



APPLICATIONS OF THE LEVINSON THEOREM

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Requirements for the Degree**

of

MASTER OF SCIENCE

**Under the Supervision of
Prof. A.S.A Alhammadi**

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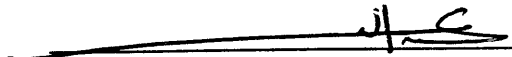
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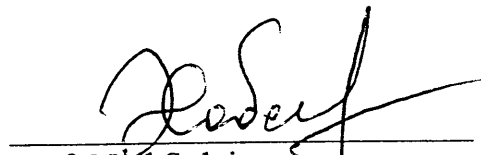
“ Applications of the Levinson Theorem ”

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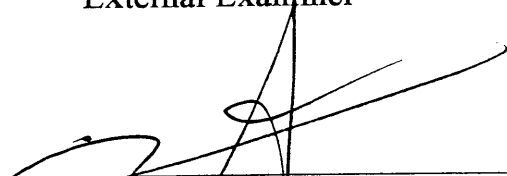
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الخلاصة

تطبيقات لنظرية Levinson

تهدف هذه الأطروحة لعرض بعض التطبيقات لنظرية Levinson لإيجاد الحل المقارب للنظام التفاضلي الخطي

$$Y'(x) = A(x)Y(x) \quad (1.1)$$

هنا $A(x)$ عبارة عن مصفوفة مربعة $n \times n$ حيث أن أي كمية موجهة للحل $Y(x)$ لا يمكن كتابتها بتعبير واضح ودقيق مساويا لها و هذا بالطبع من الصعوبات التي تخلق نوع من التحدي من أجل تطوير تشكيله من الأساليب للتحقق من صحة الحلول بالنسبة لهذه المعادلة .

إذ أن الهدف العام من النظريات المقاربة هو التعرف على تلك الأنواع من النظم التي تقارب حلولها (1.1) بحيث يكون الانطلاق من النظام التفاضلي

$$Y'(x) = A_0(x)Y(x) \quad (1.2)$$

حيث هنا $A_0(x)$ عبارة عن مصفوفة $n \times n$ ذات أرقام ثابتة ($n \times n$ Constant Matrix) والذي يمكن التعبير عن حلولها بشكل واضح ودقيق (Exact Solutions).

لذا نناقش في الفصل الأول متى يكون حلول (1.2) تمثل حلول مقاربة للنظام (1.1) وتحت أي الشروط وكما يشمل الفصل الأول على أهم النظريات في هذا الجانب لنظرية Levinson وإثباتها بالإضافة لنظرية Eastham. ويمثل الفصل الثاني تطبيقات للجانب النظري الوارد في الفصل الأول.

و في النهاية نستنتج انه من الممكن تطبيق نظرية Levinson لإيجاد الحلول التقريبية للمعادلات التفاضلية ذات الترتيب الأعلى.

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Introduction

The asymptotic theorem which provides the basis for the theory in this thesis was proved by Levinson and published in 1948. At that time, the theorem itself represented a considerable advance on existing related results but, in contrast, initial applications of the theorem were only of a very straightforward kind. It was not until some 25 years later that a real appreciation of the significance and range of applications of the theorem started to become evident through the work of Harris and Lutz (1974).

The thesis consists of two chapters . Below a brief description of the contents of the thesis is given .

In chapter one, we discuss the Levinson theorem and its proof, we state the Hartman -Wintner theorem, and Eastham theorem .

Chapter two represents applications of the theory of chapter one .

Chapter 1

The Asymptotic Solution of Linear Differential Systems

1.1 Introduction:

The basic problem posed by a differential system

$$Y'(x) = A(x)Y(x) \tag{1.1}$$

is that the solution vectors $Y(x)$ cannot normally be written as an explicit expression involving the entries of the given square matrix $A(x)$. It is, of course, the difficulty which creates the challenge and interest in developing a wide range of techniques for investigating the properties of solutions, where (1.1) alone is taken as the starting point. The general aim of the asymptotic theory of (1.1) is to identify those types of system whose solutions can nevertheless be approximated by such explicit expressions as x approaches some limiting value, the various types being specified by some given property of $A(x)$ in the neighborhood of the limiting value. Throughout this theses, we consider the asymptotic situation where $x \rightarrow \infty$, and therefore, we take (1.1) to be defined on some x - interval $[a, \infty)$.

An obvious starting point for the asymptotic theory of (1.1) is to examine the case where (1.1), though not itself soluble explicitly, does for large x approximate to a simple system

$$Y'(x) = A_0(x)Y(x) \quad (1.2)$$

which can be solved explicitly, where A_0 is a square constant matrix. The question then is whether the solutions of (1.1) approximate to the known solutions of (1.2) for large x . As we shall see, the answer to this question is not always straightforward and unexpected features do arise.

1.1.2 Notation and Basic Theory

In this section, we state some of important notation and basic theory which will be useful in this thesis.

1.1.3 Notation

O. If $|f| \leq \text{cons.} |g|$ as $x \rightarrow \infty$, we write $f(x) = O\{g(x)\}$.

o. If $\frac{f(x)}{g(x)} \rightarrow 0$ as $x \rightarrow \infty$, ($g \neq 0$), we say that $f(x) = o\{g(x)\}$.

$\sim$. We say that $f(x)$ is asymptotic to $g(x)$ and we write

$$f \sim g \text{ if } \frac{f(x)}{g(x)} \rightarrow 1 \text{ as } x \rightarrow a. \quad (g \neq 0)$$

Special Cases:

(1) $f(x) = O(1) \Rightarrow |f(x)| \leq \text{cons.} (x \rightarrow \infty)$ i.e f is bounded .

(2) $f(x) = o(1) \quad (as \ x \rightarrow \infty) \Rightarrow f = 0 \quad as \ x \rightarrow \infty$.

(3) $\frac{f}{g} \rightarrow 1 \quad (as \ x \rightarrow a) \Rightarrow \frac{f}{g} = 1 + o(1) \quad (as \ x \rightarrow a)$.

With $n \times n$ matrix $A = (a_{ij})$, we define the modulus $|A|$ by

$$|A| = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|$$

Also $|AB| \leq |A||B|$

Let A be a square matrix. We denote by $\text{dg } A$ or, morefully

$$\text{dg } (a_{11}, \dots, a_{nn})$$

the diagonal matrix formed by the diagonal entries a_{ii} in A .

We also denote by A^t and A^* the transposed and adjoint matrices defined by

$$A^t = (a_{ji}),$$

$$A^* = (\overline{a_{ji}}).$$

For matrices whose entries are differentiable functions of x , we define

$$A'(x) = (a'_{ij}(x)).$$

And similarly for integral of x over an interval (a, b) if $a < b$, then we have

$$\left| \int_a^b A(x) dx \right| \leq \int_a^b |A(x)| dx.$$

A statement such as

$$A(x) \in L^p(a, \infty) \text{ and } A(x) = o(x^{-1})(x \rightarrow \infty)$$

means that every entry $a_{ij}(x)$ has the relevant property.

We now introduce two simple systems of (1.2) which form the basis for the theory in this chapter.

1.1.4 The Constant Coefficient System

Here $A_0(x)$ is a constant matrix C , and (1.2) is

$$Y'(x) = CY(x) \quad (1.3)$$

Let λ be an eigenvalue of C with corresponding eigenvector u .

Then a solution of (1.3) is

$$Y(x) = ue^{\lambda x} \quad (1.4)$$

$$Y'(x) = \lambda ue^{\lambda x} = Cue^{\lambda x} = CY(x)$$

We regard (1.4) as an explicit expression for a solution of (1.3) in terms of the matrix C . If C is an $n \times n$ matrix and if there are n linearly independent eigenvectors u_k corresponding to eigenvalues λ_k ($1 \leq k \leq n$), (1.4) gives n linearly independent solutions of (1.3) in the form :

$$Y_K(x) = u_K e^{\lambda_K x} \quad (1.5)$$

1.1.5 The Diagonal System

Here $A_0(x)$ is a diagonal matrix $\Lambda(x)$ and (1.2) is

$$Y'(x) = \Lambda(x)Y(x) \quad (1.6)$$

We write

$$\Lambda(x) = \text{dg}(\lambda_1(x), \dots, \lambda_n(x)).$$

Then, denoting the components of $Y(x)$ by $y_i(x)$ ($1 \leq i \leq n$), we can split up (1.6) into the first order equations

$$y'_i(x) = \lambda_i(x)y_i(x)$$

for each $y_i(x)$ separately. Hence

$$y_i(x) = c_i \exp\left(\int_a^x \lambda_i(t) dt\right),$$

where the c_i are arbitrary constants. If we now take any integer k ($1 \leq k \leq n$) and choose $c_k = 1$ and other c_i zero, we obtain the solution

$$Y_k(x) = e_k \exp\left(\int_a^x \lambda_k(t) dt\right) \quad (1.7)$$

of (1.6), where e_k is the coordinate vector whose k^{th} component is unity and other components are zero. Again, as k varies from 1 to n , we have explicit expressions (1.7) for n linearly independent solutions of (1.6).

Moving on now to system (1.1) which approximate to (1.3) or (1.6) when x is large, we consider

$$Y'(x) = \{C + R(x)\}Y(x) \quad (1.8)$$

Or

$$Y'(x) = \{\Lambda(x) + R(x)\}Y(x) \quad (1.9)$$

where $R(x)$ is $n \times n$ matrix which is small in the same sense as $x \rightarrow \infty$. We describe these systems as asymptotically constant and asymptotically diagonal respectively. Note that, (1.9) is more fundamental than (1.8) because, later in the chapter, we obtain the results for (1.8) from those of (1.9) by means of a simple transformation.

There are various ways in which $R(x)$ can be thought of as small when $x \rightarrow \infty$, the most obvious being

$$R(x) \rightarrow 0 \text{ as } x \rightarrow \infty \quad (1.10)$$

This condition on $R(x)$ is not, however, a suitable one to work with at the outset because, as we shall see, it does allow a considerable variety in the asymptotic behaviour of solutions of (1.9) with more detailed properties of $R(x)$ also influencing the results.

In the theorem on which the asymptotic theory of (1.9) rests, the condition on $R(x)$ which arises naturally is

$$\int_a^\infty |R(x)| dx < \infty, \quad (1.11)$$

by which we mean that each entry in $R(x)$ has an absolutely convergent infinite integral. This theorem was proved by Levinson (1948) and it states that, subject to (1.11) and condition on $\Lambda(x)$ which will be introduced into the present account in 1.3, (1.9) has solutions $Y_k(x)$ ($1 \leq k \leq n$) such that

$$Y_K(x) = \{e_k + o(1)\} \exp \left(\int_a^x \lambda_k(t) dt \right) \quad (1.12)$$

As $x \rightarrow \infty$. Thus, comparing (1.12) with the right-hand side in (1.7), we see that the Levinson theorem gives condition on $R(x)$ and $\Lambda(x)$ under the solutions of the (1.9) do approximate to those of (1.6) for large x .

This result provides the starting point for the asymptotic theory of (1.1) that mentioned at the beginning of this section.

We refer to a system (1.9) in which (1.11) holds as having the Levinson form, and we will prove the Levinson theorem itself in sections 1.3 - 1.4. Then in the later sections of this chapter, we give a number of consequences we deal with solutions not only of (1.8), but also (1.9) where (1.11) is replaced by some alternative smallness condition on $R(x)$.

1.2 Linear Differential Systems

Definition :

A solution matrix for (1.1) is an $n \times n$ matrix $\Phi(x)$ whose columns are solutions $Y_i(x)$ ($1 \leq i \leq n$) of (1.1)

Then $\Phi(x)$ satisfies the corresponding matrix differential equation

$$\Phi'(x) = A(x) \Phi(x) \quad (2.1)$$

further , $\Phi(x)$ is either singular $\forall x \in I$ ($I \subseteq R$) or non - singular $\forall x \in I$ according as the $Y_i(x)$ are linearly dependent or linearly independent in I . In the latter case , $\Phi(x)$ is said to be a fundamental matrix for (1.1).

One of the basic properties of a fundamental matrix $\Phi(x)$ is that any solution of (1.1) can be expressed as

$$Y_i(x) = \Phi(x) c ,$$

where c is a constant vector .If we also consider a second differential system

$$Z'(x) = \{A(x) + B(x)\}Z(x) \quad (2.2)$$

along with (1.1) , any solution of (2.2) can be related to given linearly independent solution $Y_i(x)$ of (1.1) by the formula

$$Z(x) = \Phi(x)c + \Phi(x) \int_a^x \Phi^{-1}(t)B(t)Z(t) dt \quad (2.3)$$

where $a \in I$, and c is a suitable constant vector. The formula (2.3) is called variation of parameters. A useful generalization of (2.3) is

$$Z(x) = \Phi(x)c + \Phi_1(x) \int_a^x \Phi^{-1}(t)B(t)Z(t) dt - \Phi_2(x) \int_x^b \Phi^{-1}(t)B(t)Z(t) dt \quad (2.4)$$

where a and b are given points in I , and the fundamental matrix $\Phi(x)$ is split up into any two solution matrices for (1.1) in the form

$$\Phi(x) = \Phi_1(x) + \Phi_2(x).$$

$$\text{Now let } Z(x) = \Phi(x)W(x) \quad (2.5)$$

in the (2.2), then the corresponding W -system is giving by

$$\Phi W' + \Phi' W = (A + B)\Phi W$$

Thus the effect of (2.6) on (2.3) is to remove the matrix $A(x)$.

The determinate of Φ satisfies the Liouville-Jacobi formula

$$\det \Phi(x) = \det \Phi(a) \exp\left(\int_a^x \text{tr} A(t) dt\right) \quad (2.6)$$

$$\text{where } \text{tr} A = \sum_{i=1}^n a_{ii},$$

and two simple consequences of this formula will be required later.
First, let Ψ be a given non-singular matrix and define

$$A = \Psi' \Psi^{-1}. \quad (2.7)$$

Then $\Psi' = A\Psi$ and we can take $\Phi = \Psi$ in (2.7). This gives

$$\frac{d}{dx} \log \det \Psi = \text{tr} \Psi' \Psi^{-1} = -\text{tr} \Psi (\Psi^{-1})' \quad (2.8)$$

Second, let C be a given matrix, and then introduce x as a parameter and define $\Psi(x) = \exp(xC)$.

Then (2.7) gives $A=C$, and it follows from (2.6) that
 $\det \exp(xC) = \exp(x \text{tr} C)$

for any matrix C .

1.3 The Levinson Theorem

In this section, we state the Levinson theorem on asymptotically diagonal systems together with a number of explanatory comments on the conditions that are imposed in the theorem. The actual proof will follow in 1.4

Theorem 3.1 (Levinson, N(1948))

Let $\Lambda(x)$ be an $n \times n$ diagonal matrix,

$$\Lambda(x) = \text{dg}(\lambda_1(x), \dots, \lambda_n(x)),$$

which satisfies the following two parts of condition L.

Condition L : for each pair of integers of i and j in $[1, n]$ ($i \neq j$) and for all x and t such that $a \leq t \leq x < \infty$, either

$$(a) \quad \int_a^x \text{Re}\{\lambda_i(s) - \lambda_j(s)\} ds \leq K_1 \quad (3.1)$$

or

$$(b) \quad \int_a^x \text{Re}\{\lambda_i(s) - \lambda_j(s)\} ds \geq K_2 \quad (3.2)$$

where k_1 and K_2 are constants.

Let the $n \times n$ matrix $R(x)$ satisfying

$$\int_a^x |R(x)| dx < \infty \quad (3.3)$$

Then, as $x \rightarrow \infty$, the system

$$\dot{Y}(x) = \{\Lambda(x) + R(x)\} Y(x) \quad (3.4)$$

has solutions $Y_K(x)$ ($1 \leq k \leq n$) with the asymptotic form

$$Y_K(x) = \{e_k + o(1)\} \exp\left(\int_a^x \lambda_k(t) dt\right) \quad (3.5)$$

where $e_k = (C_1, C_2, C_3, C_4, \dots, C_K, C_{K+1}, \dots, C_n)^T$

And $C_i = 0$ for all $i \neq k$ and $C_k = 1$

Discussion

To gain some idea of the nature of Condition L, let us note a particular situation which the condition definitely excludes. This situation is where

$$\liminf_a^x \int_a^x \operatorname{Re}\{\lambda_i(s) - \lambda_j(s)\} ds = -\infty$$

and

$$\limsup_a^x \int_a^x \operatorname{Re}\{\lambda_i(s) - \lambda_j(s)\} ds = \infty$$

both occur for one and same pair (i, j) . In terms of (3.5), therefore, Condition L implies that we are dealing with systems where the ratio

$$|Y_i(x)| / |Y_j(x)| \quad (3.6)$$

of two given solutions does not satisfy both

$$\liminf |Y_i(x)| / |Y_j(x)| = 0$$

and

$$\limsup |Y_i(x)| / |Y_j(x)| = \infty.$$

That is, the ratio does not oscillate between large and small values as $x \rightarrow \infty$, the condition, therefore, represents a regularity situation for the relative sizes of any two solutions as $x \rightarrow \infty$.

As they stand, (3.1) and (3.2) are not mutually exclusive, but they can be rewritten in a way which does exhibit this feature if (3.1), for example, is subdivided. Let

$$I_{ij}(x) = \int_a^x \operatorname{Re}\{\lambda_i(s) - \lambda_j(s)\} ds.$$

Then (3.1) is

$$I_{ij}(x) - I_{ij}(t) \leq k_1,$$

we have either $I_{ij}(x) \rightarrow -\infty$ As $x \rightarrow \infty$ or $I_{ij}(x) \geq (\text{constant})$ for all $x \geq a$.

Putting the latter alternative together with (3.2), we can write condition L in the following equivalent form $\acute{L}$.

Condition $\acute{L}$: For each pair of integers i and j in $[1, n]$ ($i \neq j$) and all x and t such that $a \leq t \leq x < \infty$, either

$$(a') \quad \int_a^x \operatorname{Re}\{\lambda_i(s) - \lambda_j(s)\} ds \rightarrow -\infty \text{ as } x \rightarrow \infty \quad (3.7)$$

$$\int_t^x \operatorname{Re}\{\lambda_i(s) - \lambda_j(s)\} ds \leq K_1 \quad (3.8)$$

or

$$(b') \quad \int_t^x \operatorname{Re}\{\lambda_i(s) - \lambda_j(s)\} ds \geq K_2 \quad (3.9)$$

With a given j , condition $\acute{L}$ divides the integers i in $[1,n]$ into two classes according to λ_j falls under part (a') or part (b'). Correspondingly, in terms of (3.6), the solutions $Y_i(x)$ are divided into those which are small compared to $Y_j(x)$ as $x \rightarrow \infty$ and those which are not. For this reason, condition $\acute{L}$ is known as dichotomy condition, and we shall also use the same terminology for condition L . It is possible to subdivide (3.9) further into the two cases where $I_{ij}(x) \rightarrow -\infty$ and $I_{ij}(x) \geq (\text{constant})$, and it is in the resulting three parts form that the dichotomy condition (though not under that name of course) was originally introduced by Levinson (1948)

As the theory develops, we shall, on occasions, meet λ_j of the form

$$\lambda_j = p'_j / p_j \quad (3.10)$$

In this case, the integration in (3.1) and (3.2) can be carried out, and the dichotomy condition then states that either

$$|p_i(x) / p_j(x)| \leq K' |p_i(t) / p_j(t)| \quad (3.11)$$

or

$$|p_i(x) / p_j(x)| \geq K'' |p_i(t) / p_j(t)| \quad (3.12)$$

For each i and j , and that $a \leq t \leq x < \infty$, where K' and K'' are constants.

Next we mention a stronger form of the dichotomy condition which is often adequate in applications as well as being easy to verify.

Condition L'' : for each pair (i, j) , let

$$\operatorname{Re} \{\lambda_i(x) - \lambda_j(x)\} = f(x) + g(x) \quad (3.13)$$

Where $g(x)$ is $L(a, \infty)$ and either $f(x) \geq 0$ in $[a, \infty)$ or $f(x) \leq 0$ in $[a, \infty)$.

It is clear that either (3.1) or (3.2) holds when (3.13) does.

The necessity of the dichotomy condition in theorem (3.1) can be illustrated by the following example in which $n=2$, (3.3) holds, Condition L is not satisfied, and the asymptotic formula (3.5) does not hold.

Example :

we define

$$P(x) = x^2(1 - \sin x) + 1$$

$$\Lambda = dg(0, p'(x)/p(x))$$

and

$$R(x) = \begin{bmatrix} 0 & x^{-2} \\ 0 & 0 \end{bmatrix}$$

in $[a, \infty)$. With

$$Y = \begin{pmatrix} u \\ v \end{pmatrix}$$

the system (3.4) is now

$$u' = x^{-2}v, \quad v' = \left(\frac{p'}{p} \right) v$$

Hence

$$v = cp(x),$$

where c is a constant, and then $u(x) = c(1 - \sin x) + cx^{-2}$, giving

$$u(x) = c(x + \cos x) - cx^{-1} + d,$$

where d is a further constant . taking $c=0$ and $d=1$, we obtain the solution $Y_1(x)=e_1$, which does of course have the form (3.5) since $\lambda_1(x)=0$ here . however , turning to a solution corresponding to

$$\lambda_2 = p'(x)/p(x)$$

we take $d = 0$ and $c=P(a)$ to obtain the solution

$$Y_2(x) = \left\{ e_2 + \begin{bmatrix} w(x) \\ 0 \end{bmatrix} \right\} p(x)$$

where

$$W(x) = (x + \cos x - x^{-1}) / \{ x^2(1 - \sin x) + 1 \}.$$

It is clear that $w(x) \neq o(1)$ as $x \rightarrow \infty$ and hence there is no solution $Y_2(x)$ of the form (3.5) . Condition L fails to hold in this example because here

$$\int_a^x \operatorname{Re}\{\lambda_i(s) - \lambda_j(s)\} ds = \log \{p(x)/P(t)\}$$

$$= \log \{ [x^2(1 - \sin x) + 1] / [t^2(1 - \sin t) + 1] \},$$

and the choices $x = (2N+1)\pi$, $t = (2N+1/2)\pi$, and $x = (2N+1/2)\pi$, $t = 2N\pi$ (large N) show that neither (3.1) nor (3.2) is satisfied .

1.4 Proof of the Levinson Theorem

We consider a particular k in (3.5) and by making the change of variables

$$Y(x) = Z(x) \exp\left(\int_a^x \lambda_k(t) dt\right)$$

in (3.4). Then

$$Y'(x) = Z'(x) \exp\left(\int_a^x \lambda_k(t) dt\right) + \lambda_k Y(x)$$

so (3.4) gives

$$Z'(x) \exp\left(\int_a^x \lambda_k(t) dt\right) + \lambda_k(x) Z(x) \exp\left(\int_a^x \lambda_k(t) dt\right)$$

$$= \{\Lambda(x) + R(x)\} Z(x) \exp\left(\int_a^x \lambda_k(t) dt\right)$$

$$\therefore Z'(x) = \{\Lambda(x) - \lambda_k(x)I + R(x)\} Z(x)$$

$$Z'(x) = \{\Lambda_1(x) + R(x)\} Z(x) \tag{4.1}$$

where

$$\Lambda_1(x) = \Lambda(x) - \lambda_k(x)I$$

and the theorem will be proved if we show that there is a solution $Z(x)$ of (4.1) such that

$$Z(x) = e_k + o(1) \tag{4.2}$$

as $x \rightarrow \infty$.

Working now with (4.1), we write

$$\Lambda_1(x) = dg(\mu_1, \dots, \mu_n)$$

$$\mu_i(x) = \lambda_i(x) - \lambda_k(x)$$

and , in particular

$$\mu_k = 0 \tag{4.3}$$

Referring to the dichotomy condition in the form (3.7)-(3.9), we take $j=k$ with a re-ordering if necessary, we arrange that part (a') holds for $1 \leq i \leq N-1$ and part (b') for $N \leq i \leq n$, with

$$\mu_N(x) = 0 \tag{4.4}$$

corresponding to (4.3). The values $N=1$ and $N=n+1$ are allowed with the obvious meaning . If we define

$$v_i(x) = \exp\left(\int_a^x \mu_i(s) ds\right) \tag{4.5}$$

then by L' we have

$$v_i(x) \rightarrow 0 \quad (x \rightarrow \infty, 1 \leq i \leq N-1) \tag{4.6}$$

$$\left| \frac{v_i(x)}{v_i(t)} \right| \leq k' \quad (a \leq t \leq x \leq \infty, 1 \leq i \leq N-1) \tag{4.7}$$

$$\left| \frac{v_i(t)}{v_i(x)} \right| \leq k'' \quad (a \leq t \leq x \leq \infty, N \leq i \leq n) \tag{4.8}$$

where k' and k'' are constants.

We complete the initial stages of the proof by introducing the fundamental matrix $\Phi(x)$ of the diagonal system

$$Z'(x) = \Lambda_1(x)Z(x) \quad (4.9),$$

is defined by

$$\Phi(x) = dg(v_1(x), v_2(x), \dots, v_n(x))$$

we write

$$\Phi(x) = \Phi_1(x) + \Phi_2(x)$$

$$\text{where } \Phi_1(x) = dg(v_1(x), \dots, v_{N-1}(x), 0, 0, \dots, 0) \quad (4.10)$$

$$\text{and } \Phi_2(x) = dg(0, 0, \dots, v_N(x), \dots, v_n(x))$$

Then (4.7) implies that

$$|\Phi_1(x)\Phi^{-1}(t)| \leq c_1 \quad (a \leq t \leq x \leq \infty) \quad (4.11)$$

while (4.8) with x and t are interchanged, implies that

$$|\Phi_2(x)\Phi^{-1}(t)| \leq c_2 \quad (a \leq t \leq x \leq \infty) \quad (4.12)$$

where c_1 and c_2 are constants.

Moving to the main part of the proof, the aim is to show that the integral equation

$$Z(x) = e_N + \Phi_1(x) \int_a^x \Phi^{-1}(t)R(t)Z(t)dt - \int_x^\infty \Phi_2(x)\Phi^{-1}(t)R(t)Z(t)dt \quad (4.13)$$

has solution $Z(x)$ and then $Z(x)$ satisfies (4.2) with $K=N$. It is clear by differentiating (4.12) satisfies (4.1), in fact (4.13) suggested by variation of parameter formula (4.11) as applied to (4.1) and (4.9). The important difference between (4.11) and (4.13) is that we have upper limit ∞ in the second integral so we have a convergence question to be considered.

Applying the method of successive approximation to (4.13), we define $\{Z_m(x)\}$ (where $m=1,2,\dots$) by $Z_1(x)=e_N$ and

$$Z_{m+1}(x) = e_N + \Phi_1(x) \int_a^x \Phi^{-1}(t) R(t) Z_m(t) dt - \int_x^\infty \Phi_2(x) \Phi^{-1}(t) R(t) Z_m(t) dt \quad (\text{where } m \geq 1) \quad (4.14)$$

First, we prove that $Z_m(x)$ is bounded in $[a, \infty)$, then this with (4.11), (4.12) and (3.3), ensures that the infinite integral in (4.14) converges we assume that $|Z_m(x)| \leq C_m$ for some constant C_m , then by (4.11) -(4.14), we have

$$\begin{aligned} |Z_{m+1}(x)| &\leq 1 + c_1 \int_a^x |R(t)| |Z_m(t)| dt + c_2 \int_x^\infty |R(t)| |Z_m(t)| dt \\ |Z_{m+1}(x)| &\leq 1 + (c_1 + c_2) C_m \int_a^\infty |R(t)| dt = C_{m+1} \\ \therefore |Z_{m+1}(x)| &\leq C_{m+1} \end{aligned}$$

Since $Z_1(x) = e_N$, then the boundedness of $Z_m(x)$ for each m follows. Next we prove that $Z_m(x)$ converges uniformly on $[a, \infty)$ to limiting function $Z(x)$. Again by (4.11-4.14) we have

$$|Z_{m+2}(x) - Z_{m+1}(x)| \leq (c_1 + c_2) \int_a^\infty |R(t)| |Z_{m+1}(x) - Z_m(x)| dt$$

Hence, by induction ,

$$|Z_{m+1}(x) - Z_m(x)| \leq \left((c_1 + c_2) \int_a^\infty |R(t)| dt \right)^m \quad (4.15)$$

By making a suitable choice for a in (4.14) so that

$$(c_1 + c_2) \int_a^\infty |R(t)| dt \leq 1 \quad (4.16)$$

Then, by comparison with a geometric, the series

$$\sum \{Z_{m+1}(x) - Z_m(x)\}$$

converges uniformly in $[a, \infty)$. Hence we define

$$Z(x) = Z_1(x) + \sum_{l=1}^{\infty} (Z_{l+1} - Z_l) = \lim_{m \rightarrow \infty} Z_m(x) \quad (4.17)$$

and it follows from (4.15) & (4.16) that $Z(x)$ bounded in $[a, \infty)$. Thus

$$|Z(x)| \leq K \quad (4.18)$$

for some constant K . Letting $m \rightarrow \infty$ in (4.14), we obtain (4.13).

Now we need to prove that $Z(x)$ satisfies (4.2) with $K=N$ i.e.

$$Z(x) = e_N + o(1)$$

We write (4.12) as

$$Z(x) = e_N + \Phi_1(x) \int_a^X \Phi^{-1}(t) R(t) Z(t) dt + \Psi(x) \quad (4.19)$$

Where X will be chosen suitably in $[a, x]$ and

$$\Psi(x) = \Phi_1(x) \int_X^x \Phi^{-1}(t) R(t) Z(t) dt - \int_x^\infty \Phi_2(x) \Phi^{-1}(t) R(t) Z(t) dt$$

As before, using (4.11), (4.12) and (4.18), we have

$$|\Psi(x)| \leq (c_1 + c_2) K \int_X^\infty |R(t)| dt$$

$$\text{given } \varepsilon > 0, \text{ choose } X \text{ s.t. } |\Psi(x)| \leq \varepsilon$$

Also by (4.6) and (4.10)

$$\Phi_1(x) \rightarrow 0 \text{ as } x \rightarrow \infty$$

$$\therefore |Z(x) - e_N| < 2\varepsilon \text{ for sufficiently large } x.$$

This completes the proof for 4.2 as well as 3.5.

1.5 The Hartman-Wintner Theorem and Eastham Theorem

Here we state two of most important theorem in the theory of asymptotic solution of linear differential equation as we will see in our application in chapter 2 .

Theorem 5.1 (The Hartman -Wintner Theorem)

Let $\Lambda(x)$ be $n \times n$ diagonal matrix

$$\Lambda(x) = dg(\lambda_1(x), \dots, \lambda_n(x)),$$

and let λ_i be a constant such that

$$|\operatorname{re}\{\lambda_i(x) - \lambda_j(x)\}| \geq \delta \quad (5.1)$$

In some interval $[a, \infty)$ for each unequal pair of integers i and j in $[1, n]$.

Let the matrix $R(x)$ satisfies

$$\int_a^\infty |R(x)|^p dx < \infty \quad (5.2)$$

For some p such that

$$1 < p \leq 2 \quad .$$

Then the system

$$Y'(x) = \{\Lambda(x) + R(x)\}Y(x) \quad (5.3)$$

has solutions $Y_k(x)$ ($1 \leq k \leq n$) with asymptotic form as $x \rightarrow \infty$

$$Y_k(x) = \{e_k + o(1)\} \exp \left(\int_a^x \lambda_k(t) + r_{kk}(t) dt \right) \quad .$$

Theorem 5.2 (Eastham, M.S.P (1985))

Let $\Lambda(x)$ be $n \times n$ diagonal matrix

$$\Lambda(x) = dg(\lambda_1(x), \dots, \lambda_n(x)), \quad (5.5)$$

in which $\{\lambda_i(x) - \lambda_j(x)\}$ is nowhere zero in some interval $[a, \infty)$ for each pair of unequal integers i and j in $[1, n]$. Let $n \times n$ matrix $R(x)$ satisfies

$$(i) \{ \lambda_i(x) - \lambda_j(x) \}^{-1} R(x) \rightarrow 0 \text{ as } x \rightarrow \infty, \quad (5.6)$$

$$(ii) (\{ \lambda_i(x) - \lambda_j(x) \}^{-1} R(x))' \in L(a, \infty), \quad (5.7)$$

for each unequal i and j in $[1, n]$. Also, let the eigenvalues $\mu_k(x)$ ($1 \leq k \leq n$) satisfy the dichotomy condition L in section 1.3 then, as $x \rightarrow \infty$, (5.3) has solutions $Y_k(x)$ with the asymptotic form

$$Y_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t) dt\right). \quad (5.8)$$

Chapter Two

Applications of the Levinson Theorem and Eastham Theorem

2.1 Introduction

The main theorems in chapter one are applied to asymptotically diagonal systems (1.9). The aim now is to consider other types of systems which nevertheless be transformed into the asymptotically diagonal cases covered in sections 1.3-1.5 . The asymptotic form of the solutions for these wider classes of systems is then obtained by applying the theorems in chapter one.

2.2 Solutions of Linear Differential Systems with Constant Coefficient

Consider the vector differential equation

$$Y'(x) = A_0(x)Y(x) \quad , \quad (6.1).$$

which was introduced in (1.1), where A_0 is an $n \times n$ real constant matrix. A_0 defined in (6.1) is called the coefficient matrix of (6.1). In this section we will discuss the general solution of (6.1) in different cases .

2.2.1 Case n Distinct and Real Characteristic Values

Where A_0 is an $n \times n$ real matrix, suppose that each of the n characteristic values $\lambda_1, \lambda_2, \dots, \lambda_n$ of A_0 is real and distinct; and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be a set of n respective corresponding characteristic vectors of A_0 . Then on every real interval $[a, b]$, the n functions defined by

$$\alpha_1 e^{\lambda_1 x}, \alpha_2 e^{\lambda_2 x}, \dots, \alpha_n e^{\lambda_n x},$$

form a linearly independent set (fundamental set) of solutions of (6.1) ;
and

$$Y = c_1 \alpha_1 e^{\lambda_1 x} + c_2 \alpha_2 e^{\lambda_2 x} + \dots + c_n \alpha_n e^{\lambda_n x},$$

where $c_1, \dots, c_n$ are n arbitrary numbers, is a general solution of (6.1) on $[a, b]$.

Example 1

Find the general solution for the given system

$$Y' = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} Y \quad (6.2).$$

Solution

$$\text{Here } A = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix},$$

we see that the characteristic equation of A may be written

$$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0,$$

or in factored form ,

$$(\lambda - 2)(\lambda - 3)(\lambda - 5) = 0.$$

Thus the roots of the characteristic equation are

$$\lambda_1 = 2, \quad \lambda_2 = 3 \quad \text{and} \quad \lambda_3 = 5.$$

A characteristic vector corresponding to $\lambda_1 = 2$ is non-zero vector

$$v_1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \text{ such that ,}$$

$$\begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 2 \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

which was found to be ,

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

In the same way we find $v_2 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix}$,

thus a fundamental set of solutions of (6.2) is

$$v_1 e^{2x}, v_2 e^{3x} \text{ and } v_3 e^{5x},$$

or ,rewriting slightly,

$$\begin{pmatrix} e^{2x} & e^{3x} & 3e^{5x} \\ -e^{2x} & -2e^{3x} & -6e^{5x} \\ -e^{2x} & -e^{3x} & -2e^{5x} \end{pmatrix}$$

and A general solution of the system may by thus be expressed as

$$Y(x) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \begin{pmatrix} e^{2x} & e^{3x} & 3e^{5x} \\ -e^{2x} & -2e^{3x} & -6e^{5x} \\ -e^{2x} & -e^{3x} & -2e^{5x} \end{pmatrix},$$

where c_1, c_2 and c_3 are arbitrary numbers .

2.2.2 Case n Complex Characteristic Values

The complex characteristic values may be present. However ,since A_0 is a real matrix, any complex characteristic values must occur in conjugate pairs. Suppose $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$ form such a pair. Then the corresponding solutions are

$$\alpha_1 e^{(a+bi)x} \text{ and } \alpha_2 e^{(a-bi)x},$$

and these solutions are complex solutions.

Thus, if one or more conjugate complex pairs of characteristic values occur, the fundamental set can be defined by $\alpha_i e^{\lambda_i x}, i = 1, 2, \dots, n$, contains complex functions.

Example2

Solve the initial value problem

$$\begin{cases} \frac{dx}{dt} = x + 4y \\ \frac{dy}{dt} = -3x + 2y \end{cases} \quad Y_0(1, -1) \quad (6.3)$$

Solution

First, note that the matrix coefficient is

$$\begin{pmatrix} 1 & 4 \\ -3 & 2 \end{pmatrix}.$$

Next, we need to find the eigenvalues which are given as roots of the characteristic equation

$$\begin{vmatrix} 1-\lambda & 4 \\ -3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) + 12 = \lambda^2 - 3\lambda + 14 = 0.$$

This is a quadratic equation. Its roots are given by the quadratic formulas

$$\lambda = \frac{3 \pm \sqrt{9 - 56}}{2} = \frac{3 \pm i\sqrt{47}}{2},$$

$$\text{Set } \lambda = \frac{3+i\sqrt{47}}{2},$$

and find an associated eigenvector V . Set

$$V = \begin{pmatrix} x_o \\ y_o \end{pmatrix}$$

The vector V must satisfy the equation

$$\begin{pmatrix} 1 & 4 \\ -3 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_o \\ y_o \end{pmatrix} = \frac{3+i\sqrt{47}}{2} \begin{pmatrix} x_o \\ y_o \end{pmatrix}.$$

From the quadratic equation we get :

$$\left(1 - \frac{3+i\sqrt{47}}{2}\right) \cdot \left(2 - \frac{3+i\sqrt{47}}{2}\right) = -12,$$

then we have

$$\frac{-3}{2 - \frac{3+i\sqrt{47}}{2}} = \frac{1 - \frac{3+i\sqrt{47}}{2}}{4}.$$

This clearly implies that the two equations of the system are the same. Therefore, we use only the first to get

$$y_o = -\frac{1 - \frac{3+i\sqrt{47}}{2}}{4} x_o = \frac{(1+i\sqrt{47})}{8} x_o.$$

Hence, we have

$$\begin{pmatrix} x_o \\ y_o \end{pmatrix} = \frac{x_o}{8} \begin{pmatrix} 8 \\ 1+i\sqrt{47} \end{pmatrix},$$

choose

$$V = \begin{pmatrix} 8 \\ 1+i\sqrt{47} \end{pmatrix}.$$

The independent solutions which will generate the general solution to the system are the real and the imaginary parts of the complex solution

$$e^{\left(\frac{3+i\sqrt{47}}{2}\right)t} \begin{pmatrix} 8 \\ 1+i\sqrt{47} \end{pmatrix}.$$

Since

$$e^{\left(\frac{3+i\sqrt{47}}{2}\right)t} = e^{\frac{3}{2}t} \left(\cos\left(\frac{\sqrt{47}}{2}t\right) \pm i \sin\left(\frac{\sqrt{47}}{2}t\right) \right),$$

and

$$(\cos \beta t + i \sin \beta t)(1+i\sqrt{47}) = \cos \beta t - \sqrt{47} \sin \beta t + i(\sin \beta t + \sqrt{47} \cos \beta t),$$

where $\beta = \frac{\sqrt{47}}{2}$, we have

$$e^{\left(\frac{3+i\sqrt{47}}{2}\right)t} \begin{pmatrix} 8 \\ 1+i\sqrt{47} \end{pmatrix} = Y_1 + Y_2,$$

where

$$Y_1 = e^{\frac{3}{2}t} \begin{pmatrix} 8 \cos \beta t \\ \cos \beta t - \sqrt{47} \sin \beta t \end{pmatrix},$$

and

$$Y_2 = e^{\frac{3}{2}t} \begin{pmatrix} 8 \sin \beta t \\ \sin \beta t + \sqrt{47} \cos \beta t \end{pmatrix}.$$

Therefore, the general solution is

$$Y = k_1 Y_1 + k_2 Y_2.$$

The initial condition $Y_0 = (1, -1)$ implies

$$\begin{pmatrix} 8k_1 \\ k_1 + \sqrt{47} k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which gives

$$k_1 = \frac{1}{8} \text{ and } k_2 = \frac{-9}{8\sqrt{47}}.$$

Therefore, the desired particular solution is

$$Y = \frac{1}{8} e^{\frac{3}{2}t} \begin{pmatrix} 8 \cos \beta t \\ \cos \beta t - \sqrt{47} \sin \beta t \end{pmatrix} - \frac{9}{8\sqrt{47}} e^{\frac{3}{2}t} \begin{pmatrix} 8 \sin \beta t \\ \sin \beta t + \sqrt{47} \cos \beta t \end{pmatrix},$$

where $\beta = \frac{\sqrt{47}}{2}$.

2.2.3 Case Repeated Characteristic Values

If the two roots λ_1 and λ_2 of the characteristic equation are real and equal, let us denote their common value λ then we will have solution of the form

$$Y = c_1 \alpha e^{\lambda x} + c_2 x \beta e^{\lambda x}$$

where c_1 and c_2 are arbitrary constants, α is a characteristic vector corresponding to λ and β is a vector which satisfies the equation $(A_0 - \lambda I)\beta = \alpha$.

Example 3

Consider the system

$$\begin{cases} \frac{dx}{dt} = 2x + y \\ \frac{dy}{dt} = -x + 4y \end{cases} \quad (6.4)$$

1. Find the general solution.
2. Find the solution which satisfies the initial condition

$$Y_0 = (1, -1)$$

Solution

The matrix coefficient of the system is

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$$

In order to find the eigenvalues, consider the characteristic polynomial

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0.$$

Since, we have a repeated eigenvalues equal to 3, let us find the associated Eigenvector V_1 .

Set

$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then we must have $AV_1 = 3V_1$ which is translated into

$$\begin{cases} 2x + y = 3x \\ -x + 4y = 3y \end{cases}.$$

This reduces to $y=x$. Hence, we may take

$$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Next we look for the second vector V_2 . The equation giving this vector is

$$AV_2 = \lambda_1 V_2 + V_1,$$

which translates into the algebraic system

$$\begin{cases} 2x + y = 3x + 1 \\ -x + 4y = 3y + 1 \end{cases}$$

where

$$V_2 = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Clearly the two equations are reduced to the equation $y - x = 1$ or $y = 1 + x$, where x may be chosen to be any number. So if we take $x=0$ for example, we get

$$V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, the two independent solutions are

$$Y_1 = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$Y_2 = e^{3t} \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

The general solution will then be

$$Y = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

In order to find the solution which satisfies the initial condition

$$Y_0 = (1, -1)$$

we must have

$$Y(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This implies $c_1 = 1$ and $c_2 = -2$. Hence, the solution is

$$Y = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2e^{3t} \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = e^{3t} \begin{pmatrix} 1-3t \\ -1-2t \end{pmatrix}$$

2.2.4 Case Zero Characteristic Values

If we have a characteristic value $\lambda_i (1 \leq i \leq n)$ equal to zero with the corresponding characteristic vector α_i , Then the corresponding solution is $c\alpha_i$.

Example 4

Find the general solution to

$$\begin{cases} \frac{dx}{dt} = 2x - y \\ \frac{dy}{dt} = -2x + y \end{cases} \quad (6.5)$$

solution

The characteristic polynomial of this system is

$$\lambda^2 - (2+1)\lambda + 0 = 0$$

which reduces to

$$\lambda^2 - 3\lambda = 0$$

The eigenvalues are $\lambda = 0$ and $\lambda = 3$. Let us find the associated eigenvectors.

For $\lambda = 0$, set

$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix}$$

The equation $AV = 0V$ is translated into

$$\begin{cases} 2x - y = 0 \\ -2x + y = 0 \end{cases}$$

The two equations are the same. So we have $y = 2x$. Hence an eigenvector is

$$V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda = 3$, set

$$V_2 = \begin{pmatrix} x \\ y \end{pmatrix}$$

The equation $AV = 3V$ is translated into

$$\begin{cases} 2x - y = 3x \\ -2x + y = 3y \end{cases}$$

The two equations are the same (as $-x-y=0$). So we have $y = -x$. Hence an eigenvector is

$$V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Therefore the general solution is

$$Y = k_1 e^{0t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= k_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

2.3 The Asymptotic Constant System

Turning now to the asymptotically constant system

$$Y'(x) = \{C + R(x)\}Y(x) \tag{7.1}$$

which was introduced in (1.8) , we consider the situation where (7.1) can be transformed into an asymptotically diagonal system of the type already investigated. The form of the solutions of (7.1) as $x \rightarrow \infty$ is then obtained by applying the theorems of sections 1.3 and 1.5 and transforming back .

We denote the eigenvalues of constant $n \times n$ matrix C by λ_k and the eigenvectors by u_k . The basic requirement concerning C in this section is that there are n linearly independent eigenvectors u_k . Then C can be expressed in the diagonal form

$$T^{-1}CT = \Lambda,$$

where $\Lambda = \text{dg}(\lambda_1, \lambda_2, \dots, \lambda_n)$

with the eigenvalues being repeated according to their multiplicity, and T is a non singular matrix such that

$$T = (u_1, u_2, \dots, u_n).$$

Example 5

Consider the system

$$Y'(x) = \{C + R(x)\} Y(x) \quad (7.2)$$

where C an $n \times n$ constant matrix with n linearly independent eigenvectors u_k . Let the $n \times n$ matrix $R(x)$ satisfies

$$\int_a^\infty \|R(x)\| dx < \infty$$

proves that (7.2) has solutions Y_k ($1 \leq k \leq n$) with the asymptotic form

$$y_k(x) = \{u_k + o(1)\} \exp(\lambda_k x)$$

where λ_k are the eigenvalues of the matrix C .

Solution

We will use the transformation

$$Y = TZ$$

Where $T = (u_1, u_2, \dots, u_n)$, and u_k are the eigenvectors .

So

$$Y' = TZ' = CTZ + RTZ$$

$$\Rightarrow Z' = T^{-1}CTZ + T^{-1}RTZ$$

$$\therefore Z' = DZ + R_1Z \quad (7.3)$$

where

$$D = dg(\lambda_1, \lambda_2, \dots, \lambda_n),$$

and $R_1 = T^{-1}RT$ (since T is a constant matrix).

therefore,

$$\begin{aligned} \int_a^\infty |R_1(x)| dx &= \int_a^\infty |T^{-1}RT| dx \\ &\Leftrightarrow \int_a^\infty |R(x)| dx < \infty \end{aligned}$$

Therefore, (7.3) is in the Levinson form and the Dichotomy condition (3.13) is clearly satisfied since λ_i are constants ($1 \leq i \leq n$),

so by Levinson Theorem (7.3) has a solution

$$\begin{aligned} Z_k(x) &= \{e_k + o(1)\} \exp\left(\int_a^x \lambda_k(x) dx\right) \\ \Rightarrow Z_k(x) &= \{e_k + o(1)\} \exp(\lambda_k x) \quad (\text{where } 1 \leq k \leq n) \end{aligned}$$

and transforming back to Y by the mean of $T = (u_1, u_2, \dots, u_n)$,

we obtain the corresponding solutions

$$Y_k = TZ_k \Rightarrow Y_k(x) = \{u_k + o(1)\} \exp(\lambda_k x).$$

Example 6

Consider the system

$$Y'(x) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 3x^{-2} & 0 \\ 4x^{-4} & 0 \end{pmatrix} \right\} Y(x) \quad (x > 0)$$

Find the asymptotic form of $Y(x)$.

Solution

Here let

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ \& } R(x) = \begin{pmatrix} 3x^{-2} & 0 \\ 4x^{-4} & 0 \end{pmatrix}$$

Thus

$$\int_a^\infty |R(x)| dx < \infty, \text{ since}$$

$$\int_a^\infty |3x^{-2}| dx < \infty \text{ \& } \int_a^\infty |4x^{-4}| dx < \infty$$

C is diagonal matrix

We will solve the eigen-value problem: $\det(C - \lambda I) = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_1 = 1 \text{ \& } \lambda_2 = -1$$

to find the corresponding eigenvectors

we solve $\det(C - \lambda_i I) u_i = 0$

Now with $\lambda_i = \lambda_1 = 1 \Rightarrow a$ is arbitrary ,say $a=1$ and $b=0$

$\therefore u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is eigenvector corresponding to $\lambda_1 = 1$

Now with $\lambda_i = \lambda_2 = -1 \Rightarrow b$ is arbitrary ,say $b=1$ and $a=0$

$\therefore u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is eigenvector corresponding to $\lambda_2 = -1$

Therefore, as an application of example 5 we will have

$$Y_k(x) = \{u_k + o(1)\} \exp(\lambda_k x) \quad (k=1,2)$$

$$Y_1(x) = \{e_1 + o(1)\} \exp(x)$$

$$Y_2(x) = \{e_2 + o(1)\} \exp(-x)$$

are the asymptotic solutions for (7.2).

Note: This question can be solved directly as application of Levinson Theorem, since C is a diagonal matrix with λ_1, λ_2 are constants.

Example 7

Consider the system

$$Y'(x) = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} f & 0 \\ 3f & 4f \end{pmatrix} \right\} Y(x)$$

where $f(x) = kg\{1 + o(1)\} \quad (x \rightarrow \infty)$

such that $\int_a^\infty |g(t)| dt < \infty$

Show that the Levinson's theorem is true and find the asymptotic form of $Y(x)$. Give an example for this system.

Solution

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \text{ \& } R(x) = \begin{pmatrix} f & 0 \\ 3f & 4f \end{pmatrix}$$

C here is a diagonal matrix with $\lambda_1=2, \lambda_2=4$
which satisfies dichotomy condition (3.1)

$$\operatorname{Re}(\lambda_1 - \lambda_2) = 2 - 4 = -2 < 0$$

$$\int_a^\infty |R(x)| \, dx < \infty \Leftrightarrow \int_a^\infty |f(x)| \, dx < \infty$$

$$\Leftrightarrow \int_a^\infty |k(1 + o(1))g(x)| \, dx < \infty \Leftrightarrow \int_a^\infty |g(x)| \, dx < \infty$$

So Levinson theorem is true. Thus

$$Y_k = \{e_k + o(1)\} \exp(\lambda_k x) \quad (\text{where } k=1,2)$$

$$Y_1 = \{e_1 + o(1)\} \exp(2x)$$

$$Y_2 = \{e_2 + o(1)\} \exp(4x)$$

are solutions for our system.

As an example for this system

$$f(x) = kx^{-2}\{1 + o(1)\} \text{ as } x \rightarrow \infty$$

$$\text{where } g(x) = x^{-2} \Rightarrow g(x) \in L(a, \infty)$$

2.4 The Second-Order Differential Equation

The Liouville-Green Asymptotic Formulae

We will discuss the familiar second-order equation

$$(p(x)y'(x))' - q(x)y(x) = 0$$

Where the equation is written here in the usual Sturm -Liouville form, we emphasize that there is no assumption that p and q are necessarily real-valued. The main asymptotic theorems in chapter one can be applied after expressing (8.1) as first-order system .

Example 8

Consider the second-order DE.

$$(a) \quad (p(x)y'(x))' - q(x)y(x) = 0 \quad (8.1)$$

Write (8.1) as a first-order system

$$Y'(x) = A(x)Y(x) \quad (8.2)$$

Find T matrix such that the transformation

$$Y = TZ \quad (8.3)$$

Transform (8.2) into the system

$$Z' = (\Lambda + R)Z \quad (8.4)$$

where $\Lambda(x) = \text{dg} \left\{ \left(\frac{q}{p} \right)^{1/2}, - \left(\frac{q}{p} \right)^{1/2} \right\}$

and

$$R(x) = -\frac{1}{4} \frac{(pq)'}{pq} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(b) Now let p and q be nowhere zero and $C^{(1)}[a, \infty)$

$$\text{let } (pq)' / pq = o\{(p/q)^{1/2}\} \quad (x \rightarrow \infty) \quad (8.5)$$

$$\text{and let } \{p^{-1/2} q^{-3/2} (pq)'\}' \in L(a, \infty) \quad (8.6)$$

let $\text{Re}(\frac{q}{p} + r^2)^{1/2}$ have one sign in $[a, \infty)$.

$$\text{where } r = \frac{1}{4} (pq)' / pq$$

Prove that (8.1) has solutions y_1 and y_2 such that

$$y_1 \sim (pq)^{-1/4} \exp \left(\int_a^x (q/p + r^2)^{1/2} dt \right), \quad (8.8)$$

$$py_1' \sim (pq)^{1/4} \exp \left(\int_a^x (q/p + r^2)^{1/2} dt \right), \quad (8.9)$$

With similar formula for y_2 containing $- (...)^{1/2}$ in the exponential term.

Solution

(a) We consider the system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & p^{-1} \\ q & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\text{Let } A = \begin{pmatrix} 0 & p^{-1} \\ q & 0 \end{pmatrix}$$

(i) let A be expressed in its diagonal form

$$\begin{aligned}\text{solve } \det (A-\lambda I)=0 &\Rightarrow \lambda_1 = \sqrt{p^{-1}q} \\ \lambda_2 &= -\sqrt{p^{-1}q}\end{aligned}$$

To find the corresponding eigenvectors u_i we set $Au_i=\lambda_i u_i$

we get

$$u_1 = \begin{pmatrix} 1 \\ \sqrt{p^{-1}q} \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ -\sqrt{p^{-1}q} \end{pmatrix}$$

using the transformation

$$Y=TZ$$

where

$$T = \begin{pmatrix} 1 & 1 \\ (pq)^{1/2} & -(pq)^{1/2} \end{pmatrix},$$

we get

$$\therefore Z' = (T^{-1}AT - T^{-1}T')Z$$

$$\Lambda = T^{-1}AT$$

$$\therefore R = -T^{-1}T'$$

$$Z'=(\Lambda+R)Z$$

$$\text{where } \Lambda(x)=\text{dg} \left\{ \left(\frac{q}{p} \right)^{1/2}, -\left(\frac{q}{p} \right)^{1/2} \right\}$$

and we need to show

$$R(x) = -\frac{1}{4} \frac{(pq)'}{pq} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

To get T^{-1} and T'

$$T' = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \frac{d}{dx} \left(\sqrt{p^{-1}q} \right)$$

$$T^{-1} = -\frac{1}{2} (p^{-1}q)^{-1/2} \begin{pmatrix} -\sqrt{p^{-1}q} & -1 \\ -\sqrt{p^{-1}q} & 1 \end{pmatrix}$$

Simplifying $\frac{d}{dx} \left(\sqrt{p^{-1}q} \right)$ in $T' \Rightarrow$

$$\text{Thus } -T^{-1}T' = \left\{ \frac{pq' + qp'}{-4pq} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\}$$

$$R = -T^{-1}T' = \left\{ \frac{(pq)'}{-4pq} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\}$$

(b) check the Eastham's Theorem condition, which are

1-distinct eigenvalues

2- $\{\lambda_1 - \lambda_2\}^{-1} R(x) \rightarrow 0$ as $x \rightarrow \infty$

3- $[(\lambda_1 - \lambda_2)^{-1} R(x)]' \in L(a, \infty)$

4- Dichotomy condition hold for eigenvalues of $\Lambda + R$

2&3 hold using the given conditions (8.5) & (8.6) to find the eigenvalues of

$$\Lambda + R = C$$

$$C = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} - \frac{1}{4} \frac{(pq)'}{pq} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} \lambda_1 - r & r \\ r & \lambda_2 - r \end{pmatrix} \text{ where } r = \frac{1}{4} \frac{(pq)'}{pq}$$

$$\det(C - \mu I) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda_1 - r - \mu & r \\ r & \lambda_2 - r - \mu \end{vmatrix} = 0$$

$$\Rightarrow \mu = -r \pm (r^2 + q/p)^{1/2}$$

Now check the dichotomy condition

$$Re(\mu_1 - \mu_2) = Re 2(r^2 + q/p)^{1/2}$$

$$\Rightarrow \int_t^x Re 2(r^2 + q/p)^{1/2} du \geq K_1$$

or

$$\int_t^x Re 2(r^2 + q/p)^{1/2} du \leq K_2$$

Thus by Eastham's Theorem

$Z' = (\Lambda + R)Z$ has a solution of the form

$$Z_k = (e_k + o(1)) \exp \int_a^x \mu_k(t) dt$$

$$\therefore Z_1 = (e_1 + o(1)) \exp \int_a^x (-r + (r^2 + q/p)^{1/2} dt$$

$$\therefore Z_2 = (e_2 + o(1)) \exp \int_a^x (-r - (r^2 + q/p)^{1/2} dt$$

$$\therefore Y = TZ$$

$$Y_1 = \begin{pmatrix} 1 + o(1) \\ P\lambda_1 + o(1) \end{pmatrix} \exp \left\{ \int_a^x (-r + (r^2 + q/p)^{1/2}) dt \right\}$$

$$Y_2 = \begin{pmatrix} 1 + o(1) \\ P\lambda_2 + o(1) \end{pmatrix} \exp \left\{ \int_a^x (-r - (r^2 + q/p)^{1/2}) dt \right\}$$

From Y_1 , first component given by

$$y_1 = (1 + o(1)) \exp\left\{\int_a^x (-r + (r^2 + q/p)^{1/2}) dt\right\}$$

$$y_1 \sim \exp\left\{\int_a^x (-r + (r^2 + q/p)^{1/2}) dt\right\}$$

$$y_1 \sim \exp\left\{\int_a^x (r^2 + q/p)^{1/2} dt\right\} \cdot \exp\left\{\int_a^x -r dt\right\}$$

$$\text{where } r = \frac{1}{4} \frac{(pq)'}{pq}.$$

$$\text{So } \exp\left\{\int_a^x -r dt\right\} = \text{const}(pq)^{-1/4}$$

$$\text{Therefore } y_1 \sim (pq)^{-1/4} \exp\left\{\int_a^x (r^2 + q/p)^{1/2} dt\right\}$$

And so

$$py_1' = (p\lambda_1 + o(1)) \exp\left\{\int_a^x (r^2 + q/p)^{1/2} dt\right\} \cdot \exp\left\{\int_a^x -r dt\right\},$$

$$\text{Where } \lambda_1 = (p^{-1}q)^{1/2}$$

$$py_1' = (p(p^{-1}q)^{1/2} + o(1)) \exp\left\{\int_a^x (r^2 + q/p)^{1/2} dt\right\} \cdot \text{const}(pq)^{-1/4},$$

$$py_1' \sim (pq)^{1/4} \exp\left\{\int_a^x (r^2 + q/p)^{1/2} dt\right\}$$

Similarly for y_2

$$y_2 \sim (pq)^{-1/4} \exp\left\{-\int_a^x (r^2 + q/p)^{1/2} dt\right\}$$

$$(py_2') \sim (pq)^{1/4} \exp\left\{-\int_a^x (r^2 + q/p)^{1/2} dt\right\}$$

Example 9

Let p and q as in exercise 8(a) and let (8.5) and (8.6) of Ex 8(b) hold. In addition, let

$$p^{-3/2}q^{-5/2}(pq')^2 \in L(a, \infty) \quad (8.10)$$

Let

$$\operatorname{Re}(q/p)^{1/2} \text{ have one sign in } [a, \infty) \quad (8.11)$$

Prove that the solution of the D.E. (8.1) in Ex. (8) has solutions y_1 and y_2 such that

$$y_1 \sim (pq)^{-1/4} \exp\left\{\int_a^x (q/p)^{1/2} dt\right\}, \quad (8.12)$$

$$(py_1') \sim (pq)^{1/4} \exp\left\{\int_a^x (q/p)^{1/2} dt\right\}. \quad (8.13)$$

With similar formula for y_2 containing $-(q/p)^{1/2}$ in the exponential term.

Solution

From ex.8 we put equation (8.1) in the form

$$Z' = (\Lambda + R)Z$$

Where $Y = TZ$

$$\Lambda = dg((p/q)^{1/2}, -(p/q)^{1/2}), T = \begin{pmatrix} 1 & 1 \\ (p/q)^{1/2} & -(p/q)^{1/2} \end{pmatrix}$$

$$\text{and } R = -\frac{1}{4} \frac{(pq)'}{pq} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \text{ and } Y = \begin{pmatrix} y \\ py' \end{pmatrix}$$

$$\lambda_i - \lambda_j = \pm 2((q/p)^{1/2}) \neq 0 \text{ for } i, j = 1, 2 \text{ and } i \neq j$$

Since p and q are nowhere zero

$$\therefore \lambda_1 - \lambda_2 \neq 0 \quad \forall x \in [a, \infty)$$

$$\lambda_i - \lambda_j = \pm 2((q/p)^{1/2} \neq 0 \text{ for } i, j=1, 2 \& i \neq j$$

$$\lambda_1 - \lambda_2 \neq 0 \quad \forall x \in [a, \infty)$$

check the Eastham's Theorem condition

(i) condition (1) holds clearly .

$$\begin{aligned} (ii) [\{\lambda_1(x) - \lambda_2(x)\}^{-1} R(x)] &= \{2(q/p)^{1/2}\}^{-1} \left\{ -\frac{1}{4} \frac{(pq)'}{pq} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\} \\ &= -\frac{1}{8} (q/p)^{-1/2} o(q/p)^{1/2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

Using (8.5)

$$[\{\lambda_1(x) - \lambda_2(x)\}^{-1} R(x)] = -\frac{1}{8} o(1) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = o(1) \rightarrow 0 \text{ as } x \rightarrow \infty$$

So condition (2) is also hold .

$$\begin{aligned} (iii) [\{\lambda_1(x) - \lambda_2(x)\}^{-1} R(x)]' &= [\{2(q/p)^{1/2}\}^{-1} \left\{ -\frac{1}{4} \frac{(pq)'}{pq} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\}]' \\ &= -\frac{1}{8} \{p^{-1/2} q^{-3/2} (pq)'\}' \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \in L(a, \infty) \text{ (by (8.6))} \end{aligned}$$

and in a similar way as in Ex.(8), we can find the eigenvalues of

$$\Lambda(x) + R(x)$$

Which are

$$\mu_1 = -r + ((q/p) + r^2)^{1/2}$$

$$\mu_2 = -r - ((q/p) + r^2)^{1/2}$$

$$\text{where } r = \frac{1}{4} \frac{(pq)'}{pq}$$

Therefore,

$$\operatorname{Re}(\mu_1 - \mu_2) = 2 \operatorname{Re}(q/p) + r^2)^{1/2}$$

On using the binomial theorem ,we get

$$\begin{aligned} (q/p + r^2)^{1/2} &= (q/p)^{1/2} (r^2)^0 + 1/2 (q/p)^{1/2-1} (r^2)^1 \\ &+ \frac{(1/2)(1/2-1)}{2!} (q/p)^{1/2-2} (r^2)^2 + \frac{(1/2)(1/2-1)(1/2-2)}{3!} (q/p)^{1/2-3} (r^2)^3 + \dots \end{aligned}$$

Consequently, we write

$$\begin{aligned} (q/p + r^2)^{1/2} &= (q/p)^{1/2} + \frac{1}{2} (q/p)^{-1/2} r^2 - \frac{1}{8} (q/p)^{-3/2} r^4 \\ &+ \frac{1}{16} (q/p)^{-5/2} r^6 - \dots \end{aligned}$$

$$\begin{aligned} &= (q/p)^{1/2} + \frac{1}{2^5} (q/p)^{-1/2} \left(\frac{(pq)'}{(pq)} \right)^2 - \frac{1}{2^{11}} (q/p)^{-3/2} \left(\frac{(pq)'}{(pq)} \right)^4 \\ &+ \frac{1}{2^{16}} (q/p)^{-5/2} \left(\frac{(pq)'}{(pq)} \right)^6 - \dots \end{aligned}$$

$$\begin{aligned} &= (q/p)^{1/2} + \frac{1}{2^5} p^{-3/2} q^{-5/2} ((pq)')^2 - \frac{1}{2^{11}} p^{-5/2} q^{-11/2} ((pq)')^4 \\ &+ \frac{1}{2^{16}} p^{-7/2} q^{-17/2} ((pq)')^6 - \dots \end{aligned}$$

$$= (q/p)^{1/2} + \frac{1}{2^5} p^{-3/2} q^{-5/2} ((pq)')^2 \left[1 - \frac{1}{2^6} p^{-1} q^{-3} ((pq)')^2 + \frac{1}{2^{11}} p^{-2} q^{-6} ((pq)')^4 - \dots \right]$$

$$\text{let } Z = \frac{1}{2^6} p^{-1} q^{-3} ((pq)')^2 \text{ then}$$

$$(q/p + r^2)^{1/2} = (q/p)^{1/2} + \frac{1}{2^5} p^{-3/2} q^{-5/2} ((pq)')^2 [1 - Z + 2Z^2 - 4Z^3 + \dots]$$

$[1 - Z + 2Z^2 - 4Z^3 + \dots]$ is a geometric power series with

$$a = -Z, r = -2Z \text{ and } \sum = \frac{a}{1-r}$$

$$[1 - Z + 2Z^2 - 4Z^3 + \dots] = 1 + \frac{-Z}{1+2Z} = \frac{1+Z}{1+2Z} \quad \text{if } -1 < Z < 1$$

and to show that $-1 < Z < 1$

we have

$$\begin{aligned} Z &= \frac{1}{2^6} p^{-1} q^{-3} ((pq)')^2 = \frac{1}{2^6} \frac{((pq)')^2}{pq^3} \\ &= \frac{1}{2^6} \frac{p}{q} \left(\frac{(pq)'}{pq} \right)^2 = \frac{1}{2^6} \frac{p}{q} o \left\{ \left(\frac{q}{p} \right)^{1/2} \right\}^2 = o(1). \end{aligned}$$

$$\text{Since } \frac{(pq)'}{pq} = o \left\{ \left(\frac{q}{p} \right)^{1/2} \right\} \text{ using (8.5)}$$

So

$$(q/p + r^2)^{1/2} = (q/p)^{1/2} + \frac{1}{2^5} p^{-3/2} q^{-5/2} ((pq)')^2 \left[\frac{1+Z}{1+2Z} \right],$$

where $Z = o(1)$.

Therefore,

$$\frac{1}{2^5} \left[\frac{1+Z}{1+2Z} \right] = \frac{1}{2^5} \left[\frac{1+o(1)}{1+2o(1)} \right] = o(1).$$

Thus

$$(q/p + r^2)^{1/2} = (q/p)^{1/2} + o(p^{-3/2}q^{-5/2}((pq)')^2).$$

We will have

$$\begin{aligned} \operatorname{Re}(\mu_1 - \mu_2) &= 2 \operatorname{Re}(q/p + r^2)^{1/2} \\ &= 2 \operatorname{Re}(q/p)^{1/2} + 2 \operatorname{Re}\{o(p^{-3/2}q^{-5/2}((pq)')^2)\} \\ &= f(x) + g(x) \end{aligned}$$

where

$$f(x) = 2 \operatorname{Re}(q/p)^{1/2},$$

$$g(x) = 2 \operatorname{Re}\{o(p^{-3/2}q^{-5/2}((pq)')^2)\}.$$

Note that

$$g(x) \in L(a, \infty) \text{ (by (8.10))}.$$

Therefore, L which is dichotomy condition is satisfied for eigenvalues of

$$\Lambda(x) + R(x).$$

So by Eastham's Theorem we have

$$Z_k(x) = \{e^k + o(1)\} \exp\left(\int_a^x \mu_k(t) dt\right)$$

$$\text{and } Y_k(x) = TZ_k(x)$$

Then we use similar process as in Ex. 8

$$y_1 \sim (pq)^{-1/4} \exp\left\{\int_a^x (r^2 + q/p)^{1/2} dt\right\}$$

$$py_1' \sim (pq)^{1/4} \exp\left\{\int_a^x (r^2 + q/p)^{1/2} dt\right\}$$

$$y_2 \sim (pq)^{-1/4} \exp\left\{-\int_a^x (r^2 + q/p)^{1/2} dt\right\}$$

$$(py_2') \sim (pq)^{1/4} \exp\left\{-\int_a^x (r^2 + q/p)^{1/2} dt\right\}$$

Now consider

$$\begin{aligned} \int_a^x (q/p + r^2)^{1/2} dt &= \int_a^x (q/p)^{1/2} dt \\ &\quad + \int_a^x o(p^{-3/2} q^{-5/2} (pq')^2) dt \end{aligned}$$

$$\because p^{-3/2} q^{-5/2} (pq')^2 \in L(a, \infty) \quad (\text{by (8.5)})$$

We have

$$\int_a^x o(p^{-3/2} q^{-5/2} (pq')^2) dt < \infty \quad \text{as } x \rightarrow \infty$$

Thus

$$\int_a^x o(p^{-3/2} q^{-5/2} (pq')^2) dt \leq K \int_a^x (p^{-3/2} q^{-5/2} (pq')^2) dt$$

Therefore,

$$\int_a^x o(p^{-3/2} q^{-5/2} (pq')^2) dt = L + o(1) \quad \text{as } x \rightarrow \infty$$

Where L is a constant .Therefore,

$$\exp\left\{\int_a^x (q/p + r^2)^{1/2} dt\right\} = \exp\left\{\int_a^x (q/p)^{1/2} dt\right\} e^L e^{o(1)}$$

$$\because e^L \text{ is a constant and } e^{o(1)} = 1 + o(1)$$

$$\therefore \exp\left\{\int_a^x (q/p + r^2)^{1/2} dt\right\} = \exp\left\{\int_a^x (q/p)^{1/2} dt\right\} C^a$$

where $C^a = \text{constant}(1+o(1))$

$k=1,2$ on Y_k a constant multiple adjusting by

$$y_1 \sim (pq)^{-1/4} \exp\left\{\int_a^x (q/p)^{1/2} dt\right\}$$

$$py'_1 \sim (pq)^{1/4} \exp\left\{\int_a^x (q/p)^{1/2} dt\right\}$$

With similar formula for y_2 containing $-(q/p)^{1/2}$ in the exponential term

2.5 The Third Order Differential Equation

In this section we will discuss the third order differential equation

$$y'''(x) - \{q(x)\}y(x) = 0 \quad (0 \leq x < \infty), \quad (9.1)$$

which can be transform into the form

$$Y'(x) = A(x)Y(x).$$

If we let

$$y_1 = y \rightarrow y'_1 = y',$$

$$y_2 = y' \rightarrow y'_2 = y'' = y_3,$$

$$y_3 = y'' \rightarrow y'''_3 = \{q(x)\}y_1.$$

$$\text{Now } Y(x) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

so

$$Y'(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ q(x) & 0 & 0 \end{pmatrix} Y(x) \quad (9.2)$$

Example 10

Consider the system

$$Y'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ q(x) & 0 & 0 \end{bmatrix} Y(x) \quad (9.3)$$

Where $q(x) = 1 + kx^{-\delta}$ with constants k and δ , $\delta > 0$. Show that the system can be transformed into the form

$$Z'(x) = \{A(x) + R(x)\}Z(x)$$

with $A(x)$ is a diagonal matrix, and deduce the asymptotic form of three linearly independent solutions $Y(x)$

Solution

We find the eigenvalues of A_0

$$\text{where } A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ q(x) & 0 & 0 \end{pmatrix}$$

$$\lambda_1 = q^{1/3},$$

$$\lambda_2 = -q^{1/3}w$$

$$\text{and } \lambda_3 = -q^{1/3}\bar{w}$$

which form distinct eigenvalues.

where

$$w = \frac{1+i\sqrt{3}}{2}$$

To find the eigenvectors we set

$$Au_i = \lambda_i u_i$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ q(x) & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \lambda_i \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\Rightarrow u_2 = \lambda_i u_1$$

$$u_3 = \lambda_i u_2 \quad ,$$

$$qu_1 = \lambda_i u_3$$

$$\text{take } u_1 = 1 \Rightarrow u_2 = \lambda_i \Rightarrow u_3 = \lambda_i^2$$

$$T = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ q^{1/3} & -wq^{1/3} & -\overline{w}q^{1/3} \\ q^{2/3} & -\overline{w}q^{2/3} & -wq^{2/3} \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^{1/3} & 0 \\ 0 & 0 & q^{2/3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -w & -\overline{w} \\ 1 & -\overline{w} & -w \end{pmatrix}$$

Let $T = D\Omega$ where

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^{1/3} & 0 \\ 0 & 0 & q^{2/3} \end{pmatrix} \text{ and } \Omega = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -w & -\bar{w} \\ 1 & \bar{w} & w \end{pmatrix}$$

Take $Y = TZ \Rightarrow Y' = T'Z + TZ'$

$$\Rightarrow Z' = (T^{-1}A_0T - T^{-1}T')Z$$

$$Z' = (A + R)Z \quad (9.4)$$

(where $A = T^{-1}A_0T$ and $R = -T^{-1}T'$)

$$R = -\Omega^{-1}D^{-1}D'\Omega$$

$$\int_a^\infty |R| \, dx < \infty \Leftrightarrow$$

$$\int_a^\infty |D^{-1}D'| \, dx < \infty \Leftrightarrow \int_a^\infty \left| \frac{q'}{q} \right| \, dx < \infty \text{ where } \left| \frac{q'}{q} \right| \leq \text{constant} \cdot x^{-\delta-1}$$

$$\text{Also } \operatorname{Re}(\lambda_1 - \lambda_2) = \frac{3}{2} q^{1/3}$$

$$\operatorname{Re}(\lambda_2 - \lambda_3) = 0$$

$$\operatorname{Re}(\lambda_1 - \lambda_3) = \frac{3}{2} q^{1/3}$$

$$\operatorname{Re}(\lambda_i - \lambda_j) \geq 0 \forall i \neq j \text{ or}$$

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