

**University of Bahrain
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CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER

By

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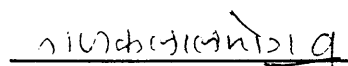
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ABSTRACT

In this thesis we study certain subclasses of analytic functions of complex order. In particular, we discuss certain results concerning starlike functions of complex order obtained by Nasr and Aouf. A class of univalent functions related to complex order by Dixit and Pal, and a generalised Ruscheweyh class of analytic functions of complex order by Ahuja are also studied. Further, we introduced and study a class of univalent analytic functions related to complex order analogous to Dixit and Pal.

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SYNOPSIS

The theory of univalent analytic functions has been one of the most fertile fields of investigation by eminent mathematicians who have enriched it by way of introduction of numerous new concepts, ingenious and powerful methods. It is concerned primarily with the interplay of analytic structure and geometric behaviour of the analytic functions. The settlement of sixty eight years old Bieberbach conjecture in 1984 by De Branges [15] not only created a land mark in the history of the subject but also aroused curiosity and interest in this branch of study.

The aim of the present thesis is to study certain subclasses of analytic functions of complex order. In particular, we discuss the results of Nasr and Aouf [44], Dixit and Pal [16], and Ahuja [3]. We also introduce and study a class of univalent analytic functions related to complex order analogous to Dixit and Pal [16]. The thesis consists of five chapters.

The basic concepts and principal results concerning the subject have been given in Chapter I.

In Chapter II, we study the results obtained by Nasr and Aouf [44] regarding starlike functions of complex order. Section 2.2 is devoted to obtain coefficient estimates for functions in the class S^{1-b} of starlike functions of complex order $1 - b$. In Section 2.3, we determine distortion and rotation theorems for functions in S^{1-b} . The sharp radius of starlikeness of the class S^{1-b} is the content of Section 2.4.

Chapter III is devoted to study the class $R^{1-b}(A, B)$ of univalent functions related to complex order introduced and studied by Dixit and Pal [16]. In Section 3.2, we discuss the coefficient bounds for functions in $R^{1-b}(A, B)$. A sufficient condition in terms of coefficients for functions in $R^{1-b}(A, B)$ is the content of Section 3.3. In Section 3.4, we determine distortion theorem for the class $R^{1-b}(A, B)$. Section 3.5 is devoted to discuss the maximization of $|a_3 - \mu a_2^2|$ over the class $R^{1-b}(A, B)$.

In Chapter IV, we introduce and study the class $S^{1-b}(A, B)$ of univalent functions related to complex order analogous to that of Dixit and Pal [16]. In Section 4.2, we establish coefficient estimates for functions in $S^{1-b}(A, B)$. Section 4.3 is devoted to obtain sufficient condition in terms of coefficients for functions in $S^{1-b}(A, B)$. In Section 4.4, we determine distortion theorem for the class $S^{1-b}(A, B)$. Section 4.5 deals with the maximization of $|a_3 - \mu a_2^2|$ over the class $S^{1-b}(A, B)$.

In Chapter V, we discuss generalised Ruscheweyh class $H_\lambda^b[A, B]$ of analytic functions of complex order introduced by Ahuja [3]. This class unifies several subclasses of analytic univalent functions including those of starlike, convex, prestarlike, Ruscheweyh class of order α , γ -spirallike, γ -convex spirallike and starlike of complex order. Section 5.2 is devoted to obtain coefficient bounds for functions in $H_\lambda^b[A, B]$.

CHAPTER I

PRELIMINARIES

In this chapter, we shall give those concepts and results of the theory of univalent functions and related topics that we shall need in the subsequent chapters. These results have been given in the form of definitions and theorems. Detailed proofs and further discussions on the subject may be found in the standard texts of Goluzin [25], Duren [17], Goodman [26, 27] etc. and in their respective research papers.

1.1. Introduction

Definition 1.1.1. A single-valued function f is said to be univalent in a domain D in the complex plane, if it takes no values more than once in D . Equivalently, if $f(z_1) = f(z_2)$ then $z_1 = z_2 \forall z_1, z_2 \in D$.

Note that every univalent function need not be analytic in a given domain D . For example, let

$$f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0.$$

This function is univalent but it is not analytic at $z = -d/c$ if it is one of the values taken by z in D . On the other hand, an analytic function need not be univalent. For example, $f(z) = e^z$ is analytic in any domain D but is not univalent in D since $f(z + 2n\pi i) = f(z) \forall z \in D$ and n is an integer.

An analytic univalent function is called a conformal mapping because of its angle preserving property. In the study of univalent analytic functions, the Riemann mapping theorem plays a crucial role. The theorem states that if D is a simply connected domain properly contained in the complex plane and z_0 is a given point in D

then $\exists$ a unique univalent analytic function f which maps D conformally onto the open unit disc $U = \{z : |z| < 1\}$ and has the properties $f(z_0) = 0$, $f'(z_0) > 0$. Therefore in view of Riemann mapping theorem, instead of studying the properties of univalent functions in an arbitrary simply connected domain, we shall confine ourselves in the case in which D is the open unit disc U .

Let $\mathbf{A}$ denote the class of all those functions f which are analytic in U and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, let

$$S = \{f \in \mathbf{A} : f \text{ is univalent in } U\}.$$

Thus, each $f \in S$ has a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \quad (1.1.2)$$

The leading example of a function of the class S is the Koebe function

$$k(z) = z(1-z)^{-2} = z + \sum_{n=2}^{\infty} n z^n \quad (1.1.3)$$

which maps the unit disc U onto the entire complex plane minus the part of the negative real axis from $-1/4$ to $-\infty$. This is best seen by writing

$$k(z) = \frac{1}{4} \left(\frac{1+z}{1-z} \right)^2 - \frac{1}{4}$$

and observing that the function

$$w = \frac{1+z}{1-z}$$

maps U conformally onto the right half-plane $\operatorname{Re}\{w\} > 0$.

Other simple examples of functions in S are

- (i) $f(z) = z$, the identity mapping;
- (ii) $f(z) = z(1-z)^{-1}$, which maps U conformally onto the half-plane $\operatorname{Re}\{f(z)\} > -1/2$;

- (iii) $f(z) = z(1 - z^2)^{-1}$, which maps U onto the entire plane minus the two half-lines $1/2 \leq u < \infty$ and $-\infty < u \leq -1/2$;
- (iv) $f(z) = \frac{1}{2} \log[(1+z)/(1-z)]$, which maps U onto the horizontal strip $-\pi/4 < \text{Im}\{f(z)\} < \pi/4$;
- (v) $f(z) = z - 1/2 z^2 = 1/2[1 - (1 - z)^2]$, which maps U onto the interior of a cardioid.

It is easy to check that the sum, difference, product and quotient of two functions in S need not be in S . The same is true for the derivative and the integral of a function $f \in S$. However, the class S is preserved under a number of elementary transformations. Here is a partial list:

- (i) **Conjugation.** If $f \in S$ and $h(z) = f(\bar{z}) = z + \square_2 z^2 + \square_3 z^3 + \dots$, then $h \in S$.
- (ii) **Rotation.** If $f \in S$ and $h(z) = e^{-i\theta} f(ze^{i\theta})$, then $h \in S$.
- (iii) **Disc automorphism.** If $f \in S$ and

$$h(z) = \frac{f\left(\frac{z+\alpha}{1+\bar{\alpha}z}\right) - f(\alpha)}{(1-|\alpha|^2)f'(\alpha)}, \quad |\alpha| < 1$$

then $h \in S$.

- (iv) **Range transformation.** If $f \in S$ and g is a function analytic and univalent on the range of f , with $g(0) = 0$ and $g'(0) = 1$, then $h = g \circ f \in S$.
- (v) **Omitted-value transformation.** If $f \in S$ and $f(z) \neq w$, then $h = wf/(w-f) \in S$.
- (vi) **Square-root transformation.** If $f \in S$ and $h(z) = \sqrt{f(z^2)}$, then $h \in S$.

The beginning of the theory of univalent analytic functions may be assigned to a research paper by Koebe, who in 1907, showed that $\exists$ a constant k (called Koebe's constant) such that the distance of the boundary of $f(U)$ for any $f \in S$ is not less than k from $w = 0$.

In 1916, Bieberbach [8] proved that the Koebe's constant $k = 1/4$. Other results proved by Bieberbach are as follows.

Theorem 1.1.4. (Koebe Covering Theorem). The range of every function of the class S contains the disc $\{w : |w| < 1/4\}$.

Theorem 1.1.5. (Distortion Theorem). For each $f \in S$,

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \quad (|z| = r < 1). \quad (1.1.6)$$

Theorem 1.1.7. (Growth Theorem). For each $f \in S$,

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \quad (|z| = r < 1). \quad (1.1.8)$$

Theorem 1.1.9. For each $f \in S$,

$$\frac{1-r}{1+r} \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \frac{1+r}{1-r}, \quad (|z| = r < 1). \quad (1.1.10)$$

The results in Theorems 1.1.5 , 1.1.7 and 1.1.9 are sharp for the Koebe function. Bieberbach [8] also proved that $|a_2| \leq 2 \forall f \in S$. The Koebe function was acting as an extremal function for most of the results he proved and since for this function $|a_n| = n \forall n \geq 2$, he made the following conjecture.

Bieberbach conjecture [8]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$, then $|a_n| \leq n \forall n \geq 2$.

This famous problem stood as a challenge until 1984 and was settled by De Branges [15] by giving a surprisingly short proof in view of the rich and continuing history of the problem. However the Bieberbach conjecture has inspired the development of many important and ingenious methods which now form the backbone of the subject. In fact, several subclasses of S have been introduced and studied. Most of these subclasses are determined by a fixed geometric property of the image domain. It is interesting to note that some of these subclasses of S play an important role in other branches of mathematics. For example, starlike functions play an

important role in the solution of certain differential equations (see [20],[51] etc.) whereas spirallike functions have been the source of useful counter examples in geometric function theory (see [18, 34]). Some of these classes are as follows.

1.2. Convex and Starlike Functions

Definition 1.2.1. A function $f \in \mathbf{A}$ is said to be convex in U if the image of U under f is a convex region; i.e., the line segment joining any two points of $f(U)$ lies entirely in $f(U)$.

We shall denote the class of all functions $f \in \mathbf{A}$ which are convex and univalent in U by C .

Theorem 1.2.2. A function $f \in \mathbf{A}$ is in C if, and only if,

$$\operatorname{Re} \left\{ 1 + z \frac{f''(z)}{f'(z)} \right\} > 0, \quad z \in U. \quad (1.2.3)$$

The condition (1.2.3) is a necessary and sufficient condition for a function $f \in C$ and was given by Robertson [50]. He also introduced the concept of order for functions in C as follows .

Definition 1.2.4. A function $f \in \mathbf{A}$ is said to be convex of order α ($0 \leq \alpha < 1$) if

$$\operatorname{Re} \left\{ 1 + z \frac{f''(z)}{f'(z)} \right\} > \alpha, \quad z \in U. \quad (1.2.5)$$

Let $C(\alpha)$ be the class of convex univalent functions of order α . It is clear that $C(0) = C$. Some of the results for convex univalent functions are as follows .

Theorem 1.2.6 [17]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C$, then $|a_n| \leq 1$ for $n = 2, 3, \dots$. The result is sharp with the extremal functions of the form

$$f(z) = z(1 - xz)^{-1}, \quad |x| = 1.$$

Theorem 1.2.7 [17]. For every positive number $r \leq 2 - \sqrt{3}$, each function $f \in S$ maps the disc $|z| < r$ onto a convex region. This is false for every $r > 2 - \sqrt{3}$.

Theorem 1.2.8 [17]. If $f \in C$, then $f(U) \supset \left\{ w : |w| < \frac{1}{2} \right\}$.

A subclass of S which is wider than C is the class of starlike functions defined as follows.

Definition 1.2.9. A function $f(z)$ in $\mathbf{A}$ is said to be starlike with respect to the origin in U if the image of U under f containing the point $w = 0$ is a starlike region. $f(U)$ is called a starlike region if the line segment joining $w = 0$ to any point of $f(U)$ lies entirely in $f(U)$. Let S^* be the class of all functions in $\mathbf{A}$ which are univalent and starlike with respect to the origin in U .

The following theorem due to Robertson [50] gives a necessary and sufficient condition for a function $f \in S^*$.

Theorem 1.2.10. A function $f \in \mathbf{A}$ is in S^* if, and only if,

$$\operatorname{Re} \left\{ z \frac{f'(z)}{f(z)} \right\} > 0, \quad z \in U. \quad (1.2.11)$$

The concept of order for starlike functions was also introduced by Robertson [50].

Definition 1.2.12. A function $f \in \mathbf{A}$ is said to be starlike of order α ($0 \leq \alpha < 1$) in U if

$$\operatorname{Re} \left\{ z \frac{f'(z)}{f(z)} \right\} > \alpha, \quad z \in U. \quad (1.2.13)$$

We shall denote by $S(\alpha)$, the class of starlike functions of order α ($0 \leq \alpha < 1$) in U . Clearly $S(0) = S^*$.

It is clear from (1.2.5) and (1.2.13) that

$$f \in C(\alpha) \text{ if, and only if, } zf' \in S(\alpha) \quad (1.2.14)$$

or

$$f \in S(\alpha) \text{ if, and only if, } \int_0^z \frac{f(t)}{t} dt = F(z) \in C(\alpha). \quad (1.2.15)$$

Some of the results of starlike functions are as follows.

Theorem 1.2.16 [17]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*$, then f is convex in $|z| < r$ where $r = 2 - \sqrt{3}$. The result is sharp.

The number r in the above theorem is called the radius of convexity for functions in S^* .

Theorem 1.2.17 [60]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(\alpha)$, then

$$|a_n| \leq \frac{\prod_{k=2}^n (k - 2\alpha)}{(n-1)!}, \quad (n = 2, 3, \dots). \quad (1.2.18)$$

Theorem 1.2.19 [60]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(\alpha)$, then

$$\frac{r}{(1+r)^{2(1-\alpha)}} \leq |f(z)| \leq \frac{r}{(1-r)^{2(1-\alpha)}}, \quad (|z| = r < 1). \quad (1.2.20)$$

The results in Theorem 1.2.17 and Theorem 1.2.19 are sharp with the extremal function $k_\alpha(z) = z(1-z)^{2\alpha-2}$.

Theorem 1.2.21 [60]. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathbf{A}$. If for some α ($0 \leq \alpha < 1$),

$$\sum_{n=2}^{\infty} (n-\alpha) a_n \leq 1-\alpha \quad (1.2.22)$$

then $f \in S(\alpha)$.

The concept of "type" for starlike functions of order α ($0 \leq \alpha < 1$) was introduced by Juneja and Mogra [30] as follows.

Definition 1.2.23. A function $f \in \mathbf{A}$ is said to be starlike of order α and type β if the condition

$$\left| \frac{z \frac{f'(z)}{f(z)} - 1}{z \frac{f'(z)}{f(z)} + 1 - 2\alpha} \right| < \beta$$

is satisfied for some α, β ($0 \leq \alpha < 1, 0 < \beta \leq 1$) and $\forall z \in U$.

The class of starlike functions of order α ($0 \leq \alpha < 1$) and type β ($0 < \beta \leq 1$) shall be denoted by $S(\alpha, \beta)$. Clearly $S(\alpha, 1) = S(\alpha)$.

1.3. Functions With Positive Real Part

Closely related to the classes C and S^* is the class P of all functions

$$p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$$

which are analytic and satisfy $\operatorname{Re}\{p(z)\} > 0$ for $z \in U$. Considerable work has been done to study various aspects of this class. Some of the results for the class P are as follows.

Theorem 1.3.1 [17]. If $p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \in P$, then $|b_n| \leq 2$ for $n = 1, 2, 3, \dots$. The result is sharp for the function

$$p(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n, \quad z \in U.$$

Theorem 1.3.2 [17]. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C$ if, and only if, $\left(1 + zf''(z)/f'(z)\right) \in P$.

Theorem 1.3.3 [17]. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*$ if, and only if, $\frac{zf'(z)}{f(z)} \in P$.

MacGregor [40] introduced the following class of functions $f \in \mathbf{A}$ such that $f'(z) \in P$, and obtained several properties for this class.

Definition 1.3.4. A function $f \in \mathbf{A}$ is said to be in R if

$$\operatorname{Re}\{f'(z)\} > 0, \quad z \in U. \quad (1.3.5)$$

Ezrohi [19] extended the results of MacGregor [40] by introducing the following class.

Definition 1.3.6. A function $f \in \mathbf{A}$ is said to be in the class $R(\alpha)$ if

$$\operatorname{Re}\{f'(z)\} > \alpha, \quad (1.3.7)$$

for some α , $0 \leq \alpha < 1$ and $z \in U$.

A subclass of R analogous to starlike functions of order α & type β has been introduced by Juneja and Mogra [31] as follows.

Definition 1.3.8. A function $f \in \mathbf{A}$ is said to be in the class $R(\alpha, \beta)$ if the condition

$$\left| \frac{f'(z) - 1}{f'(z) + 1 - 2\alpha} \right| < \beta \quad (1.3.9)$$

is satisfied for some α, β ($0 \leq \alpha < 1, 0 < \beta \leq 1$) and $\forall z \in U$.

Several other subclasses of R have been introduced and studied by different researchers (see [9], [13], [47], etc.)

1.4. Close-to-Convex and Spirallike Functions

A subclass of S which contains S^* is the class of close-to-convex functions, introduced by Kaplan [32] in 1952.

Definition 1.4.1. A function $f \in \mathbf{A}$ is said to be close-to-convex in U if there exists a convex function $\phi \in C$ such that

$$\operatorname{Re}\left\{\zeta \frac{f'(z)}{\phi'(z)}\right\} > 0, \quad z \in U, |\zeta| = 1. \quad (1.4.2)$$

We shall denote the class of close-to-convex functions in U by K^* .

In [32], Kaplan proved that every close-to-convex function is univalent; thus $K^* \subset S$. Every convex function is obviously close-to-convex. Also, it can be seen easily that every starlike function is close-to-convex in U . These remarks are summarized by the chain of proper inclusions

$$C \subset S^* \subset K^* \subset S.$$

The order α ($0 \leq \alpha < 1$) for $f \in K^*$ has been introduced by Libera [37] as follows .

Definition 1.4.3. A function $f \in \mathbf{A}$ is said to be close-to-convex of order α , $f \in K^*(\alpha)$, if $\exists \phi \in C$ such that

$$\operatorname{Re} \left\{ \frac{f'(z)}{\phi'(z)} \right\} > \alpha, \quad (1.4.4)$$

for some α ($0 \leq \alpha < 1$) and for all $z \in U$. Clearly $K^*(0) = K^*$.

Another interesting class which is wider than the class of starlike functions is the class of spirallike functions introduced and studied by Špacěk [69] in 1933 .

Definition 1.4.5. A function $f \in \mathbf{A}$ is said to be spirallike in U if

$$\operatorname{Re} \left\{ \zeta z \frac{f'(z)}{f(z)} \right\} > 0, \quad z \in U \quad (1.4.6)$$

for some ζ such that $|\zeta| = 1$.

It was shown by Špacěk [69] that spirallike functions are univalent in U . Taking $\zeta = e^{i\gamma}$, $|\gamma| < \pi/2$, Libera [38] introduced the class $SP(\gamma)$ of γ -spirallike functions as follows .

Definition 1.4.7. A function $f \in \mathbf{A}$ is said to be a γ -spirallike function, $f \in SP(\gamma)$, if

$$\operatorname{Re} \left\{ e^{i\gamma} z \frac{f'(z)}{f(z)} \right\} > 0, \quad (1.4.8)$$

for some γ ($|\gamma| < \pi/2$) and for all $z \in U$.

In [38], Libera introduced and studied the notion of order α ($0 \leq \alpha < 1$) for γ -spirallike functions in U , as follows.

Definition 1.4.9. A function $f \in \mathbf{A}$ is said to be a γ -spirallike function of order α in U if

$$\operatorname{Re} \left\{ e^{i\gamma} z \frac{f'(z)}{f(z)} \right\} > \alpha \cos \gamma, \quad z \in U, \quad (1.4.10)$$

for some real γ , ($|\gamma| < \pi/2$) and for some α ($0 \leq \alpha < 1$).

The class of γ -spirallike functions of order α , shall be denoted by $SP(\gamma, \alpha)$. Clearly $SP(0, \alpha) = S(\alpha)$, $SP(0, 0) = S^*$ and $SP(\gamma, 0) = SP(\gamma)$.

A class analogous to the class of γ -spirallike functions was introduced and studied by Chichra [10].

Definition 1.4.11. A function $f \in \mathbf{A}$ is said to be a γ -Robertson or γ -convex spirallike function of order α in U if

$$\operatorname{Re} \left\{ e^{i\gamma} \left(1 + z \frac{f''(z)}{f'(z)} \right) \right\} > \alpha \cos \gamma, \quad z \in U \quad (1.4.12)$$

for some real γ ($|\gamma| < \pi/2$) and for some α ($0 \leq \alpha < 1$).

We shall denote the class of γ -Robertson functions of order α by $RB(\gamma, \alpha)$.

1.5. Hadamard Product (Convolutions) of Analytic Functions

Definition 1.5.1. Given two functions $f, g \in \mathbf{A}$, where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, their Hadamard product or convolution $(f * g)(z)$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U.$$

This product is associative, commutative, and distributive over addition. Note that the function $I(z) = z(1-z)^{-1} = z + \sum_{n=2}^{\infty} z^n$ acts as the identity element under convolution.

Pólya and Schoenberg [48] conjectured that if $f \in C$ and $g \in C$, then $f * g \in C$. This problem has a long and illustrious history. Many researchers attacked this problem and obtained partial results. The first major result towards verifying the conjecture was due to Suffridge [71]. He showed that Hadamard product of two convex functions is close-to-convex, and hence univalent in U . The conjecture was finally proved by Ruscheweyh and Sheil-Small [59]. In fact, they proved the following results.

Theorem 1.5.2. If $f \in C$ and $g \in C$, then $f * g \in C$.

An equivalent formulation of Theorem 1.5.2 is as follows.

Theorem 1.5.3. If $f \in C$ and $g \in S^*$, then $f * g \in S^*$.

Theorem 1.5.4. If $f \in C$ and $g \in K^*$, then $f * g \in K^*$.

1.6. Prestarlike Functions

Using the Hadamard product, Ruscheweyh [58] introduced the class of prestarlike functions as follows.

Definition 1.6.1. A function $f \in \mathbf{A}$ is said to be prestarlike of order α , $\alpha \leq 1$, that is, $f \in G(\alpha)$ if, and only if,

$$\left\{ \begin{array}{l} \frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in S(\alpha) \quad \text{for } \alpha < 1 \\ \operatorname{Re} \left\{ \frac{f(z)}{z} \right\} > 1/2, \quad z \in U \quad \text{for } \alpha = 1. \end{array} \right. \quad (1.6.2)$$

$$\left\{ \begin{array}{l} \operatorname{Re} \left\{ \frac{f(z)}{z} \right\} > 1/2, \quad z \in U \quad \text{for } \alpha = 1. \end{array} \right. \quad (1.6.3)$$

The function $k_\alpha(z) = z(1-z)^{2\alpha-2}$ is the well-known extremal function

for the class $S(\alpha)$. If we let

$$c(\alpha, n) = \frac{\prod_{k=2}^n (k - 2\alpha)}{(n-1)!}, \quad (n = 2, 3, \dots), \quad (1.6.4)$$

then $k_\alpha(z)$ can be written as

$$k_\alpha(z) = z + \sum_{n=2}^{\infty} c(\alpha, n) z^n. \quad (1.6.5)$$

Note that $G(0) = C$ and $G(1/2) = S(1/2)$. In [58], Ruscheweyh proved several results for functions in $G(\alpha)$. Some of them are as follows.

Theorem 1.6.6 [58]. If $f, g \in G(\alpha)$, $\alpha \leq 1$, then $f * g \in G(\alpha)$.

Theorem 1.6.7 [58]. For $\alpha \leq \beta \leq 1$, we have $G(\alpha) \subset G(\beta)$.

The functions of $G(\alpha)$ need not be univalent. However, Silverman and Silvia [64] proved the following result.

Theorem 1.6.8 [64]. $G(\alpha) \subset S$ if and only if $\alpha \leq 1/2$.

Note that Theorem 1.6.7 generalizes the well-known result $C \subset S^*(1/2)$ whereas Theorem 1.6.6 generalizes the Polya-Schoenberg conjecture [48].

The concept of type for prestarlike functions was introduced as follows.

Definition 1.6.9. Given $\alpha < 1$ and $\beta < 1$, a function $f \in \mathbf{A}$ is said to be prestarlike of order α and type β , that is, $f \in G(\alpha, \beta)$ if $f * k_\alpha \in S(\beta)$.

Note that $G(\alpha, \alpha) \equiv G(\alpha)$, $\alpha < 1$. The class $G(\alpha, \beta)$ was investigated in [4].

1.7. Subordination and Ruscheweyh Derivatives

The concept of subordination can be traced to Lindelof (1904) but Littlewood [39] and Rogosinski [55,56] introduced the term and discovered the basic relations. Over the years a substantial theory has been developed, and subordination now plays an important role in complex analysis.

Definition 1.7.1. Let $f(z)$ and $g(z)$ be analytic functions in the unit disc U with $f(0) = g(0)$. Then g is subordinate to f , denoted by $g \ll f$ if

$$g(z) = f(w(z)), \quad z \in U \quad (1.7.2)$$

for some analytic function w with $w(0) = 0$ and $|w(z)| \leq |z|$. Some of the basic results regarding subordination are as follows.

Theorem 1.7.3. (Littlewood Subordination Theorem). Let f and g be analytic in the unit disc U , and suppose $g \ll f$. Then for $0 < p < \infty$,

$$M_p(r, g) \leq M_p(r, f), \quad 0 \leq r < 1 \quad (1.7.4)$$

where

$$M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}.$$

Strict inequality holds for $0 < r < 1$ unless f is constant or $w(z) = \alpha z$, $|\alpha| = 1$.

Theorem 1.7.5. (Rogosinski's Theorem). Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belong to **A** and suppose that $g \ll f$. Then

$$\sum_{k=2}^n |b_k|^2 \leq \sum_{k=2}^n |a_k|^2, \quad n = 2, 3, \dots \quad (1.7.6)$$

If f is convex or starlike in U , then we have the following result.

Theorem 1.7.7 [17]. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belong to $\mathbf{A}$. If $g \ll f$, and

(i) If $f \in C$, then $|b_n| \leq 1$, $n = 2, 3, \dots$

(ii) If $f \in S^*$, then $|b_n| \leq n$, $n = 2, 3, \dots$

Rogosinski Conjecture. If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is analytic in U and $g \ll f$ for some $f \in S$, then $|b_n| \leq n$, $n = 2, 3, \dots$

This conjecture is also known as generalized Bieberbach conjecture.

Ruscheweyh Derivatives.

Let $f \in \mathbf{A}$. Denote $D^\lambda : \mathbf{A} \rightarrow \mathbf{A}$ the operator defined by

$$D^\lambda f = \frac{z}{(1-z)^{\lambda+1}} * f, \lambda \geq -1, z \in U. \quad (1.7.8)$$

Note that $D^{-1}f = z$, $D^0f = f$, $D^1f = zf'$, $D^2f = z(f' + zf''/2)$.

Ruscheweyh [57] observed that $D^n f = z(z^{n-1}f(z))^{(n)}/n!$ where $n \in N_0 = \{1, 2, 3, \dots\}$. The symbol $D^n f(z)$, $n \in N_0$ was called the n th Ruscheweyh derivative of $f(z)$ by Al-Amiri [7].

Using the symbol $D^n f(z)$, Ruscheweyh [57] gave a new criterion for univalent functions by introducing the following class.

Definition 1.7.9. A function $f \in \mathbf{A}$ is said to be in the class L_n if it satisfies the condition

$$\operatorname{Re} \left(\frac{D^{n-1}f(z)}{D^n f(z)} \right) > 1/2 \quad (1.7.10)$$

for $n \in N_0$ and for all $z \in U$.

Ruscheweyh [57] proved that $L_{n-1} \subset L_n$, $n = 0, 1, 2, \dots$. Since L_0 is the class of starlike functions of order $1/2$, all functions in L_n are univalent.

Goel and Sohi [24] introduced and studied the following class using Ruscheweyh derivatives.

Definition 1.7.11. A function $f \in \mathbf{A}$ is said to be in the class $M_n(\alpha)$ if it satisfies the condition

$$\operatorname{Re}\left(\frac{D^{n+1}f(z)}{z}\right) > \alpha, \quad (1.7.12)$$

$0 \leq \alpha < 1$, $n \in N_0$ and $z \in U$.

They proved that

$$M_{n+1}(\alpha) \subset M_n(\alpha), \quad 0 \leq \alpha < 1, \quad n \in N_0. \quad (1.7.13)$$

Since $M_0(\alpha) = R(\alpha)$, it follows from (1.7.13) that all functions in $M_n(\alpha)$ are univalent. They also study integrals of functions in $M_n(\alpha)$.

Using Ruscheweyh derivatives, several other classes have been introduced and studied (see [42], [45], [70] etc.).

We close this chapter with a remark that other well-known results, which may be needed in the following chapters, will be presented along with the definitions and motivations in the chapters.

CHAPTER II

STARLIKE FUNCTIONS OF COMPLEX ORDER

2.1. Introduction

Let $\mathbf{A}$ denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disc $U = \{z : |z| < 1\}$. Denote by P the class of functions $p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ which are analytic and satisfy the condition $\operatorname{Re}(p(z)) > 0$ for $z \in U$. Nasr and Aouf [44] introduced the concept of complex order for starlike functions as follows.

Definition 2.1.1. A function $f \in \mathbf{A}$ is said to be starlike of order $1 - b$ ($b \neq 0, \text{complex}$), that is, $f \in S^{(1-b)}$ if, and only if, $f(z)/z \neq 0$ in U and

$$1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \in P, \quad z \in U. \quad (2.1.2)$$

It follows from the above definition that $f \in S^{(1-b)}$ if, and only if, there exists a function $p \in P$ such that

$$\frac{zf'(z)}{f(z)} = b[p(z) - 1] + 1, \quad z \in U. \quad (2.1.3)$$

Let Ω be the class of bounded analytic functions $\omega(z)$ in U satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| \leq |z|$ for $z \in U$. It is well-known that $p \in P$ if, and only if, there exists a function $\omega \in \Omega$ such that

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)}, \quad z \in U.$$

Using this fact it is easy to check that, $f \in S^{(1-b)}$ if, and only if, there exists a function $\omega \in \Omega$ such that

The purpose of this chapter is to discuss certain properties of the class $S^{(1-b)}$. In Section 2.2, we obtain coefficient estimates for functions in $S^{(1-b)}$. Section 2.3 is devoted to obtain distortion and rotation theorems for functions in $S^{(1-b)}$. Finally the sharp radius of starlikeness of the class $S^{(1-b)}$ is determined in Section 2.4.

2.2. Coefficient Estimates

In this section we obtain sharp coefficient bounds and maximize $|a_3 - \mu a_2^2|$ over the class $S^{(1-b)}$.

We need the following lemma due to Keogh and Merkes [33] in our discussion.

Lemma 2.2.1 [33]. Let $\omega(z) = \sum_{n=1}^{\infty} c_n z^n \in \Omega$. If v is any complex number then

$$|c_2 - v c_1^2| \leq \max\{1, |v|\}. \quad (2.2.2)$$

Equality may be attained with the functions $w(z) = z^2$ and $w(z) = z$.

Theorem 2.2.3. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^{(1-b)}$ and μ is any complex number, then

$$|a_3 - \mu a_2^2| \leq |b| \max\{1, |4b\mu - 2b - 1|\}. \quad (2.2.4)$$

This inequality is sharp for each μ .

Proof : Since $f \in S^{(1-b)}$, we get from (2.1.4) that

$$zf'(z).(1 - \omega(z)) = f(z).[1 + (2b - 1)\omega(z)].$$

Using $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\omega(z) = c_1 z + c_2 z^2 + \dots$, we obtain

$$\begin{aligned}
& (z + \sum_{n=2}^{\infty} na_n z^n)(1 - c_1 z - c_2 z^2 - c_3 z^3 - \dots) \\
&= (z + \sum_{n=2}^{\infty} a_n z^n)[1 + (2b - 1)(c_1 z + c_2 z^2 + c_3 z^3 + \dots)].
\end{aligned}$$

Equating the coefficients of z^2 and z^3 , we have

$$a_2 = 2bc_1, \quad (2.2.5)$$

and

$$a_3 = b[(2b + 1)c_1^2 + c_2]. \quad (2.2.6)$$

Using (2.2.5), (2.2.6) and (2.2.2), we get

$$\begin{aligned}
|a_3 - \mu a_2^2| &= |b\{c_2 - (4b\mu - 2b - 1)c_1^2\}| \\
&= |b| |c_2 - (4b\mu - 2b - 1)c_1^2| \\
&\leq |b| \max\{1, |4b\mu - 2b - 1|\}
\end{aligned}$$

which is (2.2.4). Since (2.2.2) is sharp then (2.2.4) should also be sharp.

We prove the following theorem by using the method of Clunie[11].

Theorem 2.2.7. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^{(1-b)}$, $\operatorname{Re}\{b\} > 0$, $z \in U$,

then

$$|a_n| \leq \frac{1}{(n-1)!} \prod_{m=0}^{n-2} |2b + m|, \quad n = 2, 3, \dots \quad (2.2.8)$$

Equality is attained in (2.2.8) for the function

$$f^*(z) = \frac{z}{(1-z)^{2b}} = z + \sum_{n=2}^{\infty} \prod_{m=0}^{n-2} \left(\frac{2b+m}{m+1}\right) z^n. \quad (2.2.9)$$

Proof : Since $f \in S^{(1-b)}$, we have

$$\frac{zf'(z)}{f(z)} = \frac{1 + (2b-1)\omega(z)}{1 - \omega(z)}. \quad (2.2.10)$$

Simplifying and rewriting (2.2.10), we get

$$\left[zf'(z) + (2b-1)f(z) \right] \omega(z) = zf'(z) - f(z).$$

Using $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $zf'(z) = z + \sum_{k=2}^{\infty} k a_k z^k$ and $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ in the above expression, we obtain

$$\left[2bz + \sum_{k=2}^{n-1} (k+2b-1)a_k z^k \right] \omega(z) = \sum_{k=2}^n (k-1)a_k z^k + \sum_{k=n+1}^{\infty} d_k z^k$$

where d_k 's are some complex numbers. Then, since $|\omega(z)| < 1$, we have

$$\left| 2bz + \sum_{k=2}^{n-1} (2b+k-1)a_k z^k \right| \geq \left| \sum_{k=2}^n (k-1)a_k z^k + \sum_{k=n+1}^{\infty} d_k z^k \right|. \quad (2.2.11)$$

Squaring both sides of (2.2.11) and integrating round $|z| = r < 1$, we get after taking the limit when $r \rightarrow 1$, that

$$(n-1)^2 |a_n|^2 \leq |2b|^2 + \sum_{k=2}^{n-1} (|2b+k-1|^2 - (k-1)^2) |a_k|^2. \quad (2.2.12)$$

Since $\operatorname{Re}\{b\} > 0$, for $n = 2$, (2.2.12) gives

$$|a_2|^2 \leq 4|b|^2 \quad \text{or} \quad |a_2| \leq 2|b|.$$

Also, for $n = 3$, (2.2.12) gives

$$4|a_3|^2 \leq 4|b|^2 + (|2b+1|^2 - 1)|a_2|^2 = |2b|^2 |2b+1|^2.$$

It follows that

$$|a_3| \leq \frac{|2b||2b+1|}{2!}$$

which is true. Suppose that (2.2.8) is true for $n = p$, then

$$|a_p| \leq \frac{1}{(p-1)!} \prod_{m=0}^{p-2} |2b+m|.$$

For $n = p+1$, equation (2.2.12) gives

$$\begin{aligned} p^2 |a_{p+1}|^2 &\leq 4|b|^2 + \sum_{k=2}^p (|2b+k-1|^2 - (k-1)^2) |a_k|^2 \\ &\leq 4|b|^2 + (|2b+1|^2 - 1)4|b|^2 + (|2b+2|^2 - 4) \left(\frac{1}{2} \prod_{m=0}^1 |2b+m| \right)^2 \\ &\quad + \dots + (|2b+p-1|^2 - (p-1)^2) \left(\frac{1}{(p-1)!} \prod_{m=0}^{p-2} |2b+m| \right)^2 \\ &= \frac{|2b|^2 |2b+1|^2 \dots |2b+p-2|^2 |2b+p-1|^2}{((p-1)!)^2} \\ &= \frac{1}{((p-1)!)^2} \prod_{m=0}^{p-1} |2b+m|^2 \end{aligned}$$

which gives

$$|a_{p+1}| \leq \frac{\prod_{m=0}^{p-1} |2b+m|}{p!}.$$

Thus if the result (2.2.8) is true for $n = p$, it is also true for $n = p+1$. Hence by mathematical induction the estimates (2.2.8) are true for all $n \geq 2$, that is,

$$|a_n| \leq \frac{1}{(n-1)!} \prod_{m=0}^{n-2} |2b+m|, \quad n = 2, 3, 4, \dots$$

This completes the proof of the theorem.

2.3. Distortion and Rotation Theorems

In this section we obtain distortion and rotation theorems for functions in $S^{(1-b)}$.

Theorem 2.3.1. If $f \in S^{(1-b)}$, then the set of all possible values of $\log\left(\frac{f(z)}{z}\right)$ for a fixed z , lies in the image, which is convex, of the disc $|z| \leq r$, $0 < r < 1$ under the mapping

$$w = \log(1-z)^{-2b}. \quad (2.3.2)$$

The function $f^* \in S^{(1-b)}$ defined by

$$f^*(z) = \frac{z}{(1-z)^{2b}} \quad (2.3.3)$$

shows that the result is sharp in the sense that the boundary points are attained by this function.

Proof: Since $f \in S^{(1-b)}$, from (2.1.6) we have

$$\log\left(\frac{f(z)}{z}\right) = \left\{ \int_0^{2\pi} \log(1 - ze^{-it})^{-2b} d\mu(t) \right\}. \quad (2.3.4)$$

Let

$$q(z, t) = \log(1 - ze^{-it})^{-2b}, \quad t \in [0, 2\pi].$$

Then since

$$\operatorname{Re} \left\{ 1 + \frac{zq''(z, t)}{q'(z, t)} \right\} = \operatorname{Re} \left\{ \frac{1}{1 - ze^{-it}} \right\} > \frac{1}{2},$$

the function $q(z, t)$ maps the disc $|z| \leq r < 1$ univalently onto a convex domain U^x , and hence they lie in the image of $|z| \leq r$ under the mapping

$$w = \log(1 - \epsilon z)^{-2b}, \quad |\epsilon| = 1$$

which gives (2.3.2). The extremal function is given by

$$\log\left(\frac{f^*(z)}{z}\right) = \log(1 - z)^{-2b} \quad (2.3.5)$$

which in view of (2.3.4) gives (2.3.3).

Theorem 2.3.6. If $f \in S^{(1-b)}$, then for $|z| = r < 1$,

$$\Phi_2(r) \leq \log\left|\frac{f(z)}{z}\right| \leq \Phi_1(r), \quad (2.3.7)$$

$$\Psi_2(r) \leq \arg\left(\frac{f(z)}{z}\right) \leq \Psi_1(r), \quad (2.3.8)$$

where

$$\begin{aligned} \Phi_{1,2}(r) = & -2 \operatorname{Re}\{b\} \log\left\{\left[1 - \left(\frac{r \operatorname{Im}\{b\}}{|b|}\right)^2\right]^{1/2} \mp r \frac{\operatorname{Re}\{b\}}{|b|}\right\} \\ & \pm 2 \operatorname{Im}\{b\} \sin^{-1}\left(\frac{r \operatorname{Im}\{b\}}{|b|}\right) \end{aligned} \quad (2.3.9)$$

and

$$\begin{aligned} \Psi_{1,2}(r) = & -2 \operatorname{Im}\{b\} \log\left\{\left[1 - \left(\frac{r \operatorname{Re}\{b\}}{|b|}\right)^2\right]^{1/2} \mp \frac{r \operatorname{Im}\{b\}}{|b|}\right\} \\ & \pm 2 \operatorname{Re}\{b\} \sin^{-1}\left(r \frac{\operatorname{Re}\{b\}}{|b|}\right). \end{aligned} \quad (2.3.10)$$

Equality is attained in both the left and right sides of (2.3.7) and (2.3.8) for the function $f^*(z)$ defined by (2.3.3) for $z = re^{i\theta_j}$, $j = 1, 2$ and $z = re^{i\theta_j}$, $j = 3, 4$ respectively, where

$$\theta_{1,2} = \sin^{-1} \left\{ \frac{r \operatorname{Im}\{b\} \operatorname{Re}\{b\}}{|b|^2} \pm \frac{\operatorname{Im}\{b\}}{|b|} \left[1 - \left(\frac{r \operatorname{Im}\{b\}}{|b|} \right)^2 \right]^{1/2} \right\} \quad (2.3.11)$$

and

$$\theta_{3,4} = -\sin^{-1} \left\{ \frac{r \operatorname{Im}\{b\} \operatorname{Re}\{b\}}{|b|^2} \pm \frac{\operatorname{Re}\{b\}}{|b|} \left[1 - \left(\frac{r \operatorname{Re}\{b\}}{|b|} \right)^2 \right]^{1/2} \right\}. \quad (2.3.12)$$

Proof: From Theorem 2.3.1, we know that the region of variability of $\log(\frac{f(z)}{z})$ for a fixed z , $|z| \leq r < 1$, coincides with the image of $|z| \leq r$ under the mapping $\log(1-z)^{-2b}$. We first show that the function f^* defined as

$$\log\left(\frac{f^*(z)}{z}\right) = \log(1-z)^{-2b} \quad (2.3.13)$$

belongs to $S^{(1-b)}$. In fact, from (2.3.13), we have

$$1 + \frac{1}{b} \left(\frac{zf^{*'}(z)}{f^*(z)} - 1 \right) = \frac{1+z}{1-z}$$

and so

$$\operatorname{Re} \left[1 + \frac{1}{b} \left(\frac{zf^{*'}(z)}{f^*(z)} - 1 \right) \right] = \operatorname{Re} \left(\frac{1+z}{1-z} \right) > 0, \text{ for } |z| \leq r < 1.$$

Thus

$$1 + \frac{1}{b} \left(\frac{zf^{*'}(z)}{f^*(z)} - 1 \right) \in P$$

and consequently $f^* \in S^{(1-b)}$. Since the function f^* defined by (2.3.13) is in $S^{(1-b)}$, we see that, for any $f \in S^{(1-b)}$

$$\log \left| \frac{f(z)}{z} \right| = \operatorname{Re} \left\{ \log \left(\frac{f(z)}{z} \right) \right\} \quad \text{and} \quad \arg \left(\frac{f(z)}{z} \right) = \operatorname{Im} \left\{ \log \left(\frac{f(z)}{z} \right) \right\}$$

will lie between the extreme values of $\Phi(\theta)$ and $\Psi(\theta)$ respectively where

$$\begin{aligned} \Phi(\theta) &= \log \left| \frac{f^*(z)}{z} \right| = \operatorname{Re} \{ \log(1-z)^{-2b} \} \\ &= \operatorname{Re} [-2b \{ \log|1 - re^{i\theta}| + i \arg(1 - re^{i\theta}) \}] \\ &= \operatorname{Re} [-2b \{ \log|(1 - r \cos \theta) - ir \sin \theta| + i \arg((1 - r \cos \theta) - ir \sin \theta) \}] \\ &= \operatorname{Re} \left[-b \log(1 + r^2 - 2r \cos \theta) - 2ib \tan^{-1} \left\{ \frac{-r \sin \theta}{1 - r \cos \theta} \right\} \right] \\ &= -\operatorname{Re}\{b\} \log(1 - 2r \cos \theta + r^2) + 2 \operatorname{Im}\{b\} \tan^{-1} \left\{ \frac{-r \sin \theta}{1 - r \cos \theta} \right\}, \quad (2.3.14) \end{aligned}$$

and

$$\begin{aligned} \Psi(\theta) &= \arg \left(\frac{f^*(z)}{z} \right) = \operatorname{Im} \{ \log(1-z)^{-2b} \} \\ &= \operatorname{Im} \left[-b \log(1 - 2r \cos \theta + r^2) - 2ib \tan^{-1} \left\{ \frac{-r \sin \theta}{1 - r \cos \theta} \right\} \right] \\ &= -\left\{ \operatorname{Im}\{b\} \log(1 - 2r \cos \theta + r^2) + 2 \operatorname{Re}\{b\} \tan^{-1} \left(\frac{-r \sin \theta}{1 - r \cos \theta} \right) \right\}. \quad (2.3.15) \end{aligned}$$

Putting $t = \tan \theta/2$ in (2.3.14), we see that $\frac{d}{d\theta} \Phi(\theta) = 0$ when

$$(1+r) \operatorname{Im}\{b\} \tan^2 \theta/2 - 2 \operatorname{Re}\{b\} \tan \theta/2 - (1-r) \operatorname{Im}\{b\} = 0$$

and hence

$$t = t_{1,2} = \frac{\frac{\operatorname{Re}\{b\}}{|b|} \mp \sqrt{1 - \left(\frac{r \operatorname{Im}\{b\}}{|b|} \right)^2}}{(1+r) \frac{\operatorname{Im}\{b\}}{|b|}}. \quad (2.3.16).$$

Now it is easy to check that the maximum of $\Phi(\theta)$ is attained for $t = t_1 = \tan \theta_1/2$ and for this value of t , the value of $\Phi(t)$ is seen to be given by $\Phi = \Phi_1(r)$; while the minimum of $\Phi(\theta)$ is attained for $t = t_2 = \tan \theta_2/2$ and for this value of t the value of $\Phi(t)$ is seen to be given by $\Phi = \Phi_2(r)$. This gives (2.3.7) and simple calculation shows the case of equality for the function given by (2.3.3) at the points $z_1 = re^{i\theta_1}$ and $z_2 = re^{i\theta_2}$ respectively where θ_1 and θ_2 are given by (2.3.11).

In a similar manner, we see that $\frac{d\Psi(\theta)}{d\theta} = 0$ when

$$\tan \theta/2 = t = t_{3,4} = \frac{\frac{-\operatorname{Im}\{b\}}{|b|} \mp \sqrt{1 - \left(\frac{r\operatorname{Re}\{b\}}{|b|}\right)^2}}{(1+r)\frac{\operatorname{Re}\{b\}}{|b|}}. \quad (2.3.17)$$

Again it is easy to check that the maximum of $\Psi(\theta)$ is attained for $t = t_3 = \tan \theta_3/2$ and for this value of t , the value of $\Psi(t)$ is seen to be given by $\Psi = \Psi_1(r)$, while the minimum of $\Psi(\theta)$ is attained for $t = t_4 = \tan \theta_4/2$ and for this value of t , the value of $\Psi(t)$ is seen to be given by $\Psi = \Psi_2(r)$. This gives (2.3.8) and equality occurs for $f^*(z)$ given by (2.3.3) at the points $z_3 = re^{i\theta_3}$ and $z_4 = re^{i\theta_4}$ respectively where θ_3 and θ_4 are given by (2.3.12).

Theorem 2.3.18. If $f \in S^{(1-b)}$, then for $|z| = r < 1$,

$$\left| z \frac{f'(z)}{f(z)} \right| \leq 1 + \frac{2|b|r}{1-r}. \quad (2.3.19)$$

Equality in (2.3.19) is attained for the function $f^*(z)$ given by (2.3.3) at

$$z = r\sqrt{\bar{b}/b} \left[1 - r \left(1 - \sqrt{\bar{b}b} \right) \right]^{-1}.$$

Proof: Since $f \in S^{(1-b)}$, it follows from (2.1.6), that

$$z \frac{f'(z)}{f(z)} = 1 + 2b \int_0^{2\pi} \frac{ze^{-it}}{1 - ze^{-it}} d\mu(t)$$

and so for $|z| = r < 1$,

$$\left| z \frac{f'(z)}{f(z)} \right| = \left| 1 + 2b \int_0^{2\pi} \frac{ze^{-it}}{1 - ze^{-it}} d\mu(t) \right| \leq 1 + \frac{2|b|r}{1-r}. \quad (2.3.20)$$

To prove the equality, consider

$$f^*(z) = \frac{z}{(1-z)^{2b}}.$$

Then

$$\frac{zf^{*'}(z)}{f^*(z)} = 1 + \frac{2bz}{1-z}$$

and hence the equality in (2.3.19) will hold only if

$$\frac{2bz}{1-z} = \frac{2|b|r}{1-r},$$

which gives

$$z = r\sqrt{b/b} \left[1 - r \left(1 - \sqrt{b/b} \right) \right]^{-1}.$$

2.4. Radius of Starlikeness for Functions in $S^{(1-b)}$.

In this section we determine the sharp radius of starlikeness and hence univalence for the functions in $S^{(1-b)}$. We need the following lemmas.

Lemma 2.4.1. If $f(z) \in S^{(1-b)}$, $a \in U$, then

$$F(z) = \frac{azf\left(\frac{z+a}{1+\bar{a}z}\right)}{f(a)(z+a)(1+\bar{a}z)^{2b-1}} \in S^{(1-b)}. \quad (2.4.2)$$

Proof: Since $f \in S^{(1-b)}$, by Corollary 2.1.7 there exists a function $f_1 \in S^*$ such that

$$f(z) = z \left[\frac{f_1(z)}{z} \right]^b. \quad (2.4.3)$$

It is well known [50] that if $f_1 \in S^*$, then F_1 defined by $F_1(0) = 0$ and

$$F_1(z) = \frac{azf_1\left(\frac{z+a}{1+\bar{a}z}\right)}{f_1(a)(z+a)(1+\bar{a}z)^{2b-1}}, \quad z \neq 0. \quad (2.4.4)$$

belongs to S^* . Therefore from (2.4.3) and (2.4.4) it follows that the function $F(z)$ defined by $F(0) = 0$ and

$$F(z) = z \left[\frac{F_1(z)}{z} \right]^b = \frac{azf\left(\frac{z+a}{1+\bar{a}z}\right)}{f(a)(z+a)(1+\bar{a}z)^{2b-1}}, \quad z \neq 0$$

also belongs to $S^{(1-b)}$, which proves the lemma.

Lemma 2.4.5. For $|z| \leq r$ and f ranging over $S^{(1-b)}$, the domain of values of $zf'(z)/f(z)$ is the disc with center

$$\left(\frac{1 + (2 \operatorname{Re}\{b\} - 1)r^2}{1 - r^2}, \frac{2 \operatorname{Im}\{b\}r^2}{1 - r^2} \right)$$

and radius $\frac{2|b|r}{1 - r^2}$.

Proof: Suppose $f \in S^{(1-b)}$. Then by Lemma 2.4.1, $F(z)$ defined by (2.4.2) is also in $S^{(1-b)}$. It follows from (2.2.8) that

$$\left| \frac{F''(0)}{2} \right| = \left| \frac{f'(a)}{f(a)}(1 - |a|^2) - \frac{1 + (2b - 1)|a|^2}{a} \right| \leq 2|b|, \text{ for } |a| < 1.$$

Since a is an arbitrary complex number in U , we can replace a by z in the above inequality and write

$$\left| \frac{zf'(z)}{f(z)} - \frac{1 + (2b - 1)r^2}{1 - r^2} \right| \leq \frac{2|b|r}{1 - r^2}, \text{ for } |z| < 1. \quad (2.4.6)$$

Now, let $\frac{zf'(z)}{f(z)} = u + iv$ and $b = \operatorname{Re}\{b\} + i\operatorname{Im}\{b\}$. Then (2.4.6) becomes

$$\left| u + iv - \frac{1 + [(2\operatorname{Re}\{b\} - 1) + i2\operatorname{Im}\{b\}]r^2}{1 - r^2} \right| \leq \frac{2|b|r}{1 - r^2}$$

which is equivalent to

$$\left(u - \frac{1 + (2\operatorname{Re}\{b\} - 1)r^2}{1 - r^2} \right)^2 + \left(v - \frac{2\operatorname{Im}\{b\}r^2}{1 - r^2} \right)^2 \leq \left(\frac{2|b|r}{1 - r^2} \right)^2.$$

Therefore, the domain of values of $\frac{zf'(z)}{f(z)}$ is the disc with center

$$\left(\frac{1 + (2\operatorname{Re}\{b\} - 1)r^2}{1 - r^2}, \frac{2\operatorname{Im}\{b\}r^2}{1 - r^2} \right) \text{ and radius } \frac{2|b|r}{1 - r^2}, \text{ and the}$$

proof is completed.

Theorem 2.4.7. The sharp radius of starlikeness of the class $S^{(1-b)}$ is

$$r_s = \left\{ |b| + [|b|^2 - 2\operatorname{Re}\{b\} + 1]^{1/2} \right\}^{-1}. \quad (2.4.8)$$

Proof: Since $f \in S^{(1-b)}$, from (2.4.6), we have

$$\left| \frac{zf'(z)}{f(z)} - \frac{1 + (2b - 1)r^2}{1 - r^2} \right| \leq \frac{2|b|r}{1 - r^2}, \quad |z| = r < 1$$

which gives

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \frac{1 + [(2\operatorname{Re}\{b\} - 1) + i2\operatorname{Im}\{b\}]r^2}{1 - r^2} \right\} \geq \frac{-2|b|r}{1 - r^2}$$

and so

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \frac{1 - 2|b|r + (2\operatorname{Re}\{b\} - 1)r^2}{1 - r^2}, \quad |z| = r.$$

Thus $\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0$ for $r < r_s$ where

$$r_s = \left\{ |b| + [|b|^2 - 2\operatorname{Re}\{b\} + 1]^{1/2} \right\}^{-1}.$$

To show that (2.4.8) is sharp, we let

$$f^*(z) = \frac{z}{(1 - z)^{2b}} \text{ and } w = r \left(r - \sqrt{\frac{\bar{b}}{b}} \right) / \left(1 - r \sqrt{\frac{\bar{b}}{b}} \right),$$

then we have

$$w \frac{f^{*'}(w)}{f^*(w)} = \frac{1 - 2|b|r + (2b - 1)r^2}{1 - r^2}.$$

Now it is easy to check that at $r = \{|b| + [|b|^2 - 2\operatorname{Re}\{b\} + 1]^{1/2}\}^{-1}$,

$$\operatorname{Re}\left(w \frac{f^{*'}(w)}{f^*(w)}\right) = \operatorname{Re}\left\{\frac{i \operatorname{Im}\{b\}}{|b|^2 - \operatorname{Re}\{b\} + |b|\sqrt{|b|^2 - 2\operatorname{Re}\{b\} + 1}}\right\} = 0.$$

Hence the radius of starlikeness of the class $S^{(1-b)}$ given by (2.4.8) is sharp. This completes the proof of the theorem.

CHAPTER III

ON A CLASS OF UNIVALENT FUNCTIONS RELATED TO COMPLEX ORDER I

3.1. Introduction

Let Ω denote the class of functions

$$\omega(z) = \sum_{n=1}^{\infty} c_n z^n$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$ and satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| \leq |z| < 1$ for $z \in U$. In this chapter we study the following class of univalent functions related to complex order introduced and studied by Dixit and Pal [16].

Let $R^{1-b}(A, B)$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in U and satisfying the condition

$$\left| \frac{f'(z) - 1}{b(A - B) - B\{f'(z) - 1\}} \right| < 1, \quad z \in U \quad (3.1.1)$$

where A and B are arbitrary fixed real numbers such that $-1 \leq B < A \leq 1$ and b is non-zero complex number. Clearly a function $f(z)$ belongs to $R^{1-b}(A, B)$ if, and only if, there exists a function $\omega(z)$ in Ω such that

$$1 + \frac{1}{b} \{f'(z) - 1\} = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad z \in U. \quad (3.1.2)$$

By giving specific values to b, A, B in (3.1.1), we obtain the

following important subclasses introduced and studied by various researchers in earlier works.

(i) Putting $b = e^{-i\gamma} \cos \gamma$, $|\gamma| < \pi/2$ in (3.1.1) we obtain the class $R^{1-e^{-i\gamma} \cos \gamma}(A, B) = R_\gamma(A, B)$ of functions $f(z)$ satisfying the condition

$$\left| \frac{e^{i\gamma} \{f'(z) - 1\}}{Be^{i\gamma} f'(z) - (A \cos \gamma + iB \sin \gamma)} \right| < 1, z \in U$$

studied by Dashrath [13].

(ii) $b = 1$, $A = \beta$ and $B = -\beta$ in (3.1.1) gives the class $R^0(\beta, -\beta) = R[\beta]$ of functions $f(z)$ satisfying the condition

$$\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta, z \in U$$

studied by Caplinger and Causey [9] and Padmanabhan [47].

(iii) Fixing $b = 1$, $A = (1 - 2\alpha)\beta$, $B = -\beta$ in (3.1.1) we obtain the class $R^0((1 - 2\alpha)\beta, -\beta) = R(\alpha, \beta)$ of functions $f(z)$ satisfying the condition

$$\left| \frac{f'(z) - 1}{f'(z) + 1 - 2\alpha} \right| < \beta, z \in U$$

where $0 \leq \alpha < 1$, $0 < \beta \leq 1$, studied by Juneja and Mogra [31].

(iv) $b = 1$ in (3.1.1) leads to the class $R^0(A, B) \equiv R^*(A, B)$ of functions $f(z)$ satisfying the condition

$$\left| \frac{f'(z) - 1}{Bf'(z) - A} \right| < 1, z \in U$$

studied by Goel and Mehrotra [23].

In this chapter, we study the various properties of the class $R^{1-b}(A, B)$. In Section 3.2, we obtain coefficient bounds for functions in $R^{1-b}(A, B)$. Section 3.3 is devoted to study sufficient condition in terms of coefficients for functions to be in $R^{1-b}(A, B)$. In Section 3.4, we determine distortion theorem for the class $R^{1-b}(A, B)$. Finally, in Section 3.5 we discuss maximization of $|a_3 - \mu a_2^2|$ over the class $R^{1-b}(A, B)$.

3.2. Coefficient Estimates for Functions in $R^{1-b}(A, B)$

Theorem 3.2.1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R^{1-b}(A, B)$, then

$$|a_n| \leq \frac{(A-B)|b|}{n}. \quad (3.2.2)$$

The estimates are sharp.

Proof: Since $f \in R^{1-b}(A, B)$, we have

$$1 + \frac{1}{b} \{f'(z) - 1\} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (3.2.3)$$

where $\omega(z)$ is analytic in U and satisfies the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in U$. From (3.2.3) we have

$$\left[(B-A) + \frac{B}{b} \sum_{n=2}^{\infty} n a_n z^{n-1} \right] \omega(z) = -\frac{1}{b} \sum_{n=2}^{\infty} n a_n z^{n-1}$$

which is equivalent to

$$\left[(A-B) - \frac{B}{b} \sum_{n=2}^{\infty} n a_n z^{n-1} \right] \left[\sum_{n=1}^{\infty} c_n z^n \right] = \frac{1}{b} \sum_{n=2}^{\infty} n a_n z^{n-1}. \quad (3.2.4)$$

Equating corresponding coefficients in (3.2.4) we observe that the coefficient a_n on the right of (3.2.4) depends only on $a_2, a_3, \dots, a_{n-1}$ on the left of (3.2.4). Hence for $n \geq 2$, it follows from (3.2.4) that

$$\left[(A - B) - \frac{B}{b} \sum_{n=2}^{k-1} n a_n z^{n-1} \right] \omega(z) = \frac{1}{b} \sum_{n=2}^k n a_n z^{n-1} + \sum_{n=k+1}^{\infty} d_n z^{n-1}.$$

Since $|\omega(z)| \leq 1$ for $z \in U$, this yields

$$\left| (A - B) - \frac{B}{b} \sum_{n=2}^{k-1} n a_n z^{n-1} \right| \geq \left| \frac{1}{b} \sum_{n=2}^k n a_n z^{n-1} + \sum_{n=k+1}^{\infty} d_n z^{n-1} \right|. \quad (3.2.5)$$

Squaring both sides of (3.2.5) and integrating round $|z| = r$, $0 < r < 1$, we obtain

$$\begin{aligned} (A - B)^2 + \frac{B^2}{|b|^2} \sum_{n=2}^{k-1} n^2 |a_n|^2 r^{2n-2} &\geq \frac{1}{|b|^2} \sum_{n=2}^k n^2 |a_n|^2 r^{2n-2} \\ &+ \sum_{n=k+1}^{\infty} |d_n|^2 r^{2n-2}. \end{aligned} \quad (3.2.6)$$

Letting $r \rightarrow 1$ in (3.2.6), we conclude that

$$(A - B)^2 + \frac{B^2}{|b|^2} \sum_{n=2}^{k-1} n^2 |a_n|^2 \geq \frac{1}{|b|^2} \sum_{n=2}^k n^2 |a_n|^2.$$

This inequality is equivalent to

$$(1 - B)^2 \sum_{n=2}^{k-1} n^2 |a_n|^2 + n^2 |a_n|^2 \leq (A - B)^2 |b|^2. \quad (3.2.7)$$

Since $-1 \leq B < 1$, we obtain from (3.2.7)

$$n^2 |a_n|^2 \leq (A - B)^2 |b|^2$$

which gives

$$|a_n| \leq \frac{(A - B)|b|}{n}, \quad n = 2, 3, \dots$$

The sharpness of the result follows for the function

$$f(z) = \int_0^z \left[1 + \frac{(A-B)bt^{n-1}}{1+Bt^{n-1}} \right] dt$$

for $n \geq 2$ and $z \in U$.

Remark 3.2.8. Taking different choices of b, A and B as stated in Section 3.1, the above theorem gives coefficient bounds for functions in the classes $R_\gamma(A, B)$, $R[\beta]$, $R(\alpha, \beta)$, $R^*(A, B)$ etc.

3.3. Sufficient Conditions in terms of the Coefficients for Functions in $R^{1-b}(A, B)$.

Theorem 3.3.1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in U . If for some A, B , $-1 \leq B < A \leq 1$,

$$\sum_{n=2}^{\infty} (1 + |B|) n |a_n| \leq (A - B) |b| \quad (3.3.2)$$

then $f \in R^{1-b}(A, B)$. The result is sharp.

Proof: We prove this theorem by the technique of Clunie and Keogh [12]. Suppose that (3.3.2) holds and that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

then for $|z| = r < 1$,

$$\begin{aligned} & \left| f'(z) - 1 \right| - \left| b(A - B) - B \{ f'(z) - 1 \} \right| \\ &= \left| \sum_{n=2}^{\infty} n a_n z^{n-1} \right| - \left| b(A - B) - B \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \\ &\leq \sum_{n=2}^{\infty} n |a_n| r^{n-1} - \left\{ |b|(A - B) - |B| \sum_{n=2}^{\infty} n |a_n| r^{n-1} \right\} \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=2}^{\infty} n|a_n|r^{n-1} - |b|(A-B) + |B| \sum_{n=2}^{\infty} n|a_n|r^{n-1} \\
&< \sum_{n=2}^{\infty} n|a_n| - |b|(A-B) + |B| \sum_{n=2}^{\infty} n|a_n| \\
&= \sum_{n=2}^{\infty} n(1+|B|)|a_n| - (A-B)|b| \\
&\leq 0, \quad \text{by (3.3.2).}
\end{aligned}$$

Hence it follows that

$$\left| \frac{f'(z) - 1}{b(A-B) - B\{f'(z) - 1\}} \right| < 1, \quad z \in U.$$

Therefore $f \in R^{1-b}(A, B)$.

The function

$$f(z) = z + \frac{(A-B)b}{(1+|B|)n} z^n$$

shows that the result is sharp for $n \geq 2$ and $z \in U$.

Remark 3.3.3. For suitable choices of b , A and B as stated in Section 3.1, the above theorem leads to sufficient conditions in terms of coefficients for functions in $R_\gamma(A, B)$, $R[\beta]$, $R(\alpha, \beta)$, $R^*(A, B)$ etc.

3.4. Distortion Theorems for the Class $R^{1-b}(A, B)$

Theorem 3.4.1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R^{1-b}(A, B)$, then

$$\operatorname{Re}\left(f'(z)\right) \geq \frac{1 - ABr^2 \operatorname{Re}\{b\} - B^2 r^2 \operatorname{Re}\{1-b\} - (A-B)|b|r}{1 - B^2 r^2} \quad (3.4.2)$$

and

$$\operatorname{Re}\left(f'(z)\right) \leq \frac{1 - AB r^2 \operatorname{Re}\{b\} - B^2 r^2 \operatorname{Re}\{1 - b\} + (A - B)|b|r}{1 - B^2 r^2}. \quad (3.4.3)$$

The bounds are sharp.

Proof: Since $f \in R^{1-b}(A, B)$, we have

$$1 + \frac{1}{b} \{f'(z) - 1\} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (3.4.4)$$

where $\omega \in \Omega$. Simplifying (3.4.4) gives

$$\omega(z) = \frac{f'(z) - 1}{b(A - B) - B(f'(z) - 1)}. \quad (3.4.5)$$

Since $\omega(z) \in \Omega$, by Schwarz Lemma,

$$|\omega(z)| \leq |z| = r, \quad z \in U. \quad (3.4.6)$$

Thus from (3.4.5) and (3.4.6) it follows that

$$\left| f'(z) - 1 \right| \leq r \left| b(A - B) - B(f'(z) - 1) \right|. \quad (3.4.7)$$

Letting $f'(z) = u + iv$ and $b = b_1 + ib_2$ in (3.4.7) and simplifying, we obtain

$$\left(u - \frac{1 - B[b_1(A - B) - B]r^2}{1 - B^2 r^2} \right)^2 + \left(v + \frac{b_2 B(A - B)r^2}{1 - B^2 r^2} \right)^2 \leq \left(\frac{|b|(A - B)r}{1 - B^2 r^2} \right)^2$$

which yields

$$\left| f'(z) - \frac{1 - ABbr^2 + B^2 r^2(b - 1)}{1 - B^2 r^2} \right| \leq \frac{(A - B)|b|r}{1 - B^2 r^2}.$$

Hence

$$\operatorname{Re}\left(f'(z)\right) \geq \frac{1 - ABr^2 \operatorname{Re}\{b\} - B^2 r^2 \operatorname{Re}\{1 - b\} - (A - B)|b|r}{1 - B^2 r^2}$$

and

$$\operatorname{Re}\left(f'(z)\right) \leq \frac{1 - ABr^2 \operatorname{Re}\{b\} - B^2 r^2 \operatorname{Re}\{1 - b\} + (A - B)|b|r}{1 - B^2 r^2}.$$

To show that the result is sharp, let

$$f(z) = \frac{B + (A - B)b}{B} z - \frac{(A - B)b}{B^2 e^{i\gamma}} \log(1 + Bze^{i\gamma}) \quad (3.4.8)$$

where $e^{i\gamma} = (|b| - Bzb)/(b - Bz|b|)$. Then

$$f'(z) = \frac{1 + (B + (A - B)b)ze^{i\gamma}}{1 + Bze^{i\gamma}}. \quad (3.4.9)$$

Substituting $e^{i\gamma} = (|b| - Bzb)/(b - Bz|b|)$ in (3.4.9) and simplifying, we get

$$f'(z) = \frac{1 + (A - B)|b|z - bABz^2 - B^2 z^2(1 - b)}{1 - B^2 z^2}.$$

Now it is easy to verify that the equality in (3.4.2) and (3.4.3) hold at $z = \pm r$.

Remark 3.4.10. Choosing appropriate values of b , A , and B in the above theorem as stated in Section 3.1, we get distortion theorems for the classes $R_\gamma(A, B)$, $R[\beta]$, $R(\alpha, \beta)$, $R^*(A, B)$ etc.

3.5. Maximization of $|a_3 - \mu a_2^2|$ over the class $R^{1-b}(A, B)$.

Theorem 3.5.1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $R^{1-b}(A, B)$ and μ is any complex number, then

$$\left| a_3 - \mu a_2^2 \right| \leq \frac{|b|(A-B)}{3} \max \left\{ 1, \frac{|4B + 3b\mu(A-B)|}{4} \right\}. \quad (3.5.2)$$

The result is sharp.

Proof: Since $f \in R^{1-b}(A, B)$, we have

$$1 + \frac{1}{b} \left\{ f'(z) - 1 \right\} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (3.5.3)$$

where $\omega(z) = \sum_{k=1}^{\infty} c_k z^k$ is analytic in U and satisfies the condition $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in U$ and $-1 \leq B < A \leq 1$.

From (3.5.3) we have

$$\begin{aligned} \omega(z) &= \frac{f'(z) - 1}{b(A-B) - B(f'(z) - 1)} \\ &= \frac{\sum_{n=2}^{\infty} n a_n z^{n-1}}{b(A-B) - B \sum_{n=2}^{\infty} n a_n z^{n-1}} \\ &= \frac{\sum_{n=2}^{\infty} n a_n z^{n-1}}{b(A-B)} \left[1 + \frac{B}{b(A-B)} \sum_{n=2}^{\infty} n a_n z^{n-1} + \dots \right]. \end{aligned}$$

Comparing the coefficients of z and z^2 on both sides, we have

$$c_1 = \frac{2a_2}{b(A-B)}$$

and

$$c_2 = \frac{1}{b(A-B)} \left[3a_3 + \frac{4Ba_2^2}{b(A-B)} \right].$$

Thus

$$a_2 = \frac{b(A-B)c_1}{2}$$

and

$$a_3 = \frac{b(A-B)c_2}{3} - \frac{4Ba_2^2}{3b(A-B)}.$$

Hence

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{b(A-B)c_2}{3} - \frac{4Ba_2^2}{3b(A-B)} - \mu a_2^2 \\ &= \frac{b(A-B)}{3} \left[c_2 - \left\{ B + \frac{3\mu b(A-B)}{4} \right\} c_1^2 \right]. \end{aligned}$$

Therefore

$$\left| a_3 - \mu a_2^2 \right| = \frac{|b|(A-B)}{3} \left[\left| c_2 - \left\{ \frac{4B + 3b\mu(A-B)}{4} \right\} c_1^2 \right| \right]. \quad (3.5.4)$$

Using Lemma 2.2.1 in (3.5.4) we obtain

$$\left| a_3 - \mu a_2^2 \right| \leq \frac{|b|(A-B)}{3} \max \left\{ 1, \frac{|4B + 3b\mu(A-B)|}{4} \right\}$$

which is (3.5.2).

To show that the result is sharp, if $\left| \frac{4B + 3b\mu(A-B)}{4} \right| > 1$, we choose the function

$$\begin{aligned} f(z) &= \frac{B + (A-B)b}{B} z - \frac{(A-B)b}{B^2} \log(1 + Bz) \\ &= \frac{B + (A-B)b}{B} z - \frac{(A-B)b}{B^2} \left[Bz - \frac{1}{2}B^2z^2 + \frac{1}{3}B^3z^3 - \dots \right] \end{aligned}$$

$$= z + \frac{1}{2}b(A-B) z^2 - \frac{1}{3}bB(A-B) z^3 + \dots$$

so that

$$\begin{aligned} \left| a_3 - \mu a_2^2 \right| &= \left| -\frac{1}{3}bB(A-B) - \mu \frac{b^2(A-B)^2}{4} \right| \\ &= \frac{|b|(A-B)}{3} \left| B + \frac{3\mu b(A-B)}{4} \right| \\ &= \frac{|b|(A-B)}{3} \left| \frac{4B + 3\mu b(A-B)}{4} \right| \end{aligned}$$

which gives the equality in (3.5.2).

If $\left| \frac{4B + 3\mu b(A-B)}{4} \right| < 1$, we choose the function

$$\begin{aligned} f(z) &= \frac{B + (A-B)b}{B} z - \frac{(A-B)b}{B} \int_0^z \frac{dt}{1+Bt^2} \\ &= \frac{B + (A-B)b}{B} z - \frac{(A-B)b}{B} \int_0^z (1 - Bt^2 + B^2t^4 - \dots) dt \\ &= \frac{B + (A-B)b}{B} z - \frac{(A-B)b}{B} \left[z - \frac{1}{3}Bz^3 + \frac{1}{5}B^2z^5 - \dots \right] \\ &= z + \frac{1}{3}(A-B) bz^3 - \frac{1}{5}(A-B) bBz^5 + \dots \end{aligned}$$

Therefore,

$$\left| a_3 - \mu a_2^2 \right| = \left| \frac{1}{3}(A-B) b - 0 \right| = \frac{1}{3}(A-B)|b|$$

which gives the equality in (3.5.2). Hence the result is sharp.

Remark 3.5.5. Different choices of b , A and B in the above theorem as stated in Section 3.1 gives maximization of $|a_3 - \mu a_2^2|$ over the classes $R_\gamma(A, B)$, $R[\beta]$, $R(\alpha, \beta)$, $R^*(A, B)$ etc.

CHAPTER IV

ON A CLASS OF UNIVALENT FUNCTIONS RELATED TO COMPLEX ORDER II

4.1. Introduction

Let Ω denote the class of functions

$$\omega(z) = \sum_{n=1}^{\infty} c_n z^n$$

which are analytic in the unit disc U and satisfy the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in U$.

In this chapter, we introduce and study the following class of univalent analytic functions related to complex order analogous to the class $R^{1-b}(A, B)$ studied in chapter III. The results thus obtained not only generalize some of the results obtained in Chapter II but also give results studied by Dasrath & Shukla [14], Juneja & Mogra [30], Padmanabhan [46], etc.

Let $S^{1-b}(A, B)$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in U and satisfy the condition

$$\left| \frac{\frac{z f'(z)}{f(z)} - 1}{b(A - B) - B \left\{ \frac{z f'(z)}{f(z)} - 1 \right\}} \right| < 1, \quad z \in U \quad (4.1.1)$$

where A and B are arbitrary fixed real numbers such that

$-1 \leq B < A \leq 1$ and b is non-zero complex number. Clearly a function $f(z)$ belongs to $S^{1-b}(A, B)$ if, and only if, there exists a function $\omega \in \Omega$ such that

$$1 + \frac{1}{b} \left\{ z \frac{f'(z)}{f(z)} - 1 \right\} = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad z \in U. \quad (4.1.2)$$

By giving specific values of b , A and B in (4.1.1) or (4.1.2) we obtain the following important subclasses of starlike and γ -spirallike functions introduced and studied by various researchers.

(i) Substituting $A = 1$ and $B = -1$ in (4.1.2) we obtain the class S^{1-b} of starlike functions of complex order $1 - b$ studied in Chapter II.

(ii) Fixing $b = 1$, $A = (1 - 2\alpha)\beta$, $B = -\beta$ in (4.1.1) we get the class $S(\alpha, \beta)$ of functions $f(z) \in \mathbf{A}$ satisfying the condition

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1 - 2\alpha} \right| < \beta$$

for some α, β ($0 \leq \alpha < 1$, $0 < \beta \leq 1$) and $z \in U$. The class $S(\alpha, \beta)$ was introduced and studied by Juneja and Mogra [30]. The functions of the class $S(\alpha, \beta)$ are called starlike functions of order α and type β .

(iii) $b = 1$, $A = \beta$ and $B = -\beta$ in (4.1.1) leads to the class $S[\beta]$ of functions $f(z) \in \mathbf{A}$ satisfying the condition

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1} \right| < \beta$$

for some β , $0 < \beta \leq 1$, $z \in U$; introduced and studied by Padmanabhan [46].

(iv) Putting $b = e^{-i\gamma} \cos \gamma$ in (4.1.2) we obtain the class $S^{1-e^{-i\gamma} \cos \gamma}(A, B) \equiv SP(\gamma, A, B)$ of functions $f(z)$ satisfying

$$e^{i\gamma} z \frac{f'(z)}{f(z)} = \cos \gamma \frac{1 + A\omega(z)}{1 + B\omega(z)} + i \sin \gamma, \quad z \in U$$

where $|\gamma| < \frac{\pi}{2}$ and $-1 \leq B < A \leq 1$; introduced and studied by Dashrath and Shukla [14].

In this chapter we establish various properties for the class $S^{1-b}(A, B)$. Section 4.2 is devoted to obtain coefficient estimates for functions in $S^{1-b}(A, B)$. In Section 4.3, we obtain sufficient condition in terms of coefficients for functions in $S^{1-b}(A, B)$. Distortion theorem for the class $S^{1-b}(A, B)$ is determined in Section 4.4. Finally in Section 4.5, we discuss maximization of $|a_3 - \mu a_2^2|$ over the class $S^{1-b}(A, B)$.

4.2. Coefficient Estimates

We need the following lemmas in our further discussion.

Lemma 4.2.1 [53]. Let

$$g(z) = \sum_{p=q}^{\infty} d_p z^p, \quad G(z) = \sum_{p=q}^{\infty} D_p z^p, \quad q \geq 0.$$

If $g(z) = \omega(z)G(z)$ where $\omega(0) = 0$ and $|\omega(z)| < 1$ in U , then $d_q = 0$ and

$$\sum_{p=q-1}^k |d_p|^2 \leq \sum_{p=q}^{k-1} |D_p|^2, \quad (k = q+1, q+2, \dots).$$

Lemma 4.2.2. For a fixed integer m , $m \geq 3$, let

$$M_j = \frac{|(A-B)b - (j-2)B|^2}{(j-1)^2}, \quad (j = 2, 3, \dots, m),$$

then

$$\left(\frac{1}{m-1}\right)^2 ((A-B)^2 |b|^2 + \sum_{p=2}^{m-1} \{ |(A-B)b - (p-1)B|^2 - (p-1)^2 \})$$

$$\prod_{j=2}^p M_j = \prod_{j=2}^m M_j \quad (4.2.3)$$

Proof: We shall prove (4.2.3) by mathematical induction on m . A brief calculation shows that (4.2.3) holds for $m = 3$. Assume that (4.2.3) is valid for $m = 4, 5, \dots, t-1$, then for $m = t$, the left side of (4.2.3) gives

$$\begin{aligned} & \frac{1}{(t-1)^2} ((A-B)^2 |b|^2 + \sum_{p=2}^{t-2} \{ |(A-B)b - (p-1)B|^2 - (p-1)^2 \}) \prod_{j=2}^p M_j \\ & + \{ |(A-B)b - (t-2)B|^2 - (t-2)^2 \} \prod_{j=2}^{t-1} M_j \\ & = \frac{1}{(t-1)^2} \left((t-2) \prod_{j=2}^{t-1} M_j + \{ (t-1)^2 M_t - (t-2)^2 \} \prod_{j=2}^{t-1} M_j \right) = \prod_{j=2}^t M_j. \end{aligned}$$

This concludes the proof of (4.2.3).

Theorem 4.2.4. Let $f \in S^{1-b}(A, B)$, and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $z \in U$.

Then

$$|a_2| \leq (A-B)|b| \quad (4.2.5)$$

and if $|(A-B)b - B| \leq 1$ and $k \geq 3$, then

$$|a_k| \leq \frac{(A-B)|b|}{k-1}. \quad (4.2.6)$$

Furthermore, if $|(A-B)b - (k-2)B| > (k-2)$, $k \geq 3$, let

$$M = \left[\frac{|(A-B)b - (k-2)B|}{k-2} \right]$$

be the greatest integer less than or equal to the expression within the square bracket. Then

$$|a_k| \leq \frac{1}{(k-1)!} \prod_{j=2}^k |(A-B)b - (j-2)B| \quad (4.2.7)$$

for $k = 3, 4, \dots, M+2$; and

$$|a_k| \leq \frac{1}{(M+1)!(k-1)} \prod_{j=2}^{M+3} |(A-B)b - (j-2)B| \quad (4.2.8)$$

for $k > M+2$. The bounds in (4.2.5) to (4.2.7) are sharp for all admissible values of b, A, B , and for each k .

Proof: From (4.1.2), we have

$$1 + \frac{1}{b} \left\{ \frac{zf'(z)}{f(z)} - 1 \right\} = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad z \in U \quad (4.2.9)$$

for some analytic function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$ in U . On simplification, (4.2.9) yields

$$(1 + B\omega(z))zf'(z) = \{1 + ((A-B)b + B)\omega(z)\}f(z),$$

that is,

$$(zf'(z) - f(z)) = \left(\{(A-B)b + B\}f(z) - Bzf'(z) \right)\omega(z). \quad (4.2.10)$$

Since

$$f(z) = z + \sum_{p=2}^{\infty} a_p z^p \quad \text{and} \quad z f'(z) = z + \sum_{p=2}^{\infty} p a_p z^p,$$

equation (4.2.10) is equivalent to

$$\sum_{p=2}^{\infty} (p-1) a_p z^p = \left(\sum_{p=1}^{\infty} \{ (A-B) b - (p-1)B \} a_p z^p \right) \omega(z) \quad (4.2.11)$$

where $a_1 = 1$. Using Lemma 4.2.1 we have

$$\sum_{p=2}^k (p-1)^2 |a_p|^2 \leq \sum_{p=1}^{k-1} \left| (A-B) b - (p-1)B \right|^2 |a_p|^2, \quad (4.2.12)$$

which simplifies to

$$|a_k|^2 \leq \left(\frac{1}{k-1} \right)^2 \left((A-B)^2 |b|^2 + \sum_{p=2}^{k-1} \left\{ \left| (A-B) b - (p-1)B \right|^2 - (p-1)^2 \right\} |a_p|^2 \right) \quad (4.2.13)$$

for every $k = 2, 3, \dots$.

For $k = 2$, we have

$$|a_2|^2 \leq (A-B)^2 |b|^2$$

which proves (4.2.5).

Suppose $\left| (A-B) b - B \right| \leq 1$ and $k \geq 3$. Then it follows that

$$\left| (A-B) b - (k-2)B \right| = \left| ((A-B) b - B) - (k-3)B \right|$$

$$\leq |(A-B)b - B| + (k-3)|B|$$

$$\leq k-2, \quad k \geq 3.$$

Since all the terms under the summation in (4.2.13) are non-positive, we obtain

$$|a_k| \leq \frac{(A-B)|b|}{k-1}, \quad k \geq 3$$

which establishes (4.2.6). However if

$$|(A-B)b - (k-2)B| > (k-2), \quad k \geq 3,$$

then all the terms under the summation in (4.2.13) are positive. We shall establish (4.2.7) for integer $k \geq 3$ and $k \leq M+2$, from (4.2.13), by mathematical induction. For $k=3$, (4.2.13) contributes

$$\begin{aligned} |a_3|^2 &\leq \frac{1}{2^2} \left[(A-B)^2|b|^2 + \left\{ |(A-B)b - B|^2 - 1 \right\} |a_2|^2 \right] \\ &\leq \frac{1}{2^2} \left[(A-B)^2|b|^2 + \left\{ |(A-B)b - B|^2 - 1 \right\} (A-B)^2|b|^2 \right] \\ &= \left(\frac{(A-B)|b| |(A-B)b - B|}{2} \right)^2 \end{aligned}$$

which proves (4.2.7) for $k=3$. Suppose (4.2.7) holds for $k=4, 5, \dots, m-1$. Then for $k=m$, (4.2.13) yields

$$|a_m|^2 \leq \left(\frac{1}{m-1} \right)^2 \left[(A-B)^2|b|^2 + \sum_{p=2}^{m-1} \left\{ |(A-B)b - (p-1)B|^2 - (p-1)^2 \right\} |a_p|^2 \right]$$

$$\begin{aligned}
&\leq \left(\frac{1}{m-1}\right)^2 ((A-B)^2 |b|^2 + \sum_{p=2}^{m-1} \left\{ |(A-B) b - (p-1)B|^2 - (p-1)^2 \right\} \\
&\quad \cdot \prod_{j=2}^p \frac{|(A-B) b - (j-2)B|^2}{(j-1)^2}) \\
&= \prod_{j=2}^m \frac{|(A-B) b - (j-2)B|^2}{(j-1)^2}
\end{aligned}$$

by Lemma 4.2.2. It is now easy to show that (4.2.7) holds for $k \leq M+2$. Finally, suppose $k > M+2$. Then we may write (4.2.13) as

$$\begin{aligned}
|a_k|^2 &\leq \left(\frac{1}{k-1}\right)^2 ((A-B)^2 |b|^2 + \sum_{p=2}^{M+2} \left\{ |(A-B) b - (p-1)B|^2 - (p-1)^2 \right\}) \cdot |a_p|^2 \\
&\quad + \sum_{p=M+3}^{k-1} \left\{ |(A-B) b - (p-1)B|^2 - (p-1)^2 \right\} |a_p|^2 \\
&\leq \left(\frac{1}{k-1}\right)^2 ((A-B)^2 |b|^2 + \sum_{p=2}^{M+2} \left\{ |(A-B) b - (p-1)B|^2 - (p-1)^2 \right\}) |a_p|^2 \\
&\leq \left(\frac{1}{k-1}\right)^2 ((A-B)^2 |b|^2 + \sum_{p=2}^{M+2} \left\{ |(A-B) b - (p-1)B|^2 - (p-1)^2 \right\} \\
&\quad \cdot \prod_{j=2}^p \frac{|(A-B) b - (j-2)B|^2}{(j-1)^2} \quad \text{(by (4.2.7))} \\
&\leq \left(\frac{1}{k-1}\right)^2 \left\{ (M+2)^2 \prod_{j=2}^{M+3} \frac{|(A-B) b - (j-2)B|^2}{(j-1)^2} \right\} \\
&\quad \text{(by Lemma 4.2.2)} \\
&= \left(\frac{1}{(M+1)!(k-1)}\right)^2 \prod_{j=2}^{M+3} |(A-B) b - (j-2)B|^2
\end{aligned}$$

which is (4.2.8). The equality in (4.2.5) and (4.2.6) are attained for

the function f_k given by

$$f_k(z) = \begin{cases} z(1 + Bz^{k-1})^{(A-B)b/B(k-1)}, & B \neq 0 \\ z \exp Abz^{k-1}/(k-1), & B = 0 \end{cases}$$

where $|(A-B)b - B| \leq 1$. Finally, the inequality (4.2.7) is sharp for the function f given by

$$f(z) = \begin{cases} z(1 + Bz)^{(A-B)b/B}, & B \neq 0 \\ z \exp(Abz), & B = 0 \end{cases}$$

where $|(A-B)b - (k-2)B| > k-2$, $k \geq 3$.

Corollary 4.2.14. Let $f \in SP(\gamma, A, B)$, and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $z \in U$. Then

$$|a_2| \leq (A-B) \cos \gamma \quad (4.2.15)$$

and if $|(A-B)e^{-i\gamma} \cos \gamma - B| \leq 1$ and $k \geq 3$, then

$$|a_k| \leq \frac{(A-B) \cos \gamma}{k-1}. \quad (4.2.16)$$

Furthermore, if $|(A-B)e^{-i\gamma} \cos \gamma - (k-2)B| > k-2$, $k \geq 3$, let

$$M = \left[\frac{|(A-B)e^{-i\gamma} \cos \gamma - (k-2)B|}{k-2} \right]$$

be the greatest integer less than or equal to the expression within the square bracket. Then

$$|a_k| \leq \frac{1}{(k-1)!} \prod_{j=2}^k |(A-B)e^{-i\gamma} \cos \gamma - (j-2)B| \quad (4.2.17)$$

for $k = 3, 4, \dots, M+2$; and

$$|a_k| \leq \frac{1}{(M+1)!(k-1)} \prod_{j=2}^{M-3} \left| (A-B)e^{-i\gamma} \cos \gamma - (j-2)B \right| \quad (4.2.18)$$

for $k > M+2$. The bounds in (4.2.15) to (4.2.17) are sharp for all admissible values of A, B and for each k .

The above corollary can be obtained from Theorem 4.2.4 by fixing $b = e^{-i\gamma} \cos \gamma$ in the theorem.

Remark 4.2.19. Fixing $A = 1$ and $B = -1$ in the Theorem 4.2.4 lead to the coefficient estimates for functions in S^{1-b} obtained in Chapter II.

Remark 4.2.20. On choosing appropriate values of b, A and B in the Theorem 4.2.4 as mentioned in Section 4.1, we may obtain the sharp coefficient estimates for functions in the classes $S(A, B)$, $S[\beta]$ and others.

4.3. Sufficient Conditions in Terms of the Coefficients for Functions in $S^{1-b}(A, B)$.

Theorem 4.3.1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in U . If for some A, B , $-1 \leq B < A \leq 1$,

$$\sum_{n=2}^{\infty} \left\{ (n-1) + \left| b(A-B) - B(n-1) \right| \right\} |a_n| \leq |b| (A-B) \quad (4.3.2)$$

then $f \in S^{1-b}(A, B)$. The result is sharp.

Proof: We prove this theorem by the technique of Clunie and Keogh [12]. Suppose that (4.3.2) holds and that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

then for $|z| = r < 1$,

$$\begin{aligned}
& \left| z f'(z) - f(z) \right| - \left| b(A - B) f(z) - B \left\{ z f'(z) - f(z) \right\} \right| \\
&= \left| \sum_{n=2}^{\infty} (n-1) a_n z^n \right| - \left| b(A - B) \left\{ z + \sum_{n=2}^{\infty} a_n z^n \right\} - B \sum_{n=2}^{\infty} (n-1) a_n z^n \right| \\
&= \left| \sum_{n=2}^{\infty} (n-1) a_n z^n \right| - \left| b(A - B) z + \sum_{n=2}^{\infty} (b(A - B) - B(n-1)) a_n z^n \right| \\
&\leq \sum_{n=2}^{\infty} (n-1) |a_n| r^n - \left\{ |b|(A - B) r - \sum_{n=2}^{\infty} |b(A - B) - B(n-1)| |a_n| r^n \right\} \\
&= r \left[\sum_{n=2}^{\infty} \{ (n-1) + |b(A - B) - B(n-1)| \} |a_n| r^{n-1} - |b|(A - B) \right] \\
&< \sum_{n=2}^{\infty} \{ (n-1) + |b(A - B) - B(n-1)| \} |a_n| - |b|(A - B) \leq 0, \\
&\hspace{25em} \text{by (4.3.2).}
\end{aligned}$$

Hence it follows that

$$\left| \frac{\frac{z f'(z)}{f(z)} - 1}{b(A - B) - B \left\{ \frac{z f'(z)}{f(z)} - 1 \right\}} \right| < 1, \quad z \in U.$$

Therefore $f \in S^{1-b}(A, B)$.

The function

$$f(z) = z + \frac{|b|(A - B)}{(n-1) + |b(A - B) - B(n-1)|} z^n$$

shows that the result is sharp for $n \geq 2$ and $z \in U$.

Corollary 4.3.3. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in U . If for some A, B , $-1 \leq B < A \leq 1$ and $|\gamma| < \pi/2$,

$$\sum_{n=2}^{\infty} \{(n-1) + |(A-B)e^{-i\gamma} \cos \gamma - B(n-1)|\} |a_n| \leq (A-B) \cos \gamma$$

then $f \in SP(\gamma, A, B)$. The result is sharp for a function

$$f(z) = z + \frac{(A-B) \cos \gamma}{(n-1) + |(A-B)e^{-i\gamma} \cos \gamma - B(n-1)|} z^n,$$

for $n \geq 2$, $|\gamma| < \pi/2$ and $z \in U$.

The above result can be obtained from Theorem 4.3.1 by fixing $b = e^{-i\gamma} \cos \gamma$ in the theorem.

Remark 4.3.4. Taking suitable choices of b , A and B as stated in Section 4.1, the above theorem gives sufficient conditions in terms of the coefficients for functions in S^{1-b} , $S(A, B)$, $S[\beta]$ and others.

4.4. Distortion Theorem for the Class $S^{1-b}(A, B)$

Theorem 4.4.1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^{1-b}(A, B)$, then

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) \geq \frac{1 - ABr^2 \operatorname{Re}\{b\} - B^2 r^2 \operatorname{Re}\{1-b\} - (A-B)|b|r}{1 - B^2 r^2} \quad (4.4.2)$$

and

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) \leq \frac{1 - ABr^2 \operatorname{Re}\{b\} - B^2 r^2 \operatorname{Re}\{1-b\} + (A-B)|b|r}{1 - B^2 r^2} \quad (4.4.3)$$

The bounds are sharp.

Proof: Since $f \in S^{1-b}(A, B)$, we have

$$1 + \frac{1}{b} \left\{ \frac{zf'(z)}{f(z)} - 1 \right\} = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad (4.4.4)$$

where $\omega \in \Omega$. Simplifying (4.4.4) gives

$$\omega(z) = \frac{\frac{zf'(z)}{f(z)} - 1}{b(A-B) - B\left(\frac{zf'(z)}{f(z)} - 1\right)}. \quad (4.4.5)$$

Since $\omega(z) \in \Omega$, by Schwarz Lemma,

$$|\omega(z)| \leq |z| = r, \quad z \in U. \quad (4.4.6)$$

Thus from (4.4.5) and (4.4.6), it follows that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq r \left| b(A-B) - B\left(\frac{zf'(z)}{f(z)} - 1\right) \right|. \quad (4.4.7)$$

Letting $\frac{zf'(z)}{f(z)} = u + iv$ and $b = b_1 + ib_2$ in (4.4.7) and simplifying, we obtain

$$\left(u - \frac{1 - B[b_1(A-B) + B]r^2}{1 - B^2r^2} \right)^2 + \left(v + \frac{b_2B(A-B)r^2}{1 - B^2r^2} \right)^2 \leq \left(\frac{|b|(A-B)r}{1 - B^2r^2} \right)^2$$

which yields

$$\left| \frac{zf'(z)}{f(z)} - \frac{1 - ABbr^2 + B^2r^2(b-1)}{1 - B^2r^2} \right| \leq \frac{(A-B)|b|r}{1 - B^2r^2}.$$

Hence

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) \geq \frac{1 - ABr^2 \operatorname{Re}\{b\} - B^2r^2 \operatorname{Re}\{1-b\} - (A-B)|b|r}{1 - B^2r^2}$$

and

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \leq \frac{1 - ABr^2 \operatorname{Re}\{b\} - B^2r^2 \operatorname{Re}\{1 - b\} + (A - B)|b|r}{1 - B^2r^2}.$$

To prove that the result is sharp, let

$$f(z) = z(1 + Bze^{i\gamma})^{(A-B)b/B}$$

where $e^{i\gamma} = (|b| - Bzb)/(b - Bz|b|)$. Then

$$\frac{zf'(z)}{f(z)} = \frac{1 + (B + (A - B)b)ze^{i\gamma}}{1 + Bze^{i\gamma}}. \quad (4.4.8)$$

Substituting $e^{i\gamma} = (|b| - Bzb)/(b - Bz|b|)$ in (4.4.8) and simplifying we get

$$\frac{zf'(z)}{f(z)} = \frac{1 + (A - B)|b|z - bABz^2 - B^2z^2(1 - b)}{1 - B^2z^2}.$$

Now it is easily seen that the equality in (4.4.2) and (4.4.3) holds at $z = \pm r$.

Remark 4.4.9. Putting appropriate choices of b , A and B in Theorem 4.4.1 as stated in Section 4.1, we get the corresponding distortion theorems for the classes $SP(\gamma, A, B)$, S^{1-b} , $S(\alpha, \beta)$, $S[\beta]$ and others.

4.5. Maximization of $|a_3 - \mu a_2^2|$ over the class $S^{1-b}(A, B)$.

Theorem 4.5.1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $S^{1-b}(A, B)$ and μ is any complex number, then

$$|a_3 - \mu a_2^2| \leq \frac{|b|(A - B)}{2} \max\left\{1, |B - b(A - B)(1 - 2\mu)|\right\} \quad (4.5.2)$$

The result is sharp.

Proof: Since $f \in S^{1-b}(A, B)$, we have

$$1 + \frac{1}{b} \left\{ \frac{zf'(z)}{f(z)} - 1 \right\} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (4.5.3)$$

where $\omega(z) = \sum_{k=1}^{\infty} c_k z^k$ is regular in U and satisfies the condition $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in U$. From (4.5.3) we have

$$\begin{aligned} \omega(z) &= \frac{zf'(z) - f(z)}{b(A-B)f(z) - B(zf'(z) - f(z))} \\ &= \frac{\sum_{n=2}^{\infty} (n-1)a_n z^n}{b(A-B) \left\{ z + \sum_{n=2}^{\infty} a_n z^n \right\} - B \sum_{n=2}^{\infty} (n-1)a_n z^n} \\ &= \frac{\sum_{n=2}^{\infty} (n-1)a_n z^{n-1}}{b(A-B) + \sum_{n=2}^{\infty} [b(A-B) - B(n-1)]a_n z^{n-1}} \\ &= \frac{\sum_{n=2}^{\infty} (n-1)a_n z^{n-1}}{b(A-B)} \left[1 + \sum_{n=2}^{\infty} \left[\frac{B(n-1)}{b(A-B)} - 1 \right] a_n z^{n-1} + \dots \right]. \end{aligned}$$

Comparing the coefficients of z and z^2 on both sides, we have

$$c_1 = \frac{a_2}{b(A-B)}$$

and

$$c_2 = \frac{2a_3}{b(A-B)} + \frac{1}{b(A-B)} \left[\frac{B}{b(A-B)} - 1 \right] a_2^2.$$

Thus

$$a_2 = b(A - B)c_1$$

and

$$a_3 = \frac{b(A - B)c_2}{2} + \frac{1}{2} \left[1 - \frac{B}{b(A - B)} \right] a_2^2.$$

Hence

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{b(A - B)c_2}{2} + \frac{1}{2} \left[1 - \frac{B}{b(A - B)} \right] a_2^2 - \mu a_2^2 \\ &= \frac{b(A - B)}{2} [c_2 - \{B - b(A - B)(1 - 2\mu)\} c_1^2]. \end{aligned}$$

Therefore

$$|a_3 - \mu a_2^2| = \frac{|b|(A - B)}{2} [|c_2 - \{B - b(A - B)(1 - 2\mu)\} c_1^2|]. \quad (4.5.4)$$

Using Lemma 2.1.1 in (4.5.4) we obtain

$$|a_3 - \mu a_2^2| \leq \frac{|b|(A - B)}{2} \max \left\{ 1, |B - b(A - B)(1 - 2\mu)| \right\}$$

which is (4.5.2).

To show that the result is sharp; if $|B - b(A - B)(1 - 2\mu)| > 1$, then we choose the function

$$\begin{aligned} f(z) &= z(1 + Bz)^{(A-B)b/B} \\ &= z + (A - B)bz^2 + \frac{(A - B)b[(A - B)b - B]}{2} z^3 + \dots \end{aligned}$$

so that

$$\begin{aligned} \left| a_3 - \mu a_2^2 \right| &= \left| \frac{(A-B)b[(A-B)b-B]}{2} - \mu(A-B)^2 b^2 \right| \\ &= \frac{(A-B)|b|}{2} |B - b(A-B)(1-2\mu)| \end{aligned}$$

which gives the equality in (4.5.2).

If $|B - b(A-B)(1-2\mu)| < 1$, choose the function

$$\begin{aligned} f(z) &= z(1 + Bz^2)^{(A-B)b/2B} \\ &= z - \frac{1}{2}b(A-B)z^3 + \dots \end{aligned}$$

Therefore,

$$|a_3 - \mu a_2^2| = \left| -\frac{1}{2}b(A-B) - 0 \right| = \frac{1}{2}|b|(A-B)$$

which gives the equality in (4.5.2). Hence the result is sharp.

CHAPTER V

ON THE GENERALISED RUSCHEWEYH CLASS OF ANALYTIC FUNCTIONS OF COMPLEX ORDER

5.1. Introduction

Recall that $\mathbf{A}$ be the class of functions f which are analytic in the unit disc $U = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, and S be the subclass of $\mathbf{A}$ consisting of functions that are univalent in U . An analytic function f is said to be subordinate to an analytic function g , denoted by $f \ll g$ if $f(z) = g(\omega(z))$, $z \in U$, for some analytic function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in U . The Hadamard product of two functions f , $g \in \mathbf{A}$, where

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U.$$

The object of this chapter is to identify various classes of analytic functions which are starlike, convex, prestarlike, Ruscheweyh class of order α , γ -spirallike, γ -convex spirallike, starlike of complex order, convex of complex order, and others as special cases of a family of analytic functions of complex order in the unit disc U . Thus we study the following generalised Ruscheweyh class of analytic functions of complex order introduced by Ahuja [3].

Denote by $D^\lambda : \mathbf{A} \rightarrow \mathbf{A}$ the operator defined by

$$D^\lambda f = \frac{z}{(1-z)^{\lambda+1}} * f, \quad \lambda \geq -1, \quad z \in U. \quad (5.1.1)$$

Notice that $D^{-1}f = z$, $D^0f = f$, $D^1f = zf'$, $D^2f = z(f' + zf''/2)$.

Let $H_\lambda^b[A, B]$ denote the class of functions f in $\mathbf{A}$ satisfying the condition

$$1 + \frac{(\lambda+1)}{b} \left(\frac{D^{\lambda+1}f}{D^\lambda f} - 1 \right) \ll \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1, \quad z \in U \quad (5.1.2)$$

where $\lambda > -1$ and $b \neq 0$ is an arbitrary fixed complex number. The class $H_\lambda^b[A, B]$ was called generalised Ruscheweyh class of analytic functions of complex order b by Ahuja [3] due to the fact that the class

$$K_\lambda = \left\{ f \in \mathbf{A} : \operatorname{Re} \left(\frac{D^{\lambda+1}f}{D^\lambda f} \right) > \frac{1}{2}, \quad z \in U \right\} \equiv H_\lambda^{(1+\lambda)/2} [1, -1] \quad (5.1.3)$$

was first introduced and studied by Ruscheweyh [57].

A function $f \in \mathbf{A}$ is said to be in $S[A, B]$ if $zf'(z)/f(z) \ll (1+Az)/(1+Bz)$, and in $C[A, B]$ if $(1 + zf''(z)/f'(z)) \ll (1+Az)/(1+Bz)$ for some arbitrary fixed real numbers A, B ($-1 \leq B < A \leq 1$) and for all z in U . These classes were investigated in [22, 28, 29, 63, 65]. In view of (5.1.1) and (5.1.2), it follows that

$$f \in S[A, B] \Leftrightarrow f \in H_0^1[A, B]; \quad (5.1.4)$$

$$f \in C[A, B] \Leftrightarrow f \in H_1^1[A, B]. \quad (5.1.5)$$

For α , $\alpha \leq 1$, let

$$S(\alpha) = \left\{ f \in \mathbf{A} : \operatorname{Re}(zf'(z)/f(z)) > \alpha, \quad z \in U \right\};$$

$$C(\alpha) = \left\{ f \in \mathbf{A} : \operatorname{Re}(1 + zf''(z)/f'(z)) > \alpha, \quad z \in U \right\}.$$

The functions in $S(\alpha)$ and $C(\alpha)$ are called, respectively, starlike of order α and convex of order α in U . It is well-known that $C(\alpha) \subset S(\alpha) \subset S$ whenever $0 \leq \alpha \leq 1$. We observe that for $\alpha < 1$,

$$S(\alpha) \equiv H_0^{1-\alpha}[1, -1], \quad C(\alpha) \equiv H_1^{1-\alpha}[1, -1]. \quad (5.1.6)$$

In fact, by setting $b = 1$ and for different choices of A, B ($-1 \leq B < A \leq 1$), the class $H_\lambda^b[A, B]$ will give rise to various subclasses of $S(0) \equiv S^*$; for instance, the class

$$H_0^1[1, -1 + 1/\alpha] \equiv \left\{ f \in \mathbf{A} : \left| z \frac{f'(z)}{f(z)} - \alpha \right| < \alpha, z \in U \right\}, \alpha > 1/2 \quad (5.1.7)$$

was introduced in [67]. For other such classes which may be obtained from $H_\lambda^b[A, B]$, one may refer to [46, 63, 65].

Other larger sets are formed by γ -spirallike functions of order α , and γ -Robertson (or γ -convex spirallike) functions of order α , respectively denoted by $SP(\gamma, \alpha)$ and $RB(\gamma, \alpha)$. Here α & γ are arbitrary fixed real numbers satisfying the conditions $|\gamma| < \pi/2$ and $0 \leq \alpha < 1$. We recall that

$$SP(\gamma, \alpha) = \left\{ f \in \mathbf{A} : \operatorname{Re} \left(e^{i\gamma} z \frac{f'(z)}{f(z)} \right) > \alpha \cos \gamma, z \in U \right\}$$

$$RB(\gamma, \alpha) = \left\{ f \in \mathbf{A} : \operatorname{Re} \left(e^{i\gamma} \left(1 + z \frac{f''(z)}{f'(z)} \right) \right) > \alpha \cos \gamma, z \in U \right\}.$$

These classes were introduced, respectively, by Libera [38] & Chichra [10]. Equivalent conditions are

$$f \in SP(\gamma, \alpha) \Leftrightarrow f \in H_0^{e^{-i\gamma} \cos \gamma} [1 - 2\alpha, -1] \quad (5.1.8)$$

$$f \in RB(\gamma, \alpha) \Leftrightarrow f \in H_1^{e^{-i\gamma} \cos \gamma} [1 - 2\alpha, -1]. \quad (5.1.9)$$

In fact, by fixing $b = e^{-i\gamma} \cos \gamma$, $|\gamma| < \pi/2$ and $\lambda = 0$, we may obtain several known subclasses of $SP(\gamma)$ from the family $H_\lambda^b[A, B]$. For instance, for $-1 \leq B < A \leq 1$, $\alpha > 1/2$, $c + d \geq 1$, $d \leq c \leq d + 1$, we have

$$H_0^{e^{-i\gamma} \cos \gamma}[A, B] \\ \equiv \left\{ f \in \mathbf{A} : e^{i\gamma} z \frac{f'(z)}{f(z)} \ll \frac{1 + Az}{1 + Bz} \cos \gamma + i \sin \gamma, z \in U \right\}, \quad (5.1.10)$$

$$H_0^{e^{-i\gamma} \cos \gamma}[1, -1 + 1/\alpha] \\ \equiv \left\{ f \in \mathbf{A} : \left| e^{i\gamma} z \frac{f'(z)}{f(z)} - (\alpha \cos \gamma + i \sin \gamma) \right| < \alpha \cos \gamma, z \in U \right\}, \quad (5.1.11)$$

and

$$H_0^{e^{-i\gamma} \cos \gamma} \left[\frac{d^2 - c^2 + c}{d}, \frac{1 - c}{d} \right] \\ \equiv \left\{ f \in \mathbf{A} : \left| e^{i\gamma} z \frac{f'(z)}{f(z)} - (c \cos \gamma + i \sin \gamma) \right| < d \cos \gamma, z \in U \right\}. \quad (5.1.12)$$

Denote the classes defined in (5.1.10) to (5.1.12), respectively, by

$SP^*(\gamma, A, B)$, $SP^*(\gamma, \alpha)$ and $SP^*(\gamma, c, d)$. These classes were introduced and studied in [14], [36] and [66], respectively. Analogous subclasses of $RB(\gamma, 0)$, denoted by $RB^*(\gamma, A, B)$, $RB^*(\gamma, \alpha)$, and $RB^*(\gamma, c, d)$ may be obtained by fixing $b = e^{-i\gamma} \cos \gamma$, $\lambda = 1$ and choosing appropriate values of A & B . In fact, several such classes studied in [1, 21, 41, 43, 54] may be related to the family $H_\lambda^b[A, B]$.

Let $G(\alpha)$ be the class of prestarlike functions of order α ; that is, $f \in G(\alpha)$ if, and only if,

$$\begin{cases} \frac{z}{(1-z)^{2-2\alpha}} * f(z) \in S(\alpha) & , \alpha < 1 \\ \operatorname{Re}(f(z)/z) > 1/2 & , \alpha = 1, z \in U. \end{cases}$$

Note that

$$k_\alpha(z) = z/(1-z)^{2-2\alpha} \quad (5.1.13)$$

is the well-known extremal function in $S(\alpha)$. Ruscheweyh [58] observed that

$$f \in G(\alpha) \Leftrightarrow \operatorname{Re} \left(\frac{D^{2-2\alpha}f}{D^{1-2\alpha}f} \right) > 1/2, \quad z \in U, \quad \alpha \leq 1.$$

Equivalently,

$$f \in G(\alpha) \Leftrightarrow f \in H_{1-2\alpha}^{1-\alpha}[1, -1], \quad \alpha < 1. \quad (5.1.14)$$

It is known [64] that $G(\alpha) \subset S$ if and only if $\alpha \leq 1/2$.

Given $\alpha < 1$ and $\beta < 1$, a function $f \in \mathbf{A}$ is said to be prestarlike of order α and type β , $f \in G(\alpha, \beta)$ if $f * k_\alpha \in S(\beta)$. Note that $G(\alpha, \alpha) \equiv G(\alpha)$, $\alpha < 1$. The class $G(\alpha, \beta)$ was investigated in [62]. Using (5.1.1) and the identity

$$z(D^\lambda f)' = (\lambda + 1)D^{\lambda-1}f - \lambda D^\lambda f, \quad \lambda > -1 \quad (5.1.15)$$

we may write

$$G(\alpha, \beta) \equiv \left\{ f \in \mathbf{A} : \operatorname{Re} \left(\frac{z(D^{1-2\alpha}f)'}{D^{1-2\alpha}f} \right) > \beta, \quad z \in U \right\},$$

and thus it follows that

$$f \in G(\alpha, \beta) \Leftrightarrow f \in H_{1-2\alpha}^{1-\beta}[1, -1], \quad \alpha < 1, \quad \beta < 1. \quad (5.1.16)$$

Let $Q_\lambda(\alpha)$ denote the class of functions f in $\mathbf{A}$ satisfying the condition

$$\operatorname{Re} \frac{z(D^\lambda f(z))'}{D^\lambda f(z)} > \alpha$$

for some $\lambda > -1$, $\alpha < 1$, and for all $z \in U$. The class $Q_\lambda(\alpha)$ is called Ruscheweyh class of order α . In view of (5.1.3) and the identity (5.1.15), we have $Q_\lambda\left(\frac{(1-\lambda)}{2}\right) \equiv K_\lambda$ for $\lambda > -1$. For $\lambda = n \in \{0, 1, 2, \dots\}$ and $0 \leq \alpha < 1$, the classes $Q_n(0)$ and $Q_n(\alpha)$ were, respectively, introduced in [68] and [2]. Also, note that $D^n f = (z(z^{n-1}f)^{(n)})/n!$, $n \in \{0, 1, 2, \dots\}$. Further, we observe that

$$f \in Q_\lambda(\alpha) \Leftrightarrow f \in H_\lambda^{1-\alpha}[1, -1], \quad \alpha < 1, \lambda > -1, \quad (5.1.17)$$

and

$$Q_\lambda(\alpha) \equiv H_\lambda^1[1-\alpha, -1] \equiv H_\lambda^{1-\alpha}[1, -1], \quad 0 \leq \alpha < 1. \quad (5.1.18)$$

Some such classes involving the Ruscheweyh derivatives and related to $H_\lambda^b[A, B]$ were studied in [4, 5, 6, 7, 61].

Nasr and Aouf [44] introduced the class S^{1-b} of starlike functions of complex order $1-b$ ($b \neq 0$ and complex), that is, $f \in S^{1-b}$ if, and only if,

$$\operatorname{Re} \left(1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right) > 0, \quad z \in U. \quad (5.1.19)$$

The class S^{1-b} was studied in Chapter II. An analogous class C^{1-b} of convex functions of complex order $1-b$ is defined by replacing $zf'(z)/f(z)$ in (5.1.19) by $1 + zf''(z)/f'(z)$; (see [44]). Equivalently, we have

$$f \in S^{1-b} \Leftrightarrow f \in H_0^b[1, -1] \quad (5.1.20)$$

$$f \in C^{1-b} \Leftrightarrow f \in H_1^b[1, -1]. \quad (5.1.21)$$

In Chapter IV, we have introduced and studied the class $S^{1-b}(A, B)$ of functions $f \in \mathbf{A}$ satisfying the condition

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{b(A-B) - B \left\{ \frac{zf'(z)}{f(z)} - 1 \right\}} \right| < 1, z \in U \quad (5.1.22)$$

where A and B are arbitrary fixed real numbers such that $-1 \leq B < A \leq 1$ and b is non-zero complex number. Equivalently, we have

$$f \in S^{1-b}[A, B] \Leftrightarrow f \in H_0^b[A, B]. \quad (5.1.23)$$

In view of the relations witnessed in (5.1.3) to (5.1.12), (5.1.14), (5.1.16) to (5.1.18), (5.1.20), (5.1.21) and (5.1.23) we conclude that the notion of the generalised Ruscheweyh class, $H_\lambda^b[A, B]$, of analytic functions of complex order b unifies several subclasses of **A**. Thus a study of $H_\lambda^b[A, B]$ may lead to information about K_λ , $S[A, B]$, $C[A, B]$, $S(\alpha)$, $C(\alpha)$, $SP(\gamma, \alpha)$, $RB(\gamma, \alpha)$, $SP^*(\gamma, A, B)$, $SP^*(\gamma, \alpha)$, $SP^*(\gamma, c, d)$, $RB^*(\gamma, A, B)$, $RB^*(\gamma, \alpha)$, $RB^*[\gamma, c, d]$, $G(\alpha)$, $G(\alpha, \beta)$, $Q_\lambda(\alpha)$, S^{1-b} , C^{1-b} , $S^{1-b}(A, B)$ and those obtained by choosing $\lambda \geq 2$ and suitable values of the parameters b , A & B . Finally, we obtain sharp coefficient estimates of the functions in $H_\lambda^b[A, B]$.

5.2. Coefficient Estimates.

We need the following lemma.

Lemma 5.2.1. For a fixed integer m , $m \geq 3$, let

$$M_j = \frac{|(A-B)b - (j-2)B|^2}{(\lambda+j-1)^2}, \quad j = (2, 3, \dots, m),$$

and

$$\delta(\lambda, p) = \frac{(\lambda+1)_{p-1}}{(p-1)!} = \frac{(\lambda+1)(\lambda+2)\dots(\lambda+p-1)}{(p-1)!} \quad (p = 2, 3, \dots).$$

Then

$$\left(\frac{1}{(m-1)\delta(\lambda, m)} \right)^2 ((A-B)^2|b|^2 + \sum_{p=2}^{m-1} \{|(A-B)b - (p-1)B|^2 - (p-1)^2\} \cdot (\delta(\lambda, p))^2 \prod_{j=2}^p M_j) = \prod_{j=2}^m M_j. \quad (5.2.2)$$

Proof: We shall prove (5.2.2) by mathematical induction on m . A brief calculation shows that (5.2.2) holds for $m = 3$. Assume that (5.2.2) is valid for $m = 4, 5, \dots, t-1$; then for $m = t$, the left side of (5.2.2) gives

$$\begin{aligned} & \left(\frac{1}{(t-1)\delta(\lambda, t)} \right)^2 ((A-B)^2|b|^2 + \sum_{p=2}^{t-2} \{|(A-B)b - (p-1)B|^2 - (p-1)^2\} \cdot (\delta(\lambda, p))^2 \prod_{j=2}^p M_j \\ & + \{|(A-B)b - (t-2)B|^2 - (t-2)^2\} (\delta(\lambda, t-1))^2 \prod_{j=2}^{t-1} M_j) \\ & = \left(\frac{1}{(t-1)\delta(\lambda, t)} \right)^2 (((t-2)\delta(\lambda, t-1))^2 \prod_{j=2}^{t-1} M_j + \{(\lambda + t - 1)^2 M_t - (t-2)^2\} \cdot (\delta(\lambda, t-1))^2 \prod_{j=2}^{t-1} M_j) \\ & = (\delta(\lambda, t-1))^2 \prod_{j=2}^{t-1} M_j = \prod_{j=2}^t M_j. \end{aligned}$$

This concludes the proof of (5.2.2).

Theorem 5.2.3. Let $f \in H_\lambda^b[A, B]$, and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $z \in U$. Then

$$|a_2| \leq \frac{(A-B)|b|}{\lambda + 1}; \quad (5.2.4)$$

and if $|(A-B)b - B| \leq 1$ and $k \geq 3$, then

$$|a_k| \leq \frac{(A-B)|b|(k-2)!}{(\lambda + 1)_{k-1}}. \quad (5.2.5)$$

Furthermore, if $|(A - B)b - (k - 2)B| > (k - 2)$, $k \geq 3$, let

$$M = \left[\frac{|(A - B)b - (k - 2)B|}{k - 2} \right]$$

be the greatest integer less than or equal to the expression within the square bracket. Then

$$|a_k| \leq \left(\frac{1}{(\lambda + 1)_{k-1}} \right) \prod_{j=2}^k |(A - B)b - (j - 2)B| \quad (5.2.6)$$

for $k = 3, 4, \dots, M + 2$; and

$$|a_k| \leq \frac{(k - 2)!}{(M + 1)!(\lambda + 1)_{k-1}} \prod_{j=2}^{M-3} |(A - B)b - (j - 2)B| \quad (5.2.7)$$

for $k > M + 2$. The bounds in (5.2.4) to (5.2.6) are sharp for all admissible b , λ , A , B , and for each k .

Proof: From (5.1.2), we have

$$1 + \frac{\lambda + 1}{b} \left(\frac{D^{\lambda+1} f}{D^{\lambda} f} - 1 \right) = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad z \in U \quad (5.2.8)$$

for some analytic function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$ in U . On simplification, (5.2.8) yields

$$(\lambda + 1)(1 + B\omega)D^{\lambda+1}f = \{\lambda + 1 + \{(A - B)b + (\lambda + 1)B\}\omega\}D^{\lambda}f,$$

that is,

$$\begin{aligned} & (\lambda + 1)(D^{\lambda+1}f - D^{\lambda}f) \\ &= (\{(A - B)b + (\lambda + 1)B\}D^{\lambda}f - B(\lambda + 1)D^{\lambda+1}f)\omega(z). \end{aligned} \quad (5.2.9)$$

Since

$$D^\lambda f = z + \sum_{p=2}^{\infty} \delta(\lambda, p) a_p z^p$$

and $\delta(\lambda + 1, p) = ((\lambda + p) / (\lambda + 1)) \delta(\lambda, p)$, it follows that (5.2.9) is equivalent to

$$\sum_{p=2}^{\infty} (p-1) \delta(\lambda, p) a_p z^p = \left(\sum_{p=1}^{\infty} \{(A-B)b - (p-1)B\} \delta(\lambda, p) a_p z^p \right) \omega(z),$$

where $a_1 = 1$. Using Lemma 4.2.1, we have

$$\sum_{p=2}^k (p-1)^2 (\delta(\lambda, p))^2 |a_p|^2 \leq \sum_{p=1}^{k-1} |(A-B)b - (p-1)B|^2 (\delta(\lambda, p))^2 |a_p|^2,$$

which simplifies to

$$|a_k|^2 \leq \left(\frac{1}{(k-1)\delta(\lambda, k)} \right)^2 \left((A-B)^2 |b|^2 + \sum_{p=2}^{k-1} \{ |(A-B)b - (p-1)B|^2 - (p-1)^2 \} \right. \\ \left. \cdot (\delta(\lambda, p))^2 |a_p|^2 \right) \quad (5.2.10)$$

for every $k = 2, 3, \dots$. For $k = 2$, we have

$$|a_2|^2 \leq \left(\frac{(A-B)|b|}{\lambda + 1} \right)^2,$$

which proves (5.2.4). Suppose $|(A-B)b - B| \leq 1$ and $k \geq 3$. Then it follows that

$$|(A-B)b - (k-2)B| \leq k-2, \quad k \geq 3.$$

Since all the terms under the summation in (5.2.10) are

non-positive, we obtain

$$|a_k| \leq \frac{(A-B)|b|}{(k-1)\delta(\lambda, k)}, \quad k \geq 3$$

which establishes (5.2.5). However, if

$$|(A-B)b - (k-2)B| > k-2, \quad k \geq 3,$$

then all the terms under the summation (5.2.10) are positive. We shall establish (5.2.6) for integer $k \geq 3$ and $k \leq M+2$, from (5.2.10), by mathematical induction. For $k = 3$, (5.2.10) contributes

$$|a_3|^2 \leq \left(\frac{(A-B)|b| |(A-B)b - B|}{(\lambda+1)(\lambda+2)} \right)^2,$$

which proves (5.2.6) for $k = 3$. Suppose (5.2.6) holds for $k = 4, 5, \dots, m-1$. Then for $k = m$, (5.2.10) yields

$$\begin{aligned} |a_m|^2 &\leq \left(\frac{1}{(m-1)\delta(\lambda, m)} \right)^2 ((A-B)^2|b|^2 + \sum_{p=2}^{m-1} \{ |(A-B)b - (p-1)B|^2 - (p-1)^2 \} \\ &\quad \cdot (\delta(\lambda, p))^2 |a_p|^2) \\ &\leq \left(\frac{1}{(m-1)\delta(\lambda, m)} \right)^2 ((A-B)^2|b|^2 + \sum_{p=2}^{m-1} \{ |(A-B)b - (p-1)B|^2 - (p-1)^2 \} \\ &\quad \cdot (\delta(\lambda, p))^2 \prod_{j=2}^p \frac{|(A-B)b - (j-2)B|^2}{(\lambda+j-1)^2} \\ &= \prod_{j=2}^m \frac{|(A-B)b - (j-2)B|^2}{(\lambda+j-1)^2}, \end{aligned}$$

by Lemma 5.2.1. It is now easy to show that (5.2.6) holds for $k \leq M+2$. Finally, suppose $k > M+2$. Then we may write (5.2.10) as

$$\begin{aligned}
|a_k|^2 &\leq \left(\frac{1}{(k-1)\delta(\lambda, k)} \right)^2 ((A-B)^2|b|^2 + \sum_{p=2}^{M+2} \{|(A-B)b - (p-1)B|^2 - (p-1)^2\} \\
&\quad \cdot (\delta(\lambda, p))^2 |a_p|^2 + \sum_{p=M+3}^{k-1} \{|(A-B)b - (p-1)B|^2 - (p-1)^2\} (\delta(\lambda, p))^2 |a_p|^2) \\
&\leq \left(\frac{1}{(k-1)\delta(\lambda, k)} \right)^2 ((A-B)^2|b|^2 + \sum_{p=2}^{M+2} \{|(A-B)b - (p-1)B|^2 - (p-1)^2\} \\
&\quad \cdot (\delta(\lambda, p))^2 |a_p|^2) \\
&\leq \left(\frac{1}{(k-1)\delta(\lambda, k)} \right)^2 ((A-B)^2|b|^2 + \sum_{p=2}^{M+2} \{|(A-B)b - (p-1)B|^2 - (p-1)^2\} \\
&\quad \cdot (\delta(\lambda, p))^2 \prod_{j=2}^p \frac{|(A-B)b - (j-2)B|^2}{(\lambda + j - 1)^2}) \\
&= \left(\frac{(M+2)\delta(\lambda, M+3)}{(k-1)\delta(\lambda, k)} \right)^2 \prod_{j=2}^{M+3} \frac{|(A-B)b - (j-2)B|^2}{(\lambda + j - 1)^2} \\
&= \left(\frac{(k-2)!}{(M+1)!(\lambda+1)(\lambda+2)\dots(\lambda+k-1)} \right)^2 \prod_{j=2}^{M+3} |(A-B)b - (j-2)B|^2
\end{aligned}$$

by an application of Lemma 5.2.1; and thus (5.2.7) immediately follows from the above. The equality in (5.2.4) and (5.2.5) are attained for the functions f_k given by

$$f_k * \frac{z}{(1-z)^{\lambda+1}} = \begin{cases} z(1+Bz^{k-1})^{(A-B)b \cdot B(k-1)} & , B \neq 0 \\ z \exp(Abz^{k-1}/(k-1)) & , B = 0 \end{cases}$$

where $|(A-B)b - B| \leq 1$. Finally, the inequality (5.2.6) is sharp for the function f given by

$$f * \frac{z}{(1-z)^{\lambda+1}} = \begin{cases} z(1+Bz)^{(A-B)b \cdot B}, & B \neq 0 \\ z \exp(Abz) & , B = 0 \end{cases}$$

where $|(A-B)b - (k-2)B| > k-2$, $k \geq 3$.

Remark 5.2.11. Fixing $\lambda = 0$, $A = 1$ and $B = -1$ in Theorem 5.2.3 lead to the coefficient estimates for the functions in the class S^{1-b} of starlike functions of order $1-b$, obtained in Chapter II.

Remark 5.2.12. The coefficient estimates established in Chapter IV for functions in the class $S^{1-b}(A, B)$ of univalent analytic functions related to complex order may be obtained from Theorem 5.2.3 by fixing $\lambda = 0$.

Remark 5.2.13. On choosing appropriate values of the parameters b , λ , A , B in Theorem 5.2.3, we may obtain the sharp estimates for all coefficients of functions in several subclasses of **A**. For instant, using the relations (5.1.3) to (5.1.6), (5.1.8) to (5.1.12), (5.1.14), (5.1.16) to (5.1.18) and (5.1.21), the theorem yields the corresponding coefficient estimates of the functions in the classes K_λ , $S[A, B]$, $C[A, B]$, $S(\alpha)$, $C(\alpha)$, $SP(\gamma, \alpha)$, $RB(\gamma, \alpha)$, $SP^*(\gamma, A, B)$, $SP^*(\gamma, \alpha)$, $SP^*[\gamma, c, d]$, $G(\alpha)$, $G(\alpha, \beta)$, $Q_\lambda(\alpha)$ and C^{1-b} , respectively. In particular, the above theorem generalizes the results for the classes studied in [14, 22, 38, 43, 63] and several others.

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الخلاصة

" فئات جزئية محددة للدوال التحليلية ذات الرتب العقدية "

تعتبر دراسة الدوال أحادية القيمة من أكثر المجالات الخصبة بالبحوث و الدراسات التي أجريت بواسطة رياضيين بارزين من خلال تقديم العديد من المفاهيم الجديدة و الطرق المبتكرة. و هذه الدراسة معنية أساسا بالتفاعل بين البناء التحليلي و السلوك الهندسي للدوال التحليلية.

الهدف الأساسي من هذه الرسالة هو دراسة فئات جزئية محددة من الدوال التحليلية العقدية. و على وجه الخصوص، تم دراسة نتائج نصر و عوف [٤٤]، ديكست و بال Dixit and Pal [١٦]، و أهيوجا Ahuja [٣]. كما تم تعريف و دراسة الدوال التحليلية أحادية القيمة ذات الرتب العقدية المشابهة لتلك التي درسها كل من ديكست و بال Dixit and Pal [١٦].

الرسالة تتكون من خمسة فصول.

الفصل الأول: يعرض المفاهيم الأساسية و النتائج المهمة المتعلقة بموضوع الرسالة.

الفصل الثاني: يدرس نتائج كل من نصر و عوف [٤٤] في ما يتصل بالدوال النجمية ذات الرتب العقدية S^{1-b} (starlike functions of complex order 1-b).

الفصل الثالث: خصص لدراسة فئات الدوال أحادية القيمة ذات الرتب العقدية $R^{1-b}(A,B)$ التي درسها كل من ديكست و بال Dixit and Pal [١٦].

الفصل الرابع: تم تعريف و مناقشة فئات الدوال أحادية القيمة ذات الرتب العقدية $S^{1-b}(A,B)$ المشابهة لتلك التي درسها كل من ديكست و بال Dixit and Pal.

الفصل الخامس: تم مناقشة فئة راستشوي المعممة $H^b[A,B]$ للدوال التحليلية ذات الرتب العقدية و التي تمت دراستها من قبل أهيوجا [٣]. و هذه الفئة توحد أجزاء الفئات للدوال التحليلية أحادية القيمة من أمثال فئات الدوال

(starlike, convex, prestarlike, Ruscheweyh class of order α , γ -spirallike, γ -convex spirallike and starlike of complex order).