :

$$\rho \int_R^\infty \left(\frac{R^2 \ddot{R}}{r^2} + \frac{2 R \dot{R}^2}{r^2} - \frac{2 R^4 \dot{R}^2}{r^5} \right) dr = - \int_R^\infty \left(\frac{\partial P}{\partial r} \right) dr$$

$$\rho \left[-\frac{R^2 \ddot{R}}{r} - \frac{2 R \dot{R}^2}{r} + \frac{R^4 \dot{R}^2}{2 r^4} \right]_R^\infty = [-P]_R^\infty$$

$$\rho \left[\left(-\frac{R^2 \ddot{R}}{\infty} - \frac{2R \dot{R}^2}{\infty} + \frac{R^4 \dot{R}^2}{\infty} \right) - \left(-\frac{R^2 \ddot{R}}{R} - \frac{2R \dot{R}^2}{R} + \frac{R^4 \dot{R}^2}{2R^4} \right) \right]$$

$$= -[P(\infty) - P(R)]$$

$$\rho \left[R \ddot{R} + 2 \dot{R}^2 - \frac{\dot{R}^2}{2} \right] = P(R) - P(\infty)$$

$$\rho \left[R \ddot{R} + \frac{3}{2} \dot{R}^2 \right] = P(R) - P(\infty)$$

$$\Rightarrow R \ddot{R} + \frac{3}{2} \dot{R}^2 = \frac{P(R) - P(\infty)}{\rho}$$

Which called the Rayleigh equation .

1.3 The collapse of bubbles in an incompressible ideal liquid

Rayleigh (1917) considered the total collapse of the bubbles from $R = R_0$ to $R = 0$ under the assumption of liquid incompressibility, and that the cavity is of spherical symmetry with neglect of surface tension and liquid viscosity.

The time of the total collapse of the bubble is found by using Beta and Gamma functions.

since

$$R \ddot{R} + \frac{3}{2} \dot{R}^2 = \frac{P(R) - P(\infty)}{\rho}$$

assuming that $\frac{P(\infty) - P(R)}{\rho} = C = \text{constant}$

Rayleigh supposed that the cavity is empty (i.e. $P(R) = 0$)

Therefore
$$R \ddot{R} + \frac{3}{2} \dot{R}^2 = -C \quad (1.3.1)$$

Assuming that $C > 0$ which corresponds to bubble contraction whereas $C < 0$ corresponds to the bubbles growth and it is not our concern.

Changing the variables in equation (1.3.1) by assuming that

$$\dot{R} = w \quad (1.3.2)$$

Therefore

$$\frac{d^2 R}{dt^2} = \frac{dW}{dR} \cdot \frac{dR}{dt}$$

$$\ddot{R} = \frac{dR}{dt} \cdot \frac{dW}{dR}$$

$$\ddot{R} = \dot{R} \cdot \frac{dW}{dR}$$

$$\ddot{R} = W \cdot \frac{dW}{dR} \quad (1.3.3)$$

Substituting (1.3.2) and (1.3.3) into (1.3.1) :

$$R W \cdot \frac{d W}{d R} + \frac{3}{2} W^2 = -C$$

$$\frac{1}{2 R^2} \cdot \frac{d}{d R} \left[R^3 W^2 \right] = -C \quad (1.3.4)$$

$$\frac{d}{d R} \left[R^3 W^2 \right] = -2 R^2 C$$

Integrating both sides implies :

$$R^3 W^2 = -\frac{2 C R^3}{3} + D \quad (1.3.5)$$

Where D is constant

Using the initial conditions to find D :

$$(\text{i.e. when } t = 0 \Rightarrow R = R_0, \quad W = 0 \Rightarrow D = \frac{2}{3} C R_0^3)$$

$$R^3 W^2 = -\frac{2}{3} C R^3 + \frac{2}{3} C R_0^3$$

$$R^3 W^2 = -\frac{2}{3} C [R^3 - R_0^3]$$

$$R^3 W^2 = -\frac{2}{3} C R^3 \left[1 - \frac{R_0^3}{R^3} \right]$$

Divide both sides by R^3 implies :

$$W^2 = -\frac{2}{3} C \left[1 - \frac{R_0^3}{R^3} \right]$$

Therefore equation (1.3.5) become :

$$W = \pm \sqrt{-\frac{2}{3} C \left(1 - \left(\frac{R_0}{R} \right)^3 \right)} \quad (1.3.6)$$

Where " + " sign corresponds to bubble growth :

$$\lim_{R \rightarrow \infty} W = \sqrt{-\frac{2}{3} C}$$

And the " - " sign corresponds to bubble contraction

$$\lim_{R \rightarrow 0} W = \infty$$

Determination of $R(t)$ from (1.3.6) through the elementary functions seems to be unsuccessful , however in the contraction problem the time of the bubble completely vanishing may be determined .

In the contraction problem equation (1.3.6) become :

$$\frac{dR}{dt} = -\sqrt{\frac{2}{3} C \left(-1 + \left(\frac{R_0}{R} \right)^3 \right)}$$

$$dt \left[-\sqrt{\frac{2}{3} C \left(-1 + \left(\frac{R_0}{R} \right)^3 \right)} \right] = dR$$

$$dt = - \frac{dR}{\sqrt{\frac{2}{3} C \left(-1 + \left(\frac{R_0}{R} \right)^3 \right)}}$$

Now integrating both sides :

$$\int_{t=0}^{t_{TC}} dt = - \int_{R_0}^0 \frac{dR}{\sqrt{\frac{2}{3} C \left(-1 + \left(\frac{R_0}{R} \right)^3 \right)}}$$

$$[t] \Big|_0^{t_{TC}} = - \int_{R_0}^0 \frac{dR}{\sqrt{\frac{2}{3} C \left(-1 + \left(\frac{R_0}{R} \right)^3 \right)}}$$

$$t_{TC} = - \int_{R_0}^0 \frac{dR}{\sqrt{\frac{2}{3} C \left(-1 + \left(\frac{R_0}{R} \right)^3 \right)}}$$

$$t_{TC} = \int_0^{R_0} \frac{dR}{\sqrt{\frac{2}{3} C \left(-1 + \left(\frac{R_0}{R} \right)^3 \right)}} \quad (1.3.7)$$

to integrate the equation (1.3.7) a change of variables is used .

$$\text{let } \left(\frac{R}{R_0} \right)^3 = x$$

$$\frac{R}{R_0} = x^{1/3}$$

$$R = R_0 x^{1/3}$$

$$R = 0 \quad \Rightarrow \quad x = 0$$

$$R = R_0 \quad \Rightarrow \quad x = 1$$

$$\text{But } \left(\frac{R}{R_0} \right)^3 = x$$

$$\Rightarrow \frac{R^3}{R_0^3} = x \quad \text{differentiate both side :}$$

$$\frac{3 R^2}{R_0^3} d R = d x$$

$$d R = \frac{R_0^3}{3 R^2} d x$$

$$d R = \frac{R_0^3}{3 R_0^2 x^{2/3}} d x, \quad \text{since } R^2 = R_0^2 x^{2/3}$$

$$d R = \frac{R_0}{3 x^{2/3}} d x \quad (1.3.8)$$

Substituting (1.3.8) into (1.3.7) yields :

$$t_{TC} = \int_0^1 \frac{R_0 \, dx}{3x^{2/3} \sqrt{\frac{2}{3} C \left(-1 + \frac{1}{x} \right)}}$$

$$t_{TC} = \int_0^1 \frac{R_0 \, dx}{x^{2/3} \sqrt{6 C \frac{(-x+1)}{x}}}$$

$$t_{TC} = \int_0^1 \frac{R_0 \, dx}{x^{2/3} \left(6C \frac{(1-x)}{x} \right)^{\frac{1}{2}}}$$

$$t_{TC} = \int_0^1 \frac{R_0 \, dx}{\sqrt{6C} \, x^{2/3} x^{-1/2} (1-x)^{\frac{1}{2}}}$$

$$t_{TC} = \int_0^1 \frac{R_0 \, dx}{\sqrt{6C} \, x^{1/6} (1-x)^{1/2}}$$

$$t_{TC} = \int_0^1 \frac{R_0 \, dx}{\sqrt{6C} \, x^{1/6} (1-x)^{1/2}}$$

$$t_{TC} = \int_0^1 \frac{R_0}{\sqrt{6C}} \, x^{-1/6} (1-x)^{-1/2} \, dx$$

$$t_{TC} = \frac{R_0}{\sqrt{6C}} \int_0^1 x^{-1/6} (1-x)^{-1/2} \, dx$$

Since Beta function $B(p, q)$ is define as :

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx$$

$p > 0, q > 0$

$$\Rightarrow t_{TC} = \frac{R_0}{\sqrt{6C}} B\left(\frac{5}{6}, \frac{1}{2}\right)$$

$$\text{But } B(p, q) = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)}$$

Where Γ is Gamma Function defined by :

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx$$

$p > 0$

$$\text{Therefore } t_{TC} = \frac{R_0}{\sqrt{6C}} \frac{\Gamma\left(\frac{5}{6}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{6} + \frac{1}{2}\right)}$$

$$t_{TC} = \frac{R_0}{\sqrt{C}} \frac{\Gamma\left(\frac{5}{6}\right) \Gamma\left(\frac{1}{2}\right)}{\sqrt{6} \Gamma\left(\frac{4}{3}\right)}$$

$$t_{TC} = 0.915 \frac{R_0}{\sqrt{C}}$$

Therefore the time t_{TC} required for total collapse of the bubbles from

$R=R_0$ to $R=0$ is given by :

$$t_{TC} = 0.915 \frac{R_0}{\sqrt{C}}$$

and note that $\lim_{R \rightarrow 0} \frac{dR}{dt} = \infty$

$$\text{where } C = \frac{P(\infty) - P(R)}{\rho}$$

CHAPTER 2 :

THE COLLAPSE OF BUBBLES IN A VISCOUS LIQUID

2.1 Introduction :

We will be concerned with the filling of an empty spherical bubble in a viscous liquid. The existence of two different types of motion is discovered: bubbles, which are smaller than a critical size, are filled slowly in an infinitely long time; the filling of large bubbles takes place rapidly with an unlimited accumulation of energy during collapse.

A quantitative formula is obtained for the critical radius of the bubble .

Assume that for some reason a bubble has formed inside the liquid, which in the future can again be filled under the action of the surrounding pressure. The problem of the filling of a spherical bubble in an in viscid incompressible liquid was studied by Rayleigh . He found, in particular, that the velocity of motion its surface toward the center grows indefinitely as $r^{-3/2}$, as the bubble completes the process of collapsing ,i.e. an unlimited accumulation of energy takes place. This phenomenon is assumed

to be a possible for the rapid wear in screw propellers and turbines which operate under cavitations conditions.

The collapse of bubbles at a metal surface can damage it severely.

Now let us study the Rayleigh problem for the viscous liquid. Such a statement of the problem corresponds more nearly to the actual conditions, although it still does not give an accurate description of the phenomenon, since it does not take into account the compressibility of the liquid, the inevitable presence of its vapor in the bubble and the possible instability of the slope of the bubble.

2.2 Applies of differential equations for finding radius of bubble :

Assume that a spherical bubble of radius r was formed in a liquid density ρ , pressure P_0 (far from bubble) and viscosity η . There is no pressure inside the bubble and the initial velocity is zero.

The motion of the bubble will be spherically symmetric and can be described by the Navier-Stokes equations of the form

$$\frac{\partial u}{\partial r} + \frac{2u}{r} = 0, \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0 \quad (2.2.1)$$

Here $u(r, t)$ is the velocity, $p(r, t)$ is the pressure.

The viscosity does not enter Equation (2.2.1) because in the general equation it occurs in the term :

$$\eta(\text{grad div } u - \text{Curl Curl } u)$$

Which in our case is identically equal to zero. This is so because $\mathbf{div} \mathbf{u} = 0$, since the liquid is incompressible and $\mathbf{Curl} \mathbf{u} = 0$ because of spherical symmetry.

The normal stress σ_{rr} at the free surface of the bubble is zero (boundary with a vacuum), and since

$$\sigma_{rr} = -p + 2\mu \frac{du}{dr} \quad \text{then}$$

$$P_1 = 2\mu \left(\frac{\partial u}{\partial r} \right)_1$$

i.e. the viscosity has entered the boundary condition. Here and in the following, the index 1 denotes a value at the boundary. The second boundary condition is

$$P = P_0 \quad \text{for} \quad r = \infty$$

from (2.2.1) we obtain

$$u(r, t) = \frac{q(t)}{r^2} \quad \left(q(t) = u_1 r_1^2 \right)$$

This is substituted into the second equation of (2.2.1) , integrated from r_1 to ∞ , taking into account the boundary condition on p, and after some simple calculation we obtain

$$\frac{d u_1}{d r_1} + \frac{3}{2} \frac{u_1}{r_1} + \frac{P_0}{\rho u_1 r_1} + \frac{4 \nu}{r_1^2} = 0 \quad \left(\nu = \frac{\mu}{\rho} \right) \quad (2.2.2)$$

Here ν is the kinematic viscosity. Let us introduce non-dimensional variable

$$\xi = \frac{u_1}{\sqrt{\frac{P_0}{\rho}}} \quad , \quad \alpha = \frac{r_1 R}{a} \quad \left(R = \frac{a}{\nu} \sqrt{\frac{P_0}{\rho}} \right)$$

Where R is the Reynolds number. The Equation (2.2.2) become

$$\frac{d \xi}{d \alpha} + \frac{3}{2} \frac{\xi}{\alpha} + \frac{1}{\xi \alpha} + \frac{4}{\alpha^2} = 0 \quad (2.2.3)$$

The initial condition is of form $\xi(R) = 0$ (i.e. $u_1(a) = 0$) . As we see, the problem under study has a family of solution which depends on the number R. One of them,

$R = \infty$ ($\nu = 0$) , coincides with Rayleigh solution. For this family, the velocity near the center grows as $\xi \sim a^{-3/2}$.

Let us study the behavior of the velocity near the center in the case of nonzero viscosity. For studying the value ξ^{-1} we represent Equation (2.2.3) in the form

$$\frac{d\xi^{-1}}{d\alpha} = \frac{\xi^{-2}}{\alpha} \left(\frac{3}{2}\xi + \xi^{-1} + \frac{4}{\alpha} \right) \quad (2.2.4)$$

The point ($\alpha = 0$, $\xi^{-1} = 0$) for this equation is a multiple singularity, shown in Fig. 1.

The loci of the zeros and infinities are shown in Fig .1 by means of heavy lines and identified by $\mathbf{0}$ and ∞ , whereas the integral curves are denoted by the light lines . the dot-dash line $\mathbf{OA}$ is the only integral curve which emanates from the origin with a finite slope. Its slope $\mathcal{X}$ is determined from (2.2.4) by substitution of the solution $\xi^{-1} = x\alpha$ near $\alpha = 0$, and it become $x = \frac{1}{8}$.The locus $\mathbf{OB}$ of the zeros passes by lower ; its slope is $-\frac{3}{8}$.

The integral curves above OA enter a node ,

$$\text{where } \xi \sim a^{-3/2} .$$

Below OA they represent a saddle where

$$\xi \rightarrow -\frac{1}{4} \alpha , \quad \text{or} \quad u_1 \rightarrow -\frac{r_1 P_0}{4\mu}$$

$$\text{as } a \rightarrow 0 .$$

Both of these results are verified by, substitution in (2.2.4) with $\alpha \rightarrow 0$.The curve which corresponds to the solution is that which leads towards point corresponding to the initial condition ,
i.e. $\alpha = R , \quad \xi = 0$

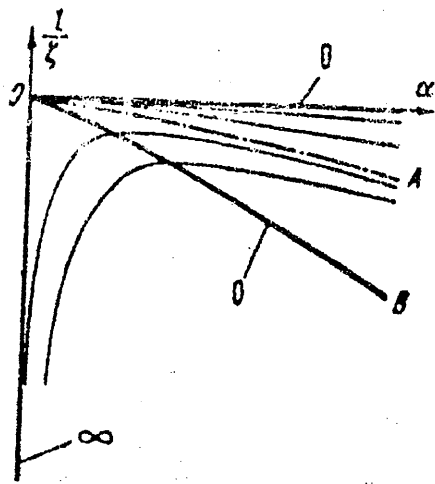


Fig. 1.

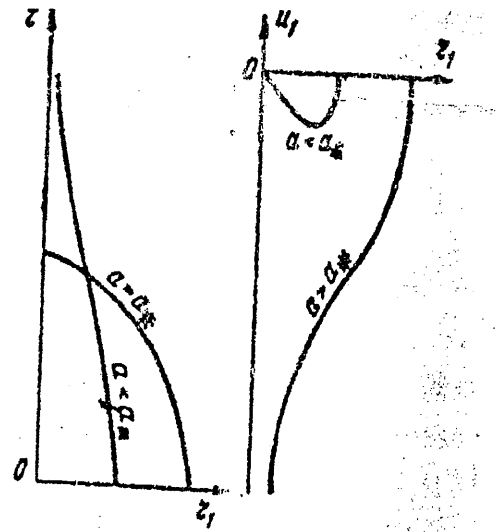


Fig. 2.

Generally speaking, for different number R the solution may belong to different families: in the case of the node they correspond to an unlimited increase of the velocity $\xi \sim a^{-3/2}$; in the case of the saddle point they correspond to a decelerated motion of the bubble with $\xi \sim a$.

as a result of which its filling occurs only after an infinite time.

In the intermediate case, which corresponds to the dividing line OA , the filling occurs after a finite time, and the velocity near the center grows as $\xi \sim a^{-1}$. The Reynolds number corresponding to the dividing line is the critical Reynolds number. It separates two fundamentally different classes of solution. Its

critical value $R = R_*$ can be determined by construction the separatrix from $\alpha = 0$, (near that point its asymptotic representation is known : $\xi = -\frac{1}{8} \alpha^{-1}$) to : $\xi = 0$, where $\alpha = R$. Its construction, which is accomplished with the aid of numerical integration (2.2.3), yields

$$R_* = 8.4 \quad \left(R = \frac{a}{\nu} \sqrt{\frac{P_0}{\rho}} \right)$$

For $R < 8.4$ the filling of the bubble takes place slowly in an infinitely long time. The accumulation of energy is completely dissipated by the viscosity.

For $R > 8.4$ the velocity near the focus grows indefinitely as in the Rayleigh problem (without viscosity), i.e. as $\text{const } r_1^{-\frac{3}{2}}$, but with a smaller value of $|\text{const}|$. In the interim case for $R_* = 8.4$ the bubble is filling in finite time, and the velocity of the focus grows indefinitely, but more weakly, as r_1^{-1} .

The schemes of all three cases of motion are shown in Fig.2. For given P_0 , ρ and ν one can speak about a critical radius a_* of the bubble.

$$a_* = 8.4 \nu \sqrt{\frac{\rho}{P_0}}$$

For $a < a_*$ the energy accumulation is completely dissipated by the velocity .

The velocity at the end of the filling of small bubbles

$$u_1 = - \frac{r_1 P_0}{4 \mu}$$

does not depend on the density and on their initial radius a . In practice, the critical radius a_* is very small ,i.e. the viscosity dissipates the accumulation only in bubbles of very small size; for

instance, for $P_0 = 1 \text{ atm} = 10^6 \text{ bar}$ in water

($\nu = 0.01$, $\rho = 1$) $a_* = 0.8 \text{ micron}$, and in glycerin

($\nu = 6.8$, $\rho = 0.8$) $a_* = 0.5 \text{ mm}$.

CHAPTER 3 :

THE PROBLEM OF SMALL OSCILLATIONS OF GAS BUBBLES IN A FLUID

3.1 Introduction

Let us examine a gas bubble pulsating in a perfect incompressible fluid on assumption of spherical symmetry of the process and a uniform pressure in the bubble. Such a pressure occurs when the length of sound wave in the gas is much in excess of the bubble dimension.

Considering the assumptions adopted, the questions of heat influx and of continuity for gaseous phase in the spherical Euler coordinates (r, t) are of the form [3 , 4]

$$\rho C_p \left(\frac{\partial T}{\partial t} + V \frac{\partial T}{\partial r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\lambda r^2 \cdot \frac{\partial T}{\partial r} \right) + \frac{dp}{dt}$$
$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho V) = 0 \quad (3.1.1)$$

Where ρ - is density; T is temperature, V is velocity; C_p is heat capacity at a constant pressure; λ - is thermal conductivity.

As shown in [3.1.1], the boundary condition for the temperature on the bubble surface may be presented as

$$r = R(t) ; T = T_0 = \text{Const} \quad (3.1.2)$$

Where R is the bubble radius ; T_0 is temperature of the fluid which acts as a thermostat .

The boundary condition at the bubble center is

$$r = 0 : \frac{\partial T}{\partial r} = 0$$

The dynamics of the radial oscillations of the bubble is described by the Reyleigh equation [3.1.2].

$$R \frac{d^2 R}{dt^2} + \frac{3}{2} \left(\frac{dR}{dt} \right)^2 = \frac{p - p_\infty - 2\sigma/R}{\rho_L} \quad (3.1.4)$$

Where p_∞ - is pressure in fluid far from the bubble ; σ - is coefficient of surface tension ; ρ_L is density of the fluid.

The equation of state for gas is of the form

$$P = \rho B T \quad (3.1.5)$$

Where B is gas constant

The condition of uniform pressure in the bubble permits (3.1.1) and (3.1.5) to be used to obtain the relation

$$\text{Since } \rho C_p \left(\frac{\partial T}{\partial t} + V \frac{\partial T}{\partial r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\lambda r^2 \cdot \frac{\partial T}{\partial r} \right) + \frac{dp}{dt}$$

$$\text{Where } \left(\frac{\partial T}{\partial t} + V \frac{\partial T}{\partial r} \right) = \frac{dT}{dt}$$

$$\text{Then : } C_p \rho \frac{dT}{dt} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\lambda r^2 \cdot \frac{\partial T}{\partial r} \right) + \frac{dp}{dt}$$

$$\text{But : } P = \rho B T$$

$$\frac{dP}{dt} = B \left(\frac{d\rho}{dt} T + \rho \frac{dT}{dt} \right)$$

$$\rho \frac{dT}{dt} = \frac{1}{B} \frac{dP}{dt} - T \frac{d\rho}{dt}$$

Then :

$$C_p \rho \frac{dT}{dt} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\lambda r^2 \cdot \frac{\partial T}{\partial r} \right) + \frac{dp}{dt}$$

$$C_p \left(\frac{1}{B} \frac{dP}{dt} - T \frac{d\rho}{dt} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\lambda r^2 \cdot \frac{\partial T}{\partial r} \right) + \frac{dp}{dt}$$

$$r^2 \left(\frac{C_p}{B} - 1 \right) \frac{dP}{dt} = \frac{\partial}{\partial r} \left(\lambda r^2 \cdot \frac{\partial T}{\partial r} \right) + r^2 C_p T \frac{d\rho}{dt}$$

$$\text{Sine : } \frac{d\rho}{dt} = -\rho \operatorname{div}(\bar{V}) = -\frac{\rho}{r^2} \frac{\partial}{\partial r} (r^2 V)$$

$$r^2 \left(\frac{C_p}{B} - 1 \right) \frac{dP}{dt} = \frac{\partial}{\partial r} \left(\lambda r^2 \cdot \frac{\partial T}{\partial r} \right) - \rho C_p T \frac{\partial}{\partial r} (r^2 V)$$

integrate both side from $r=0$ to $r=R$

$$\left(\frac{C_p}{B} - 1 \right) \frac{dP}{dt} \int_0^R r^2 dr = \int_0^R \frac{\partial}{\partial r} \left(\lambda r^2 \cdot \frac{\partial T}{\partial r} \right) dr - \rho C_p T \int_0^R \frac{\partial}{\partial r} (r^2 V) dr$$

$$\left(\frac{C_p}{B} - 1 \right) \frac{dP}{dt} \frac{R^3}{3} = \lambda R^2 \frac{\partial T}{\partial r} \Big|_R - \rho C_p T R^2 V$$

$$\frac{d P}{d t} = \frac{3}{R^3} \left(\frac{B}{c_p - B} \right) \left[\lambda R^2 \frac{\partial T}{\partial r} \Big|_R - c_p \frac{(\rho B T)}{B} R^2 V \right]$$

$$\frac{d P}{d t} = \frac{3}{R^3} \left(\frac{B/c_v}{c_p/c_v - B/c_v} \right) \lambda R^2 \frac{\partial T}{\partial r} \Big|_R - \frac{c_p}{c_p - B} \frac{3P}{R^3} R^2 V$$

Since :

$$B = c_p - c_v$$

$$\frac{B}{c_v} = \frac{c_p}{c_v} - 1 = \gamma - 1$$

$$\frac{d P}{d t} = \frac{3}{R^3} \left(\frac{\gamma - 1}{\gamma - (\gamma - 1)} \right) \lambda R^2 \frac{\partial T}{\partial r} \Big|_R - \frac{c_p/c_v}{c_p/c_v - B/c_v} \frac{3P}{R} V$$

$$\frac{d P}{d t} = \frac{3(\gamma - 1)}{R^3} \lambda R^2 \frac{\partial T}{\partial r} \Big|_R - \frac{\gamma}{\gamma - (\gamma - 1)} \frac{3P}{R} \dot{R}$$

$$\frac{dp}{dt} = \frac{3(\gamma-1)}{R} \lambda \left. \frac{\partial T}{\partial r} \right|_R - \frac{3\gamma p}{R} \dot{R} \quad (3.1.6)$$

Where γ is the adiabatic exponent of gas; λ - is thermal conductivity.

In case of small oscillations the bubble radius may be described by the real part of the expression

$$R = R_0 [1 + \delta \exp (h t)] \quad (3.1.7)$$

The subscript 0 corresponds to the parameters in unperturbed state ; δ is a complex number corresponding to the condition $|\delta| \ll 1$;

the oscillation frequency

$$f = \frac{\omega}{2\pi} = \frac{\text{Im}(h)}{2\pi}.$$

The set of basic equation (3.1.1) – (3.1.6) is linearized. The minor deviations of pressure and temperature from the equilibrium state are assumed to be of the from [1-3]:

$$p = P_o [1 + P \exp (h t)] ,$$

$$T = T_o [1 + \theta(r) \exp (h t)] \quad (3.1.8)$$

The set of the basic equation takes the following form after linearization and transition to dimensionless values :

$$H \theta = \nabla^2 \theta + \left(1 - \frac{1}{\gamma} \right) H p \quad (3.1.9)$$

$$P = \frac{\delta H^2}{N} - S \delta \quad (3.1.10)$$

$$H P = 3\gamma \left[\frac{\partial \theta}{\partial \xi} \Big|_1 - \delta H \right] \quad (3.1.11)$$

$$\xi = \frac{r}{R_0} , \quad \nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{2}{\xi} \frac{\partial}{\partial \xi} , \quad S = \frac{2\sigma}{R_0 P_0}$$

$$N = \frac{P_0 R_0^2}{\rho_L a^2} , \quad a = \frac{\lambda}{\rho C_p} , \quad H = \frac{h R_0^2}{a}$$

The solution for the equation (3.1.9) on the boundary condition (3.1.2) and (3.1.3) is

$$\theta = \left(1 - \frac{1}{\gamma}\right) P \left[1 - \frac{\text{Sinh}(\sqrt{H} \xi)}{\xi \text{Sinh}(\sqrt{H})} \right] \quad (3.1.12)$$

The condition of the existence of a nontrivial solution for the set of linear equation (3.1.9) - (3.1.12) leads to the characteristic equation in H .

In case of sufficiently large bubbles; when the capillary effect can be disregarded, the characteristic equation in H takes the form

$$H + \frac{B}{H} = \varepsilon \left(\sqrt{H} \coth \sqrt{H} - 1 \right) \quad (3.1.13)$$

$$B = 3 \gamma N, \quad \varepsilon = -3 (\gamma - 1)$$

Examine the structure of the roots of (3.1.13). At $\varepsilon = 0$ (heat exchange is absent), equation has two roots, $H = \pm \sqrt{B_1}$ i . which corresponds to nonattenuating adiabatic oscillation of the bubble.

3.2 Structure of the roots of transcendental equation :

Similarly as it was done in [1] we shall prove that the roots of the equation (except two complex-conjugate roots) are located on a real straight line with negative values. After findings, at first, all the roots of the form

$$H = -y^2 \quad (y > 0) \quad (3.2.1)$$

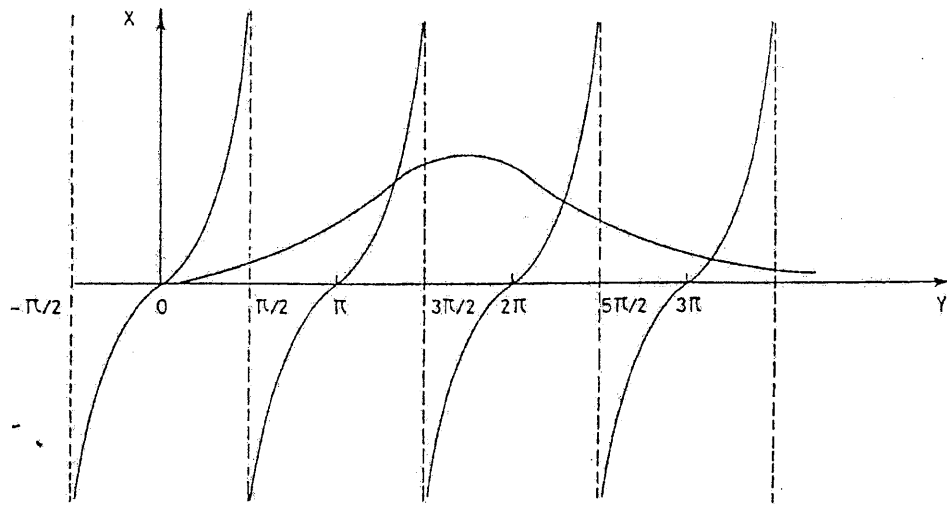
We obtain ,

$$-y^2 - \frac{B}{y^2} = \varepsilon (y \coth y - 1) \quad (3.2.2)$$

Or

$$\tanh y = - \frac{\varepsilon y}{y^2 + \frac{B}{y^2} - \varepsilon} \quad (3.2.3)$$

Figure 1

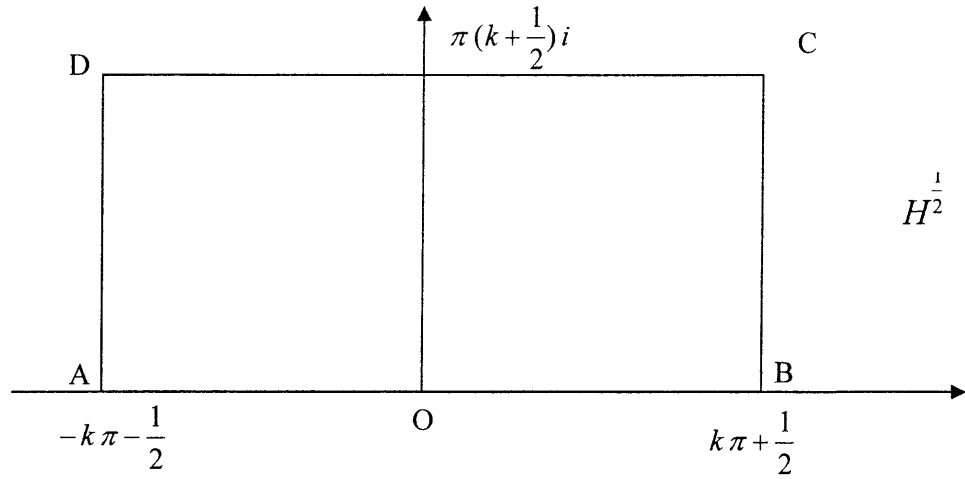


The shapes of the curves are shown schematically in Fig. 1. The following iteration formula can be constructed to find the roots of equation (3.2.3).

$$y(0)_k = k\pi, \quad k = 1, 2, 3, \dots$$

$$y(n+1)_k = k\pi - \arctan h \left| \frac{\varepsilon y}{y^2 + \frac{B}{y^2} - \varepsilon} \right|_{y=y(n)_k} \quad (3.2.4)$$

Let a rectangular contour ADCB be taken on the complex half-plane H (see Fig.2) .



The formula :

$$\coth(X + iy) = \frac{\sinh 2X - i \sin 2y}{\cosh 2X - \cos 2y}$$

Proof :

According to the definition :

$$\coth(X + iy) = \frac{\cosh(X + iy)}{\sinh(X + iy)} = \frac{\cosh X \cosh(iy) + \sinh X \sinh(iy)}{\sinh X \cosh(iy) + \cosh X \sinh(iy)}$$

But :

$$\sin(iy) = i \sinh y \quad \text{so} \quad \sinh(iy) = i \sin y$$

$$\cos(iy) = \cosh y \quad \text{so} \quad \cosh(iy) = \cos y$$

Then :

$$\begin{aligned}
\coth(X + iy) &= \frac{\cosh X \cos y + i \sinh X \sin y}{\sinh X \cos y + i \cosh X \sin y} \\
&= \frac{\cosh X \cos y + i \sinh X \sin y}{\sinh X \cos y + i \cosh X \sin y} \cdot \frac{\cosh 2X - \cos 2y}{\sinh 2X - i \sin 2y} \cdot \frac{\sinh 2X - i \sin 2y}{\cosh 2X - \cos 2y} \\
&= \frac{\cosh X \cos y + i \sinh X \sin y}{\sinh X \cos y + i \cosh X \sin y} \cdot \frac{2 \sinh^2 X + 1 - (1 - 2 \sin^2 y)}{2 \sinh X \cosh X - 2i \sin y \cos y} \cdot \frac{\sinh 2X - i \sin 2y}{\cosh 2X - \cos 2y} \\
&= \frac{\cosh X \cos y + i \sinh X \sin y}{\sinh X \cos y + i \cosh X \sin y} \cdot \frac{\sinh^2 X + \sin^2 y}{\sinh X \cosh X - i \sin y \cos y} \cdot \frac{\sinh 2X - i \sin 2y}{\cosh 2X - \cos 2y} \\
&= \frac{(\cosh X \cos y + i \sinh X \sin y)(\sinh^2 X + \sin^2 y)}{(\sinh X \cos y + i \cosh X \sin y)(\sinh X \cosh X - i \sin y \cos y)} \cdot \frac{\sinh 2X - i \sin 2y}{\cosh 2X - \cos 2y} \\
&= \frac{\sinh^2 X \cosh X \cos y + \cosh X \sin^2 y \cos y + i \sinh^3 X \sin y + i \sinh X \sin^3 y}{\sinh^2 X \cosh X \cos y - i \sinh X \sin y \cos^2 y + i \sinh X \cosh^2 X \sin y + \cosh X \sin^2 y \cos y} \cdot \frac{\sinh 2X - i \sin 2y}{\cosh 2X - \cos 2y} \\
&= \frac{\cosh X \cos y (\sinh^2 X + \sin^2 y) + i \sinh X \sin y (\sinh^2 X + \sin^2 y)}{\cosh X \cos y (\sinh^2 X + \sin^2 y) + i \sinh X \sin y (-\cos^2 y + \cosh^2 X)} \cdot \frac{\sinh 2X - i \sin 2y}{\cosh 2X - \cos 2y}
\end{aligned}$$

But :

$$\sin^2 y + \cos^2 y = 1 = \cosh^2 X - \sinh^2 X$$

$$\text{so} \quad \sin^2 y + \sinh^2 X = -\cos^2 y + \cosh^2 X$$

$$\coth(X + i y) =$$

$$\frac{\cosh X \cos y (\sinh^2 X + \sin^2 y) + i \sinh X \sin y (\sinh^2 X + \sin^2 y)}{\cosh X \cos y (\sinh^2 X + \sin^2 y) + i \sinh X \sin y (-\cos^2 y + \cosh^2 X)} \cdot \frac{\sinh 2X - i \sin 2y}{\cosh 2X - \cos 2y}$$

$$= \frac{\cosh X \cos y (\sinh^2 X + \sin^2 y) + i \sinh X \sin y (\sinh^2 X + \sin^2 y)}{\cosh X \cos y (\sinh^2 X + \sin^2 y) + i \sinh X \sin y (\sinh^2 X + \sin^2 y)} \cdot \frac{\sinh 2X - i \sin 2y}{\cosh 2X - \cos 2y}$$

$$\coth(X + i y) = \frac{\sinh 2X - i \sin 2y}{\cosh 2X - \cos 2y} \quad (3.2.5)$$

From the last formula , it is following that :

$$| \coth(X + i y) | < 1 \quad (3.2.6)$$

On DC,

$$| \coth(X + i y) | < 1, \quad 1 \quad \text{at} \quad k \geq 1 \quad (3.2.7)$$

On AD and BC, so

$$\left| \varepsilon (H^{\frac{1}{2}} + \cosh H^{\frac{1}{2}} - 1) \right| < |\varepsilon| (1, \sqrt{H} + 1) < |\varepsilon| (1 + 2k\pi) \quad (3.2.8)$$

On the contour ADCB.

$$\text{If } |H| > \sqrt{B}, \text{ then } \left| H + \frac{B}{H} \right| > \frac{1}{2} |H| \quad (3.2.9)$$

Whence, after taking

$$k > \max \left(\frac{\sqrt[4]{B}}{\pi}, \frac{4|\varepsilon| + 1}{\pi} \right) \quad (3.2.10)$$

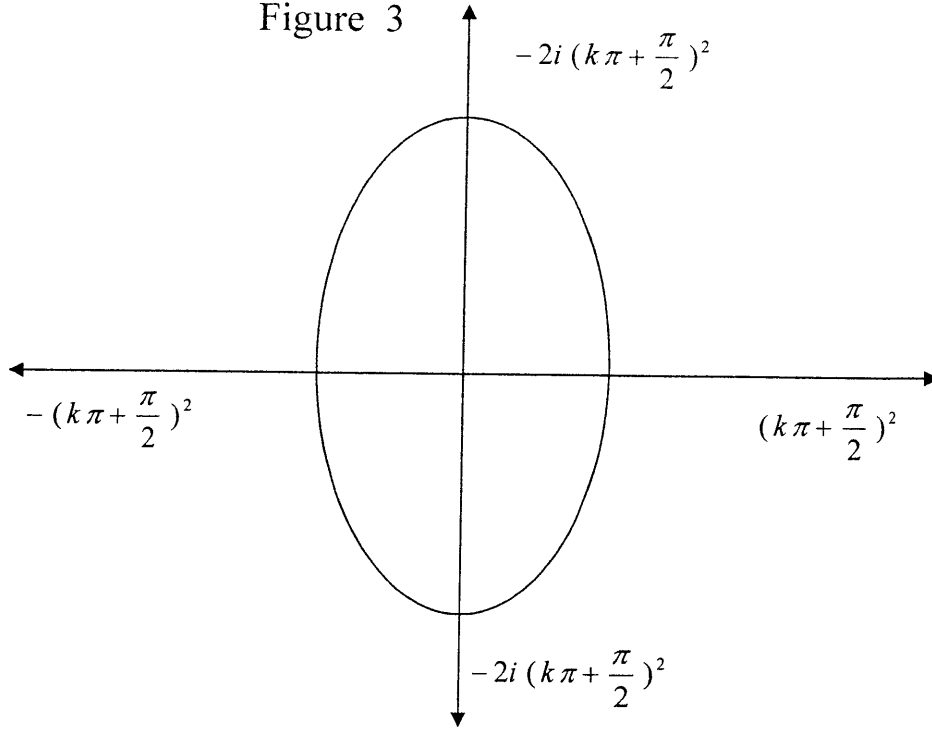
We obtain on the contour ADCB:

$$\left| H + \frac{B}{H} \right| > \left| \varepsilon (H^{\frac{1}{2}} \coth H^{\frac{1}{2}} - 1) \right| \quad (3.2.11)$$

Therefore; the theorem for meromorphic function [3.1.4] can be used to obtain the following relation for the region H (see Fig.3) formed out of two symmetric parabolas.

$$N - P = N_0 - P_0 \quad (3.2.12)$$

Figure 3



Where N is the number of zeros; P is the number of the poles of the function $\varphi = H + \frac{B}{H} - \varepsilon (H^{1/2} \coth H^{1/2} - 1)$; N_0 is the numbers of zeros; P_0 is the numbers of the poles of the function $(H + \frac{B}{H})$ in the given region.

A similar approach was used in Argument principle [3.1.5], whence it follows that

$$N - p = 1 \quad (3.2.13)$$

It is well-known that $P = k + 1$, whence

$$N = k + 2 \quad (3.2.14)$$

k -roots have been already found. Therefore, apart from the above-mentioned roots. The approximate equations for all the can be written as follows.

Let $\frac{|\varepsilon|}{K\pi} \ll 1$. Using, then the expansion

$$\tan^{-1} X = X - \frac{X^3}{3} + \frac{X^5}{5} - \frac{X^7}{7} + \dots, \quad |X| < 1$$

We obtain

$$y = -\frac{\varepsilon y}{y^2 + \frac{B}{y^2} - \varepsilon} + \frac{1}{3} \left(\frac{\varepsilon y}{y^2 + \frac{B}{y^2} - \varepsilon} \right)^3 - \frac{1}{5} \left(\frac{\varepsilon y}{y^2 + \frac{B}{y^2} - \varepsilon} \right)^5 + \dots + K\pi \quad (3.2.15)$$

$$= \frac{-\varepsilon y}{y^2 + \frac{B}{y^2}} \left[1 + \frac{\varepsilon}{y^2 + \frac{B}{y^2}} + \left(\frac{\varepsilon}{y^2 + \frac{B}{y^2}} \right)^2 + \dots \right] \\ + \frac{1}{3} \left(\frac{\varepsilon y}{y^2 + \frac{B}{y^2}} \right)^3 \left[1 + \frac{\varepsilon}{y^2 + \frac{B}{y^2}} + \dots \right]^3 + \dots + K\pi$$

Whence

$$y_0 = K\pi \quad ; \quad y_1 = K\pi - \frac{\varepsilon K\pi}{K^2\pi^2 + \frac{B^2}{K^2\pi^2}} ; \dots \dots \dots (3.2.16)$$

Two complex-conjugate roots at $\left| \frac{\varepsilon}{\sqrt{B}} \right| \ll i$ can be calculated in a similar manner :

$$H = \frac{H - \frac{B}{H} + \varepsilon (H^{\frac{1}{2}} \coth H^{\frac{1}{2}} - 1)}{2} \quad (3.2.17)$$

Let $b = \sqrt{2} B^{1/4}$ be introduced. Assuming that $H_0 = i \sqrt{B}$, we obtain

$$\begin{aligned} \frac{H_1}{\sqrt{B}} &= \frac{2i + \frac{\varepsilon}{\sqrt{B}} \left[\frac{b}{2} (1+i) \coth \frac{b}{2} (1+i) - 1 \right]}{2} \\ &= i + \frac{\varepsilon}{2\sqrt{B}} \left[\frac{b}{2} (1+i) \frac{\sinh b - i \sin b}{\cosh b - \cos b} - 1 \right] \quad (3.2.18) \end{aligned}$$

Therefore, the roots can approximately be founded by the formula

$$\begin{aligned} \frac{H}{\sqrt{B}} &= \pm i \left(1 + \frac{\varepsilon}{2b} \frac{\sinh b - \sin b}{\cosh b - \cos b} \right) \\ &\quad + \frac{\varepsilon}{b^2} \left[\frac{\frac{b}{2} (\sinh b + \sin b) - \cosh b + \cos b}{\cosh b - \cos b} \right] \quad (3.2.19) \end{aligned}$$

It can be readily verified that the expression in square brackets is always positive. Therefore, the real parts of the roots are of the same sign as $\mathcal{E}$, i.e., they are negative.

The absolute values of all real roots are much higher than those of the real parts of the complex roots by the results obtained in [3.1,3.2], where the characteristic equations were solved numerically, while the problem concerning the structure and number of the roots was not analyzed, have been demonstrated to be correct.

CHAPTER 4 :

ON THE DYNAMICS OF A GAS BUBBLE IN A VISCOUS INCOMPRESSIBLE LIQUID

4.1 Introduction :

It is generally agreed that intense oscillations of cavity bubbles without collapse . constitute one of the main reasons for the cavity erosion of the materials . when the dimension of a cavity bubbles reaches a certain limiting value , strong pressure pulses may occur in the surrounding liquid , which can cause erosion by local cyclic loads [5 - 9] .

Oscillations of the cavity bubbles in a viscous liquid , exhibit a number of distinctive features caused by the viscosity .Authors of [10 and 6] noted the fundamental influence of viscosity while investigating the behavior of a spherical cavity in a viscous , incompressible liquid. The existence of two different types of motion was discovered :bubble which are smaller than a critical size, are filled slowly in an infinitely long time, the filling of large bubbles takes place rapidly with an unlimited accumulation of energy during collapse.

Below we find that , when the bubble in a viscous incompressible liquid is filled with gas , then two modes of motion exist , depending on the initial radius of the bubble , oscillatory or monotonically a periodic .

Authors of [2] use dimensional analysis to derive a qualitative formula defining the critical bubble size D_c , separating the inertial and interfiles mode of expansion of a gaseous sphere in a viscous liquid

$$D_c = \left(\frac{\mu \tau_*}{\rho} \right)^{1/2}$$

Where μ and ρ are the dynamic viscosity and density of the liquid , respectively , and τ_* is the characteristic time of the process , determined experimentally , below we derive a formula for the critical diameter of the gas bubble

$$R \frac{d^2 R}{dt^2} + \frac{3}{2} \left(\frac{dR}{dt} \right)^2 + \frac{4\mu}{\rho R} + \frac{2\sigma}{\rho R} + \frac{P_0 - P'}{\rho} = 0 \quad (4.1.1)$$

and the initial conditions are

$$R = R_0 , \quad \frac{dR}{dt} = 0 \quad \text{when } t = 0$$

Here $R = R(t)$ the radius of the bubble, σ denotes the surface tension of the liquid, P_0 denotes constant pressure of the liquid away from the bubble and p' denotes the pressure of gas within the bubble.

Assuming that the process of expansion and contraction of gas within the bubble is adiabatic, we can derive the following relation between p' and R at any instant

$$P' = P_0' \left(\frac{R_0}{R} \right)^{3n} \quad (4.1.2)$$

when n is the adiabatic index and P_0' is the initial pressure of gas within the bubble. In the case of adiabatic oscillations [8] of a bubble in water, we have $n = \frac{4}{3}$. We also assume, that there is no diffusion of gas through the boundary of the bubble.

The process of obtaining the general solution of (4.1.1) is fairly difficult even with various simplifying assumptions, Some conclusions concerning the appearance of cavitation described by this equation can, however, be drawn by perusing some general properties of the integral curves of (4.1.1) on the phase plane, This method gives a measure of success, in the qualitative investigations of nonlinear differential equations.

4.2 Study of differential equation :

Let us introduce the following dimensionless variables:

$$u = \frac{R}{R_0} , \quad \tau = \frac{t}{R_0} \left(\frac{P_0'}{\rho} \right)^{1/2} \quad (4.2.1)$$

Then Eq,(4.1.1) becomes

$$u u'' + \frac{3}{2} u^2 + \frac{4\mu}{R_0 \sqrt{\rho P_0'}} \frac{u'}{u} + \frac{2\sigma}{R_0 P_0'} \frac{1}{u} + \frac{P_0}{P_0'} - u^{-4} = 0 \quad (4.2.2)$$

Assuming that

$$u' = \frac{du}{d\tau} = y(u) , \quad \alpha = \frac{4\mu}{R_0 \sqrt{\rho P_0'}} , \quad b = \frac{2\sigma}{R_0 P_0'} , \quad c = \frac{P_0}{P_0'} \quad (4.2.3)$$

we can reduce (4.2.2) to the following first order differential equation:

$$\frac{dy}{du} = \frac{1 - c u^4 - b u^3 - \alpha y u^3 - 1.5 y^2 u^4}{y u^5} \quad (4.2.4)$$

which has one isolated singularity [7]

$$y = 0 , \quad u = u_* \quad (4.2.5)$$

Here u_* is the only positive root of

$$1 - b u^3 - c u^4 = 0 \quad (4.2.6)$$

We note that u_* can be > 1 , or < 1 .

Near the singularity we can write Eq.(4.2.4) as

$$\frac{dy}{dx} = \frac{-(4c u_*^3 + 3b u_*^3) x - \alpha y u_*^3 + P_2(x, y)}{y u_*^5 + Q_2(x, y)} \quad (4.2.7)$$

where $x = u - u_*$ while $P_2(x, y)$ and $Q_2(x, y)$ are , functions which are second order infinitesimals , when x , y are first order infinitesimals.

Since in the present case

$$\left(3b + 4cu_* \right) u_*^2 \neq 0$$

we shall use the following characteristic equation

$$\lambda^2 + \alpha u_* \lambda + \left(3b + 4cu_* \right) u_*^2 = 0 \quad (4.2.8)$$

to determine the character of the singularity. The discriminate of (4.2.8) is equal to

$$d_* = \left(3b + 4cu_* \right) u_*^3 - \frac{1}{4} \alpha^2 u_*^2 \quad (4.2.9)$$

Investigating the roots of (4.2.8) we find that two basically distinct possibilities may exist.

They are:

1) roots of the characteristic equation are real, distinct and of the same sign ($d < 0$), consequently the singularity is a node. All integral curves in the neighborhood of this singularity enter it at the same angle, and the common tangent is given by

$$y + kx = 0, \quad k = \frac{\alpha + \left[\alpha^2 - 4u_* (3b + 4cu_*) \right]^{\frac{1}{2}}}{2u_*^2}$$

As $x \rightarrow 0$, the velocity of the boundary of the bubble tends also to zero in a linear manner.

2) roots of the characteristic equation are complex conjugate . The singular point is a focus, each integral curve is contained between two logarithmic spirals. And approaches the point $y = 0$, $u = u_*$, circumscribing it an infinite number of times . The gas bubble oscillates infinitely long with diminishing amplitude.

When $d = 0$. we have the intermediate case when the characteristic equation (4.2.8) has a multiple root and the singularity is a degenerate node. All integral curves enter the singularity and are tangent at this point to

$$y + \frac{\alpha x}{2 u_*^2} = 0$$

The critical bubble radius separating two distinct modes of behavior, is obtained by solving the following pair of equations

$$d = 0 , \quad 1 - b u_*^3 - c u_*^4 = 0 \quad (4.2.10)$$

If the initial radius is larger than the critical one we have a focus if it is smaller we have a node . In the case of a perfect fluid ($\mu = 0$) Eq. (4.1.4) can be integrated to yield

$$y^2 u^4 + \frac{2 P_0'}{\rho} R_0^4 + \frac{2 P_0}{3 \rho} u^4 + \frac{2 \sigma}{\rho} u^3 + c_1 u = 0$$

Where c_1 is a constant of integration .

Considering this integral near the singularity and neglecting the third and higher order infinitesimals , we obtain an equation of the ellipse

$$y^2 u_*^4 + u^2 \left(3b u_* + 4c u_*^2 \right) + c_2 y + c_3 u + c_4 = 0 .$$

$$\left(c_2, c_3, c_4 = const ; c_4 < 0 \right)$$

$$(4.2.11)$$

With the singularity situated at its center .Here the gas bubble performs a steady oscillation.

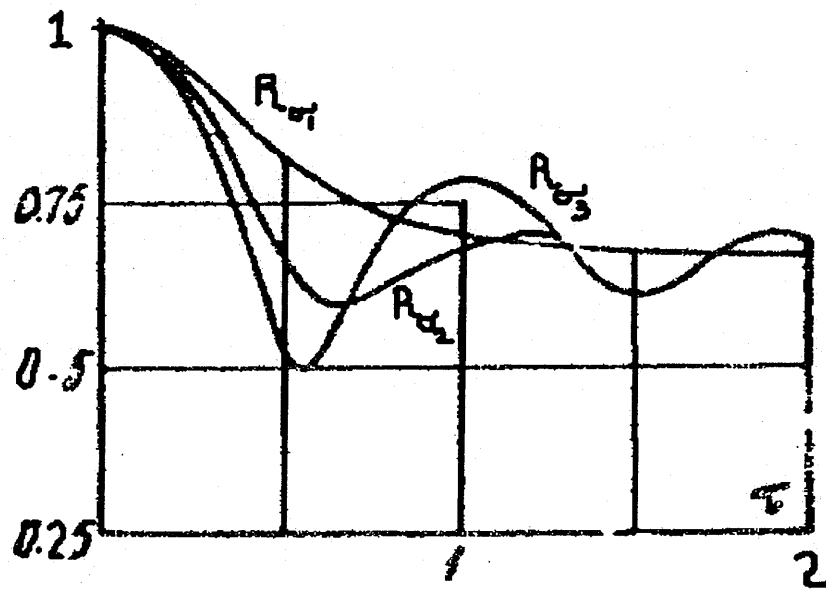


Fig. 1

It should be noted that there is a distinct difference between the process of collapsing of a bubble in a viscous liquid and that of expansion /contraction of a gaseous bubble discussed above . In the first case the surface tension has a qualitative effect on the collapse of a cavity (duration of the process remaining finite for any value of the viscosity coefficient [6]), while in the latter case two distinct modes of behavior are observed even in the absence of surface tension , the only variable being the critical radius of the bubble .

In particular, putting $b = 0$ in (4.2.10),we obtain

$$R_{0*} = \frac{\mu}{\sqrt{\rho} (P_0 P'_0)^{0.25}} \quad (4.2.12)$$

The actual value of the critical radius is fairly small , e.g.

when $P_0 = 1 \text{ atm}$ and $P'_0 = 2 \text{ atm}$,we have for water

$$R_{0*} = 0.178 * 10^{-3} \text{ mm} \text{ and for glycerin , } R_{0*} = 0.24 \text{ mm} .$$

Fig . 1 shows the results of a direct computation using Equation (4.1) , with $b = 0$, for the values

$$R_{0*} = 0.15 * 10^{-3} \text{ mm} < R_{0*} , R_{03} = 1 * 10^{-3} \text{ mm} > R_{0*} \text{ and}$$

$$R_{03} = 0.4 * 10^{-3} \text{ mm} > R_{0*}$$

NOMENCLATURE

R	bubble radius
$\dot{R}$	time derivative of the radius = radial velocity at the surface of the bubble
$\ddot{R}$	acceleration of the radius
r	radial coordinate
t	time
P	pressure
V	velocity
K	gas constant
$f(t)$	function of time
H	exponential index
$P(\infty)$	pressure far away from the bubble
$P(R)$	pressure inside the bubble
$C = \frac{P(\infty) - P(R)}{\rho}$	constant
D	constant of integration
m	mass of the bubble
$B(p, q)$	Beta function

B non dimensional parameter containing viscosity

$y = \frac{R}{R_0}$ non dimensional radius

$t_* = \frac{R_o}{\sqrt{C}}$ characteristic time

P amplitude of the pressure of bubble

t_{TC} time required for total collapse of the bubble

Greek letters

σ surface tension of the liquid

σ_{rr} normal stress

a_* critical radius of bubble

δ amplitude of the radius of the bubble

ν kinematic viscosity

ρ density

$\tau = \frac{t}{t_0}$ non dimensional time

μ dynamic viscosity

$\Gamma(p)$ Gamma function

λ thermal conductivity

γ adiabatic exponent of gas

Subscript

O at equilibrium

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الملخص

تهتم هذه الأطروحة بدراسة بعض السمات الرياضية لأثر الضغط في حجم فقاعة داخل سائل لزج غير قابل للانضغاط .

أولاً تم تحليل تقلص فقاعة في سائل غير لزج باستخدام معادلة ريليه :

$$R \ddot{R} + \frac{3}{2} \left(\dot{R} \right)^2 = \frac{P(R) - P(\infty)}{\rho}$$

حيث تم التوصل للوقت الكلي لتقلص فقاعة بـ :

$$t_{TC} = 0.915 \frac{R_0}{\sqrt{C}}$$

$$C = \frac{P(\infty) - P(R)}{\rho}$$

حيث

أما بالنسبة للسائل اللزج فإن زمن التقلص يكون منتهياً حينما $R > 8.4$ بينما زمن

$$\left(R = \frac{a}{\nu} \sqrt{\frac{P_0}{\rho}} \right) \text{ حيث } R < 8.4 \text{ عندما يكون غير منتهياً عند}$$

حيث في ديناميكية الفقاعة الغازية تعتمد حركة الفقاعة على قطرها الابتدائي .

أخيراً أُختبر تذبذب قطر الفقاعة في السائل ، حيث دُرست بالتفصيل هيئة توزيع جذور المعادلة ضمن التبادل الحراري بين المراحل .

المعادلة أظهرت وجود جذرين مترافقين - غير حقيقيين - وعدد غير محدود من الجذور الحقيقية ، كل الجذور وجدت بالجانب الأيسر لنصف المستوى الديكارتي للأعداد المركبة ، صعوداً لتخفيف التذبذب القطري للفقاعة .