

University of Bahrain

Schatten Classes on Banach Spaces

By

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A Thesis Submitted to the Department of Mathematics
as a Partial Fulfillment of the Requirement for the
Degree of Master of Science

Under the Supervision of

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June 1996

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX



12 January 1997

The Thesis Defence Committee considers the thesis entitled:

"Schatten Classes on Banach Spaces"

submitted by *Mariam Al-Hawaj* to the Department of Mathematics
satisfactory and acceptable for the Master of Science Degree in
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ACKNOWLEDGEMENT

I would like to express my sincere thanks to Professor Roshdi Khalil for his supervision. I would also like to thank my family, friends and especially my fiance for their real support.

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PREFACE

Let H be a separable Hilbert space and $\mathcal{L}(H)$ the space of bounded linear operators on H . The Schatten class of index p on H is the class of operators: $\mathcal{C}_p(H) = \{T \in \mathcal{L}(H) : \sup \sum_{n=1}^{\infty} | \langle T e_n, f_n \rangle |^p < \infty\}$ where the supremum is taken over all orthonormal sequences $(e_n), (f_n)$ in H .

The class $\mathcal{C}_p(H)$ was generalized to Banach spaces via $(p, 2)$ -summing operators, as it is done by Pietsch in his book "Operator Ideals".

The object of this thesis is to give a generalization of $\mathcal{C}_p(H)$ to Banach space case, via p -orthonormal sequences.

The thesis consists of three chapters. In chapter I, we present the basic material on $\mathcal{C}_p(H)$. Orthogonality in Banach spaces is presented in chapter II, where we present different types of orthogonality known in the literature.

The main work in this thesis is presented in chapter III, where we introduce new classes of operators " $\mathcal{C}_r(X, Y)$ " and study the basic properties of these classes. The case $X = Y = l^p$ is given some attention.

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CHAPTER I

SCHATTEN CLASSES ON HILBERT SPACES

This chapter deals with basic materials needed in our work. It consists of three sections. Section 1 includes basic definitions and notations. Compact operators is the topic of section 2, while section 3 deals with Schatten p -classes.

1.1 Basic Definitions and Notations.

Throughout this thesis H denotes a separable Hilbert space. For $x, y \in H$, $\langle x, y \rangle$ denotes the inner product of x and y . For $x \in H$, $\|x\|^2 = \langle x, x \rangle$. By an orthonormal sequence in H , we mean a sequence (x_n) such that

$$\langle x_n, x_m \rangle = \begin{cases} 0 & \text{for } n \neq m \\ 1 & \text{for } n = m. \end{cases}$$

The best examples of separable Hilbert spaces are:

(1) $l^2 = \left\{ (x_n) : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$. For $x = (x_n)$ and $y = (y_n)$ in l^2 ,

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \bar{y}_n$$

and

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2}$$

(2) $\mathcal{L}^2(I)$ = the space of Lebesgue square integrable functions (equivalence classes) on $I = [0, 1]$ with values in $\mathbb{R}$ (or $\mathbb{C}$). For $f, g \in \mathcal{L}^2(I)$,

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt,$$

and

$$\|f\| = \left(\int_0^1 |f(t)|^2 dt \right)^{1/2}$$

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Theorem 1.2.4. $\mathcal{K}(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$.

Proof: See [13, p.98]. $\square$

A non trivial example of a compact operator is the following :

Example 1.2.5. Let $F \in \mathcal{C}(I \times I)$, $I = [0, 1]$. Define

$$T : \mathcal{C}(I) \rightarrow \mathcal{C}(I)$$

by

$$T(f)x = \int_0^1 F(x, y)f(y)dy$$

For the proof that this is compact, we refer to [8, p.455]. $\square$

Lemma 1.2.6. Let $T \in \mathcal{K}(X)$. If $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and $T - \lambda I$ is one-to-one, then $\mathcal{R}(T - \lambda I)$ is closed.

Proof: Let (x_n) be a sequence in X , such that $(T - \lambda I)x_n$ converge to y , say. We claim that: $y = (T - \lambda I)x$ for some $x \in X$.

There are two cases for x_n :

(i) Let (x_n) be bounded. Since T is compact, then (Tx_n) has a convergent subsequence (Tx_{n_k}) converges to z , say. But $(T - \lambda I)x_{n_k}$ converges to y , implies that (λx_{n_k}) converges. Therefore, x_{n_k} converges to x , say. Now, since $T - \lambda I$ is continuous we get: $(T - \lambda I)x_{n_k}$ converges to $(T - \lambda I)x$. Hence, $y = (T - \lambda I)x$.

(ii) Let (x_n) be a sequence with no bounded subsequence. Then $\|x_n\|$ goes to ∞ . Now we have: $(T - \lambda I)x_n$ converges to y and $\frac{1}{\|x_n\|}$ converges to 0, imply that $\frac{(T - \lambda I)x_n}{\|x_n\|}$ converges to 0. But since T is compact and $\frac{x_n}{\|x_n\|}$ is bounded, then $\frac{Tx_n}{\|x_n\|}$ converges to w , say and $(\lambda \frac{x_n}{\|x_n\|})$ converges to x , say. Now we have: $\frac{(T - \lambda I)x_n}{\|x_n\|}$ converges to $(T - \lambda I)x$. Hence, $(T - \lambda I)x = 0$. That implies that $T - \lambda I$ is not one-to-one, which contradicts the condition on T . Hence (x_n) must contain a bounded subsequence. Hence the required. $\square$

Theorem 1.2.7. Let X be Banach spaces and $T \in \mathcal{K}(X)$. Then :

- (i) $\sigma(T)$ is countable and contains only eigenvalues of T together with 0.
- (ii) $\mathcal{R}(T - \lambda I)$ is closed for all $\lambda \neq 0$.

Proof:

(i) We need to prove that: If $\lambda \in \sigma(T)$ and $\lambda \neq 0$, then λ is an eigenvalue of T . Let $\lambda \in \sigma(T)$. Then $T - \lambda I$ has no inverse. If $T - \lambda I$ is not one-to-one, then there is $x \in \ker(T - \lambda I)$ and so $(T - \lambda I)x = 0$. This implies that $Tx = \lambda x$ and λ is an eigenvalue. Now, if $T - \lambda I$ is one-to-one, then by Lemma 1.2.5, we get: $\mathcal{R}(T - \lambda I)$ is closed. If $\mathcal{R}(T - \lambda I) = X$ then the Open Mapping Theorem implies that $T - \lambda I$ is invertible which contradicts the assumption $\lambda \in \sigma(T)$. If $\mathcal{R}(T - \lambda I) = W \subset X$. Then by the Hahn-Banach Theorem, there is $x^* \in X^*$ such that $x^*(\omega) = 0$ for all $\omega \in W$. Hence, $\langle (T - \lambda I)x, x^* \rangle = 0$ for all $x \in X$. Then $\langle x, (T^* - \bar{\lambda}I)x^* \rangle = 0$. But again if $x(f) = 0$ for all $x \in X$ and some $f \in X^*$, then $f = 0$. So $(T^* - \bar{\lambda}I)x^* = 0$. Then we have $T^*x^* - \bar{\lambda}Ix^* = 0$, which implies that $\bar{\lambda}$ is an eigenvalue for T^* . But it is known, [1, p.344] that $\rho(T) = \overline{\rho(T^*)}$ and $\sigma(T) = \overline{\sigma(T^*)}$. Hence λ is an eigenvalue for T .

For the proof of the rest of (i) we refer to [8, p.421].

(ii) Suppose that $(T - \lambda I)(X)$ is not closed. Then there is a $y \in \overline{(T - \lambda I)(X)}$, $y \notin (T - \lambda I)(X)$ and a sequence (x_n) in X such that

$$y_n = (T - \lambda I)x_n \rightarrow y.$$

Since $(T - \lambda I)(X)$ is a vector space, $0 \in (T - \lambda I)(X)$. But $y \notin (T - \lambda I)(X)$, so that $y \neq 0$. This implies $y_n \neq 0$ and $x_n \notin \mathcal{N}(T - \lambda I)$ for all sufficiently large n . Without loss of generality we may assume that this holds for all n . Since $\mathcal{N}(T - \lambda I)$ is closed, the distance δ_n from x_n to $\mathcal{N}(T - \lambda I)$ is positive, that is,

$$\delta_n = \inf_{z \in \mathcal{N}(T - \lambda I)} \|x_n - z\| > 0.$$

By the definition of an infimum there is a sequence (z_n) in $\mathcal{N}(T - \lambda I)$ such that

$$a_n = \|x_n - z_n\| < 2\delta_n.$$

Now we need to show that

$$a_n = \|x_n - z_n\| \rightarrow \infty \quad \text{as } n \rightarrow \infty. \quad (1)$$

Suppose (1) does not hold. Then $(x_n - z_n)$ has a bounded subsequence. Since T is compact, it follows from Theorem 1.2.3 that $(T(x_n - z_n))$ has a convergent subsequence. Now from $\lambda \neq 0$ we have $I = \lambda^{-1}[T - (T - \lambda I)]$. Using $(T - \lambda I)z_n = 0$ [remember that $z_n \in \mathcal{N}(T - \lambda I)$], we thus obtain

$$\begin{aligned} x_n - z_n &= \frac{1}{\lambda}[T - (T - \lambda I)](x_n - z_n) \\ &= \frac{1}{\lambda}[T(x_n - z_n) - (T - \lambda I)x_n]. \end{aligned}$$

$(T(x_n - z_n))$ has a convergent subsequence and $((T - \lambda I)x_n)$ converges since $y_n = (T - \lambda I)x_n \rightarrow y$; hence $(x_n - z_n)$ has a convergent subsequence, say, $x_{n_k} - z_{n_k} \rightarrow v$. Since T is compact, T is continuous and so is $T - \lambda I$. Hence

$$(T - \lambda I)(x_{n_k} - z_{n_k}) \rightarrow (T - \lambda I)v.$$

Here $(T - \lambda I)z_{n_k} = 0$ because $z_n \in \mathcal{N}(T - \lambda I)$, so that from $y_n \rightarrow y$ we also have

$$(T - \lambda I)(x_{n_k} - z_{n_k}) = (T - \lambda I)x_{n_k} \rightarrow y,$$

Hence $(T - \lambda I)v = y$. Thus $y \in (T - \lambda I)(X)$, which contradicts $y \notin (T - \lambda I)(X)$, mentioned at the beginning of the proof. This contradiction resulted from our assumption that (1) does not hold, so that (1) is now proved. Using again a_n as defined in (1) and setting

$$w_n = \frac{1}{a_n}(x_n - z_n) \quad (2)$$

we have $\|w_n\| = 1$. Since $a_n \rightarrow \infty$ whereas $(T - \lambda I)z_n = 0$ and $((T - \lambda I)x_n)$ converges, it follows that

$$(T - \lambda I)w_n = \frac{1}{a_n}(T - \lambda I)x_n \rightarrow 0. \quad (3)$$

Using again $I = \lambda^{-1}[T - (T - \lambda I)]$, we obtain

$$w_n = \frac{1}{\lambda}[Tw_n - (T - \lambda I)w_n]. \quad (4)$$

Since T is compact and (w_n) is bounded, (Tw_n) has a convergent subsequence. Furthermore, $((T - \lambda I)w_n)$ converges by (3). Hence (4) shows that (w_n) has a convergent subsequence, say

$$w_{n_j} \rightarrow w.$$

A comparison with (3) implies that $(T - \lambda I)w = 0$. Hence $w \in \mathcal{N}(T - \lambda I)$. Since $z_n \in \mathcal{N}(T - \lambda I)$, also

$$u_n = z_n + a_n w \in \mathcal{N}(T - \lambda I).$$

Hence for the distance from x_n to u_n we must have

$$\|x_n - u_n\| \geq \delta_n.$$

Writing u_n out and using (2) and $a_n < 2\delta_n$, we thus obtain

$$\begin{aligned} \delta_n &\leq \|x_n - z_n - a_n w\| \\ &= \|a_n w_n - a_n w\| \\ &= a_n \|w_n - w\| \\ &< 2\delta_n \|w_n - w\|. \end{aligned}$$

Dividing by $2\delta_n > 0$, we have $\frac{1}{2} < \|w_n - w\|$. This contradicts $w_{n_j} \rightarrow w$ and proves the theorem. $\square$

On Hilbert spaces, compact operators have a richer structure.

Theorem 1.2.8. $T : H \mapsto H$ is compact if and only if whenever (x_n) is weakly convergent in H , then $T(x_n)$ is convergent in H .

Proof: See [8, p.410]. $\square$

Theorem 1.2.9. $T \in \mathcal{K}(H)$ if and only if

$$\lim_{n \rightarrow \infty} \langle T e_n, f_n \rangle = 0$$

for all $(e_n), (f_n)$ orthonormal sequences.

Proof: See [12, p.58]. $\square$

Theorem 1.2.10. Let $T \in \mathcal{L}(H)$. The following are equivalent :

- (i) T is compact.
- (ii) There exists a decreasing sequence of positive real numbers (λ_n) which converge to 0, and orthonormal sequences (e_n) and (f_n) in H such that for every $x \in H$:

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle f_n.$$

Proof: See [12, p.71]. $\square$

The representation : $T = \sum_{n=1}^{\infty} \lambda_n e_n \otimes f_n$ is called the Schmidt Representation of T .

One can easily see that if $T = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n$, then $T^* = T$. The converse is also true.

Definition 1.2.11. A bounded linear operator T from X into Y is called an operator of finite rank if $\dim T(X) < \infty$.

The space of all finite rank operators on X is denoted by $\mathcal{F}(X)$.

Lemma 1.2.12. $\mathcal{F}(X, Y) \subseteq \mathcal{K}(X, Y)$, if X is a Banach space.

Proof: Since every bounded sequence in a finite dimensional normed space has a convergent subsequence. $\square$

Lemma 1.2.13.

$$\left\| \sum_{n=1}^{\infty} \lambda_n e_n \otimes f_n \right\| = \|\lambda\|_{\infty} = \sup_n |\lambda_n|$$

where λ_n is a decreasing sequence of positive real numbers.

Proof: Consider

$$\begin{aligned} \|Tx\|^2 &= \left\| \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle f_n \right\|^2 \\ &= \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, e_n \rangle|^2 \\ &\leq \sum_{n=1}^{\infty} \|\lambda_n\|_{\infty}^2 |\langle x, e_n \rangle|^2 \\ &\leq \|\lambda_n\|_{\infty}^2 \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \\ &\leq \|\lambda_n\|_{\infty}^2 \|x\|^2 \end{aligned}$$

This implies that $\|Tx\| \leq \|\lambda_n\|_{\infty} \|x\|$. Hence $\|T\| \leq \|\lambda_n\|_{\infty} = \sup_n |\lambda_n|$. But $\|\lambda_n\|_{\infty} = \sup_n |\lambda_n| = \lambda_1$. Now consider $\|Te_1\| = \left\| \sum_{n=1}^{\infty} \lambda_n \langle e_1, e_n \rangle f_n \right\| = |\lambda_1| = \sup_n |\lambda_n|$. Hence $\|T\| = \sup_n |\lambda_n|$. Hence the required. $\square$

Theorem 1.2.14. $\mathcal{F}(H)$ is dense in $\mathcal{K}(H)$.

Proof: Let $T \in \mathcal{K}(H)$. Then $T = \sum_{n=1}^{\infty} \lambda_n e_n \otimes f_n$. Define an operator T_k of finite rank by $T_k = \sum_{n=1}^k \lambda_n e_n \otimes f_n$. Now by Lemma 1.2.13, we have $\|T - T_k\| = \left\| \sum_{n=k+1}^{\infty} \lambda_n e_n \otimes f_n \right\| = |\lambda_{k+1}|$ which converges to 0, as k goes to ∞ . Hence the required. $\square$

Theorem 1.2.15. $\mathcal{K}(H)$ is a two-sided ideal in $\mathcal{L}(H)$.

Proof: Let $S, R \in \mathcal{L}(H)$ and $T \in \mathcal{K}(H)$. Let F_n be a sequence of finite rank that converge to T . Now consider:

$$\|STR - SF_nR\| = \|S(T - F_n)R\| \leq \|S\|\|T - F_n\|\|R\|$$

Since T is compact, then by Theorem 1.2.14. we have: $\|T - F_n\|$ converge to 0. Hence $\|STR - SF_nR\|$ converge to 0, and therefore $STR \in \mathcal{K}(H)$ since SF_nR is in $\mathcal{F}(H)$. Hence the required. $\square$

1.3 Schatten Classes on H .

In this section, we are going to study a very interesting class of operators on a Hilbert space H , namely the so-called Schatten classes.

Definition 1.3.1. Let H be a separable Hilbert space and let $1 \leq p < \infty$. $\mathcal{C}_p(H)$ is defined to be the set of all bounded linear operators on H such that

$$\sup_{(\theta_n)} \sum_{n=1}^{\infty} |\langle T\theta_n, \theta_n \rangle|^p < \infty$$

where the supremum is taken over all orthonormal sequences (θ_n) .

Theorem 1.3.2. $\mathcal{C}_p(H)$ is a vector space.

Proof: Let $T \in \mathcal{C}_p(H)$. Consider

$$\sum_{n=1}^{\infty} |\langle \lambda T\theta_n, \theta_n \rangle|^p = |\lambda|^p \sum_{n=1}^{\infty} |\langle T\theta_n, \theta_n \rangle|^p.$$

Thus

$$\sup_{(\theta_n)} \sum_{n=1}^{\infty} |\langle \lambda T\theta_n, \theta_n \rangle|^p < \infty.$$

Hence $\lambda T \in \mathcal{C}_p(H)$.

Now, let $T_1, T_2 \in \mathcal{C}_p(H)$. Then using the Minkowski's inequality, we get:

$$\begin{aligned} \left(\sum_{n=1}^{\infty} | \langle (T_1 + T_2)\theta_n, \theta_n \rangle |^p \right)^{1/p} &\leq \left(\sum_{n=1}^{\infty} | \langle T_1\theta_n, \theta_n \rangle + \langle T_2\theta_n, \theta_n \rangle |^p \right)^{1/p} \\ &\leq \left(\sum_{n=1}^{\infty} | \langle T_1\theta_n, \theta_n \rangle |^p \right)^{1/p} \\ &\quad + \left(\sum_{n=1}^{\infty} | \langle T_2\theta_n, \theta_n \rangle |^p \right)^{1/p} \end{aligned}$$

Now, we take the supremum over all orthonormal sequences, on both sides of the inequality, we get:

$$\begin{aligned} \sup_{(\theta_n)} \left(\sum_{n=1}^{\infty} | \langle (T_1 + T_2)\theta_n, \theta_n \rangle |^p \right)^{1/p} &\leq \sup_{(\theta_n)} \left(\sum_{n=1}^{\infty} | \langle T_1\theta_n, \theta_n \rangle |^p \right)^{1/p} \\ &\quad + \sup_{(\theta_n)} \left(\sum_{n=1}^{\infty} | \langle T_2\theta_n, \theta_n \rangle |^p \right)^{1/p} \\ &< \infty \end{aligned}$$

This implies that $T_1 + T_2 \in \mathcal{C}_p(H)$ and hence, $\mathcal{C}_p(H)$ is a vector space. $\square$

Definition 1.3.3. Let T be self-adjoint operator in $\mathcal{L}(H)$. Then T is said to be positive if and only if $\langle Tx, x \rangle \geq 0$ for all $x \in H$. The operator T^*T is positive for any $T \in \mathcal{L}(H)$.

Definition 1.3.4. $|T|$ is defined to be $(T^*T)^{1/2}$, [12].

If T is compact, then $|T|$ is compact and positive. Further, if $T = \sum_{n=1}^{\infty} \lambda_n e_n \otimes f_n$ then $|T| = \sum_{n=1}^{\infty} |\lambda_n| e_n \otimes e_n$. Hence, if T is self-adjoint, then $T = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n$ where λ_n is a sequence converging to 0.

We refer to [12] for more on $|T|$.

Definition 1.3.5. Let $T \in \mathcal{L}(H)$. We say that T is partial isometry if there is a closed subspace M of H such that

$$\|Tx\| = \|x\| \quad \text{for all } x \in M$$

and

$$Tx = 0 \quad \text{for all } x \in M^\perp.$$

Theorem 1.3.6. Let T be any operator in $\mathcal{L}(H)$. Then $T = U|T|$, where U is a partial isometry.

Proof: See [12, p.48]. $\square$

Such a decomposition is called the polar decomposition of T .

Theorem 1.3.7. Suppose that $1 \leq p < \infty$, and T be a compact self-adjoint operator on H . Let (λ_n) be the sequence of non-zero eigenvalues of T . Then $T \in \mathcal{C}_p(H)$ if and only if $(\lambda_n) \in l^p$.

Proof: Let $T \in \mathcal{C}_p(H)$. Since T is self-adjoint, then $T = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n$ where (e_n) is an orthonormal sequence. Consider

$$\begin{aligned} T e_n &= \sum_{j=1}^{\infty} \lambda_j \langle e_n, e_j \rangle e_j \\ &= \lambda_n e_n. \end{aligned}$$

So

$$| \langle T e_n, e_n \rangle |^p = | \langle \lambda_n e_n, e_n \rangle |^p = |\lambda_n|^p.$$

Hence

$$\sum_{n=1}^{\infty} | \langle T e_n, e_n \rangle |^p = \sum_{n=1}^{\infty} |\lambda_n|^p.$$

This implies that $\sum_{n=1}^{\infty} |\lambda_n|^p < \infty$. Hence $(\lambda_n) \in l^p$.

Conversely, let $(\lambda_n) \in l^p$. Consider

$$\begin{aligned} \langle T \theta_n, \theta_n \rangle &= \langle \sum_{i=1}^{\infty} \lambda_i \langle e_i, \theta_n \rangle e_i, \theta_n \rangle \\ &= \sum_{i=1}^{\infty} \lambda_i \langle e_i, \theta_n \rangle \langle e_i, \theta_n \rangle \end{aligned}$$

Then, using Hölder inequality we get:

$$\begin{aligned} | \langle T \theta_n, \theta_n \rangle | &\leq \sum_{i=1}^{\infty} |\lambda_i| | \langle e_i, \theta_n \rangle |^2 \\ &\leq \sum_{i=1}^{\infty} |\lambda_i| | \langle e_i, \theta_n \rangle |^{2/p} | \langle e_i, \theta_n \rangle |^{2/p^*} \\ &\leq \left(\sum_{i=1}^{\infty} |\lambda_i|^p | \langle e_i, \theta_n \rangle |^2 \right)^{1/p} \left(\sum_{i=1}^{\infty} | \langle e_i, \theta_n \rangle |^2 \right)^{1/p^*} \end{aligned}$$

Since (e_i) is an orthonormal sequence, we get from Bessel's inequality that:

$$\sum_{i=1}^{\infty} |\langle e_i, \theta_n \rangle|^2 \leq \|\theta_n\|^2 = 1.$$

Therefore,

$$|\langle T\theta_n, \theta_n \rangle|^p \leq \sum_{i=1}^{\infty} |\lambda_i|^p |\langle e_i, \theta_n \rangle|^2$$

Then

$$\begin{aligned} \sum_{n=1}^k |\langle T\theta_n, \theta_n \rangle|^p &\leq \sum_{n=1}^k \sum_{i=1}^{\infty} |\lambda_i|^p |\langle e_i, \theta_n \rangle|^2 \\ &= \sum_{i=1}^{\infty} \sum_{n=1}^k |\lambda_i|^p |\langle e_i, \theta_n \rangle|^2 \\ &= \sum_{i=1}^{\infty} |\lambda_i|^p \sum_{n=1}^k |\langle e_i, \theta_n \rangle|^2 \\ &\leq \sum_{i=1}^{\infty} |\lambda_i|^p \end{aligned}$$

Now, we have

$$\sum_{n=1}^k |\langle T\theta_n, \theta_n \rangle|^p \leq \|\lambda\|_p^p \quad \text{for all } k.$$

and so, $\sup_{(\theta_n)} \sum_{n=1}^k |\langle T\theta_n, \theta_n \rangle|^p \leq \|\lambda\|_p^p < \infty$. Hence, $T \in \mathcal{C}_p(H)$. $\square$

Theorem 1.3.8. Let $1 \leq p < \infty$. Then $T \in \mathcal{C}_p(H)$ if and only if

$$\sup_{(e_n), (f_n)} \sum_{n=1}^{\infty} |\langle Te_n, f_n \rangle|^p < \infty$$

where the supremum is taken over all orthonormal sequences $(e_n), (f_n)$.

Proof Let $T = \sum_{n=1}^{\infty} \lambda_n \theta_n \otimes g_n$. If $\sup_{(e_n), (f_n)} \sum_{n=1}^{\infty} |\langle Te_n, f_n \rangle|^p < \infty$, then since $|\langle T\theta_n, g_n \rangle|^p = |\lambda_n|^p$ we get: $(\lambda_n) \in l^p$ and $T \in \mathcal{C}_p(H)$ by Theorem 1.3.7. Conversely, let $T \in \mathcal{C}_p(H)$ and $T = U|T|$ be the polar decomposition of T . Then

$$\begin{aligned}
\sum_{j=1}^{\infty} | \langle T e_j, f_j \rangle |^p &= \sum_{j=1}^{\infty} | \langle U|T|e_j, f_j \rangle |^p \\
&= \sum_{j=1}^{\infty} | \langle |T|^{1/2} e_j, |T|^{1/2} U^* f_j \rangle |^p \\
&\leq \sum_{j=1}^{\infty} \| |T|^{1/2} e_j \|^p \| |T|^{1/2} U^* f_j \|^p \\
&\leq \left(\sum_{j=1}^{\infty} \| |T|^{1/2} e_j \|^{2p} \right)^{1/2} \left(\sum_{j=1}^{\infty} \| |T|^{1/2} U^* f_j \|^{2p} \right)^{1/2} \\
&= \left(\sum_{j=1}^{\infty} | \langle |T|^{1/2} e_j, |T|^{1/2} e_j \rangle |^p \right)^{1/2} \\
&\quad \left(\sum_{j=1}^{\infty} | \langle |T|^{1/2} U^* f_j, |T|^{1/2} U^* f_j \rangle |^p \right)^{1/2} \\
&= \left(\sum_{j=1}^{\infty} | \langle |T| e_j, e_j \rangle |^p \right)^{1/2} \left(\sum_{j=1}^{\infty} | \langle |T| U^* f_j, U^* f_j \rangle |^p \right)^{1/2}
\end{aligned}$$

Taking the supremum over all (e_i) and (f_i) we get that:

$\sup_{(e_j), (f_j)} \sum_{j=1}^{\infty} | \langle T e_j, f_j \rangle |^p < \infty$, noting from [12,p.85] that $|T| \in \mathcal{C}_p(H)$ and $(U^* f_j)$ is orthonormal. Hence, the required. $\square$

Definition 1.3.9. Suppose that $1 \leq p < \infty$ and $T \in \mathcal{C}_p(H)$. We set $\|T\|_p$ to denote

$$\sup_{(e_n), (f_n)} \left(\sum_{n=1}^{\infty} | \langle T e_n, f_n \rangle |^p \right)^{1/p}$$

where the supremum is taken over all $(e_n), (f_n)$ orthonormal sequences.

Lemma 1.3.10. $\|T\| \leq \|T\|_p$ for all $T \in \mathcal{C}_p(H)$.

Proof:

$$\begin{aligned} \|T\| &= \sup_{\|x\|=\|y\|=1} | \langle Tx, y \rangle | \\ &\leq \sup_{(x_i), (y_i)} \left(\sum_{i=1}^{\infty} | \langle Tx_i, y_i \rangle |^p \right)^{1/p} \end{aligned}$$

where $(x_i), (y_i)$ are orthonormal sequences. Hence $\|T\| \leq \|T\|_p$. $\square$

Theorem 1.3.11. $\|\cdot\|_p$ is a norm on $\mathcal{C}_p(H)$.

Proof: First, it is clear that $\|T\|_p \geq 0$. And if $T = 0$, it is easy to see that $\|T\|_p = 0$. Now if $\|T\|_p = 0$, then by Lemma 1.3.10, we get $\|T\| = 0$. Hence, $T = 0$. Now consider,

$$\begin{aligned} \|\lambda T\|_p &= \sup_{(e_n), (f_n)} \left(\sum_{n=1}^{\infty} | \langle \lambda T e_n, f_n \rangle |^p \right)^{1/p} \\ &= \sup_{(e_n), (f_n)} \left(\sum_{n=1}^{\infty} |\lambda|^p | \langle T e_n, f_n \rangle |^p \right)^{1/p} \\ &= |\lambda| \sup_{(e_n), (f_n)} \left(\sum_{n=1}^{\infty} | \langle T e_n, f_n \rangle |^p \right)^{1/p} \\ &= |\lambda| \|T\|_p. \end{aligned}$$

Finally, by using Minkowski's inequality we get:

$$\begin{aligned} \left(\sum_{n=1}^{\infty} | \langle (T_1 + T_2) e_n, f_n \rangle |^p \right)^{1/p} &\leq \left(\sum_{n=1}^{\infty} | \langle T_1 e_n, f_n \rangle + \langle T_2 e_n, f_n \rangle |^p \right)^{1/p} \\ &\leq \left(\sum_{n=1}^{\infty} | \langle T_1 e_n, f_n \rangle |^p \right)^{1/p} \\ &\quad + \left(\sum_{n=1}^{\infty} | \langle T_2 e_n, f_n \rangle |^p \right)^{1/p} \end{aligned}$$

Now, we take the supremum over all orthonormal sequences, on both sides of the inequality, we get:

$$\sup_{(e_n), (f_n)} \left(\sum_{n=1}^{\infty} | \langle (T_1 + T_2)e_n, f_n \rangle |^p \right)^{1/p} \leq \sup_{(e_n), (f_n)} \left(\sum_{n=1}^{\infty} | \langle T_1 e_n, f_n \rangle |^p \right)^{1/p} + \sup_{(e_n), (f_n)} \left(\sum_{n=1}^{\infty} | \langle T_2 \theta_n, \theta_n \rangle |^p \right)^{1/p}$$

Hence,

$$\|T_1 + T_2\|_p \leq \|T_1\|_p + \|T_2\|_p.$$

Hence, the required. $\square$

We need the following known inequality, and we include the proof for completeness.

Lemma 1.3.12.

$$\left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{nm} \right)^p \right)^{1/p} \leq \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} (a_{nm})^p \right)^{1/p}$$

where a_{nm} are positive real numbers.

Proof: We may assume that $\left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{nm} \right)^p \right)^{1/p} < \infty$.

Now let $\beta_n = \sum_{m=1}^{\infty} a_{nm}$. Then β_n is a sequence in l^p . and

$$\left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{nm} \right)^p \right)^{1/p} = \left(\sum_{n=1}^{\infty} (\beta_n)^p \right)^{1/p} = \|\beta\|_p$$

Since $\beta \in l^p$, then there is $\alpha \in l^{p^*}$ such that $\|\alpha\|_{p^*} = 1$ and $\langle \beta, \alpha \rangle = \|\beta\|_p$. So, we have

$$\begin{aligned} \|\beta\|_p &= \langle \beta, \alpha \rangle \\ &= \sum_{n=1}^{\infty} \alpha_n \beta_n \\ &= \sum_{n=1}^{\infty} \alpha_n \sum_{m=1}^{\infty} a_{nm} \end{aligned}$$

By Monotone Convergence Theorem, we have:

$$\|\beta\|_p = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_n a_{nm}$$

Thus

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{nm} \right)^p \right)^{1/p} &= \|\beta\|_p \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_n a_{nm} \\ &\leq \sum_{m=1}^{\infty} \left[\left(\sum_{n=1}^{\infty} \alpha_n^{p^*} \right)^{1/p^*} \left(\sum_{n=1}^{\infty} (a_{nm})^p \right)^{1/p} \right] \\ &\leq \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} (a_{nm})^p \right)^{1/p}. \quad \square \end{aligned}$$

Theorem 1.3.13. $(\mathcal{C}_p(H), \|\cdot\|_p)$ is a Banach space.

Proof: Let $T_n \in \mathcal{C}_p(H)$ such that $\sum_{n=1}^{\infty} \|T_n\|_p < \infty$. Claim: $T = \sum_{n=1}^{\infty} T_n \in \mathcal{C}_p(H)$. Since $\sum_{n=1}^{\infty} \|T_n\|_p < \infty$, then $\sum_{n=1}^{\infty} \|T_n\| < \infty$ by Lemma 1.3.10. But $\mathcal{L}(H)$ is a Banach space, therefore $T = \sum_{n=1}^{\infty} T_n \in \mathcal{L}(H)$.

Now, let $(e_n), (f_n)$ be orthonormal sequences in H . Then:

$$\begin{aligned} \left(\sum_{n=1}^{\infty} | \langle T e_n, f_n \rangle |^p \right)^{1/p} &= \left(\sum_{n=1}^{\infty} \left| \left\langle \left(\sum_{i=1}^{\infty} T_i \right) e_n, f_n \right\rangle \right|^p \right)^{1/p} \\ &\leq \left(\sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} | \langle T_i e_n, f_n \rangle |^p \right) \right)^{1/p} \\ &\leq \sum_{i=1}^{\infty} \left(\sum_{n=1}^{\infty} | \langle T_i e_n, f_n \rangle |^p \right)^{1/p} \quad (\text{by Lemma 1.3.12}) \\ &\leq \sum_{i=1}^{\infty} \|T_i\|_p < \infty \end{aligned}$$

Hence, $\|T\|_p \leq \sum_{i=1}^{\infty} \|T_i\|_p < \infty$. Further,

$$\begin{aligned} \|T - \sum_{i=1}^n T_i\|_p &= \left\| \sum_{i=n+1}^{\infty} T_i \right\|_p \\ &\leq \sum_{i=n+1}^{\infty} \|T_i\|_p \end{aligned}$$

Hence, $\|T - \sum_{i=1}^n T_i\|_p$ converges to 0 as n goes to ∞ , since $\sum_{i=1}^{\infty} \|T_i\|_p < \infty$.
Hence, the required. $\square$

Theorem 1.3.14. $\|T\|_p = \|T^*\|_p$ for all $T \in \mathcal{C}_p(H)$.

Proof: This follows from:

$$\sup_{(e_n), (f_n)} \left(\sum_{n=1}^{\infty} | \langle T e_n, f_n \rangle |^p \right)^{1/p} = \sup_{(e_n), (f_n)} \left(\sum_{n=1}^{\infty} | \langle e_n, T^* f_n \rangle |^p \right)^{1/p}. \quad \square$$

Theorem 1.3.15. If U is unitary and $T \in \mathcal{C}_p(H)$, then $UT, TU \in \mathcal{C}_p(H)$ and

$$\|UT\|_p = \|TU\|_p = \|T\|_p.$$

Proof:

$$\begin{aligned} \|UT\|_p &= \sup_{(e_n), (f_n)} \left(\sum_{n=1}^{\infty} | \langle UT e_n, f_n \rangle |^p \right)^{1/p} \\ &= \sup_{(e_n), (f_n)} \left(\sum_{n=1}^{\infty} | \langle T e_n, U^* f_n \rangle |^p \right)^{1/p} \\ &= \|T\|_p \end{aligned}$$

since $(U^* f_n)$ is an orthonormal sequence.
Similarly for TU . $\square$

Definition 1.3.16. For $T \in \mathcal{C}_1(H)$, and (θ_j) an orthonormal basis for H , define

$$\tau r(T) = \sum_{j=1}^{\infty} \langle T \theta_j, \theta_j \rangle.$$

Theorem 1.3.17. $\tau r(T)$ is independent of (θ_j) .

Proof: Clearly, $\tau r(T) < \infty$, since $|\tau r(T)| \leq \|T\|_1 < \infty$. Since any operator T can be written as $T = T_1 + iT_2$ where T_1 and T_2 are self-adjoint operators, then it is enough to prove the theorem for self-adjoint operators only. Now let $T = T^*$ and (θ_j) be any orthonormal basis. Then

$$\begin{aligned}
\sum_{j=1}^{\infty} \langle T\theta_j, \theta_j \rangle &= \sum_{j=1}^{\infty} \left(\sum_{n=1}^{\infty} \lambda_n \langle e_n, \theta_j \rangle \overline{\langle e_n, \theta_j \rangle} \right) \\
&= \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \lambda_n |\langle e_n, \theta_j \rangle|^2 \\
&= \lim_{k \rightarrow \infty} \sum_{j=1}^k \lim_{l \rightarrow \infty} \sum_{n=1}^l \lambda_n |\langle e_n, \theta_j \rangle|^2 \\
&= \lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \sum_{j=1}^k \sum_{n=1}^l \lambda_n |\langle e_n, \theta_j \rangle|^2 \\
&= \lim_{l \rightarrow \infty} \lim_{k \rightarrow \infty} \sum_{n=1}^l \sum_{j=1}^k \lambda_n |\langle e_n, \theta_j \rangle|^2 \\
&= \lim_{l \rightarrow \infty} \sum_{n=1}^l \lim_{k \rightarrow \infty} \sum_{j=1}^k \lambda_n |\langle e_n, \theta_j \rangle|^2 \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \lambda_n |\langle e_n, \theta_j \rangle|^2 \\
&= \sum_{n=1}^{\infty} \lambda_n \sum_{j=1}^{\infty} |\langle e_n, \theta_j \rangle|^2 \\
&= \sum_{n=1}^{\infty} \lambda_n \|e_n\|^2 \\
&= \sum_{n=1}^{\infty} \lambda_n
\end{aligned}$$

Hence, it is independent of (θ_j) . $\square$

Theorem 1.3.18. τr is a linear functional on $\mathcal{C}_1(H)$.

Proof: Clear. $\square$

Theorem 1.3.19. For $T \in \mathcal{C}_p(H)$,

$$\|T\|_p = \| |T| \|_p = (\tau r |T|^p)^{1/p}$$

Proof Let $T = \sum_{n=1}^{\infty} \lambda_n e_n \otimes f_n$. Then from [12,p.86], we have $\|T\|_p = (\sum_{n=1}^{\infty} |\lambda_n|^p)^{1/p}$. But $|T| = \sum_{n=1}^{\infty} |\lambda_n| e_n \otimes e_n$ So

$$\| |T| \|_p = \left(\sum_{n=1}^{\infty} |\lambda_n|^p \right)^{1/p}$$

Hence

$$\|T\|_p = \| |T| \|_p.$$

Now,

$$|T|^p = \sum_{n=1}^{\infty} |\lambda_n|^p e_n \otimes e_n$$

Then

$$\tau r |T|^p = \sum_{n=1}^{\infty} |\lambda_n|^p = \|T\|_p^p.$$

The result follows. $\square$

Theorem 1.3.20. For $T \in \mathcal{C}_p(H)$:

$$\|T\|_p = \sup_{(\theta_n)} \left(\sum_{n=1}^{\infty} \|T\theta_n\|^p \right)^{1/p} \quad \text{if} \quad p \geq 2$$

and

$$\|T\|_p = \inf_{(\theta_n)} \left(\sum_{n=1}^{\infty} \|T\theta_n\|^p \right)^{1/p} \quad \text{if} \quad 1 \leq p \leq 2$$

where the supremum and the infimum are taken over all orthonormal sequences (θ_n) .

Proof: See [11]. $\square$

Theorem 1.3.21. Suppose that $T \in \mathcal{L}(H)$ and $1 \leq p < \infty$. Then $T \in \mathcal{C}_p(H)$ if and only if there is a sequence of finite rank operators F_n , with rank not greater

than n such that

$$\sum_{n=1}^{\infty} \|T - F_n\|^p < \infty.$$

Proof: We get the following proof from [12,p.77].

Let D denote the class of all operators T on H for which such a sequence (F_n) exists. Given S and T in D and scalars α and β , choose operators F_n and G_n with finite rank not greater than n for which

$$\sum_{n=1}^{\infty} \|S - F_n\|^p < \infty, \quad \sum_{n=1}^{\infty} \|T - G_n\|^p < \infty, \quad (*)$$

and define a sequence (H_n) by

$$H_1 = 0, \quad H_{2n} = H_{2n+1} = \alpha F_n + \beta G_n \quad (n = 1, 2, \dots).$$

Then H_n has finite rank not greater than n , and

$$\begin{aligned} \sum_{n=2}^{\infty} \|\alpha S + \beta T - H_n\|^p &= 2 \sum_{n=1}^{\infty} \|\alpha(S - F_n) + \beta(T - G_n)\|^p \\ &< 2 \sum_{n=1}^{\infty} \{|\alpha| \|S - F_n\| + |\beta| \|T - G_n\|\}^p. \end{aligned}$$

The last quantity is finite, by $(*)$ and Minkowski's inequality, so $\alpha S + \beta T \in D$. Furthermore $S^* \in D$, since F_n^* has finite rank not more than n and

$$\sum_{n=1}^{\infty} \|S^* - F_n^*\|^p = \sum_{n=1}^{\infty} \|S - F_n\|^p < \infty.$$

Finally, S is compact since it is the limit in norm of the sequence (F_n) of operators of finite rank.

It follows from the results just proved that D is a linear subspace of $\mathcal{L}(H)$ which consists of compact operators and contains the self-adjoint and skew-adjoint parts of each of its members. It has already seen, [12, p.76], that $\mathcal{C}_p(H)$ has these properties. Hence each of the subspaces D and $\mathcal{C}_p(H)$ is the linear span of its self-adjoint elements; and in order to prove that $D = \mathcal{C}_p(H)$, it is sufficient to show that a compact self-adjoint operator T lies in D if and only if it lies in $\mathcal{C}_p(H)$. Given such a T , we may suppose that

$$Tx = \sum_{m=1}^{\infty} \lambda_m \langle x, \phi_m \rangle \phi_m \quad (x \in H),$$

where (ϕ_m) is an orthonormal sequence and (λ_m) is the sequence of non-zero eigenvalues of T (arranged in order of decreasing magnitude, counted according to their algebraic multiplicities, and followed by an infinite sequence of zeros if T is of finite rank).

If $T \in \mathcal{C}_p(H)$, define an operator F_p by

$$F_n x = \sum_{m=1}^n \lambda_m \langle x, \phi_m \rangle \phi_m \quad (x \in H).$$

Then F_n has finite rank not greater than n ,

$$(T - F_n)x = \sum_{m=n+1}^{\infty} \lambda_m \langle x, \phi_m \rangle \phi_m \quad (x \in H),$$

$$\|T - F_n\| = \sup_{m>n} |\lambda_m| = |\lambda_{n+1}|.$$

By Theorem 1.3.7,

$$\sum_{n=1}^{\infty} \|T - F_n\|^p = \sum_{n=1}^{\infty} |\lambda_{n+1}|^p < \infty,$$

so $T \in D$.

Suppose conversely that $T \in D$. Let G_n be an operator of finite rank not more than n ($n = 1, 2, \dots$) for which $\sum_{n=1}^{\infty} \|T - G_n\|^p < \infty$. For a fixed n , we may choose vectors x_m, y_m ($1 \leq m \leq n$) in H such that

$$G_n x = \sum_{m=1}^{\infty} \langle x, x_m \rangle y_m \quad (x \in H).$$

Then $G_n(\alpha_1 \phi_1 + \dots + \alpha_{n+1} \phi_{n+1}) = 0$ provided that the scalars $\alpha_1, \dots, \alpha_{n+1}$ satisfy the n linear equations

$$\langle \alpha_1 \phi_1 + \dots + \alpha_{n+1} \phi_{n+1}, x_m \rangle = 0 \quad (m = 1, \dots, n).$$

With $\alpha_1, \dots, \alpha_{n+1}$ (not all zero) satisfying these equations, we have

$$\begin{aligned} \|(T - G_n)(\alpha_1 \phi_1 + \dots + \alpha_{n+1} \phi_{n+1})\| &= \|T(\alpha_1 \phi_1 + \dots + \alpha_{n+1} \phi_{n+1})\| \\ &= \|\lambda_1 \alpha_1 \phi_1 + \dots + \lambda_{n+1} \alpha_{n+1} \phi_{n+1}\| \\ &= \left[|\lambda_1 \alpha_1|^2 + \dots + |\lambda_{n+1} \alpha_{n+1}|^2 \right]^{1/2} \\ &\geq |\lambda_{n+1}| [|\alpha_1|^2 + \dots + |\alpha_{n+1}|^2] \\ &= |\lambda_{n+1}| \|\alpha_1 \phi_1 + \dots + \alpha_{n+1} \phi_{n+1}\| \end{aligned}$$

Hence $|\lambda_{n+1}| \leq \|T - G_n\|$,

$$\sum_{n=1}^{\infty} |\lambda_n|^p \leq |\lambda_1|^p + \sum_{n=1}^{\infty} \|T - G_n\|^p < \infty,$$

and by Theorem 1.3.7, $T \in \mathcal{C}_p(H)$. $\square$

Theorem 1.3.22. $\mathcal{F}(H) \subseteq \mathcal{C}_p(H)$ for all $1 \leq p < \infty$.

Proof: Let $x, y \in X$. We have:

$$\begin{aligned} \|x \otimes y\|_p &= \sup_{(e_n), (f_n)} \left(\sum_{n=1}^{\infty} |\langle x, e_n \rangle \langle y, f_n \rangle|^p \right)^{1/p} \\ &\leq \sup_{(e_n), (f_n)} \left\{ \left(\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^{2p} \right)^{1/2p} \left(\sum_{n=1}^{\infty} |\langle y, f_n \rangle|^{2p} \right)^{1/2p} \right\} \\ &\leq \|x\| \|y\| \end{aligned}$$

Also, we have:

$$\|x\| \|y\| = \|x \otimes y\|_{op} \leq \|x \otimes y\|_p$$

Hence,

$$\|x \otimes y\|_p = \|x\| \|y\|.$$

And since $\mathcal{F}(H)$ is the span of $\{x \otimes y : x, y \in H\}$ then $\mathcal{F}(H) \subseteq \mathcal{C}_p(H)$. $\square$

Theorem 1.3.23. $\mathcal{F}(H)$ is dense in $(\mathcal{C}_p(H), \|\cdot\|_p)$.

Proof Let $T \in \mathcal{C}_p(H)$, then $T = \sum_{n=1}^{\infty} \lambda_n e_n \otimes f_n$ and $\|T\|_p^p = \sum_{n=1}^{\infty} |\lambda_n|^p$.
Let $F_n = \sum_{i=1}^n \lambda_i e_i \otimes f_i$, then

$$\|T - F_n\|_p = \left\| \sum_{i=n+1}^{\infty} \lambda_i e_i \otimes f_i \right\|_p = \left(\sum_{i=n+1}^{\infty} |\lambda_i|^p \right)^{1/p}$$

But since $\sum_{i=1}^{\infty} |\lambda_i|^p$ is finite, we have $\left(\sum_{i=n+1}^{\infty} |\lambda_i|^p \right)^{1/p}$ converge to 0 as n goes to ∞ . Hence, we get: $\|T - F_n\|_p$ converges to 0. $\square$

Theorem 1.3.24. $\mathcal{C}_p(H) \subseteq \mathcal{C}_q(H)$ if $p \leq q$.

Proof: See, [12, p.76]. $\square$

Theorem 1.3.25. $\mathcal{C}_p(H) \circ \mathcal{C}_q(H) = C_r$ for $1/p + 1/q = 1/r$.

Proof: See, [12, p.97]. $\square$

Definition 1.3.26. A Banach space X is called uniformly convex if whenever $\|\frac{x_n+y_n}{2}\| \rightarrow 1$ for any sequences (x_n) and (y_n) in $B_1(X)$, then $\|x_n - y_n\| \rightarrow 0$.

Theorem 1.3.27. Let $1 < p < \infty$. $\mathcal{C}_p(H)$ is uniformly convex.

Proof: For $p \geq 2$, we have the Clarkson-McCarthy inequality, [14, p.127]:

$$\|A + B\|_p^p + \|A - B\|_p^p \leq 2^{p-1} (\|A\|_p^p + \|B\|_p^p)$$

Let $A_n, B_n \in \mathcal{C}_p(H)$ such that $\|\frac{A_n+B_n}{2}\|_p \rightarrow 1$. Claim: $\|A_n - B_n\|_p \rightarrow 0$.

We can assume that $\|A_n\|_p = 1$, $\|B_n\|_p = 1$.

Now,

$$\|A + B\|_p^p + \|A - B\|_p^p \leq 2^{p-1}(2) = 2^p$$

So

$$2^p \|\frac{A_n + B_n}{2}\|_p^p + \|A_n - B_n\|_p^p \leq 2^p.$$

And by taking $\underline{\lim}$, we get:

$$2^p + \underline{\lim} \|A_n - B_n\|_p^p \leq 2^p$$

Then

$$\underline{\lim} \|A_n - B_n\|_p^p = 0$$

Similarly, we can get $\overline{\lim} \|A_n - B_n\|_p^p = 0$. Hence, $\lim \|A_n - B_n\|_p^p = 0$. $\square$

CHAPTER II

ORTHOGONALITY IN BANACH SPACES

In this chapter we present several types of orthogonality in Banach spaces. Orthogonality in Hilbert spaces is studied in section 1, while other types of orthogonality will be discussed in section 2.

2.1 Orthogonality in Hilbert Spaces.

In this section, we present some basic properties of orthogonality in Hilbert spaces. Throughout, H is a Hilbert space.

Definition 2.1.1. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Two elements $x, y \in H$ are called orthogonal, if $\langle x, y \rangle = 0$. We will write $x \perp y$ for x is orthogonal to y .

Definition 2.1.2. A sequence (x_i) in H is said to be orthonormal if:

- (i) $\langle x_i, x_j \rangle = 0$ for all $i \neq j$
- (ii) $\|x_i\| = 1$ for all i .

Theorem 2.1.3. Let $x, y \in H$, then the parallelogram law holds, that is

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Proof: The proof is immediate and will be omitted. $\square$

Theorem 2.1.4. If (x_n) is an orthonormal sequence in H , then the following are equivalent:

- (i) $\sum_{n=1}^{\infty} a_n x_n$ converges, in the norm topology of H .
- (ii) $\sum_{n=1}^{\infty} |a_n|^2 < \infty$.
- (iii) $\sum_{n=1}^{\infty} a_n \langle x_n, y \rangle$ converges, for every $y \in H$.

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Proof:

(i) $\rightarrow$ (ii).

Let $\sum_{n=1}^{\infty} a_n x_n$ converges in H . Thus the sequence $\|\sum_{n=1}^k a_n x_n\| \rightarrow \|\sum_{n=1}^{\infty} a_n x_n\|$.
 But $\|\sum_{n=1}^k a_n x_n\| = \left(\sum_{n=1}^k |a_n|^2\right)^{1/2}$, and $\left(\sum_{n=1}^k |a_n|^2\right)^{1/2}$ converges to $\left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}$.
 Hence $\|\sum_{n=1}^{\infty} a_n x_n\| = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}$ and so $\sum_{n=1}^{\infty} |a_n|^2 < \infty$.

(ii) $\rightarrow$ (iii).

Let $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. Then using Hölder's inequality, we get:

$$\begin{aligned} \sum_{n=1}^{\infty} |a_n \langle x_n, y \rangle| &\leq \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} |\langle x_n, y \rangle|^2\right)^{1/2} \\ &\leq \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2} \|y\| \quad (\text{by Bessel's inequality.}) \\ &< \infty. \end{aligned}$$

(iii) $\rightarrow$ (i).

Let $\sum_{n=1}^{\infty} a_n \langle x_n, y \rangle$ converges for all y . Now

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \langle x_n, y \rangle &= \lim_{k \rightarrow \infty} \sum_{n=1}^k a_n \langle x_n, y \rangle \\ &= \lim_{k \rightarrow \infty} \langle \sum_{n=1}^k a_n x_n, y \rangle. \end{aligned}$$

Let $T_k = \sum_{n=1}^k a_n x_n$. Then $\lim_{k \rightarrow \infty} T_k(y)$ exists for all y . Let $T(y) = \lim_{k \rightarrow \infty} T_k(y)$. Since $\|T_k\| < \infty$ for all k , and $T_k(y) \rightarrow T(y)$, then by the Uniform Boundedness Principle, T is bounded. Hence $\lim_{k \rightarrow \infty} T_k(y)$ is convergent. That is $\sum_{n=1}^{\infty} a_n x_n$ converges which completes the proof. $\square$

Definition 2.1.5. If $M \subset H$, then the orthogonal complement of M is the set $M^{\perp} = \{x \in H | x \perp y \text{ for all } y \in M\}$.

$M^{\perp}$ is readily seen to be a linear subspace of H which is closed; and the intersection $M \cap M^{\perp} = \{0\}$.

Definition 2.1.6. A vector space X is said to be the direct sum of two subspaces A and B of X , written $X = A \oplus B$ if each $x \in X$ has a unique representation $x = a + b$ where $a \in A$ and $b \in B$.

Then B is called an algebraic complement of A in X .

Lemma 2.1.7. Every non-empty closed convex set $M \subset H$ contains a unique x of minimal norm.

Proof: Let $\delta = \inf_{x \in M} \|x\|$. Choose $x_n \in M$ so that $\|x_n\| = \delta$. Since M is convex, then $\frac{1}{2}(x_n + x_m) \in M$ which implies that $\|x_n + x_m\|^2 \geq 4\delta^2$. Furthermore, we have by the parallelogram equality, $\|x_n + x_m\|^2 + \|x_n - x_m\|^2 = 2(\|x_n\|^2 + \|x_m\|^2)$. But the right side of this equality tends to $4\delta^2$. Hence (x_n) is a Cauchy sequence in H , which therefore converge to some $x \in M$, with $\|x\| = \delta$. Now to show uniqueness:

If $y \in M$ and $\|y\| = \delta$, then the sequence $(x, y, x, y, \dots)$ must converge as we just saw. Hence $y = x$. $\square$

Theorem 2.1.8. If M is a closed subspace of H , then $H = M \oplus M^\perp$.

Proof: Let $x \in H$. Then by Lemma 2.1.7, when applied to the set $x - M$, there exists $x_1 \in M$ that minimizes $\|x - x_1\|$. Set $x_2 = x - x_1$. Then $\|x_2\| \leq \|x_2 + y\|$ for all $y \in M$. Hence $x_2 \in M^\perp$ by Lemma 2.1.7. Now since $x = x_1 + x_2$ this completes the proof. $\square$

Corollary 2.1.9. If M is a closed subspace of H , then $(M^\perp)^\perp = M$.

Proof:

It is obvious that $M \subset (M^\perp)^\perp$. Since $M \oplus M^\perp = H = M^\perp \oplus (M^\perp)^\perp$, M cannot be a proper subspace of $(M^\perp)^\perp$. This completes the proof. $\square$

Definition 2.1.10. An orthonormal sequence (x_i) is said to be an orthonormal basis of H if and only if for every $x \in H$, $x = \sum_{i=1}^{\infty} \langle x, x_i \rangle x_i$ where the convergence of the series is in the norm of H .

This expansion is called the Fourier series of x .

Definition 2.1.11. Given a set I , let $F(I)$ be the set of all finite subsets of I , directed by inclusion. Then a family $(x_i)_{i \in I}$ of vectors in H is called unconditionally summable if the net $\left(\sum_{i \in M} x_i \right)_{M \in F(I)}$ converges in H .

The limit will then be denoted by $\sum_{i \in I} x_i$.

Theorem 2.1.12. For every orthonormal sequence (x_i) in H , the following are equivalent:

- (i) (x_i) is an orthonormal basis.
- (ii) For every $\lambda \in l^2(I)$, the family $(\lambda_i x_i)_{i \in I}$ is unconditionally summable in H , and

$$\phi : l^2(I) \rightarrow H : \lambda \mapsto \sum_{i \in I} \lambda_i x_i$$

is an isometric onto isomorphism.

- (iii) Every $x \in H$ can be written $x = \sum_{i=1}^{\infty} \langle x, x_i \rangle x_i$.
- (iv) For all $x, y \in H$, we have $\langle x, y \rangle = \sum_{i=1}^{\infty} \langle x, x_i \rangle \langle x_i, y \rangle$.
- (v) Every $x \in H$ satisfies the so-called Parseval's identity:

$$\|x\|^2 = \sum_{i=1}^{\infty} |\langle x, x_i \rangle|^2.$$

Proof:

(i) $\rightarrow$ (ii).

Let (x_i) be an orthonormal basis for H . Then consider $\phi : l^2 \rightarrow H$ where

$$\phi(a_i) = \sum_{i=1}^{\infty} a_i x_i$$

By Theorem , $\|\phi((a_i))\| = (\sum_{i=1}^{\infty} |a_i|^2)^{1/2} = \|(a_i)\|_2$. Hence ϕ is an isometry. Now let $x \in H$. By (i), $x = \sum_{i=1}^{\infty} a_i x_i$, and $\|x\| = (\sum_{i=1}^{\infty} |a_i|^2)^{1/2}$. Then $\phi((a_i)) = x$.

(ii) $\rightarrow$ (iii).

Let $x \in H$. Then $x = \phi(\lambda)$ for some λ since ϕ is onto. Thus $x = \sum_{i=1}^{\infty} \lambda_i x_i$. Further, $\|x\| = (\sum_{i=1}^{\infty} |\lambda_i|^2)^{1/2}$ because ϕ is an isometry. But

$$\begin{aligned} \left\| \sum_{i=1}^{\infty} \lambda_i x_i \right\| &= \lim_{k \rightarrow \infty} \left\| \sum_{i=1}^k \lambda_i x_i \right\| \\ &= \lim_{k \rightarrow \infty} \left(\left\langle \sum_{i=1}^k \lambda_i x_i, \sum_{i=1}^k \lambda_i x_i \right\rangle \right)^{1/2} \\ &= \lim_{k \rightarrow \infty} \left(\sum_{i=1}^k |\lambda_i|^2 \right)^{1/2} \left| \langle x_i, x_i \rangle \right|^{1/2} \\ &= \left(\sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{1/2} \|x_i\|. \end{aligned}$$

Therefore, $\|x_i\| = 1$ for all i . Now

$$\begin{aligned}
 \langle x, x_1 \rangle &= \langle \sum_{i=1}^{\infty} \lambda_i x_i, x_1 \rangle \\
 &= \sum_{i=1}^{\infty} \lambda_i \langle x_i, x_1 \rangle \\
 &= \lambda_1 \langle x_1, x_1 \rangle \\
 &= \lambda_1.
 \end{aligned}$$

and

$$\begin{aligned}
 \langle x, x_j \rangle &= \langle \sum_{i=1}^{\infty} \lambda_i x_i, x_j \rangle \\
 &= \sum_{i=1}^{\infty} \lambda_i \langle x_i, x_j \rangle \\
 &= \lambda_j.
 \end{aligned}$$

Hence

$$x = \sum_{i=1}^{\infty} \langle x, x_i \rangle x_i.$$

(iii) $\rightarrow$ (iv).

Let $x, y \in H$, then

$$\begin{aligned}
 \langle x, y \rangle &= \langle \sum_{i=1}^{\infty} \langle x, x_i \rangle x_i, y \rangle \\
 &= \sum_{i=1}^{\infty} \langle \langle x, x_i \rangle x_i, y \rangle \\
 &= \sum_{i=1}^{\infty} \langle x, x_i \rangle \langle x_i, y \rangle.
 \end{aligned}$$

(iv) $\rightarrow$ (v).

$$\begin{aligned}
\|x\|^2 &= \langle x, x \rangle \\
&= \sum_{i=1}^{\infty} \langle x, x_i \rangle \langle x_i, x \rangle \\
&= \sum_{i=1}^{\infty} \langle x, x_i \rangle \overline{\langle x, x_i \rangle} \\
&= \sum_{i=1}^{\infty} |\langle x, x_i \rangle|^2.
\end{aligned}$$

(v) $\rightarrow$ (i).

Let $x \in \{x_1, x_2, \dots\}^\perp$, then each term in $\sum_{i=1}^{\infty} |\langle x, x_i \rangle|^2$ equals zero; which implies that $x = 0$. Therefore, $\{x_1, x_2, \dots\}^\perp = \{0\}$ and hence (x_i) is an orthonormal basis. That ends the proof. $\square$

Corollary 2.1.13. A separable Hilbert space H is either finite-dimensional or isometrically isomorphic to l^2 .

Proof: See [4, p.317]. $\square$

Corollary 2.1.14. Every separable Hilbert space H has a countable orthonormal basis.

Proof: See [4, p.317]. $\square$

Definition 2.1.15. An operator $T \in \mathcal{L}(H)$ is called unitary if $TT^* = T^*T = I$.

Theorem 2.1.16. If (x_n) is an orthonormal sequence in H and U is unitary operator, then (Ux_n) is an orthonormal sequence.

Proof: Consider

$$\begin{aligned}
\langle Ux_n, Ux_m \rangle &= \langle U^*Ux_n, x_m \rangle \\
&= \langle Ix_n, x_m \rangle \\
&= \langle x_n, x_m \rangle
\end{aligned}$$

$$= \begin{cases} 0 & \text{for } n \neq m \\ 1 & \text{for } n = m. \end{cases}$$

Hence (Ux_n) is an orthonormal sequence. $\square$

Set $T = \sum_{n=1}^{\infty} e_n \otimes f_n$ to denote the operator $Tx = \sum_{n=1}^{\infty} \langle x, e_n \rangle f_n$. Then we have:

Theorem 2.1.17. An operator $T = \sum_{n=1}^{\infty} e_n \otimes f_n$ is:

- (i) Isometry if and only if (e_n) is a basis and (f_n) is an orthonormal sequence.
- (ii) Unitary if and only if $(e_n), (f_n)$ are orthonormal bases.
- (iii) Partial isometry if (e_n) and (f_n) are orthonormal sequences.

Proof:

First, let (e_n) and (f_n) be orthonormal sequences. (i) Let T be an isometry. Assume that (e_n) is not a basis. Consider $M = \overline{\text{span}(e_n)}$. Since M is a closed subspace of H then H can be the direct sum of M and $M^\perp$. That is $H = M \oplus M^\perp$. Let $x \neq 0$ be in $M^\perp$. Then $Tx = \sum_{n=1}^{\infty} \langle x, e_n \rangle f_n = 0$ since $\langle x, e_n \rangle = 0$. Which implies that $\|Tx\| = \|0\| = 0$. But $\|x\| \neq 0$. This contradicts T being an isometry. Thus (e_n) must be a basis.

Conversely, let (e_n) be a basis. Then consider

$$\|Tx\|^2 = \left\| \sum_{n=1}^{\infty} \langle x, e_n \rangle f_n \right\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$$

since (f_n) is orthonormal. Since (e_n) is a basis, then by definition 2.1.10, we have $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ which implies that $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$. Hence $\|Tx\|^2 = \|x\|^2$ or $\|Tx\| = \|x\|$ for all x . That shows T is isometry.

(ii) Let T be unitary, and assume that (e_n) is not a basis. Then by (i), T is not an isometry. That implies T is not unitary which contradicts our assumption. Hence, (e_n) is a basis. Now for (f_n) , let T be unitary and assume that (f_n) is not a basis. Let $M = \overline{\text{span}(f_i)}$. Since (f_n) is not a basis, then $M \subsetneq H$. But $Tx = \sum_{n=1}^{\infty} \langle x, e_n \rangle f_n \in M$ for all $x \in H$. Then $\mathcal{R}(T) \subseteq M$. That is $T : H \rightarrow M$. Consider now an element $z \in H$ which is not in M . Then there is no element $x \in H$ such that $Tx = z$, which contradicts the fact that T is onto (by the definition of unitary operators.) Hence, (f_n) must be a basis.

Conversely, let $(e_n), (f_n)$ be orthonormal bases. Then by (i) T is an isometry. We need to show that T is onto. Consider $\mathcal{R}(T) = \overline{\text{span}(f_i)} = H$. Hence the required.

(iii) To show that T is partial isometry, we need to show the following: There exists $W \subset H$ such that $H = W \oplus W^\perp$; and $T : W \rightarrow H$ is an isometry.

Moreover, $T : W^\perp \rightarrow H$ is the zero operator. Now let $W = \overline{\text{span}(e_n)}$. Take $x \in W$. We have: $Tx = \sum_{n=1}^{\infty} \langle x, e_n \rangle f_n$. Further, $\|Tx\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$. But $x \in W$ implies that $x = \sum_{n=1}^{\infty} a_n e_n$ and $\|x\|^2 = \sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$. Thus $\|Tx\|^2 = \|x\|^2$. Hence T is an isometry on W . Now let $y \in W^\perp$, then $Ty = \sum_{n=1}^{\infty} \langle y, e_n \rangle f_n = 0$ because $\langle y, e_n \rangle = 0$ for all n . That completes the proof. $\square$

Compact operators on H have a very nice representation using orthonormal sequences:

Theorem 2.1.18. $T = \sum_{n=1}^{\infty} \lambda_n e_n \otimes f_n$, where $(e_n), (f_n)$ are orthonormal sequences, λ_n converges to 0, and $\|T\| = \|(\lambda_n)\|_\infty$.

Proof: See Theorem 1.2.10. $\square$

2.2 Orthogonality in Banach Spaces.

One tries to generalize orthogonality in Hilbert spaces to Banach spaces. There are many ways in which one can define orthogonality in Banach spaces. In this section we mention some of such orthogonalities.

2.2.1 James Orthogonality.

Throughout, X is a Banach space. Although all definitions could be given for more general spaces than Banach spaces, it is sufficient for our work to consider only Banach spaces.

James, [3], had given the following:

Definition 2.2.1.1. Let x and y be elements in X . Then x is said to be orthogonal to y if $\|x + ky\| \geq \|x\|$ for all k . We write $x \perp y$ for x orthogonal to y .

An element x is orthogonal to a subset M of X if $x \perp z$ for all $z \in M$.

This orthogonality is homogeneous, that is, $x \perp y$ implies that $ax \perp by$ for all numbers a and b . However, it is neither symmetrical nor additive. That is, $x \perp y$ does not imply that $y \perp x$; and $(x \perp y$ and $x \perp z)$ does not imply that $x \perp (y + z)$, respectively.

It is clear that if (y_n) is a sequence that converges to y ; and x is orthogonal to each y_n , then x is orthogonal to y . Hence, for any x the set of all y such that $x \perp y$ is closed.

Theorem 2.2.1.2. If H is a Hilbert space then $x \perp y$ if and only if $\langle x, y \rangle = 0$.

Proof: Let $\langle x, y \rangle = 0$, then $\langle y, x \rangle = 0$. Consider

$$\begin{aligned}\|x + \lambda y\|^2 &= \|x\|^2 + \overline{\lambda} \langle x, y \rangle + \lambda \langle y, x \rangle + |\lambda|^2 \|y\|^2 \\ &= \|x\|^2 + 0 + 0 + |\lambda|^2 \|y\|^2 \\ &= \|x\|^2 + |\lambda|^2 \|y\|^2.\end{aligned}$$

Hence $\|x + \lambda y\|^2 \geq \|x\|^2$. Consequently, $x \perp y$.

Conversely, let $x \perp y$. That is, $\|x + \lambda y\|^2 \geq \|x\|^2$ for all $\lambda \in \mathbb{C}$. If $y = 0$, then the result is obvious. If $y \neq 0$, then take $\overline{\lambda} = -\frac{\langle x, y \rangle}{\|y\|^2}$. Then

$$\begin{aligned}\|x + \lambda y\|^2 &= \|x\|^2 - \frac{\overline{\langle x, y \rangle}}{\|y\|^2} \langle x, y \rangle - \frac{\langle x, y \rangle}{\|y\|^2} \overline{\langle x, y \rangle} + \frac{\langle x, y \rangle \overline{\langle x, y \rangle}}{\|y\|^2} \\ &= \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2}.\end{aligned}$$

Thus $\langle x, y \rangle = 0$. That completes the proof. $\square$

Definition 2.2.1.3. Let X be a linear space over $\mathbb{R}$ (or $\mathbb{C}$). A set $M \subset X$ is said to be a (real) linear variety if there exists a (real) linear subspace $L \subset X$ and some $x_0 \in X$ such that $M = x_0 + L$.

A (real) linear variety M is said to be a (real) hyperplane if L is a maximal (real) linear subspace.

It is known [10, p.127] that M is a hyperplane if and only if there is some $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $M = \{x : f(x) = \alpha\}$.

Lemma 2.2.1.4. Let f be a non-zero linear functional on a Banach space X and $M = \{z : f(z) = 0\}$ a hyperplane. Then $|f(x)| = \|f\| \cdot \|x\|$ if and only if $x \perp M$.

Proof: Let M be a hyperplane consisting of all elements z for which $f(z) = 0$. Suppose that $|f(x)| = \|f\| \cdot \|x\|$. Then $f(z) = 0$ implies that $|f(x + z)| = |f(x)|$.

But $|f(x+z)| \leq \|f\| \cdot \|x+z\|$ which implies that $\|x+z\| \geq \|x\|$ for all $z \in M$. Hence $x \perp M$.

Conversely, let $x \perp M$ and $|f(x)| = p\|x\|$. So we have:

$$|f(x+z)| = |f(x)| = p\|x\| \leq p\|x+z\|.$$

But M is a hyperplane through the origin, which implies that $|f(y)| \leq p\|y\|$ for all $y \in X$. That is, $p = \|f\|$ and $|f(x)| = \|f\| \cdot \|x\|$. That completes the proof. $\square$

Theorem 2.2.1.5. Any element of a Banach space is orthogonal to some hyperplane through the origin.

Proof: Since for any element x of a Banach space X there is a linear functional f such that $f(x) = \|f\|\|x\|$. Then a direct consequence of Lemma 2.2.1.4 is that: x is orthogonal to some hyperplane through the origin. $\square$

Corollary 2.2.1.6. Let x and y be elements of a Banach space X . Then there exists a number a such that $x \perp (ax+y)$. Also, $x \perp (ax+y)$ if and only if there is a linear functional f with $|f(x)| = \|f\|\|x\|$ and $a = -f(y)/f(x)$. If $x \perp (ax+y)$ then $|a| \leq \|y\|/\|x\|$.

Proof: The first part follow from Lemma 2.2.1.4 and Theorem 2.2.1.5. Now $x \perp (ax+y)$ if and only if $\|x+k(ax+y)\| \geq \|x\|$ for all k . let $k = -1/a$, then we get $|a| \leq \|y\|/\|x\|$. $\square$

2.2.2 Orthonormal Sequences in Banach Spaces.

A sequence (x_n) in a Banach space X is said to be a Schauder basis for X if and only if for every $x \in X$ there is a unique sequence of scalars (a_n) such that $x = \sum_{n=1}^{\infty} a_n x_n$, where the series converges in the norm topology of X . If the sequence (x_n) is a Schauder basis of $[x_n]$, the closed span of $\{x_n\}$ in X , it is called a basic sequence.

The following definition is given in [6].

Definition 2.2.2.1. A sequence (x_n) in X is said to be semi-orthonormal if the following hold:

- (i) $\|x_n\| = 1$ for all n ; and
- (ii) if $a_n \in \mathbb{C}$ such that $\sum_{n=1}^{\infty} a_n x_n \in X$, then $\|\sum_{n=1}^{\infty} a_n x_n\| \geq \sup_n |a_n|$.

Definition 2.2.2.2. Let (x_n) be a semi-orthonormal sequence in a Banach space X , and (x_n^*) be a sequence of corresponding coefficient functionals. The sequence (x_n) is said to be orthonormal if for any $(\lambda_n) \in l^\infty$ and any $x \in X$, the series $\sum_{n=1}^{\infty} \lambda_n \langle x, x_n^* \rangle x_n$ converges and the inequality

$$\left\| \sum_{n=1}^{\infty} \lambda_n \langle x, x_n^* \rangle x_n \right\| \leq \sup_n |\lambda_n| \|x\|$$

holds. The sequence (x_n) is said to be orthogonal if the sequence $(x_n/\|x_n\|)$ obtained by deleting trivial terms is orthonormal.

If x and y are elements in the unit sphere of a Banach space X , then x is orthogonal to y and written xOy if $\{x, y\}$ is an orthogonal set.

Lemma 2.2.2.3.

- (i) Orthogonality defined in Definition 2.2.2.2 is symmetric. That is, xOy implies that yOx .
- (ii) The orthogonality is homogenous; that is, xOy implies that $axOby$ for all complex numbers a and b .
- (iii) Every singleton $\{x\}$ forms an orthogonal set.
- (iv) The set $x^\perp = \{y \in X : xOy\}$ is closed in X .
- (v) If (x_n) is an orthonormal sequence in X , then (Ux_n) is an orthonormal sequence for every onto isometry $U \in \mathcal{L}(X)$.

Proof: The proofs of the properties (i) $\rightarrow$ (iv) are simple and will be omitted. we only prove (v).

Now let (x_n) be an orthonormal sequence in X and U an onto isometry of X . Since $\left\| \sum_{n=1}^{\infty} U(\lambda_n x_n) \right\| = \left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|$, it follows that (Ux_n) is semi-orthonormal. If (θ_n^*) is a sequence of associated coefficient functionals of (Ux_n) . Thus for $z = U\tilde{z}$, we have:

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} \lambda_n \langle z, \theta_n^* \rangle Ux_n \right\| &= \left\| \sum_{n=1}^{\infty} \lambda_n \langle U\tilde{z}, \theta_n^* \rangle Ux_n \right\| \\ &= \left\| \sum_{n=1}^{\infty} \lambda_n \langle \tilde{z}, U^* \theta_n^* \rangle Ux_n \right\| \\ &= \left\| \sum_{n=1}^{\infty} \lambda_n \langle \tilde{z}, U^* \theta_n^* \rangle x_n \right\| \\ &\leq \|(\lambda_n)\|_\infty \|\tilde{z}\| \\ &= \|(\lambda_n)\|_\infty \|U\tilde{z}\|. \end{aligned}$$

Hence (Ux_n) is an orthonormal sequence in X . $\square$

Lemma 2.2.2.4. Let (x_n) be an infinite orthonormal sequence in X and (x_n^*) be a sequence of associated coefficient functionals. Then $\lim_{n \rightarrow \infty} x_n^*(x) = 0$ for all $x \in [x_n]$.

Proof: Let $x = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle x_n \in [x_n]$, where the series converges in the norm of X . Hence $\|\sum_{i=1}^n \langle x, x_i^* \rangle x_i - \sum_{i=1}^{n+1} \langle x, x_i^* \rangle x_i\| \rightarrow 0$ as $n \rightarrow \infty$. Thus $\|\langle x, x_{n+1}^* \rangle x_{n+1}\| = |\langle x, x_{n+1}^* \rangle|$ and that converges to 0 as $n \rightarrow \infty$. This completes the proof. $\square$

Lemma 2.2.2.5. Let (x_n) be an infinite orthonormal sequence in X , then any sequence (x_n^*) of associated coefficient functionals is orthonormal in X^* .

Proof: We have $\|x_n^*\| = 1$. Now let $\lambda_n \in \mathbb{C}$ such that $\sum_{n=1}^{\infty} \lambda_n x_n^* \in X^*$. Then for each $i = 1, 2, \dots$ $|\lambda_i| = |\langle x_i, \sum_{n=1}^{\infty} \lambda_n x_n^* \rangle| \leq \|x_i\| \|\sum_{n=1}^{\infty} \lambda_n x_n^*\| = \|\sum_{n=1}^{\infty} \lambda_n x_n^*\|$. Hence, $\|\sum_{n=1}^{\infty} \lambda_n x_n^*\| \geq \sup_n |\lambda_n|$, and (x_n^*) is semi-orthonormal. Now let $(\lambda_n) \in l^\infty$. Since (x_n) is an orthonormal sequence in X , then the linear map $T : X \rightarrow X$ defined by $Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, x_n^* \rangle x_n$ is bounded. Further, for every $x \in X$ and $x^* \in X^*$ we have:

$$\lim_{k \rightarrow \infty} \sum_{n=1}^k \lambda_n \langle x, x_n^* \rangle \langle x_n, x^* \rangle = \sum_{n=1}^{\infty} \lambda_n \langle x, x_n^* \rangle \langle x_n, x^* \rangle \quad (*)$$

Hence

$$\lim_{k \rightarrow \infty} \langle T_k x, x^* \rangle = \langle T x, x^* \rangle \quad (**)$$

for all $x \in X$ and $x^* \in X^*$, where $T_k x = \sum_{n=1}^k \lambda_n \langle x, x_n^* \rangle x_n$. Consequently,

$$\lim_{k \rightarrow \infty} \langle x, T_k^* x^* \rangle = \langle x, T^* x^* \rangle.$$

This implies that for each $x^* \in X^*$, the sequence of functionals $T_k^* x^*$ converges to the functional $T^* x^*$ in the w^* -topology. But from (*) and (**), we have $T^* x^* = \sum_{n=1}^{\infty} \lambda_n \langle x_n, x^* \rangle x_n^*$. Hence $\sum_{n=1}^{\infty} \lambda_n \langle x_n, x^* \rangle x_n^* \in X^*$ for any $x^* \in X^*$. Further,

$$\begin{aligned} \left| \langle x, \sum_{n=1}^{\infty} \lambda_n \langle x_n, x^* \rangle x_n^* \rangle \right| &= \left| \langle x, T^* x^* \rangle \right| \\ &= \left| \langle T x, x^* \rangle \right| \\ &\leq \|x^*\| \|T x\| \end{aligned}$$

But $Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, x_n^* \rangle x_n$. Hence the orthogonality of (x_n) implies that

$$\|Tx\| \leq \|x\| \sup_n |\lambda_n|.$$

Thus

$$\left\| \sum_{n=1}^{\infty} \lambda_n \langle x_n, x^* \rangle x_n^* \right\| \leq \|x^*\| \sup_n |\lambda_n|,$$

and (x_n^*) is orthonormal. $\square$

Theorem 2.2.2.6. Let H be a Hilbert space and $x, y \in H$. Then xOy if and only if $\langle x, y \rangle = 0$.

Proof: If $\langle x, y \rangle = 0$, then it can be easily shown that xOy . Conversely, let xOy and x^*, y^* be the associated coefficient functionals. Without loss of generality, we can assume that $\|x\| = \|y\| = 1$. If $\langle x, y \rangle \neq 0$, then $y = ax + z$, $\|z\| < 1$ and $\langle x, z \rangle = 0$. Now $y^*(y) = 1 = y^*(ax + z) = y^*(z)$. This implies that $\|y^*\| > 1$, which is a contradiction. $\square$

Notation. For $x \in l^p$, let $\text{supp}(x) = \{n : x(n) \neq 0\}$.

Assume $1 \leq p < \infty$ in the following:

Lemma 2.2.2.7. Let $x, y \in l^p$ such that $\text{supp}(x) \cap \text{supp}(y) = \emptyset$. Then xOy .

Proof: Let $\|x\| = \|y\| = 1$. Since $\text{supp}(x) \cap \text{supp}(y) = \emptyset$, x and y are independent. Let $x^*, y^* \in l^{p^*}$, $\frac{1}{p} + \frac{1}{p^*} = 1$, such that $x^*(x) = 1, y^*(y) = 1$ and $\|x^*\| = \|y^*\| = 1$. Then $\text{supp}(x^*) = \text{supp}(x)$ and $\text{supp}(y^*) = \text{supp}(y)$. Consequently $x^*(y) = y^*(x) = 0$. Now

$$\begin{aligned} \|\lambda_1 x + \lambda_2 y\| &= (|\lambda_1|^p \|x\|^p + |\lambda_2|^p \|y\|^p)^{1/p} \\ &\geq \max\{|\lambda_1|, |\lambda_2|\}. \end{aligned}$$

Further, if $z = ax + by$ then $\|z\| = (|a|^p + |b|^p)^{1/p}$ and

$$\begin{aligned} \|\lambda_1 \langle z, x^* \rangle x + \lambda_2 \langle z, y^* \rangle y\| &= (|\lambda_1|^p |\langle z, x^* \rangle|^p + |\lambda_2|^p |\langle z, y^* \rangle|^p)^{1/p} \\ &= (|\lambda_1|^p |a|^p + |\lambda_2|^p |b|^p)^{1/p} \\ &\leq \max\{|\lambda_1|, |\lambda_2|\} (|a|^p + |b|^p)^{1/p}. \end{aligned}$$

Hence xOy . $\square$

The converse of Lemma 2.2.2.7 is not true as the following example shows:

Example 2.2.2.8. Let $p > 2$ and $x = (\frac{1}{2^{1/p}}, \frac{1}{2^{1/p}}, 0, \dots)$, $y = (\frac{1}{2^{1/p}}, -\frac{1}{2^{1/p}}, 0, \dots)$ be two elements in l^p . Then $\|x\|_p = \|y\|_p = 1$. Further $\text{supp}(x) = \text{supp}(y)$, and x, y are independent. Let $x^* = (\frac{1}{2^{1/p^*}}, \frac{1}{2^{1/p^*}}, 0, 0, \dots)$ and $y^* = (\frac{1}{2^{1/p^*}}, -\frac{1}{2^{1/p^*}}, 0, 0, \dots)$. Then $\|x^*\|_{p^*} = \|y^*\|_{p^*} = 1$, and $x^*(y) = y^*(x) = 0$ and $x^*(x) = y^*(y) = 1$. Let $z = \lambda_1 x + \lambda_2 y = (\frac{\lambda_1 + \lambda_2}{2^{1/p}}, \frac{\lambda_1 - \lambda_2}{2^{1/p}}, 0, 0, \dots)$. Then

$$\begin{aligned}\|z\|_p &= \left(\frac{|\lambda_1 + \lambda_2|^p}{2} + \frac{|\lambda_1 - \lambda_2|^p}{2} \right)^{1/p} \\ &\geq (|\lambda_1|^p + |\lambda_2|^p)^{1/p} \quad (\text{By Clarkson's Inequality}). \\ &\geq \max\{|\lambda_1|, |\lambda_2|\}.\end{aligned}$$

Hence x, y form a semi-orthonormal set in l^p . Now let $a, b \in \mathbb{R}$ and $z = \lambda_1 x + \lambda_2 y$. Then

$$\begin{aligned}\|a \langle z, x^* \rangle x + b \langle z, y^* \rangle y\| &= \left\| \left(\frac{a\lambda_1 + b\lambda_2}{2^{1/p}}, \frac{a\lambda_1 - b\lambda_2}{2^{1/p}}, 0, 0, \dots \right) \right\| \\ &= \left(\frac{|a\lambda_1 + b\lambda_2|^p}{2} + \frac{|a\lambda_1 - b\lambda_2|^p}{2} \right)^{1/p}.\end{aligned}$$

Without loss of generality we may assume that $0 < |a| < |b|$, $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Then $|a\lambda_1 + b\lambda_2|^p + |a\lambda_1 - b\lambda_2|^p = |b|^p |\lambda_1|^p [|t + s|^p + |t - s|^p]$, where $s = \frac{\lambda_1}{\lambda_2}$ and $t = \frac{a}{b}$. Further,

$$[\max\{|a|, |b|\}]^p [|\lambda_1 + \lambda_2|^p + |\lambda_1 - \lambda_2|^p] = |b|^p |\lambda_1|^p [1 + |s|^p + 1 - |s|^p].$$

But from the geometry of balls of $\mathbb{R}^2$ with the p -norm, we have:

$$\|(s, -s) - (t, t)\|_p \leq \|(s, -s) - (1, 1)\|_p.$$

This implies

$$\|a \langle z, x^* \rangle x + b \langle z, y^* \rangle y\| \leq \max\{|a|, |b|\} \|z\|.$$

Hence xOy . $\square$

Theorem 2.2.2.9. Let $f, g \in \mathcal{C}(I)$ and $\|f\| = \|g\| = 1$. If $\text{sign}(f) = \text{sign}(g)$ or $\text{sign}(f) = -\text{sign}(g)$, then fOg if and only if $\text{supp}(f) \cap \text{supp}(g) = \phi$.

Proof: Let fOg . We assume that $\text{sign}(f) = \text{sign}(g)$ (for the proof of the other case is the same).

Let $h(x) = f(x) - g(x)$, $a = 1$ and $b = -1$. Then fOg implies $\|f + g\| \leq \|f - g\|$. Choose $\tilde{h}(x) = f(x) + g(x)$, $a = 1$, $b = -1$. Then $\|f - g\| \leq \|f + g\|$. Hence

$$\|f + g\| = \|f - g\| \quad (*)$$

Since $\text{sign}(f(x)) = \text{sign}(g(x))$, it follows that $\|f + g\| > 1$ unless $\text{supp}(f) \cap \text{supp}(g) = \phi$. Similarly, $\|f - g\| < 1$ unless $\text{supp}(f) \cap \text{supp}(g) = \phi$. Hence for equation (*) to be valid, one must have $\text{supp}(f) \cap \text{supp}(g) = \phi$.

Conversely, if f and g have disjoint supports, then

$$\|\lambda_1 f + \lambda_2 g\| = \max\{|\lambda_1| \|f\|, |\lambda_2| \|g\|\}.$$

Further, if $x \in \text{supp}(f)$, $y \in \text{supp}(g)$ and $\|f\| = |f(x)|$, $\|g\| = |g(y)|$, let μ_1 and μ_2 be the Dirac measure at x and y , respectively. Then

$$\langle f, \mu_1 \rangle = 1, \langle g, \mu_2 \rangle = 1 \quad \text{and} \quad \langle f, \mu_2 \rangle = 0, \langle g, \mu_1 \rangle = 0$$

and

$$\begin{aligned} \|\lambda_1 \langle h, \mu_1 \rangle f + \lambda_2 \langle h, \mu_2 \rangle g\| &= \max\{|\lambda_1| \langle h, \mu_1 \rangle \|f\|, |\lambda_2| \langle h, \mu_2 \rangle \|g\|\} \\ &\leq \max\{|\lambda_1|, |\lambda_2|\} \|h\| \end{aligned}$$

for any $h \in [f, g]$. Hence fOg . $\square$

The condition $\text{supp}(f) \cap \text{supp}(g) = \phi$ is not necessarily needed for $\{f, g\}$ to form an orthonormal set in general as the following example shows:

Example 2.2.2.10. Let $I = [-2, 2]$. Choose $f(x) = 1$ for all $x \in I$ and

$$g(x) = \begin{cases} 1 & \text{for } 1 \leq x \leq 2 \\ x & \text{for } -1 < x < 1 \\ -1 & \text{for } -2 \leq x \leq -1. \end{cases}$$

Then both f and g are continuous. Further, simple computations show that $\|\lambda_1 f + \lambda_2 g\| = |\lambda_1| + |\lambda_2|$. Consequently, $\|\lambda_1 f + \lambda_2 g\| \geq \max\{|\lambda_1|, |\lambda_2|\}$ and

$\|a(\lambda_1 f) + b(\lambda_2 g)\| \leq \max\{|a|, |b|\} \|\lambda_1 f + \lambda_2 g\|$. Hence fOg . But $\text{supp}(f) \cap \text{supp}(g) \neq \phi$. $\square$

Another example is the following:

Example 2.2.2.11. Let $x = (1, 1, 1, \dots, 1, \dots)$ and $y = (-1, 1, 1, \dots, 1, \dots)$ be two elements in l^∞ . Then $\text{supp}(x) \cap \text{supp}(y) \neq \phi$. However, with the help of some calculations similar to those in Example 2.2.2.10. we get xOy . $\square$

Proposition 2.2.2.12. Let X be a reflexive Banach space with an orthonormal basis. Let $T \in \mathcal{L}(X)$ be compact. Then

$$\lim_{n \rightarrow \infty} \|Te_n\| = 0$$

for all orthonormal bases (e_n) in X .

Proof: Assume that $\lim_{n \rightarrow \infty} \|Te_n\| \neq 0$ for some (e_n) . Then there exists a subsequence (e_{n_k}) such that $\|Te_{n_k}\| \geq \epsilon$ for all n_k and some $\epsilon > 0$. Since (e_{n_k}) is bounded, (Te_{n_k}) has a convergent subsequence. Without loss of generality we may assume that (Te_{n_k}) converges to some $x \in X$. It follows from Lemma 2.2.2.4 and Lemma 2.2.2.5 that

$$\lim_{k \rightarrow \infty} \langle e_{n_k}, x^* \rangle = 0$$

for all $x^* \in [e_n^*]$, the space generated from the associated linear functionals of (e_n) . (Since $[e_n] = X$, the sequence (e_n^*) is uniquely determined by (e_n) .) Since X is reflexive, then it follows that $[e_n^*] = X^*$. Hence (e_{n_k}) converges weakly to zero. Consequently $Te_{n_k} \rightarrow 0$ weakly and hence $x = 0$. Thus $\|Te_{n_k}\| \rightarrow 0$ which contradicts the assumption that $\|Te_{n_k}\| \geq \epsilon$. That completes the proof. $\square$

Proposition 2.2.2.13. Let T be a compact operator in $\mathcal{L}(X)$, where X is a reflexive Banach space. Then

$$\lim_{n \rightarrow \infty} |\langle Te_n, e_n^* \rangle| = 0$$

for all orthonormal bases (e_n) in X .

Proof: If possible, assume that there exists a subsequence (e_{n_k}) of a basis (e_n) such that $|\langle Te_{n_k}, e_{n_k}^* \rangle| \geq \epsilon$ for some $\epsilon > 0$ and all k . Hence $(e_{n_k}^*)$ is the associated linear functionals. Since (e_{n_k}) is bounded and T is compact, we can assume that $Te_{n_k} \rightarrow x$ for some $x \in X$. Thus there exists k_0 such that $\|Te_{n_k} - x\| < \epsilon/2$ if $k \geq k_0$. Thus $|\langle Te_{n_k} - x, e_{n_k}^* \rangle| = |\langle Te_{n_k}, e_{n_k}^* \rangle|$

– $\langle x, e_{n_k}^* \rangle \mid \langle \epsilon/2$ for all $k \geq k_0$. But $\mid \langle Te_{n_k}, e_{n_k}^* \rangle \mid > \epsilon$. Therefore, $\mid \langle x, e_{n_k}^* \rangle \mid \geq \epsilon/2$ for all $k \geq k_0$. Hence

$$\lim_{k \rightarrow \infty} \langle x, e_n^* \rangle \neq 0,$$

which contradicts Lemma 2.2.2.4. $\square$

Definition 2.2.2.14. Let $(x_n)_{n \in \mathbb{N}}$ be an orthonormal sequence in X , and A a subset of $\mathbb{N}$. Set P_A to denote the projection:

$$P_A : [x_n] \rightarrow [x_n],$$

$$P_A \left(\sum_{n=1}^{\infty} a_n x_n \right) = \sum_{n \in A} a_n x_n.$$

Theorem 2.2.2.15. Let (e_n) be an infinite orthonormal sequence in a reflexive Banach space X . Let $T = \sum_{n=1}^{\infty} \lambda_n e_n^* \otimes e_n$. Then T is compact if and only if $\lim_{n \rightarrow \infty} |\lambda_n| = 0$.

Proof: Let T be compact. By Proposition 2.2.2.13, $\|Te_n\| \rightarrow 0$. But $Te_n = \lambda_n e_n$. Hence $\|Te_n\| = |\lambda_n| \rightarrow 0$.

Conversely, let $|\lambda_n| \rightarrow 0$. Let P_j be the projection on $\text{span}\{e_1, \dots, e_j\}$. Then each P_j is a finite-rank operator and $T - TP_j = \sum_{n=j+1}^{\infty} \lambda_n e_n^* \otimes e_n$. But $\|T - TP_j\| \leq \sup_{h > j} |\lambda_h|$. Hence $\|T - TP_j\| \rightarrow 0$. Since TP_j is a finite-rank operator, T is compact. $\square$

Definition 2.2.2.16. Let (x_n) be an infinite sequence in X . $D((x_n))$ is defined to be the set $\{(z_n^*) \text{ orthonormal in } X^* \text{ such that } \|\sum_{n=1}^{\infty} \lambda_n \langle x, z_n^* \rangle x_n\| \leq \|x\| \sup_n |\lambda_n|, x \in X, (\lambda_n) \in l^\infty\}$.

Theorem 2.2.2.17. Let $\lambda_n \in \mathbb{C}$, (e_n) be an infinite orthonormal sequence in X , and $(\theta_n^*) \in D((e_n))$. Then the following are equivalent:

- (i) $\sum_{n=1}^{\infty} \lambda_n \langle x, \theta_n^* \rangle e_n \in X$ for all $x \in X$.
- (ii) $(\lambda_n) \in l^\infty$.
- (iii) $T = \sum_{n=1}^{\infty} \lambda_n \theta_n^* \otimes e_n \in L(X)$ and $\|T\| = \|(\lambda_n)\|_\infty$.

Proof:

(i) $\rightarrow$ (ii).

Let $T_N x = \sum_{n=1}^N \lambda_n \langle x, \theta_n^* \rangle e_n$. Then $T_N \in \mathcal{L}(X)$ and the orthogonality of (e_n) implies that $\|T_N\| = \sup_{1 \leq n \leq N} |\lambda_n|$. If we set $Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \theta_n^* \rangle e_n$, then (i) implies $\|Tx\| < \infty$ for all x and $\|T_N x\| \rightarrow \|Tx\|$. Hence $(\lambda_n) \in l^\infty$.

(ii) $\rightarrow$ (iii).

For any $x \in X$, set $Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \theta_n^* \rangle e_n$. Since (e_n) is orthonormal and $(\lambda_n) \in l^\infty$, we have $Tx \in X$. Set $T_k = \sum_{n=1}^k \lambda_n \theta_n^* \otimes e_n$. Then $T_k \in L(X)$ and $\|T_k\| = \sup_{1 \leq n \leq k} |\lambda_n|$ by the orthonormality of (e_n) . Since $Tx \in X$ and $T_k x \rightarrow Tx$, it follows that T must be bounded, noting that $\sup_k \|T_k\| \leq \|(\lambda_k)\|_\infty$. Hence $T = \sum_{n=1}^{\infty} \lambda_n \theta_n^* \otimes e_n \in \mathcal{L}(X)$.

(iii) $\rightarrow$ (i).

Clear. $\square$

Theorem 2.2.2.18. Let (e_n) be an orthonormal basis in X and (e_n^*) be the associated linear functionals. Let $T = \sum_{n=1}^{\infty} \lambda_n e_n^* \otimes e_n$ for some $(\lambda_n) \in l^\infty$. Then the following are equivalent:

(i) $T^{-1} \in \mathcal{L}(X)$.

(ii) $(\lambda_n^{-1}) \in l^\infty$.

This is the case $T^{-1} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} e_n^* \otimes e_n$

Proof:

(i) $\rightarrow$ (ii).

Let $T^{-1} \in \mathcal{L}(X)$. Then $T^{-1}(Te_n) = T^{-1}(\lambda_n e_n) = e_n$. Hence $T^{-1}e_n = \frac{1}{\lambda_n} e_n$ for all n . Further, for any $x \in X$, $T^{-1}Tx = x = \sum_{n=1}^{\infty} \langle x, e_n^* \rangle e_n$, so that $\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\| \|Tx\|$. Hence, $|\lambda_n| = \|Te_n\| \geq \frac{1}{\|T^{-1}\|}$, and $\inf_n |\lambda_n| > 0$. Further, $T^{-1}x = \sum_{n=1}^{\infty} \langle x, e_n^* \rangle T^{-1}e_n = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} e_n^* \otimes e_n$.

(ii) $\rightarrow$ (i).

Let $\inf_n |\lambda_n| > 0$. Hence $(\frac{1}{\lambda_n}) \in l^\infty$, and by Theorem 2.2.2.17, the operator $A = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} e_n^* \otimes e_n \in \mathcal{L}(X)$. Further, $AT = TA = I$. Hence $A = T^{-1}$. $\square$

2.2.3 P-Orthogonality in Banach Spaces.

Throughout, X is a Banach space.

Another type of orthogonality is defined in the following:

Definition 2.2.3.1. Let X be a Banach space and x and y be elements of X . x is said to be p -orthogonal to y , written $x \perp_p y$, if and only if for all scalars a and b , $\|ax + by\|^p = |a|^p \|x\|^p + |b|^p \|y\|^p$ holds.

It is easy to see that if $x \perp_p y$ then $y \perp_p x$.

Lemma 2.2.3.2. Let $T \in \mathcal{L}(X)$ be an isometry. If $x \perp_p y$, then $Tx \perp_p Ty$.

Proof: Consider:

$$\begin{aligned} \|Tx + Ty\|^p &= \|T(x + y)\|^p \\ &= \|x + y\|^p \\ &= \|x\|^p + \|y\|^p \\ &= \|Tx\|^p + \|Ty\|^p. \end{aligned}$$

Hence $Tx \perp_p Ty$. And that completes the proof. $\square$

Definition 2.2.3.3. Let (x_n) be a sequence in X . (x_n) is called p -orthonormal if for any choice of $a_1, \dots, a_k \in \mathbb{C}$ we have

$$\left\| \sum_{n=1}^k a_n x_n \right\| = \left(\sum_{n=1}^k |a_n|^p \right)^{1/p}.$$

Theorem 2.2.3.4. Let H be a Hilbert space and x, y be two elements in H . Then $x \perp_2 y$ if and only if $\langle x, y \rangle = 0$.

Proof: Let $\langle x, y \rangle = 0$. Then

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \\ &= \|x\|^2 + 0 + \|y\|^2 \\ &= \|x\|^2 + \|y\|^2. \end{aligned}$$

Hence $x \perp_2 y$.

Conversely, let $x \perp_2 y$. Then

$$\begin{aligned}\|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \|y\|^2.\end{aligned}$$

Hence $\langle x, y \rangle = 0$. That completes the proof. $\square$

Theorem 2.2.3.5. Let x and y be elements in l^p where $1 < p < \infty$ and $p \neq 2$. Then $x \perp_p y$ if and only if $\text{supp}(x) \cap \text{supp}(y) = \phi$.

Proof: Let $\text{supp}(x) \cap \text{supp}(y) = \phi$. Then $x = (x_i)$ and $y = (y_i)$ with $y_i \neq 0$ if $x_i = 0$. Further, $x + y = (x_i + y_i)$. Then

$$\begin{aligned}\|x + y\|^p &= \sum_{i=1}^{\infty} |x_i + y_i|^p \\ &= \sum_{i \in \text{supp}(x)} |x_i|^p + \sum_{i \in [\text{supp}(x)]^c} |y_i|^p \\ &= \|x\|^p + \|y\|^p.\end{aligned}$$

Thus $x \perp_p y$.

Conversely, let $x \perp_p y$. Assume the contrary that $\text{supp}(x) \cap \text{supp}(y) \neq \phi$. Then there exists i_0 such that $x_{i_0} \neq 0$ and $y_{i_0} \neq 0$. Further,

$$\begin{aligned}\|x + y\|^p &= \sum_{i=1}^{\infty} |x_i + y_i|^p \\ &= \sum_{i \in \text{supp}(x)} |x_i|^p + \sum_{i \in \text{supp}(y)} |y_i|^p + |x_{i_0} + y_{i_0}|^p.\end{aligned}$$

But since $|x_{i_0} + y_{i_0}|^p \neq |x_{i_0}|^p + |y_{i_0}|^p$ for $1 < p < \infty$. Consequently, $\|x + y\|^p \neq \|x\|^p + \|y\|^p$ which contradicts x being p -orthogonal to y . That ends the proof. $\square$

Theorem 2.2.3.6. Let f and g be elements in $L^p, 1 < p < \infty$. Then $f \perp_p g$ if and only if $\mu(\text{supp}(f) \cap \text{supp}(g)) = 0$.

Proof: Let $f \perp_p g$. Then

$$\|f + g\|_p^p = \|f\|_p^p + \|g\|_p^p \quad (*)$$

Assume that $\text{supp}(f) \cap \text{supp}(g) = E$, with $\mu(E) > 0$. Claim: (*) can not be true. Consider,

$$\begin{aligned}\|f + g\|_p^p &= \int_I |f + g|^p d\mu \\ &= \int_{E_1} |f|^p d\mu + \int_E |f + g|^p d\mu + \int_{E_2} |g|^p d\mu\end{aligned}$$

where $E_1 = \text{supp}(f) - E$, $E_2 = \text{supp}(g) - E$. Thus $\|f + g\|_p^p \neq \|f\|_p^p + \|g\|_p^p$ which contradicts (*).

Conversely, assume $\text{supp}(f) \cap \text{supp}(g) = E$ with $\mu(E) = 0$. Then

$$\begin{aligned}\|f + g\|_p^p &= \int_I |f + g|^p d\mu \\ &= \int_{\text{supp}(f) \cup [\text{supp}(f)]^c} |f + g|^p d\mu \\ &= \int_{\text{supp}(f)} |f + g|^p d\mu + \int_{[\text{supp}(f)]^c} |f + g|^p d\mu \\ &= \int_{\text{supp}(f)} |f|^p d\mu + \int_{[\text{supp}(f)]^c} |g|^p d\mu \\ &= \int_I |f|^p d\mu + \int_I |g|^p d\mu \\ &= \|f\|_p^p + \|g\|_p^p.\end{aligned}$$

Hence $f \perp_p g$. That completes the proof. $\square$

Theorem 2.2.3.7. Let X be a Banach space. If (x_n) is p -orthonormal sequence in X , then $\left(\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^{p^*} \right)^{1/p^*} \leq \|x^*\|$.

Proof: By the Hahn-Banach Theorem, we have:

$$\begin{aligned}
\left(\sum_{n=1}^k | \langle x_n, x^* \rangle |^{p^*} \right)^{1/p^*} &= \left| \sum_{n=1}^k a_n \langle x_n, x^* \rangle \right|, \|a\| \leq 1, (a_n) \in l^p. \\
&= \left| \langle \sum_{n=1}^k a_n x_n, x^* \rangle \right| \\
&\leq \left\| \sum_{n=1}^k a_n x_n \right\| \|x^*\| \\
&= \left(\sum_{n=1}^k |a_n|^p \right)^{1/p} \|x^*\| \\
&\leq \|x^*\| \\
&= \|x^*\|.
\end{aligned}$$

Now take the limit as $k \rightarrow \infty$, we get:

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} | \langle x_n, x^* \rangle |^{p^*} \right)^{1/p^*} &= \lim_{k \rightarrow \infty} \left(\sum_{n=1}^k | \langle x_n, x^* \rangle |^{p^*} \right)^{1/p^*} \\
&\leq \lim_{k \rightarrow \infty} \|x^*\|.
\end{aligned}$$

Thus $\left(\sum_{n=1}^{\infty} | \langle x_n, x^* \rangle |^{p^*} \right)^{1/p^*} \leq \|x^*\|$. That ends the proof. $\square$

CHAPTER III

SCHATTEN CLASSES ON BANACH SPACES

We are ready in this chapter to generalize the concept of Schatten classes to Banach spaces which is the main part of the thesis. The chapter consists of two sections. In section 1 we are going to define Schatten classes on Banach spaces. In section 2 we will present some results on Schatten type classes.

Throughout, X and Y are Banach spaces.

3.1 Generalizing Schatten Classes to Banach Spaces.

In this section we will define the Schatten classes on Banach spaces.

Definition 3.1.1. Let $1 \leq p < \infty$ and $r \geq p$. $\mathcal{C}_r(X, Y)$ is defined to be the set of all linear operators $T \in \mathcal{L}(X, Y)$ such that

$$\sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|Te_n\|^r \right)^{1/r} < \infty$$

where the supremum is taken over all p -orthonormal sequences (e_n) in X .

$\mathcal{C}_r(X, Y)$ is called the Schatten class of index r .

Lemma 3.1.2. $\mathcal{C}_r(X, Y)$ is a vector space.

Proof: Let $T \in \mathcal{C}_r(X, Y)$ and λ is a scalar. Then

$$\begin{aligned} \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|(\lambda T)e_n\|^r \right)^{1/r} &= |\lambda| \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|Te_n\|^r \right)^{1/r} \\ &< \infty. \end{aligned}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{T}\mathcal{E}\mathcal{X}$

Hence $\lambda T \in \mathcal{C}_r(X, Y)$. Now let $T_1, T_2 \in \mathcal{C}_r(X, Y)$. Consider,

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \|(T_1 + T_2)e_n\|^r \right)^{1/r} &= \left(\sum_{n=1}^{\infty} \|T_1 e_n + T_2 e_n\|^r \right)^{1/r} \\ &\leq \left(\sum_{n=1}^{\infty} \|T_1 e_n\|^r \right)^{1/r} + \left(\sum_{n=1}^{\infty} \|T_2 e_n\|^r \right)^{1/r} \\ &\leq \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|T_1 e_n\|^r \right)^{1/r} + \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|T_2 e_n\|^r \right)^{1/r} \end{aligned}$$

Hence

$$\begin{aligned} \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|(T_1 + T_2)e_n\|^r \right)^{1/r} &\leq \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|T_1 e_n\|^r \right)^{1/r} + \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|T_2 e_n\|^r \right)^{1/r} \\ &< \infty. \end{aligned}$$

Hence $T_1 + T_2 \in \mathcal{C}_r(X, Y)$. Hence the required. $\square$

Definition 3.1.3. Suppose that $1 \leq p < \infty$, $r \geq p$ and $T \in \mathcal{C}_r(X, Y)$. We set $\|T\|_r$ to denote

$$\sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|T e_n\|^r \right)^{1/r}$$

where the supremum is taken over all p -orthonormal sequences (e_n) .

Lemma 3.1.4. Let $T \in \mathcal{C}_r(X, Y)$. Then $\|T\| \leq \|T\|_r$ for all $r \geq p$ and $1 \leq p < \infty$.

Proof: Every singleton $\{x\}$ is a p -orthonormal set since $\|ax\|^p = |a|^p \|x\|^p$. Thus $\|Tx\| \leq \|T\|_r$ for all x with $\|x\| = 1$. Further

$$\sup_{\|x\| \leq 1} \|Tx\| \leq \|T\|_r.$$

This implies that

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| \leq \|T\|_r.$$

That ends the proof. $\square$

Lemma 3.1.5. $\|\cdot\|_r$ is a norm on $\mathcal{C}_r(X, Y)$.

Proof: It is obvious that $\|T\|_r \geq 0$ for all $T \in \mathcal{C}_r(X, Y)$ and if $T = 0$, then $\|T\|_r = 0$. Now if $\|T\|_r = 0$, then by Lemma 3.1.4 we get $\|T\| = 0$ and hence $T = 0$. Further,

$$\begin{aligned}\|\lambda T\|_r &= \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|\lambda T e_n\|^r \right)^{1/r} \\ &= \sup_{(e_n)} \left(\sum_{n=1}^{\infty} |\lambda|^r \|T e_n\|^r \right)^{1/r} \\ &= \sup_{(e_n)} |\lambda| \left(\sum_{n=1}^{\infty} \|T e_n\|^r \right)^{1/r}\end{aligned}$$

Hence $\|\lambda T\|_r = |\lambda| \|T\|_r$. Finally, by using Minkowski's inequality we get:

$$\begin{aligned}\left(\sum_{n=1}^{\infty} \|(T_1 + T_2)e_n\|^r \right)^{1/r} &\leq \left(\sum_{n=1}^{\infty} \|T_1 e_n\|^r \right)^{1/r} + \left(\sum_{n=1}^{\infty} \|T_2 e_n\|^r \right)^{1/r} \\ &\leq \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|T_1 e_n\|^r \right)^{1/r} + \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|T_2 e_n\|^r \right)^{1/r}\end{aligned}$$

Hence

$$\sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|(T_1 + T_2)e_n\|^r \right)^{1/r} \leq \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|T_1 e_n\|^r \right)^{1/r} + \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|T_2 e_n\|^r \right)^{1/r}$$

Hence $\|T_1 + T_2\|_r \leq \|T_1\|_r + \|T_2\|_r$. And that completes the proof. $\square$

Theorem 3.1.6. $(\mathcal{C}_r(X, Y), \|\cdot\|_r)$ is a Banach space.

Proof: Let $T_n \in \mathcal{C}_r(X, Y)$ such that $\sum_{n=1}^{\infty} \|T_n\|_r < \infty$. Claim: $T = \sum_{n=1}^{\infty} T_n \in \mathcal{C}_r(X, Y)$. Since $\sum_{n=1}^{\infty} \|T_n\|_r < \infty$, then by Lemma 3.1.4 we get $\sum_{n=1}^{\infty} \|T_n\| < \infty$. But $\mathcal{L}(X, Y)$ is a Banach space, consequently $T = \sum_{n=1}^{\infty} T_n \in \mathcal{L}(X, Y)$, where $Tx = \sum_{n=1}^{\infty} T_n x$.

Now we claim $T \in \mathcal{C}_r(X, Y)$, that is $\lim_{k \rightarrow \infty} \sum_{n=1}^k T_n$ exists in $\mathcal{C}_r(X, Y)$. Indeed: By Lemma 1.3.10 we have:

$$\left(\sum_{i=1}^{\infty} \left\| \sum_{n=1}^{\infty} T_n e_n \right\|^r \right)^{1/r} \leq \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} \|T_n e_n\|^r \right)^{1/r}$$

where (e_n) is a p -orthonormal sequence in X . But

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} \|T_n e_n\|^r \right)^{1/r} \leq \sum_{n=1}^{\infty} \|T_n\|_r < \infty.$$

Hence the operator $T = \sum_{n=1}^{\infty} T_n \in \mathcal{C}_r(X, Y)$.

It remains to show that $\lim_{k \rightarrow \infty} \sum_{n=1}^k T_n$ exists in $\mathcal{C}_r(X, Y)$. That is we want to prove that $\|\sum_{n=1}^k T_n - T\|_r \rightarrow 0$. But

$$\begin{aligned} \left(\sum_{i=1}^{\infty} \left\| \sum_{n=k+1}^{\infty} T_n e_n \right\|^r \right)^{1/r} &\leq \sum_{n=k+1}^{\infty} \left(\sum_{i=1}^{\infty} \|T_n e_n\|^r \right)^{1/r} \\ &\leq \sum_{n=k+1}^{\infty} \|T_n\|_r. \end{aligned}$$

So

$$\sup_{(e_n)} \left(\sum_{i=1}^{\infty} \left\| \sum_{n=k+1}^{\infty} T_n e_n \right\|^r \right)^{1/r} \leq \sum_{n=k+1}^{\infty} \|T_n\|_r$$

Thus

$$\left\| \sum_{n=k+1}^{\infty} T_n \right\| \leq \sum_{n=k+1}^{\infty} \|T_n\|_r.$$

But $\sum_{n=k+1}^{\infty} \|T_n\|_r \rightarrow 0$ since $\sum_{n=1}^{\infty} \|T_n\|_r < \infty$. Hence the required. $\square$

Remarks 3.1.7.

(1) For $p = 2$, $X = Y = l^2$ we have:

$\mathcal{C}_r(l^2, l^2) = \{T \in \mathcal{L}(l^2) : \sum_{n=1}^{\infty} \|T e_n\|^r \leq \lambda < \infty \text{ for all 2-orthonormal sequences } (e_n) \text{ in } l^2\}$. If $r \geq 2$, then $\mathcal{C}_r(l^2, l^2)$ is just the Schatten class $\mathcal{C}_r(l^2)$ a fact which follows from the definition of $\mathcal{C}_r(l^2)$, [12, p.75].

(2) $\mathcal{C}_{\infty}(l^1, E) = \mathcal{L}(l^1, E)$ where E is a Banach space.

Proof: As in Lemma 1.2.4, $\mathcal{C}_{\infty}(l^1, E) \subseteq \mathcal{L}(l^1, E)$, and

$$\|T\| \leq \|T\|_{\infty} \quad (*)$$

Now let (e_i) be a 1-orthonormal sequence in l^1 , and $T \in \mathcal{L}(l^1, E)$. Then $\|Te_i\| \leq \|T\|\|e_i\|$ for all i . This implies that $\|(\|Te_i\|)\|_\infty \leq \|T\|$. So $\|T\|_\infty \leq \|T\|$. From (*) we get $\|T\| = \|T\|_\infty$. $\square$

Theorem 3.1.8. If $r_1 \leq r_2$, then $\mathcal{C}_{r_1}(X) \subseteq \mathcal{C}_{r_2}(X)$ and $\|T\|_{r_2} \leq \|T\|_{r_1}$.

Proof: Let $T \in \mathcal{C}_{r_1}(X)$ and (e_n) be a p -orthonormal sequence in X . Since

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \|Te_n\|^{r_2} \right)^{1/r_2} &\leq \left(\sum_{n=1}^{\infty} \|Te_n\|^{r_1} \right)^{1/r_1} \\ &\leq \|T\|_{r_1} \end{aligned}$$

We get $T \in \mathcal{C}_{r_2}(X)$ and $\|T\|_{r_2} \leq \|T\|_{r_1}$. $\square$

Theorem 3.1.9. Let $A \in \mathcal{C}_r(X)$ and $B \in \mathcal{L}(X)$. Then $BA \in \mathcal{C}_r(X)$ and $\|BA\|_r \leq \|B\|\|A\|_r$.

Proof: Consider

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \|(BA)e_n\|^r \right)^{1/r} &\leq \left(\sum_{n=1}^{\infty} \|B\|^r \|Ae_n\|^r \right)^{1/r} \\ &\leq \|B\| \left(\sum_{n=1}^{\infty} \|Ae_n\|^r \right)^{1/r} \end{aligned}$$

Take the supremum over all p -orthonormal sequences (e_n) in X , to both sides of the inequality, we get:

$$\begin{aligned} \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|(BA)e_n\|^r \right)^{1/r} &\leq \|B\| \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|Ae_n\|^r \right)^{1/r} \\ \text{and } \|BA\|_r &\leq \|B\|\|A\|_r. \end{aligned}$$

Hence $BA \in \mathcal{C}_r(X)$ which completes the proof. $\square$

Theorem 3.1.10. Let $A \in \mathcal{C}_r(X)$, $r \geq p$. If T is an isometry onto X , then $\|AT\|_r = \|A\|_r$.

Proof: Let (e_n) be any p -orthonormal sequence in X . Since T is isometric onto, then there exists (f_n) such that $Te_n = f_n$, and (f_n) is p -orthonormal, by Lemma 2.2.3.2. Now

$$\sum_{n=1}^{\infty} \|ATe_n\|^r = \sum_{n=1}^{\infty} \|Af_n\|^r \quad (*)$$

Again since T is isometric onto, then for every (f_n) p -orthonormal sequence there exists (e_n) p -orthonormal sequence such that $Te_n = f_n$. Hence $(*)$ gives

$$\sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|ATe_n\|^r \right)^{1/r} = \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|Ae_n\|^r \right)^{1/r}$$

and this implies that $\|AT\|_r = \|A\|_r$. $\square$

Theorem 3.1.11. Let $T \in \mathcal{C}_r(l^p)$. Then there exists $(\lambda_n) \in l^r$ and $g_n \in B_1(l^p)$ such that $T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes g_n$ where (δ_n) is the natural basis of l^p .

Proof: Let $x \in l^p$, then $x = \sum_{n=1}^{\infty} \langle x, \delta_n \rangle \delta_n$, where the series converges in l^p . Then

$$\begin{aligned} Tx &= T \left(\lim_{k \rightarrow \infty} \sum_{n=1}^k \langle x, \delta_n \rangle \delta_n \right) \\ &= \lim_{k \rightarrow \infty} \sum_{n=1}^k \langle x, \delta_n \rangle T\delta_n \\ &= \sum_{n=1}^{\infty} \|T\delta_n\| \langle x, \delta_n \rangle g_n, \end{aligned}$$

where $g_n = \frac{T\delta_n}{\|T\delta_n\|}$, noting that we remove the indices where $T\delta_n = 0$. Set $\lambda_n = \|T\delta_n\|$. Since $T \in \mathcal{C}_r(l^p)$, and (δ_n) is p -orthonormal, we get $(\lambda_n) \in l^r$. Thus $Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \delta_n \rangle g_n$. Consequently, $T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes g_n$. And that completes the proof. $\square$

Definition 3.1.12. Let $1 \leq p < \infty$ and E be any Banach space. We set

$$\mathcal{C}_p(l^{p^*}, E) = \{T \in \mathcal{L}(l^{p^*}, E) : \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|Te_n\|^p \right)^{1/p} < \infty\}$$

where the supremum is taken over all p^* -orthonormal sequences (e_n) in l^{p^*} .

Note here p need not be greater than p^* . We set

$$\|T\|_p = \sup_{(e_n)} \left(\sum_{n=1}^{\infty} \|Te_n\|^p \right)^{1/p},$$

where the supremum is taken over all p^* -orthonormal sequence (e_n) .

For $\mathcal{C}_p(l^{p^*}, E)$ we have a better result than that in Theorem 3.1.11. Indeed:

Theorem 3.1.13. Let $1 < p < \infty$. Then the following are equivalent:

(i) $T \in \mathcal{C}_p(l^{p^*}, E)$.

(ii) $T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes g_n$, where $(\lambda_n) \in l^p$ and $g_n \in E$ with $\|g_n\| = 1$.

In this case $\|T\|_p = \|(\lambda_n)\|_p$.

Proof:

(i) $\rightarrow$ (ii).

Let $T \in \mathcal{C}_p(l^{p^*}, E)$ and $x \in l^{p^*}$. Since (δ_n) is a basis for l^{p^*} , we have $x = \sum_{n=1}^{\infty} \langle x, \delta_n \rangle \delta_n$. So $Tx = \sum_{n=1}^{\infty} \langle x, \delta_n \rangle T\delta_n$ where the series converges strongly. Hence $T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes g_n$ where $\lambda_n = \|T\delta_n\|$ and $g_n = \frac{T\delta_n}{\|T\delta_n\|}$. But since $T \in \mathcal{C}_p(l^{p^*}, E)$ then

$$\left(\sum_{n=1}^{\infty} |\lambda_n|^p \right)^{1/p} = \left(\sum_{n=1}^{\infty} \|T\delta_n\|^p \right)^{1/p} \leq \|T\|_p.$$

So $(\lambda_n) \in l^p$ and $\|(\lambda_n)\|_p \leq \|T\|_p$.

(ii) $\rightarrow$ (i).

Conversely, Let $T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes g_n$, $(\lambda_n) \in l^p$ and $g_n \in E$ with $\|g_n\| = 1$. We claim that $T \in \mathcal{C}_p(l^{p^*}, E)$. First we prove $T \in \mathcal{L}(l^{p^*}, E)$. For this let $x \in l^{p^*}$. Then

$$\begin{aligned} \|Tx\| &\leq \sum_{n=1}^{\infty} |\lambda_n| \langle \delta_n, x \rangle \\ &\leq \|(\lambda_n)\|_p \|x\|_{p^*}. \end{aligned}$$

(by Hölder's inequality.)

Thus $T \in \mathcal{L}(l^{p^*}, E)$ and $\|T\| \leq \|(\lambda_n)\|_p$. Now let (e_k) be any p^* -orthonormal set in l^{p^*} . Then

$$\begin{aligned} \left(\sum_{k=1}^{\infty} \|Te_k\|^p \right)^{1/p} &= \left(\sum_{k=1}^{\infty} \left\| \sum_{n=1}^{\infty} \lambda_n \langle e_k, e_n \rangle g_n \right\|^p \right)^{1/p} \\ &\leq \left(\sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} |\lambda_n| |\langle e_k, e_n \rangle| \right)^p \right)^{1/p} \\ &\leq \left| \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_k |\lambda_n| \langle e_k, e_n \rangle \right| \quad \text{for } \|(a_k)\|_{p^*} = 1, \\ &\quad \text{(by the Hahn-Banach Theorem.)} \\ &\leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |a_k| |\lambda_n| |\langle e_k, e_n \rangle|. \end{aligned}$$

But $\| \sum_{n=1}^{\infty} |\lambda_n| \delta_n \|_p = (\sum_{n=1}^{\infty} |\lambda_n|^p)^{1/p}$. Since $|\delta_n| = \delta_n$. Thus $\sum_{n=1}^{\infty} |\lambda_n| \delta_n \in l^p$. Consequently, since $|e_k| \in l^{p^*}$ we have:

$$\left(\sum_{k=1}^{\infty} \|Te_k\|^p \right)^{1/p} \leq \sum_{k=1}^{\infty} |a_k| < \sum_{n=1}^{\infty} |\lambda_n| \delta_n, |e_k| >.$$

Again since (e_k) are p^* -orthonormal then $|e_k|$ is also p^* -orthonormal. So $\| \sum_{k=1}^{\infty} |a_k| e_k \|_{p^*} = \|(a_k)\|_{p^*} = 1$. So $\sum_{k=1}^{\infty} |a_k| e_k \in l^{p^*}$. Hence

$$\begin{aligned} \left(\sum_{k=1}^{\infty} \|Te_k\|^p \right)^{1/p} &\leq \left| \left\langle \sum_{n=1}^{\infty} |\lambda_n| \delta_n, \sum_{k=1}^{\infty} |a_k| e_k \right\rangle \right| \\ &\leq \left\| \sum_{n=1}^{\infty} |\lambda_n| \delta_n \right\|_p \left\| \sum_{k=1}^{\infty} |a_k| e_k \right\|_{p^*} \\ &\leq \|(\lambda_n)\|_p. \end{aligned}$$

Since (e_k) was arbitrary p^* -orthonormal sequence, it follows that $T \in \mathcal{C}_p(l^{p^*}, E)$ and $\|T\|_p \leq \|(\lambda_n)\|_p$. $\square$

Theorem 3.1.14. If $A \in \mathcal{C}_r(X, Y)$ and $B \in \mathcal{L}(Y, Z)$, then $BA \in \mathcal{C}_r(X, Z)$.

Proof:

Consider

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \|BA\delta_n\|^r \right)^{1/r} &\leq \left(\sum_{n=1}^{\infty} \|B\|^r \|A\delta_n\|^r \right)^{1/r} \\ &\leq \|B\| \left(\sum_{n=1}^{\infty} \|A\delta_n\|^r \right)^{1/r} \end{aligned}$$

Take supremum over all p -orthonormal sequences in X , to both sides of the inequality, we get:

$$\|BA\|_r \leq \|B\|\|A\|_r \quad \text{and} \quad BA \in \mathcal{C}_r(X, Z). \quad \square$$

Theorem 3.1.15. For $r \geq p^*$, $\mathcal{F}(l^p) \subseteq \mathcal{C}_r(l^p)$.

Proof:

Let $x^* \in l^{p^*}$, $y \in l^p$ and (e_n) be any p -orthonormal sequence in l^p . Then

$$\left(\sum_{n=1}^{\infty} \|(x^* \otimes y)(e_n)\|^r \right)^{1/r} = \left(\sum_{n=1}^{\infty} |\langle x^*, e_n \rangle|^r \right)^{1/r} \|y\|$$

since $r \geq p^*$. Now

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \|(x^* \otimes y)(e_n)\|^r \right)^{1/r} &= \|y\| \left(\sum_{n=1}^{\infty} |\langle x^*, e_n \rangle|^r \right)^{1/r} \\ &\leq \|y\| \left(\sum_{n=1}^{\infty} |\langle x^*, e_n \rangle|^{p^*} \right)^{1/p^*} \\ &\leq \|y\| \|x^*\|. \end{aligned}$$

Taking the supremum over all p -orthonormal sequences (e_n) , we get:
 $\|x^* \otimes y\|_r \leq \|x^*\| \|y\|$. Thus $\mathcal{F}(l^p) \subseteq \mathcal{C}_r(l^p)$. $\square$

Theorem 3.1.16. $\mathcal{C}_r(l^p) \subseteq \mathcal{K}(l^p)$ for $r \geq p \geq 2$.

Proof: $T \in \mathcal{C}_r(l^p)$ implies that $T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes g_n$. Set $T_k = \sum_{n=1}^k \lambda_n \delta_n \otimes g_n$.

Then

$$\begin{aligned} \|T - T_k\| &= \left\| \sum_{n=k+1}^{\infty} \lambda_n \delta_n \otimes g_n \right\| \\ &\leq \left(\sum_{n=k+1}^{\infty} |\lambda_n|^p \right)^{1/p} \end{aligned}$$

But $(\sum_{n=k+1}^{\infty} |\lambda_n|^p)^{1/p}$ converges to 0. Since $T_k \in \mathcal{F}(l^p)$, then $T \in \mathcal{K}(l^p)$. That completes the proof. $\square$

Theorem 3.1.17. For $r \geq p \geq 2$, $x^* \otimes y \in \mathcal{C}_r(X, Y)$ for all $x^* \in X^*$ and $y \in Y$.

Proof: Let (e_n) be a p -orthonormal sequence in X . Then $(x^* \otimes y)(e_n) = \langle e_n, x^* \rangle y$. Further,

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \|(x^* \otimes y)(e_n)\|^r \right)^{1/r} &\leq \left(\sum_{n=1}^{\infty} \|(x^* \otimes y)(e_n)\|^p \right)^{1/p} \\ &= \|y\| \left(\sum_{n=1}^{\infty} |\langle e_n, x^* \rangle|^p \right)^{1/p} \\ &\leq \|y\| \left(\sum_{n=1}^{\infty} |\langle e_n, x^* \rangle|^{p^*} \right)^{1/p^*} \\ &\leq \|y\| \|x^*\| \quad (\text{by Theorem 2.2.3.7.}) \end{aligned}$$

Hence $x^* \otimes y \in \mathcal{C}_r(X, Y)$. $\square$

Theorem 3.1.18. Let $r \geq p \geq 2$. Then $\mathcal{F}(X, Y) \subseteq \mathcal{C}_r(X, Y)$.

Proof: Since $\mathcal{F}(X, Y) = \text{span} \{x^* \otimes y : x^* \in X^*, y \in Y\}$. $\square$

Theorem 3.1.19. Let E be reflexive Banach space and $1 < p < \infty$, $p \neq 2$. Then

$$\left(\mathcal{C}_p(l^{p^*}, E) \right)^* \cong \mathcal{C}_{p^*}(l^p, E^*).$$

Proof: Let $A \in \mathcal{C}_{p^*}(l^p, E^*)$. Define

$$F_A : \mathcal{C}_p(l^{p^*}, E) \rightarrow \mathbb{R} \quad \text{by}$$

$$F_A(T) = \sum_{n=1}^{\infty} \langle A\delta_n, T\delta_n \rangle.$$

Then

$$\begin{aligned} |F_A(T)| &\leq \sum_{n=1}^{\infty} |\langle A\delta_n, T\delta_n \rangle| \\ &\leq \left(\sum_{n=1}^{\infty} \|A\delta_n\|^{p^*} \right)^{1/p^*} \left(\sum_{n=1}^{\infty} \|T\delta_n\|^p \right)^{1/p} \quad (\text{by Hölder's inequality}) \\ &\leq \|A\|_{p^*} \|T\|_p \end{aligned}$$

Hence $\|F_A\| \leq \|A\|_{p^*}$. This implies that $F_A \in (\mathcal{C}_p(l^{p^*}, E))^*$. Further

$$\begin{aligned}
\|A\|_{p^*} &= \left(\sum_{n=1}^{\infty} \|A\delta_n\|^{p^*} \right)^{1/p^*} \quad (\text{by Theorem 3.1.13.}) \\
&= \sum_{n=1}^{\infty} a_n \|A\delta_n\|, \quad \|a_n\|_p = 1 \\
&= \left| \sum_{n=1}^{\infty} a_n \langle A\delta_n, y_n \rangle \right|, \quad \|y_n\| = 1 \\
&= \sum_{n=1}^{\infty} \langle A\delta_n, \tilde{y}_n \rangle, \quad \|\tilde{y}_n\|_p = 1
\end{aligned}$$

Define

$$\begin{aligned}
T_0 : l^{p^*} &\rightarrow E \quad \text{by} \\
T_0 &= \sum_{n=1}^{\infty} \delta_n \otimes \tilde{y}_n
\end{aligned}$$

Then

$$\begin{aligned}
\|T_0 x\| &\leq \|x\|_{p^*} \|(\tilde{y}_n)\|_p \quad (\text{by Hölder's inequality.}) \\
&\leq \|x\|_{p^*}.
\end{aligned}$$

So $T_0 \in \mathcal{L}(l^{p^*}, E)$. But $(\sum_{n=1}^{\infty} \|T_0 \delta_n\|^p)^{1/p} = (\sum_{n=1}^{\infty} \|(\tilde{y}_n)\|^p)^{1/p} = 1$. Again by Theorem 3.1.13, $T_0 \in \mathcal{C}_p(l^{p^*}, E)$ and $\|T_0\|_p = 1$. Thus

$$\begin{aligned}
\|A\|_{p^*} &= \left| \sum_{n=1}^{\infty} \langle A\delta_n, T_0 \delta_n \rangle \right| \\
&= |F_A(T_0)| \\
&\leq \|F_A\| \|T_0\|_p \\
&\leq \|F_A\|
\end{aligned}$$

This implies that $\|F_A\| = \|A\|_{p^*}$.

Now define

$$J : \mathcal{C}_{p^*}(l^p, E^*) \rightarrow (\mathcal{C}_p(l^{p^*}, E))^* \quad \text{by}$$

$$J(A) = F_A.$$

J is isometry. Further we claim J is onto, Indeed: Let $F \in (\mathcal{C}_p(l^{p^*}, E))^*$. Define

$$A : l^p \rightarrow E^* \quad \text{by}$$

$$\langle Ax, y \rangle = F(x \otimes y).$$

As $x \otimes y \in \mathcal{C}_p(l^{p^*}, E)$, it follows that

$$\begin{aligned} |\langle Ax, y \rangle| &= |F(x \otimes y)| \\ &\leq \|F\| \|x \otimes y\|_p \\ &= \|F\| \|x\| \|y\|. \end{aligned}$$

Hence $\|A\| \leq \|F\|$ and $A \in \mathcal{L}(l^p, E^*)$. Now consider

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \|A\delta_n\|^{p^*} \right)^{1/p^*} &= \left| \sum_{n=1}^{\infty} a_n \langle A\delta_n, g_n \rangle \right|, \quad \|(a_n)\|_p = 1, \quad \|g_n\| = 1, \quad g_n \in E. \\ &= \sum_{n=1}^{\infty} \langle A\delta_n, \tilde{g}_n \rangle, \quad (\|(\tilde{g}_n)\|_p = 1.) \\ &= \sum_{n=1}^{\infty} F(\delta_n \otimes \tilde{g}_n) \end{aligned}$$

But by Theorem 3.1.13, $\sum_{n=1}^{\infty} \delta_n \otimes \tilde{g}_n \in \mathcal{C}_p(l^{p^*}, E)$ and $\|\sum_{n=1}^{\infty} \delta_n \otimes \tilde{g}_n\|_p = (\sum_{n=1}^{\infty} \|\tilde{g}_n\|^p)^{1/p}$. Thus

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \|A\delta_n\|^{p^*} \right)^{1/p^*} &= \left| F \left(\sum_{n=1}^{\infty} \delta_n \otimes \tilde{g}_n \right) \right| \\ &\leq \|F\| \|(\tilde{g}_n)\|_p \\ &\leq \|F\|. \end{aligned}$$

Thus by Theorem 3.1.13, $A \in \mathcal{C}_{p^*}(l^p, E^*)$. Further it is easy to see that $F = F_A$. That completes the proof. $\square$

3.2 More Results on Schatten Classes.

In this section we will study the relationship between $\mathcal{C}_r(X, Y)$ and the operator ideal of p -summing operators.

Definition 3.2.1. Let $T \in \mathcal{L}(X, Y)$. Then T is called (p, q) -summing operator if there exists $\lambda > 0$ such that

$$\left(\sum_{i=1}^n \|Tx_i\|^p \right)^{1/p} \leq \lambda \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^q \right)^{1/q} \quad (*)$$

for all $\{x_1, x_2, \dots, x_n\} \subseteq X$. We write $\Pi_{p,q}(X, Y)$ for the space of (p, q) -summing operators from X to Y . For $T \in \Pi_{p,q}(X, Y)$ we set

$$\|T\|_{\Pi(p,q)} = \inf\{\lambda : * \text{ holds}\}.$$

It is known, [11, p.230], that $\|T\|_{\Pi(p,q)}$ is a norm on $\Pi_{p,q}(X, Y)$ and $(\Pi_{p,q}(X, Y), \|\cdot\|_{\Pi(p,q)})$ is a Banach space. If $p = q$ we write $\Pi_p(X, Y)$, $\|\cdot\|_{\Pi(p)}$ for $\Pi_{p,p}(X, Y)$ and $\|\cdot\|_{\Pi(p,p)}$ respectively.

We refer to Pietsch, [11] for more on (p, q) -summing operators.

Theorem 3.2.2. $\Pi_{r,p^*}(X, Y) \subseteq \mathcal{C}_r(X, Y)$.

Proof: Let $T \in \Pi_{r,p^*}(X, Y)$ and (e_n) be any p -orthonormal sequence in X . Then

$$\begin{aligned} \left(\sum_{n=1}^k \|Te_n\|^r \right)^{1/r} &\leq \lambda \sup_{\|x^*\| \leq 1} \left(\sum_{n=1}^{\infty} |\langle e_n, x^* \rangle|^{p^*} \right)^{1/p^*} \\ &\leq \lambda \|x^*\| \quad (\text{by Lemma 2.2.3.7.}) \\ &= \lambda. \end{aligned}$$

Hence $\left(\sum_{n=1}^{\infty} \|Te_n\|^r \right)^{1/r} \leq \lambda$. So $T \in \mathcal{C}_r(X, Y)$. $\square$

For $r = p = 2$, we have the following nice result:

Theorem 3.2.3. Let X be any Banach space and $0 < s < \infty$. Then $\Pi_s(l^2, X) = \mathcal{C}_2(l^2, X)$ where equality is the set equality.

Proof: By a result of Kwapien, [9], $\Pi_s(l^2, X) = \Pi_2(l^2, X)$. Thus it is enough to prove the theorem for $s = 2$.

By Theorem 3.2.2, $\Pi_2(l^2, X) \subseteq \mathcal{C}_2(l^2, X)$. For the converse, let $T \in \mathcal{C}_2(l^2, X)$ and (e_n) be any sequence in l^2 . If

$$\sup_{\|x^*\| \leq 1} \left(\sum_{n=1}^{\infty} |\langle e_n, x^* \rangle|^2 \right)^{1/2} = \infty$$

then

$$\left(\sum_{n=1}^{\infty} \|Te_n\|^2 \right)^{1/2} \leq \sup_{\|x^*\| \leq 1} \left(\sum_{n=1}^{\infty} |\langle e_n, x^* \rangle|^2 \right)^{1/2}.$$

So assume that

$$\sup_{\|x^*\| \leq 1} \left(\sum_{n=1}^{\infty} |\langle e_n, x^* \rangle|^2 \right)^{1/2} = \lambda < \infty.$$

Define $A : l^2 \rightarrow l^2$ by $A = \sum_{n=1}^{\infty} \delta_n \otimes e_n$. Then for each $x \in l^2$, we have

$$\begin{aligned} \|Ax\| &= \left\| \sum_{n=1}^{\infty} \langle \delta_n, x \rangle e_n \right\| \\ &= \left| \sum_{n=1}^{\infty} \langle \delta_n, x \rangle \langle e_n, x^* \rangle \right| \end{aligned}$$

for some $x^* \in l^2$, $\|x^*\| = 1$ by the Hahn-Banach Theorem. So

$$\begin{aligned} \|Ax\| &\leq \left(\sum_{n=1}^{\infty} |\langle \delta_n, x \rangle|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} |\langle e_n, x^* \rangle|^2 \right)^{1/2} \quad (\text{by Schwartz inequality}) \\ &\leq \|x\| \left(\sum_{n=1}^{\infty} |\langle e_n, x^* \rangle|^2 \right)^{1/2} \\ &\leq \lambda \|x\|. \end{aligned}$$

Thus $A \in L(l^2)$ and $\|A\| \leq \lambda$. Now

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \|Te_n\|^2 \right)^{1/2} &= \left(\sum_{n=1}^{\infty} \|TA\delta_n\|^2 \right)^{1/2} \\ &= \|A\| \left(\sum_{n=1}^{\infty} \left\| T \frac{A}{\|A\|} \delta_n \right\|^2 \right)^{1/2} \end{aligned}$$

Set $\tilde{A} = \frac{A}{\|A\|}$. Then $\|\tilde{A}\| = 1$ and thus since $\tilde{A} = \sum_{i=1}^4 a_i U_i$, where U_i are unitary operators. Since unitary operators take 2-orthonormal sets to 2-orthonormal sets, we get:

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \|T e_n\|^2 \right)^{1/2} &\leq \|A\| \|T\|_2 \sum_{i=1}^4 |a_i| \\ &\leq \gamma \|T\|_2 \sup_{\|x^*\| \leq 1} \left(\sum_{n=1}^{\infty} |\langle e_n, x^* \rangle|^2 \right)^{1/2} \end{aligned}$$

Thus $T \in \Pi_2(l^2, X)$ and $\|T\|_{\Pi(2)} \leq \gamma \|T\|_2$. And that completes the proof. $\square$

Theorem 3.2.3 is not true in general if l^2 is replaced by other Banach spaces as the following example shows:

Example 3.2.4. Let $p > 2$, $(\lambda_n) \in l^p$ but $(\lambda_n) \notin l^{p^*}$. Consider the operator $T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes \delta_n$. Then $T \in \Pi_p(l^p) \subseteq \mathcal{C}_p(l^p)$. But $T \notin \Pi_{p^*}(l^p)$. Hence $\Pi_s(l^p) \neq \mathcal{C}_p(l^p)$ for all $0 < s < \infty$. $\square$

For the next theorem which is our last result, we need the following definition:

Definition 3.2.5. Let X be a Banach space. (y_n) is called unconditionally summable if there exists λ such that $\|\sum_{n=1}^{\infty} a_n y_n\| \leq \lambda \sup_n |a_n|$ for all $(a_n) \in l^\infty$.

Theorem 3.2.6. Let $r \geq p \geq 2$ and $T : l^p \rightarrow X$ such that $T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes y_n$, where (y_n) is unconditionally summable and $(\lambda_n) \in l^r$. Then $T \in \mathcal{C}_r(l^p, X)$.

Proof: First we prove that $T \in \mathcal{L}(l^p, X)$: Consider

$$\begin{aligned} \|Tx\| &= \left\| \sum_{n=1}^{\infty} \lambda_n \langle x, \delta_n \rangle y_n \right\| \\ &\leq \lambda \|\lambda_n \langle x, \delta_k \rangle\|_{\infty} \quad (y_i \text{ is unconditionally summable}) \\ &\leq \lambda \|x\| \|(\lambda_n)\|_{\infty} \end{aligned}$$

Hence $T \in \mathcal{L}(l^p, X)$. Now let $A : l^p \rightarrow l^r$ be defined by $Ax = (\lambda_n \langle x, \delta_n \rangle)$. Then

$$\begin{aligned} \|Ax\| &= \left(\sum_{n=1}^{\infty} |\lambda_n \langle x, \delta_n \rangle|^r \right)^{1/r} \\ &\leq \sup_k |\langle x, \delta_k \rangle| \|(\lambda_n)\|_r \\ &\leq \|x\| \|(\lambda_n)\|_r \end{aligned}$$

Hence $A \in \mathcal{L}(l^p, l^r)$. We claim that $A \in \Pi_r(l^p, l^r)$. Indeed:

$$\begin{aligned}
\left(\sum_{i=1}^n \|Ae_i\|^r \right)^{1/r} &= \left(\sum_{i=1}^n \|(\lambda_n < e_i, \delta_n >)\|^r \right)^{1/r} \\
&= \left(\sum_{i=1}^n \sum_{n=1}^{\infty} |\lambda_n|^r |< e_i, \delta_n >|^r \right)^{1/r} \\
&= \left(\sum_{i=1}^n |\lambda_n|^r \sum_{n=1}^{\infty} |< e_i, \delta_n >|^r \right)^{1/r} \\
&\leq \|(\lambda_n)\|_r \sup_k \left(\sum_{i=1}^n |< e_i, \delta_k >|^r \right)^{1/r} \\
&\leq \|(\lambda_n)\|_r \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |< e_i, x^* >|^r \right)^{1/r}
\end{aligned}$$

Thus $A \in \Pi_r(l^p, l^r)$. Thus by Theorem 3.2.2, $A \in \mathcal{C}_r(l^p, l^r)$.

Now define $B : l^r \rightarrow X$ by $B(a_n) = \sum_{n=1}^{\infty} a_n y_n$. Then

$$\begin{aligned}
\|B(a_n)\| &\leq \lambda \|a_n\|_{\infty} \quad (\text{since } y_n \text{ is unconditionally summable.}) \\
&\leq \lambda \|a_n\|_r.
\end{aligned}$$

That implies that $B \in \mathcal{L}(l^r, X)$ and $\|B\| \leq \lambda$. Finally $T = BA$. Now $A \in \mathcal{C}_r(l^p, l^r)$ implies that $BA \in \mathcal{C}_r(l^p, X)$. Consequently, $T \in \mathcal{C}_r(l^p, X)$ which completes the proof. $\square$

INDEX OF SYMBOLS

In each line we give the number of the page on which the symbol is defined.

Symbol	Stands for ...
$A \oplus B$	Direct sum of subspaces A and B , 24
$B_1(X)$	Unit ball of the space X , 2
$\mathbb{C}$	A set of complex numbers, 1
$\mathcal{C}(\Omega)$	Space of all continuous functions from Ω to $\mathbb{R}$ (or $\mathbb{C}$), 2
$\mathcal{C}(I)$	Space of continuous functions on I , 3
$\mathcal{C}_1(H)$	Trace class on H , 16
$\mathcal{C}_p(H)$	Schatten class of index p on H , 8
$\mathcal{C}_r(l^2)$	Schatten classes on l^2 , 48
$\mathcal{C}_2(l^2, E)$	Schatten classes of index 2 from l^2 to E , 59
$\mathcal{C}_r(X)$	Schatten classes on X , 49
$\mathcal{C}_r(X, Y)$	Schatten classes of index r , 45
$\mathcal{C}_\infty(l^1, E)$	Compact operators from l^1 to E , 48
δ_n	Kronecker delta, the sequence $(0, \dots, 0, 0, \underset{\substack{\downarrow \\ n\text{-th coordinate}}}{1}, 0, 0, \dots)$, 50
$\dim T(X)$	Dimension of $T(X)$, 6
$\mathcal{F}(X)$	Space of finite rank operators on X , 6
$F(I)$	Set of all finite subsets of I , 25
$\mathcal{F}(X, Y)$	Finite rank operators from X into Y , 7
$\overline{g(t)}$	Conjugate of the function $g(t)$, 1
$\ f\ _\infty$	Sup-norm of a function f , 2
H	A separable Hilbert space, 1

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I	An interval $[0, 1]$, 1
I	An identity operator. 4
$\ker(T)$	Kernel of the operator T , 4
$\mathcal{K}(H)$	Space of compact operators on H , 6
$\mathcal{K}(X, Y)$	Space of compact operators from X to Y , 2
l^2	A sequence space, 1
l^p	A sequence space, 10
l^∞	A sequence space, 39
$\mathcal{L}^2(I)$	Space of Lebesgue square integrable functions on I , 1
$\mathcal{L}(X)$	Space of all bounded linear operators on X , 2
$\mathcal{L}(X, Y)$	Space of bounded linear operators from X into Y , 2
λ^{-1}	Inverse of λ , 5
$\bar{\lambda}$	Conjugate of a scalar λ , 31
$M^\perp$	Orthogonal complement of a set M , 24
$\mu(E)$	Measure of a set E , 42
$\mathbb{N}$	A set of natural numbers, 39
$\mathcal{N}(T)$	Null space of an operator T . 4
P_A	Projection operator, 39
p^*	Dual exponent, $\frac{1}{p} + \frac{1}{p^*} = 1$, 10
$\Pi_{p,q}(X, Y)$	Space of (p, q) -summing operators from X to Y , 57
$\ \cdot\ _{\pi(p,q)}$	a norm on $\Pi_{p,q}(X, Y)$, 57
$\Pi_p(X, Y)$	Space of (p, p) -summing operators from X to Y , 57
$\ \cdot\ _{\pi(p)}$	A norm on $\Pi_{p,p}(X, Y)$, 57
$\mathbb{R}$	A set of real numbers, 1
$\mathcal{R}(T - I)$	range of an operator $T - I$, 3
$\rho(T)$	Resolvent of an operator T , 4
$\sigma(T)$	Spectrum of an operator T , 3
$\text{sign}(f)$	Sign of a function f , 37
$\text{supp}(x)$	Support of x , 35
$[\text{supp}(x)]^c$	Complement of the support of x , 42
$\text{span}(e_i)$	Span of the sequence (e_i) , 29
T^{-1}	Inverse of an operator T , 40
T^*	Adjoint of an operator T , 4

$ T $	Absolute value of an operator T , 9
$\ T\ $	Norm of an operator T , 2
$\ T\ _p$	p -norm of T on $\mathcal{C}_p(H)$, 12
$\ T\ _r$	r -norm of T , 46
$\tau r(T)$	Trace of the operator T , 16
$\overline{T(S)}$	Closure of $T(S)$, 2
U	A partial isometry, 10
X^*	Dual space of a normed space X , 2
$[x_n]$	Closed span of $\{x_n\}$ in X , 32
$\langle x, y \rangle$	Inner product of x and y , 1
$\overline{\langle x, y \rangle}$	Conjugate of $\langle x, y \rangle$, 17
xOy	x is orthogonal to y , defined in [6], 33
$\perp$	Orthogonal, 23
$\perp$	(James) orthogonal, 30
$\perp_p$	p -orthogonal, 41
$\perp_2$	2-orthogona, 41
$\otimes$	Tensor product of operators, 6
$\cong$	isometrically isomorphic, 54

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