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Matrix Methods for Linear Systems of Differential Equations

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ABSTRACT

Matrix Methods for Linear Systems Of Differential Equations

In the present thesis we study the transforming the differential equations to first-order systems and analyze the basic theory of turning the differential equations into a first order system and show that this approach may lead to different solvability results and fluency requirements.

We present some topics and general extensions with some examples to derive the form of solutions for our systems.

DEDICATION

This thesis is dedicated to my mother whose encouragement has meant so much to me during the pursuit of my graduate degree and when writing this thesis.

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Chapter 1

Introduction

1.1 Background and history:

In this thesis, we introduce the systems of differential equations, where our equations in the system are linear. We need to use basic linear algebra and some methods of solving differential equations. In addition, we will use some mathematical analysis to cover our theory. Our results can apply to different fields such as physics, biology, chemistry... etc.

The beginning of matrices and determinants goes back to the second century BC although traces can be seen back to the fourth century BC. However, it was not until near the end of the 17th Century that the ideas reappeared and development really got underway.

It is not surprising that the beginnings of matrices and determinants should arise through the study of systems of linear equations. The Babylonians studied problems, which lead to simultaneous linear equations, and some of these are preserved in clay tablets that survive.

The Chinese, between 200 BC and 100 BC, came much closer to matrices than the Babylonians.

In 1870 the Jordan canonical form appeared in *Treatise on substitutions and algebraic equations* by Jordan. It appears in the context of a canonical form for linear substitutions over the finite field of order a prime.

Frobenius, in 1878, wrote an important work on matrices on linear substitutions and bilinear forms although he seemed unaware of Cayley's

work. Frobenius on deals with coefficients of forms and does not use the term matrix. However, he proved important results on canonical matrices as representatives of equivalence classes of matrices. He cites Kronecker and Weierstrass as having considered special cases of his results in 1874 and 1868 respectively. Frobenius also proved the general result that a matrix satisfies its characteristic equation. And in 1878 Frobenius contains the definition of the rank of a matrix, which he used in his work on canonical forms, and the definition of orthogonal matrices.

1.2 Statement of purpose:

“To use the basic theory of transforming the differential equations to first-order systems and then obtain the general properties of solutions”.

An n^{th} order differential equation can be transform into an n -dimensional system of first order equations. There are various reasons for doing this, one being that a first order system is much easier to solve numerically (using computer software) and most differential equations we applies it in “real life” (physics, biology, chemistry, engineering ...etc) don’t have nice exact solutions.

If the equation is of order n and the unknown function is y , then set $y_1 = y$, $y_2 = y'$, ..., $y_n = y^{(n-1)}$. Note that we only go up to the $(n - 1)$ th derivative in this process.

We will look into the process of the conversion through some examples in this thesis, but before that, we have to mention why we need this kind of conversion. As shown in the following illustration, once we get a bunch of first order differential equations out of higher order equation, we can

- I. Use them more easily for numerical processing to solve the problem.
- II. Convert into a matrix form in which we can use many linear algebra tools to analyze/solve the equation.
- III. Convert into a state space model, which can be analyzed by various tools specially designed for State Space analysis tool.

We will discuss system of first order differential equations. There are many applications that involving finds several unknown functions simultaneously. Those unknown functions are related by a set of equations that involving the unknown functions and their first derivatives.

As in the case of one equation, we want to find out the general solutions for the linear first order system of equations. To this end, we first have the following results for the homogeneous equation $\mathbf{Y}'(x) = A\mathbf{Y}(x)$.

We will use a powerful method called eigenvalue method to solve the homogeneous system $\mathbf{Y}'(x) = A\mathbf{Y}(x)$,where A is a matrix with constant entry.

1.3 Justification of the problem:

Differential equations arise whenever a deterministic relation involving some continuously varying quantities - modeled by functions - and their rates of change in space and/or time - expressed as derivatives - is known or postulated. Because such relations are extremely common, differential equations play a prominent role in many disciplines including Engineering, physics, economics, and biology.

Differential equations are mathematically studied from several different perspectives, mostly concerned with their solutions the set of functions that satisfy the equation. Only the simplest differential equations admit solutions given by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers.

Therefore, the study of these systems of differential equations reduces many difficulties of natural problems in many subjects such as physics, biology, chemistry... etc.

1.4 Theoretical framework:

We will discuss the theory of linear systems in normal form and present methods for solving such systems. The theory and methods are natural extensions of the development for second order and higher-order equations. The important properties and techniques are listed below:

Homogeneous normal system: fundamental solution set, fundamental matrix, general solution to homogeneous system, homogeneous system with constant coefficients.

Nonhomogeneous normal system: general solution to nonhomogeneous system, variation of parameters.

1.4.1 METHODS AND PROCEDURES

We will transform our equation to first-order system or matrix system to find the eigenvalues and eigenvectors to obtain the general solutions.

We know that the system is equivalent to a second order homogeneous linear differential equation. As a result, we know that the general solution contains two linearly independent parts. As well, the solution will be consisted of some type of exponential functions. We see that this new equation is exactly the relation that defines eigenvalues and eigenvectors of the coefficient matrix A . Just like the solution of a second order homogeneous linear equation, there are three possibilities, depending on the number of distinct, and the type of, eigenvalues the coefficient matrix A has.

The possibilities are that A has:

- I. Two distinct real eigenvalues.
- II. Complex conjugate eigenvalues.
- III. A repeated eigenvalue.

A related note, (from linear algebra,) we know that eigenvectors that each corresponds to a different eigenvalue are always linearly independent from each other's.

1.4.2 PROPOSED CHAPTER OUTLINE

The thesis will contain four chapters:

Chapter one (Introduction)

It includes:

- a. Background and history.
- b. Statement of Purpose.
- c. Justification of the problem.
- d. Theoretical framework.
- e. Literature review.
- f. Definitions.
- g. Basic Theorems.

Chapter two (Linear Systems in Normal Form)

In this chapter, I will present the norm form of linear systems and homogeneous and nonhomogeneous systems and present some numerical examples.

Chapter three (Homogeneous Linear Systems with Constant Coefficients)

In this chapter, I will present transforming homogeneous linear systems with constant coefficients to first-order system or matrix system to find the eigenvalues and eigenvectors to obtain the general solutions.

Chapter four (Nonhomogeneous Linear Systems)

In this chapter, I will present some extension and obtain the general properties of solutions.

Discussion and Conclusions

Finally, I will present the summary and conclusions.

1.5 Literature review:

Many fundamental laws of physics and chemistry can be formulated as differential equations. In biology and economics, differential equations are used to model the behavior of complex systems. They relate the behavior of certain unknown functions (dependent variables) at a given point (time, position, etc. called independent variable) to their behavior at neighboring points. The mathematical theory of differential equations first developed together with the sciences where the equations had originated and where the results found application. However, diverse problems, sometimes originating in quite distinct scientific fields, may give rise to identical differential equations. Whenever this happens, view mathematical theory behind the equations can as a unifying principle behind diverse phenomena.

Sophus Lie's (1870) work put the theory of differential equations on a more satisfactory foundation. He showed that the integration theories of the older mathematicians can, by the introduction of what are now called Lie groups, be referred to a common source, and that ordinary differential equations that admit the same infinitesimal transformations present comparable difficulties of integration. He also emphasized the subject of transformations of contact.

Lie's group theory of differential equations has been certified, namely: (1) that it unifies the many ad hoc methods known for solving differential equations, and (2) that it provides powerful new ways to find solutions. The theory has applications to both ordinary and partial differential equations. A general approach to solve DEs uses the symmetry property of differential equations, the continuous infinitesimal transformations of solutions to solutions (Lie theory).

V. I. Arnold (1999) Linear systems are almost the only large class of differential equations for which there exists a definitive theory. This theory is essentially a branch of linear algebra, and allows us to solve all autonomous linear equations.

We study the analysis of systems of differential equations where the equations in the systems are linear. In particular, we introduce some theorems to solve the homogeneous and the non-homogeneous systems with constant or variable coefficients. In addition, we give some examples for such systems. In addition, we will use our results to find the solutions. Some properties of solutions will be given in our chapters with some extensions.

The method of variation of parameters is a general method for solving a linear nonhomogeneous system; It was first suggested by Jean Bernoulli in 1697 for solving homogeneous first-order linear equations and extended later by Lagrange to higher-order linear equation. It allows one to find the general solution in explicit form via quadrature.

The emergence of the subject came from determinants, values connected to a square matrix, studied by the founder of calculus, Leibnitz, in the late 17th century. Lagrange came out with his work regarding Lagrange multipliers, a way to “characterize the maxima and minima multivariate functions.” (Darkwing) More than fifty years later, Cramer presented his ideas of solving systems of linear equations based on determinants more than 50 years after Leibnitz (Darkwing).

Interestingly enough, Cramer provided no proof for solving an $n \times n$ system. As we see, linear algebra has become more relevant since the emergence of calculus even though it is foundational equation of $ax + b = 0$ dates back centuries. Euler brought to light the idea that a system of equations does not necessarily have to have a solution (Perotti). He recognized the need for conditions to be placed upon unknown variables to find a solution. The initial work up until this period mainly dealt with the concept of unique solutions and square matrices where the number

of equations matched the number of unknowns. With the turn into the 19th century, Gauss introduced a procedure to be used for solving a system of linear equations. His work dealt mainly with the linear equations and had yet to bring in the idea of matrices or their notations. His efforts dealt with equations of differing numbers and variables as well as the traditional pre-19th century works of Euler, Leibnitz, and Cramer. The other part, matrix multiplication or matrix algebra came from the work of Arthur Cayley in 1855. Cayley's defined matrix multiplication as, "the matrix of coefficients for the composite transformation T_2T_1 is the product of the matrix for T_2 times the matrix of T_1 " (Tucker, 1993). His work dealing with Matrix multiplication culminated in his theorem, the Cayley-Hamilton Theorem. Simply stated, a square matrix satisfies its characteristic equation. Cayley's efforts were published in two papers, one in 1850 and the other in 1858. His works introduced the idea of the identity matrix as well as the inverse of a square matrix.

He also did much to further the ongoing transformation of the use of matrices and symbolic algebra. He used the letter "A" to represent a matrix, something that had been very little before his works. His efforts were little recognized outside of England until the 1880s. Matrices at the end of the 19th century were heavily connected with Physics issues and for mathematicians, more attention was given to vectors as they proved to be basic mathematical elements. For a time, however, interest in a lot of linear algebra slowed until the end of World War II brought on the development of computers. Now instead of having to break down an enormous $n \times n$ matrix, computers could quickly and accurately solve these systems of linear algebra. With the advancement of technology using the methods of Cayley, Gauss, Leibnitz, Euler, and others determinants and linear algebra moved forward more quickly and more effective. Regardless of the technology though Gaussian elimination still proves to be the best way known to solve a system of linear equations (Tucker, 1993). The influence of Linear Algebra in the mathematical

world is spread wide because it provides an important base to many of the principles and practices. Some of the things Linear Algebra is used for are to solve systems of linear format, to find least-square best fit lines to predict future outcomes or find trends, and the use of the Fourier series expansion to solving partial differential equations.

The elevation of the matrix from mere tool to important mathematical theory owes a lot to the work of female mathematician Olga Taussky Todd (1906-1995), who began by using matrices to analyze vibrations on airplanes during World War II and became the torchbearer for matrix theory.

The asymptotic theorem, which provides the basis for the theory in this research, was proved by Levinson and published in 1984.

1.6 Definitions :

Definition 1.6.1:

An equation containing ordinary derivatives which means that there is only one independent variable is called an Ordinary Differential Equation (ODE).

Definition 1.6.2:

An equation containing partial derivatives which means that there are several independent variables is called a Partial Differential Equation (PDE).

Definition 1.6.3:

An (n^{th} -order) ordinary differential equation is a relation

$$\Phi (x, y, y', \dots, y^{(n)}) = 0 \quad (1.6.1)$$

containing the independent variable x , the dependent variable y and its derivatives $y' = \frac{dy}{dx}$, $\dots$, $y^{(n)} = \frac{d^n y}{dx^n}$.

Definition 1.6.4:

We consider the n^{th} -order linear differential equation

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \dots + a_{n-1}(x)y'(x) + a_n(x)y(x) = b(x) \quad (1.6.2)$$

where $a_i(x)$ ($0 \leq i \leq n$) and $b(x)$ are continuous functions in some interval $[a, b]$ of the real line with $a_0(x) \neq 0$ in $[a, b]$.

Since $a_0(x) \neq 0$ then, divide both side by $a_0(x)$

$$y^{(n)}(x) + \frac{a_1(x)}{a_0(x)}y^{(n-1)}(x) + \dots + \frac{a_{n-1}(x)}{a_0(x)}y'(x) + \frac{a_n(x)}{a_0(x)}y(x) = \frac{b(x)}{a_0(x)} \quad (1.6.3)$$

or

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_{n-1}(x)y'(x) + p_n(x)y(x) = g(x) \quad (1.6.4)$$

where

$$g(x) = \frac{b(x)}{a_0(x)} \quad , \quad p_i(x) = \frac{a_i(x)}{a_0(x)} \quad , \quad i = 1, 2, \dots, n .$$

Definition 1.6.5:

A function $y = \varphi(x)$ is called a solution to the differential equation (1.6.1) if equation (1.6.1) is satisfied identically in x after replacing $y, y', \dots, y^{(n)}$ by $\varphi(x), \varphi'(x), \dots, \varphi^{(n)}(x)$, respectively.

Definition 1.6.6:

The differential equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_{n-1}(x)y'(x) + p_n(x)y(x) = g(x) \quad (1.6.4)$$

$$\text{With } y(x_0) = y_0, \quad y'(x_0) = y_1, \dots, \quad y^{(n-1)}(x_0) = y_{n-1} . \quad (1.6.5)$$

where $x_0 \in [a, b]$, and $y_0, \dots, y_{n-1}$ real or complex numbers is called an initial value problem and (1.6.5) is called initial conditions.

Definition 1.6.7:

The solution $y = \varphi(x)$ to the differential equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_{n-1}(x)y'(x) + p_n(x)y(x) = g(x) \quad (1.6.4)$$

which satisfies

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \dots, \quad y^{(n-1)}(x_0) = y_{n-1} . \quad (1.6.5)$$

is called a solution of the initial value problem.

Definition 1.6.8:

A set of n function $f_1, f_2, \dots, f_n$ is said to be linearly independent in some interval I if the equation

$$\alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_n f_n = 0$$

true only for $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$, otherwise $f_1, f_2, \dots, f_n$ linearly dependent.

Definition 1.6.9:

The wronskian of $\varphi_1, \varphi_2, \dots, \varphi_n$, is define by

$$W(\varphi_1, \varphi_2, \dots, \varphi_n)(x) = W(x) = \begin{vmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_n \\ \varphi_1' & \varphi_2' & \dots & \varphi_n' \\ \vdots & \vdots & \dots & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \dots & \varphi_n^{(n-1)} \end{vmatrix}$$

where φ_i are differentiable $(n-1)$ times in I .

Remark consider $x = a$ in $W(x)$

$$W(a) = \begin{vmatrix} \varphi_1(a) & \varphi_2(a) & \dots & \varphi_n(a) \\ \varphi_1'(a) & \varphi_2'(a) & \dots & \varphi_n'(a) \\ \vdots & \vdots & \dots & \vdots \\ \varphi_1^{(n-1)}(a) & \varphi_2^{(n-1)}(a) & \dots & \varphi_n^{(n-1)}(a) \end{vmatrix}$$

if $\varphi_j^{(i-1)}(a) = b_{ij}$, $(i, j = 1, 2, \dots, n)$

$$\text{then } W(a) = \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \dots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix} = \det(b_{ij})$$

Definition 1.6.10:

We consider the n^{th} order differential equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y = 0, \quad (1.6.6)$$

as homogenous equation, our coefficients are continuous in some interval I of the real line with $a_0(x) \neq 0$ in this interval.

Definition 1.6.11:

A set of linearly independent solutions of equation (1.6.6) is said to be a fundamental set.

1.7 Basic Theorems:

Theorem 1.7.1:

The solution to the initial value problem

$$y^{(n)} = \varphi(x, y, y', \dots, y^{(n-1)})$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

Exist and Unique in a neighborhood of x_0 provided that $\varphi(x, y, y', \dots, y^{(n-1)})$ is continuously differentiable in a neighborhood of $x_0, y_0, y_1, \dots, y_{n-1}$.

Theorem 1.7.2:

Let the function $p_i(x)$ ($i = 1, 2, \dots, n$) and $g(x)$ be continuous on an open interval I , which contains x_0 .

Then for any choice of y_i ($i = 0, 1, 2, \dots, n-1$) there exists and unique solution to the initial value problem (1.6.4)

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \dots + p_{n-1}(x)y'(x) + p_n(x)y(x) = g(x),$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

on the interval I .

Theorem 1.7.3:

Let $\varphi_1, \dots, \varphi_n$ be solutions for (1.6.6) then $\varphi(x) = c_1\varphi_1 + \dots + c_n\varphi_n$ is solution for (1.6.6), where c_j ($1 \leq j \leq n$) are constants.

Theorem 1.7.4:

If $\varphi(x)$ solution for (1.6.6) such that

$$\varphi(x_0) = \varphi'(x_0) = \dots = \varphi^{(n-1)}(x_0) = 0, \text{ then } \varphi(x) \equiv 0.$$

Proof :

The first solution is $\varphi(x)$ is satisfies the initial conditions. And the second solution $\psi(x) = 0$ satisfies same initial conditions and the differential equation (1.6.6), Thus by uniqueness theorem $\varphi(x) = \psi(x)$

$$\therefore \varphi(x) = 0 \quad \blacksquare$$

Note: The above solution is called the trivial solution.

Theorem 1.7.5:

Let $b_{ij} (1 \leq i, j \leq n)$ be given real or complex numbers and let a be any fixed point in I . Let a be any fixed point which satisfies the initial conditions

$$\varphi_j^{(i-1)}(a) = b_{ij} \quad (1 \leq i, j \leq n)$$

Then a necessary and sufficient condition for $\varphi_j(x)$ to a form a fundamental set is $\det(b_{ij}) \neq 0$.

Proof:

($\Rightarrow$)

Let $\varphi_j(x)$ form a fundamental set and suppose that $\det(b_{ij}) = 0$.

There exist numbers α_j (not all zero's) such that

$$\left. \begin{array}{l} \alpha_1 b_{11} + \alpha_2 b_{12} + \dots + \alpha_n b_{1n} = 0 \\ \vdots \\ \alpha_1 b_{n1} + \alpha_2 b_{n2} + \dots + \alpha_n b_{nn} = 0 \end{array} \right\} (1.7.1)$$

Let $\varphi(x) = \alpha_1 \varphi_1(x) + \dots + \alpha_n \varphi_n(x)$ then $\varphi(x)$ solution for (1.7.1).

$$\varphi(a) = \varphi'(a) = \dots = \varphi^{(n-1)}(a) = 0$$

$\Rightarrow \varphi(x) \equiv 0$ by uniqueness theorem

$$\therefore \alpha_1 \varphi_1(x) + \dots + \alpha_n \varphi_n(x) = 0$$

$\therefore \varphi_1, \dots, \varphi_n$ are linearly dependent

which contradicts with $\varphi_1, \dots, \varphi_n$ fundamental set

$\therefore \det(b_{ij}) \neq 0$

($\Leftarrow$)

Let $\det(b_{ij}) \neq 0$, consider

$$\left. \begin{aligned} \alpha_1 \varphi_1(x) + \alpha_2 \varphi_2(x) + \dots + \alpha_n \varphi_n(x) &= 0 \\ \alpha_1 \varphi_1'(x) + \alpha_2 \varphi_2'(x) + \dots + \alpha_n \varphi_n'(x) &= 0 \\ \vdots \\ \alpha_1 \varphi_1^{(n-1)}(x) + \dots + \alpha_n \varphi_n^{(n-1)}(x) &= 0 \end{aligned} \right\} (1.7.2)$$

Let $x = a$, we get (1.7.1) since $\det(b_{ij}) \neq 0$

$$\therefore \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

i.e. $\varphi_1, \dots, \varphi_n$ is a fundamental set ■

Theorem 1.7.6: (Liouville's formula)

Let $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ be n solutions of

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y = 0, \quad (1.6.6)$$

then

$$W(\varphi_1, \varphi_2, \dots, \varphi_n)(x) = W(\varphi_1, \varphi_2, \dots, \varphi_n)(x_0) \exp\left(-\int_{x_0}^x \frac{a_1(t)}{a_0(t)} dt\right) \quad (1.7.3)$$

for $x_0 \in [a, b]$.

Proof:

Since φ_i solution for

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y = 0,$$

$$\text{Hence } a_0 \varphi_i^{(n)} + a_1 \varphi_i^{(n-1)} + \dots + a_n \varphi_i = 0. \quad (i = 1, 2, \dots, n) \quad (1.7.4)$$

Now

$$W(x) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}$$

Differentiate then we get

$$W'(x) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \varphi_n^{(n)} \end{vmatrix}$$

From (1.7.4)

$$\varphi_i^{(n)} = \frac{-a_n}{a_0} \varphi_i - \cdots - \frac{a_1}{a_0} \varphi_i^{(n-1)}$$

$$\therefore W'(x) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & \cdots & \vdots \\ \frac{-a_n}{a_0} \varphi_1 - \cdots - \frac{a_1}{a_0} \varphi_1^{(n-1)} & \cdots & \frac{-a_n}{a_0} \varphi_n - \cdots - \frac{a_1}{a_0} \varphi_n^{(n-1)} \end{vmatrix}$$

$$\frac{a_n}{a_0} \varphi_i + \text{row } n \rightarrow \text{row } n \quad i = 1, 2, \dots, n-1$$

$$\therefore W'(x) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & \cdots & \vdots \\ -\frac{a_1}{a_0} \varphi_1^{(n-1)} & \cdots & -\frac{a_1}{a_0} \varphi_n^{(n-1)} \end{vmatrix}$$

$$\therefore W'(x) = \frac{-a_1}{a_0} \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_1^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix} = \frac{-a_1}{a_0} W(x)$$

$$\therefore \frac{dW}{dx} = \frac{-a_1}{a_0} W, \text{ where } a_1, a_0 \text{ functions of } t.$$

$$\frac{dW}{W} = \frac{-a_1}{a_0} dt$$

$$\int_{x_0}^x \frac{dW}{W} = - \int_{x_0}^x \frac{a_1}{a_0} dt$$

$$\begin{aligned}
[\ln|W|]_{x_0}^x &= - \int_{x_0}^x \frac{a_1}{a_0} dt \\
\ln|W(x)| - \ln|W(x_0)| &= - \int_{x_0}^x \frac{a_1}{a_0} dt \\
\ln \left| \frac{W(x)}{W(x_0)} \right| &= - \int_{x_0}^x \frac{a_1}{a_0} dt \\
\frac{W(x)}{W(x_0)} &= \exp \left(- \int_{x_0}^x \frac{a_1}{a_0} dt \right) \\
W(x) &= W(x_0) \exp \left(- \int_{x_0}^x \frac{a_1}{a_0} dt \right) \quad \blacksquare
\end{aligned}$$

Corollary 1.7.1:

If $a_1(x) = 0$ in (1.6.6), then $W(x) = \text{constant}$.

Proof:

Substitute $a_1(x) = 0$ in Liouville's formula, we get $W(x) = W(x_0)e^0 = W(x_0)$ which is a constant.

Corollary 1.7.2:

If $a_1(x) = \alpha a_0'(x)$ in (1.6.6), then $W(x) = \text{constant } a_0^{-\alpha}(x)$.

Proof:

By Liouville's formula

$$W(x) = W(x_0) \exp \left(- \int_{x_0}^x \frac{a_1}{a_0} dt \right) = W(x_0) \exp \left(- \int_{x_0}^x \frac{\alpha a_0'}{a_0} dt \right)$$

$$W(x) = W(x_0) \exp(-\alpha [\ln|a_0(t)|]_{x_0}^x)$$

$$W(x) = W(x_0) \exp(-\alpha [\ln|a_0(x)| - \ln|a_0(x_0)|])$$

$$W(x) = W(x_0) \exp\left(-\alpha \ln \left| \frac{a_0(x)}{a_0(x_0)} \right| \right)$$

$$W(x) = W(x_0) \exp\left(\ln \left| \frac{a_0(x)}{a_0(x_0)} \right|^{-\alpha} \right)$$

$$W(x) = W(x_0) \left| \frac{a_0(x)}{a_0(x_0)} \right|^{-\alpha} = \text{const.} a_0^{-\alpha}(x) \quad \blacksquare$$

For example, the differential equation

$$(p_0 y^{(m)})^{(m)} + (p_1 y^{(m-1)})^{(m-1)} + \dots + (p_{m-1} y')' + p_m y = 0$$

$$\text{so it can be written as } p_0 y^{(2m)} + m p_0' y^{(2m-1)} + \dots = 0$$

$$\text{where } a_1 = m p_0' = m a_0'$$

$$\therefore \alpha = m,$$

$$\text{Then } W(x) = \text{constant} \cdot p_0^{-m}.$$

Theorem 1.7.7:

Let $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ be a solution of (1.6.6). Then $W(\varphi_1, \varphi_2, \dots, \varphi_n)(x)$ is either never zero or identical to zero in interval I .

Proof :

Use Liouville's formula

If $W(x_0) \neq 0$, then $W(x) \neq 0 \quad \forall x$ which mean's $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ are linearly independent.

If $W(x_0) = 0$, then $W(x) = 0 \quad \forall x$ which mean's $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ are linearly dependent.

Theorem 1.7.8:

Let $\varphi_j(x)$, $(1 \leq j \leq n)$ be a fundamental set for (1.6.6) and let $\varphi(x)$ be any solution for (1.6.6). Then there exist a unique constants $c_1, c_2, \dots, c_n$ such that $\varphi(x) = c_1\varphi_1 + \dots + c_n\varphi_n$.

Proof:

Now $\varphi(x) = c_1\varphi_1 + \dots + c_n\varphi_n$

If there exist another constants say $\varphi(x) = k_1\varphi_1 + \dots + k_n\varphi_n$

Subtract those equations we get

$$0 = (c_1 - k_1)\varphi_1 + \dots + (c_n - k_n)\varphi_n$$

$\therefore \varphi_1, \varphi_2, \dots, \varphi_n$ linearly independent then $c_i - k_i = 0 \quad \forall i$

$\therefore c_i = k_i \quad \forall i \quad \blacksquare$

Chapter 2

Linear System in Normal Form

2.1 Normal Form:

Definition 2.1.1:

First order system of differential equations is a set of n functions given by :

$$\begin{aligned} y_1' &= f_1\{x, y_1, \dots, y_n\} \\ y_2' &= f_2\{x, y_1, \dots, y_n\} \\ &\vdots \\ y_n' &= f_n\{x, y_1, \dots, y_n\} \end{aligned}$$

The system of n linear differential equations is in normal form if it expressed as

$$\mathbf{Y}'(x) = A(x)\mathbf{Y}(x) + \mathbf{b}(x) \quad (2.1.1)$$

Where $\mathbf{Y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}$, $\mathbf{b}(x) = \begin{bmatrix} b_1(x) \\ b_2(x) \\ \vdots \\ b_n(x) \end{bmatrix}$ and $A(x) = [a_{ij}(x)]$

is an $n \times n$ matrix.

Example 2.1.1:

We can write the following system in the matrix form $\mathbf{Y}'(x) = A\mathbf{Y}(x) + \mathbf{b}(x)$:

i. $x'(t) = 3x(t) - y(t) + t^2$, $y'(t) = -x(t) + 2y(t) + e^t$.

as
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t^2 \\ e^t \end{pmatrix}$$

ii. $\frac{dx}{dt} = t^2x - y - z + t$, $\frac{dy}{dt} = e^t z + 5$, $\frac{dz}{dt} = tx - y + 3z - e^t$.

as
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} t^2 & -1 & -1 \\ 0 & 0 & e^t \\ t & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} t \\ 5 \\ -e^t \end{pmatrix}$$

iii. $y''(t) - 3y'(t) - 10y(t) = \sin t$.

as
$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -10 & -3 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}$$

Definition 2.1.2:

As with a scalar linear differential equations, a system is called homogeneous when $\mathbf{b}(x) = 0$; otherwise, it is called nonhomogeneous.

Definition 2.1.3:

A linear differential equations with all elements of A are all constant, the system is called constant coefficients. Now consider the n^{th} - order differential equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x) \quad (2.1.2)$$

Can be rewrite as a first order system in normal form using the substitution

$$y_1(x) = y(x), \quad y_2(x) = y'(x), \dots, \quad y_n(x) = y^{(n-1)}(x);$$

Indeed, equation (2.1.2) is equivalent to (2.1.1)

$$\mathbf{Y}'(x) = A(x)\mathbf{Y}(x) + \mathbf{b}(x)$$

where $\mathbf{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}$, $\mathbf{b}(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(x) \end{bmatrix}$ and

$$A(x) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -p_1(x) & -p_2(x) & -p_3(x) & \cdots & -p_{n-1}(x) & -p_n(x) \end{bmatrix}$$

Example 2.1.2:

We can express

$$y^{(n)} - y = 0$$

as special case with $p_n = 0$, $p_{n-1} = 0$, ..., $p_2 = 0$, $p_1 = -1$

$$\begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$$

The theory for systems in normal form parallels very closely the theory of linear differential equations. In many cases, the proofs for scalar linear differential equations carry over to normal systems with appropriate modifications. Conversely, results for normal systems apply to scalar linear equations since any scalar linear equation can be expressed as a normal system.

Definition 2.1.4:

This is the case with the existence and uniqueness for linear differential equations, the initial value problem for the normal system (2.1.1) is the problem of

finding a differentiable vector function $\mathbf{Y}(x)$ that satisfies the system on an interval I and also satisfies the initial condition $\mathbf{y}(x_0) = \mathbf{y}_0$, where x_0 is given

point of I and $\mathbf{Y}_0 = \begin{bmatrix} y_{1,0} \\ \vdots \\ y_{n,0} \end{bmatrix}$, there exists a unique solution $\mathbf{Y}(x)$ on the whole interval I to the initial value problem $\mathbf{Y}'(x) = A(x)\mathbf{Y}(x) + \mathbf{b}(x)$, $\mathbf{y}(x_0) = \mathbf{y}_0$.

Definition 2.1.5:

The n vector function $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ are said to be linearly dependent on an interval I if there exist constant $c_1, c_2, \dots, c_n$, not all zero, such that

$$c_1\mathbf{y}_1(x) + c_2\mathbf{y}_2(x) + \dots + c_n\mathbf{y}_n(x) = \mathbf{0} \quad \text{for all } x \text{ in } I.$$

If the vectors are not linearly dependent, they are said to be linearly independent on I .

Example 2.1.3:

To show that the vector function $\mathbf{y}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\mathbf{y}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ are linearly dependent on $\mathbb{R}$. We notice that $\mathbf{y}_2$ is just twice of $\mathbf{y}_1$ and therefore $-2\mathbf{y}_1(x) + \mathbf{y}_2(x) = \mathbf{0}$ for all x . Hence, $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly dependent on $\mathbb{R}$.

Example 2.1.4:

To prove that the vector function $\mathbf{y}_1(x) = \begin{bmatrix} e^{-x} \\ 2e^{-x} \\ e^{-x} \end{bmatrix}$, $\mathbf{y}_2 = \begin{bmatrix} e^x \\ 0 \\ e^x \end{bmatrix}$, $\mathbf{y}_3 = \begin{bmatrix} e^{3x} \\ -e^{3x} \\ 2e^x \end{bmatrix}$ are linearly independent on $\mathbb{R}$. we assume c_1, c_2 and c_3 are constants for which

$$c_1\mathbf{y}_1(x) + c_2\mathbf{y}_2(x) + c_3\mathbf{y}_3(x) = \mathbf{0}$$

holds at every x in $\mathbb{R}$, and to show that $c_1 = c_2 = c_3 = 0$

$$\therefore c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 0$$

which is equivalent to the system of linear equations

$$c_1 + c_2 + c_3 = 0$$

$$2c_1 + 0 - c_3 = 0$$

$$c_1 + c_2 + 2c_3 = 0$$

by solving these three equations we obtain $c_1 = c_2 = c_3 = 0$.

Therefore the vector functions $\mathbf{y}_1, \mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent on $\mathfrak{R}$,
(in fact, on any interval containing $x = 0$) .

2.2: Homogeneous System

Theorem 2.2.1:

Let $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ be n linearly independent solution to the homogeneous system

$$\mathbf{Y}'(x) = A(x)\mathbf{Y}(x) \quad (2.2.1)$$

on the interval I , where $A(x)$ is an $n \times n$ matrix function continuous on I .

Then every solution to (2.2.1) on I can be expressed in the form

$$\mathbf{Y}(x) = c_1\mathbf{y}_1(x) + c_2\mathbf{y}_2(x) + \dots + c_n\mathbf{y}_n(x) \quad (2.2.2)$$

where $c_1, c_2, \dots, c_n$ are constants.

Definition 2.2.1:

A fundamental solution set for

$$\mathbf{Y}'(x) = A(x)\mathbf{Y}(x) \quad (2.2.1)$$

on I is a set of solution $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ that are linearly independent on I or, equivalently, whose wronskian doesn't vanish on I .

Definition 2.2.2:

The linear combination in

$$\mathbf{Y}(x) = c_1\mathbf{y}_1(x) + c_2\mathbf{y}_2(x) + \dots + c_n\mathbf{y}_n(x) \quad (2.2.2)$$

Written with arbitrary constants, is referred to as a general solution to

$$\mathbf{Y}'(x) = A(x)\mathbf{Y}(x). \quad (2.2.1)$$

Definition 2.2.3:

The vector in a fundamental solution set and let them form the columns of a matrix $\Phi(x)$, that is, $\Phi(x) = [\boldsymbol{\varphi}_1(x) \quad \boldsymbol{\varphi}_2(x) \quad \dots \quad \boldsymbol{\varphi}_n(x)]$

$$\text{Where } \boldsymbol{\varphi}_1 = \begin{pmatrix} \varphi_{11} \\ \vdots \\ \varphi_{n1} \end{pmatrix}, \boldsymbol{\varphi}_2 = \begin{pmatrix} \varphi_{12} \\ \vdots \\ \varphi_{n2} \end{pmatrix}, \dots, \boldsymbol{\varphi}_n = \begin{pmatrix} \varphi_{1n} \\ \vdots \\ \varphi_{nn} \end{pmatrix}$$

$$\boldsymbol{\Phi}(x) = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{bmatrix} \quad (2.2.3)$$

Then the matrix $\boldsymbol{\Phi}(x)$ is called a fundamental matrix for

$$\mathbf{Y}'(x) = A(x)\mathbf{Y}(x) \quad (2.2.1)$$

We can use it to express the general solution

$$\boldsymbol{\Phi}(x) = c_1 \boldsymbol{\varphi}_1(x) + c_2 \boldsymbol{\varphi}_2(x) + \cdots + c_n \boldsymbol{\varphi}_n(x) \quad (2.2.2)$$

Since $\det \boldsymbol{\Phi} \neq 0 \quad \forall x \text{ in } I$, then $\boldsymbol{\Phi}(x)$ is non-singular.

Definition 2.2.4:

The determinant of the matrix (2.2.3) is called the wronskian and denoted by $W = W(\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_n)$

Thus $W = \det \boldsymbol{\Phi}$

$$\text{i.e.} \quad W(\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_n)(x) = \begin{vmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \ddots & \vdots \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{vmatrix}$$

Theorem 2.2.2: (Liouville's formula)

If $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \dots, \boldsymbol{\varphi}_n$ are any n solutions to the $n \times n$ system

$$\mathbf{Y}'(x) = A(x)\mathbf{Y}(x).$$

then Abel's formula gives a representation for the wronskian

Namely, $W(x) = W(x_0) \exp\left(\int_{x_0}^x \text{tr} A dt\right)$, where $\text{tr} A = \sum_{i=1}^n a_{ii}$.

Proof:

$$\text{Since } W(\varphi_1, \dots, \varphi_n)(x) = \begin{vmatrix} \varphi_{11} & \dots & \varphi_{1n} \\ \vdots & \ddots & \vdots \\ \varphi_{n1} & \dots & \varphi_{nn} \end{vmatrix}$$

$$\text{Now } \varphi_{ij}' = \sum_{k=1}^n a_{ik} \varphi_{kj} \quad , \quad i, j = 1, 2, \dots, n.$$

$$\begin{aligned} \therefore W' &= \begin{vmatrix} \varphi_{11}' & \dots & \varphi_{1n}' \\ \varphi_{21}' & \dots & \varphi_{2n}' \\ \vdots & \vdots & \vdots \\ \varphi_{n1}' & \dots & \varphi_{nn}' \end{vmatrix} + \begin{vmatrix} \varphi_{11} & \dots & \varphi_{1n} \\ \varphi_{21}' & \dots & \varphi_{2n}' \\ \vdots & \vdots & \vdots \\ \varphi_{n1}' & \dots & \varphi_{nn}' \end{vmatrix} + \dots + \begin{vmatrix} \varphi_{11} & \dots & \varphi_{1n} \\ \varphi_{21} & \dots & \varphi_{2n} \\ \vdots & \vdots & \vdots \\ \varphi_{n1}' & \dots & \varphi_{nn}' \end{vmatrix} \\ W' &= \begin{vmatrix} \sum_{k=1}^n a_{1k} \varphi_{k1} & \dots & \sum_{k=1}^n a_{1k} \varphi_{kn} \\ \varphi_{21} & \dots & \varphi_{2n} \\ \vdots & \vdots & \vdots \\ \varphi_{n1} & \dots & \varphi_{nn} \end{vmatrix} + \dots + \begin{vmatrix} \varphi_{11} & \dots & \varphi_{1n} \\ \varphi_{21} & \dots & \varphi_{2n} \\ \vdots & \vdots & \vdots \\ \sum_{k=1}^n a_{nk} \varphi_{k1} & \dots & \sum_{k=1}^n a_{nk} \varphi_{kn} \end{vmatrix} \end{aligned}$$

By elementary row operation (multiply the second row by a_{12} , the third by a_{13} , etc. and subtracting the sum of the last $(n-1)$ rows from the first) so the first determinant is $a_{11} \det \Psi$, Same for others.

$$\text{We obtain } W' = \sum_{i=1}^n a_{ii} W.$$

$$\frac{W'}{W} = \sum_{i=1}^n a_{ii}$$

$$\text{Integrate both side } \int_{x_0}^x \frac{W'}{W} dt = \int_{x_0}^x \sum_{i=1}^n a_{ii} dt$$

$$\ln \left| \frac{W(x)}{W(x_0)} \right| = \int_{x_0}^x \sum_{i=1}^n a_{ii} dt$$

$$\frac{W(x)}{W(x_0)} = \exp\left(\int_{x_0}^x \text{tr} A dt\right)$$

$$W(x) = W(x_0) \exp\left(\int_{x_0}^x \text{tr} A dt\right) \blacksquare$$

Theorem 2.2.3:

The wronskian of n solutions to $\mathbf{Y}'(x) = A(x)\mathbf{Y}(x)$ on the interval I is either identically zero on interval I or never zero on I .

Proof:

$$\therefore W(x) = W(x_0) \exp\left(\int_{x_0}^x \text{tr} A dt\right) ,$$

and

$$\therefore \exp\left(\int_{x_0}^x \text{tr} A dt\right) \text{ never equal to } 0$$

If $W(x_0) = 0$ then $W(x) = 0 \quad \forall x$.

And

If $W(x_0) \neq 0$ then $W(x) \neq 0 \quad \forall x$.

Example 2.2.1:

To verify that the vector functions

$$\boldsymbol{\varphi}_1 = \begin{bmatrix} e^{3x} \\ 0 \\ e^{3x} \end{bmatrix} , \quad \boldsymbol{\varphi}_2 = \begin{bmatrix} -e^{3x} \\ e^{3x} \\ 0 \end{bmatrix} , \quad \boldsymbol{\varphi}_3 = \begin{bmatrix} -e^{-3x} \\ -e^{-3x} \\ e^{-3x} \end{bmatrix}$$

are solutions to the homogeneous system

$$\mathbf{Y}' = A\mathbf{Y} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \mathbf{Y}$$

on $\mathbb{R}$,

by substituting the first vector in the set S in to the right-hand side gives

$$A\boldsymbol{\varphi}_1 = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} e^{3x} \\ 0 \\ e^{3x} \end{bmatrix} = \begin{bmatrix} 3e^{3x} \\ 0 \\ 3e^{3x} \end{bmatrix} = \boldsymbol{\varphi}'_1$$

Hence this vector satisfies the given system for all x . Similar computations verify that the remaining vectors in S are solution to the given system on $\mathfrak{R}$.

$$A\boldsymbol{\varphi}_2 = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -e^{3x} \\ e^{3x} \\ 0 \end{bmatrix} = \begin{bmatrix} -3e^{3x} \\ 3e^{3x} \\ 0 \end{bmatrix} = \boldsymbol{\varphi}'_2$$

$$A\boldsymbol{\varphi}_3 = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -e^{-3x} \\ -e^{-3x} \\ e^{-3x} \end{bmatrix} = \begin{bmatrix} 3e^{-3x} \\ 3e^{-3x} \\ -3e^{-3x} \end{bmatrix} = \boldsymbol{\varphi}'_3$$

The fundamental set is given by

$$\Phi = \begin{bmatrix} e^{3x} & -e^{3x} & -e^{-3x} \\ 0 & e^{3x} & -e^{-3x} \\ e^{3x} & 0 & e^{-3x} \end{bmatrix}$$

To show that vectors are solutions, it's enough to observe that the wronksian

$$W(x) = \det \Phi$$

$$W(x) = \begin{vmatrix} e^{3x} & -e^{3x} & -e^{-3x} \\ 0 & e^{3x} & -e^{-3x} \\ e^{3x} & 0 & e^{-3x} \end{vmatrix}$$

$$W(x) = e^{3x} \begin{vmatrix} e^{3x} & -e^{-3x} \\ 0 & e^{-3x} \end{vmatrix} + e^{3x} \begin{vmatrix} -e^{3x} & -e^{-3x} \\ e^{3x} & -e^{-3x} \end{vmatrix} = 3e^{3x} \neq 0$$

To find a general solution to $\mathbf{Y}' = A\mathbf{Y}$.

we can be expressed as

$$\mathbf{Y}(x) = c_1 \begin{bmatrix} e^{3x} \\ 0 \\ e^{3x} \end{bmatrix} + c_2 \begin{bmatrix} -e^{3x} \\ e^{3x} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -e^{-3x} \\ -e^{-3x} \\ e^{-3x} \end{bmatrix}$$

2.3 Non-homogenous System:

Theorem 2.3.1:

Let $\mathbf{y}_\rho$ be a particular solution to the nonhomogeneous system

$$\mathbf{Y}'(x) = A(x)\mathbf{Y}(x) + \mathbf{b}(x) \quad (2.3.1)$$

on the interval I , and let $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ be a fundamental solution set on I for the corresponding homogeneous system $\mathbf{Y}'(x) = A(x)\mathbf{Y}(x)$. Then every solution to (2.3.1) on I can be expressed in the form

$$\mathbf{Y}(x) = \mathbf{y}_\rho(x) + c_1\mathbf{y}_1(x) + c_2\mathbf{y}_2(x) + \dots + c_n\mathbf{y}_n(x) \quad (2.3.2)$$

where $c_1, c_2, \dots, c_n$ are constants.

Example 2.3.1:

To verify that the vector functions

$$\mathbf{y}_1 = \begin{bmatrix} e^{3x} \\ 0 \\ e^{3x} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -e^{3x} \\ e^{3x} \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} -e^{-3x} \\ -e^{-3x} \\ e^{-3x} \end{bmatrix}, \text{ and } \mathbf{y}_\rho = \begin{bmatrix} 5x + 1 \\ 2x \\ 4x + 2 \end{bmatrix}$$

are solutions and a particular solution to the nonhomogeneous system

$$\mathbf{Y}' = A\mathbf{Y} + \mathbf{b}(x) = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} -9x \\ 0 \\ -18x \end{bmatrix}$$

By substituting the first vector to the right hand side gives

$$A\mathbf{y}_1 = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} e^{3x} \\ 0 \\ e^{3x} \end{bmatrix} = \begin{bmatrix} 3e^{3x} \\ 0 \\ 3e^{3x} \end{bmatrix} = \mathbf{y}_1'$$

Same for the second vector

$$A\mathbf{y}_2 = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -e^{3x} \\ e^{3x} \\ 0 \end{bmatrix} = \begin{bmatrix} -3e^{3x} \\ 3e^{3x} \\ 0 \end{bmatrix} = \mathbf{y}_2'$$

And the third vector

$$A\mathbf{y}_3 = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -e^{3x} \\ e^{3x} \\ 0 \end{bmatrix} = \begin{bmatrix} -3e^{3x} \\ 3e^{3x} \\ 0 \end{bmatrix} = \mathbf{y}_3'$$

And the particular solution

$$A\mathbf{y}_p + \mathbf{b}(x) = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5x+1 \\ 2x \\ 4x+2 \end{bmatrix} + \begin{bmatrix} -9x \\ 0 \\ -18x \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} = \mathbf{y}_p'$$

To find a general solution to $\mathbf{Y}' = A\mathbf{Y} + \mathbf{b}(x)$.

$$\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p = c_1 \begin{bmatrix} e^{3x} \\ 0 \\ e^{3x} \end{bmatrix} + c_2 \begin{bmatrix} -e^{3x} \\ e^{3x} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -e^{3x} \\ e^{3x} \\ 0 \end{bmatrix} + \begin{bmatrix} 5x+1 \\ 2x \\ 4x+2 \end{bmatrix}$$

Chapter 3

Homogeneous Linear Systems with Constant Coefficients

In this chapter, we introduce the first-order system and we shall study the homogeneous system $\mathbf{Y}' = A(x)\mathbf{Y}$. In particular $A(x) = A$, constant matrix in section 3.2 and 3.3.

3.1 The First-order system

Definition 3.1.1:

Now the first order system is

$$\left. \begin{aligned} y_1' &= f_1\{x, y_1, y_2, \dots, y_n\} \\ y_2' &= f_2\{x, y_1, y_2, \dots, y_n\} \\ &\vdots \\ y_n' &= f_n\{x, y_1, y_2, \dots, y_n\} \end{aligned} \right\} \quad (3.1.1)$$

Let $f_1, f_2, \dots, f_n$ be linear homogeneous functions of $y_1, y_2, \dots, y_n$

$$\left. \begin{aligned} f_1 &= a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n \\ &\vdots \\ f_n &= a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n \end{aligned} \right\} \quad (3.1.2)$$

Then (3.1.1) can be written as $\mathbf{Y}' = A(x)\mathbf{Y}$ (3.1.3)

where $A = (a_{ij})$ and $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, (3.1.3) is called linear homogeneous system.

Example 3.1.1:

We can write out the following n^{th} - order differential equation as first-order system $Y' = A(x)Y$.

$$(1) a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$$

We define $y_1 = y, y_2 = y', \dots, y_n = y^{(n-1)}$

$$Y = \begin{pmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{(n-1)} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & -\frac{a_{n-2}}{a_0} & \dots & -\frac{a_2}{a_0} & -\frac{a_1}{a_0} \end{pmatrix}$$

So it can be written as

$$\begin{pmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{n-1} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & -\frac{a_{n-2}}{a_0} & \dots & -\frac{a_2}{a_0} & -\frac{a_1}{a_0} \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{n-1} \end{pmatrix}$$

$$Y' = A(x)Y$$

(2) Consider the differential equation $(r_{n-1} \dots (r_2(r_1 y'))' \dots)' - qy = 0$ of any order $r_j \neq 0 \quad \forall j$.

We define

$$y_1 = y, y_2 = r_1 y', y_3 = r_2(r_1 y')', \dots, y_n = r_{n-1}(\dots (r_2(r_1 y'))')$$

$$Y = \begin{pmatrix} y \\ r_1 y' \\ r_2(r_1 y')' \\ r_3(r_2(r_1 y')')' \\ \vdots \\ r_{n-1} \dots (r_2(r_1 y')')' \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & r_1^{-1} & 0 & \dots & 0 \\ 0 & 0 & r_2^{-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & r_{n-1}^{-1} \\ q & 0 & 0 & \dots & 0 \end{pmatrix}$$

So it can be written as

$$\begin{pmatrix} y \\ r_1 y' \\ r_2(r_1 y')' \\ r_3(r_2(r_1 y')')' \\ \vdots \\ r_{n-1} \dots (r_2(r_1 y')')' \end{pmatrix}' = \begin{pmatrix} 0 & r_1^{-1} & 0 & \dots & 0 \\ 0 & 0 & r_2^{-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & r_{n-1}^{-1} \\ q & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} y \\ r_1 y' \\ r_2(r_1 y')' \\ r_3(r_2(r_1 y')')' \\ \vdots \\ r_{n-1} \dots (r_2(r_1 y')')' \end{pmatrix}$$

$$Y' = A(x)Y$$

(3) $Z_r^{(m_r)}(x) = f_r\{x, z_1(x), \dots, z_n(x)\} (1 \leq r \leq n)$, where the z_r functions and the m_r are positive integers.

To find an equivalent first-order system $y' = f\{x, y\}$ with a suitable $y(x)$.

$$\text{Let } \left. \begin{array}{l} z_1^{(m_1)} = f_1\{x, z_1, \dots, z_n\} \\ z_2^{(m_2)} = f_2\{x, z_1, \dots, z_n\} \\ \vdots \\ z_n^{(m_n)} = f_n\{x, z_1, \dots, z_n\} \end{array} \right\} (*)$$

Define $z_1 = y_1$, $z_1' = y_2$, $\dots$, $z_1^{(m-1)} = y_m$

Hence $y_1' = z_1' = y_2$, $y_2' = z_1'' = y_3$, $\dots$, $y_{m-1}' = z_1^{(m_1-1)} = y_{m_1}$

so $y_{m_1}' = z_1^{(m_1)}$

also define $z_2 = h_1$, $z_2' = h_2$, $\dots$, $z_2^{(m_2-1)} = h_{m_2}$

hence $h_1' = z_2' = h_2$, $h_2' = z_2'' = h_3$, $\dots$, $h_{m-1}' = z_2^{(m_2-1)} = h_{m_2}$

so $h_{m_2}' = z_2^{(m_2)}$

complete until we define $z_n = g_1$, $z_n' = g_2$, $\dots$, $z_n^{(m_2-1)} = g_{m_n}$

hence $g_1' = z_n' = g_2$, $g_2' = z_n'' = g_3$, $\dots$, $g_{m_n-1}' = z_n^{(m_n-1)} = g_{m_n}$

so $g_{m_n}' = z_1^{(m_n)}$

so (*) gives

$$y_{m_1}' = f_1\{x, y_1, h_1, \dots, g_1\}$$

$$h_{m_2}' = f_2\{x, y_1, h_1, \dots, g_1\}$$

$\vdots$

$$g_{m_n}' = f_n\{x, y_1, h_1, \dots, g_1\}$$

$$\therefore Y = \begin{pmatrix} y_{m_1} \\ h_{m_2} \\ \vdots \\ g_{m_n} \end{pmatrix}$$

Theorem 3.1.1:

Let $\varphi_1(x), \dots, \varphi_n(x)$ be n a solution for (3.1.3) $Y' = A(x)Y$,
then the $n \times n$ matrix

$$\Phi(x) = (\varphi_1(x) \quad \varphi_2(x) \quad \dots \quad \varphi_n(x)) \quad (3.1.4)$$

$$\Phi(x) = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \dots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \dots & \varphi_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \varphi_{n1} & \varphi_{n2} & \dots & \varphi_{nn} \end{pmatrix} \quad (3.1.5)$$

either non-singular for all $x \in I$ or singular for all $x \in I$.

proof:

let $\Phi(a)$ be singular for some $a \in I$,

then

$$c_1\boldsymbol{\varphi}_1(a) + c_2\boldsymbol{\varphi}_2(a) + \cdots + c_n\boldsymbol{\varphi}_n(a) = 0$$

for some $c_1, c_2, \dots, c_n$ not all zero.

Define

$$\Phi(x) = c_1\boldsymbol{\varphi}_1(x) + c_2\boldsymbol{\varphi}_2(x) + \cdots + c_n\boldsymbol{\varphi}_n(x)$$

then $\Phi(x)$ is a solution for (3.2.3) and $\Phi(a) = 0$. This initial condition is also satisfied by the identically zero solution.

Hence by uniqueness theorem, $\Phi(x) = 0$

i.e. $c_1\boldsymbol{\varphi}_1(x) + c_2\boldsymbol{\varphi}_2(x) + \cdots + c_n\boldsymbol{\varphi}_n(x) = 0, \forall x \in I$

i.e. $\Phi(x)$ is a singular for all $\forall x \in I$ ($\det \Phi = 0$) ■

Definition 3.1.2:

If Φ is non-singular, it is called a fundamental matrix for (3.1.3) and $\boldsymbol{\varphi}_j$ ($1 \leq j \leq n$) form a fundamental set for (3.1.3)

Theorem 3.1.2:

Let $\Phi(x)$ be a solution of (3.1.3) and let $\boldsymbol{\varphi}_j$ ($1 \leq j \leq n$) form a fundamental set for (3.1.3). Then there are unique constants c_j such that $\forall x \in I$

$$\Phi(x) = c_1\boldsymbol{\varphi}_1(x) + c_2\boldsymbol{\varphi}_2(x) + \cdots + c_n\boldsymbol{\varphi}_n(x)$$

$$\Phi = \varphi(x)C, \quad \text{where } C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \quad (3.1.6)$$

Proof:

If $\boldsymbol{\varphi}_j(x)$ solutions then there exist constants $c_1, \dots, c_n$ such that

$$\Phi(x) = c_1\boldsymbol{\varphi}_1(x) + c_2\boldsymbol{\varphi}_2(x) + \cdots + c_n\boldsymbol{\varphi}_n(x) \quad (1)$$

These constants are unique if not then

$$\Phi(x) = k_1 \varphi_1(x) + k_2 \varphi_2(x) + \cdots + k_n \varphi_n(x) \quad (2)$$

Then (1) – (2)

$$0 = (c_1 - k_1) \varphi_1 + \cdots + (c_n - k_n) \varphi_n$$

But φ_j are linearly independent; $c_1 - k_1 = 0, \dots, c_n - k_n = 0$

so $c_1 = k_1, \dots, c_n = k_n$

$c_i = k_i (i = 1, 2, \dots, n)$ unique. ■

Theorem 3.1.3:

Let Φ be a fundamental matrix for (3.1.3) and let Ψ satisfy $\Phi' = A\Phi$. Then Ψ is also a fundamental matrix if and only if

$$\Psi(x) = \Phi(x)C, \quad (3.1.7)$$

where C is a constant non-singular matrix.

Proof:

($\Leftarrow$) Let $\Psi(x) = \Phi(x)C$, then Ψ is non-singular matrix and so is fundamental matrix, because Φ non-singular and C constant non-singular matrix.

($\Rightarrow$) Let Ψ be a fundamental matrix, we define $\Theta(x) = \Phi^{-1}\Psi$

We need to show $\Theta = C$.

$$\Phi\Theta = \Psi$$

$$\Phi'\Theta + \Phi\Theta' = \Psi'$$

$$A\Phi\Theta + \Phi\Theta' = A\Psi$$

$$\Theta' = 0$$

$$\therefore \Theta = C$$

■

3.2 Eigenvalues and Eigenvectors:

Consider the linear homogeneous system:

$$Y' = AY, \quad (3.2.1)$$

where A is a $n \times n$ constant matrix.

The system (3.2.1) is also solved by Euler's substitution modified as follows:

$$y = e^{\lambda x} v, \lambda = \text{constant}. \quad (3.2.2)$$

where $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is an unknown constant vector to be determined .

Substitution of (3.2.2) in (3.2.1) gives the following equation for v :

$$(A - \lambda I) v = 0 \quad (3.2.3)$$

where I is the $n \times n$ identity matrix.

The system (3.2.3) has a solution $v \neq 0$ if and only if the determinant $|A - \lambda I| = \det(A - \lambda I)$ vanishes. Subsequently, the characteristic polynomial $P_n(\lambda)$ and the characteristic equation for the system (3.2.1) are defined by

$$P_n(\lambda) \equiv |A - \lambda I| = 0 \quad (3.2.4)$$

Definition 3.2.1:

The eigenvalues of the matrix A are the solutions $\lambda_1, \dots, \lambda_n$ of the characteristic equation $P_n(\lambda) \equiv |A - \lambda I| = 0$, (3.2.4)

and the corresponding $v_1, \dots, v_n$ of $(A - \lambda I)v = 0$ (3.2.3)

are termed the eigenvectors of the matrix A .

Example 3.2.1:

We can find the eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

The characteristic polynomial $p_n(\lambda)$ and the characteristic equation for the system $\mathbf{Y}' = A\mathbf{Y}$ are defined by $p_n(\lambda) \equiv |A - \lambda I| = 0$

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 4 \\ 3 & 2 - \lambda \end{vmatrix} = (2 - \lambda)(1 - \lambda) - 12 = 0$$

$$\lambda^2 - 3\lambda + 2 - 12 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$(\lambda - 5)(\lambda + 2) = 0$$

$$\lambda = 5, \lambda = -2$$

Hence, the eigenvalue are $\lambda_1 = 5$ and $\lambda_2 = -2$.

For $\lambda_1 = 5$, the system $(A - \lambda I)\mathbf{v} = 0$ is written

$$\begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -4v_1 + 4v_2 \\ 3v_1 - 3v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The solution to the equations $-4v_1 + 4v_2 = 0$, $3v_1 - 3v_2 = 0$ are $v_1 = v_2 = s$,

With an arbitrary parameter $s \neq 0$. Letting for example $s = 1$ we obtain the following eigenvector corresponding to the eigenvalue $\lambda_1 = 5$ is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -2$, the system $(A - \lambda I)\mathbf{v} = 0$ is written

$$\begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3v_1 + 4v_2 \\ 3v_1 + 4v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The solution to the equation $3v_1 + 4v_2 = 0$ are $v_1 = -4s, v_2 = 3s$,

with an arbitrary parameter $s \neq 0$. Letting for example $s=1$, we obtain the following eigenvector corresponding to the eigenvalue $\lambda_2 = -2$ is $\mathbf{v}_2 = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

Example 3.2.2:

Similarly, we can find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The characteristic polynomial $p_n(\lambda)$ and the characteristic equation for the system $\mathbf{Y}' = A\mathbf{Y}$ are defined by $p_n(\lambda) \equiv |A - \lambda I| = 0$

$$\det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & 2 & 2 \\ 0 & 3 - \lambda & 1 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = (4 - \lambda)(3 - \lambda)(1 - \lambda) = 0$$

So $\lambda = 4, \lambda = 3, \lambda = 1$

Hence, the eigenvalue are $\lambda_1 = 4$, $\lambda_2 = 3$ and $\lambda_3 = 1$.

For $\lambda_1 = 4$, the system $(A - \lambda I)\mathbf{v} = 0$ is written

$$\begin{pmatrix} 0 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2v_2 + 2v_3 \\ -v_2 + v_3 \\ -3v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Whence $v_1 = 1, v_2 = 0, v_3 = 0$. Hence we have the eigenvector corresponding

to the eigenvalue $\lambda_1 = 4$ is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

For $\lambda_2 = 3$, the system $(A - \lambda I)\mathbf{v} = 0$ is written

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} \nu_1 + 2\nu_2 + 2\nu_3 \\ -\nu_2 + \nu_3 \\ 3\nu_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The solution to the equations $\nu_1 + 2\nu_2 + 2\nu_3 = 0$, $-\nu_2 + \nu_3 = 0$ and $3\nu_3 = 0$ are $\nu_1 = 2s, \nu_2 = -s, \nu_3 = 0$, With an arbitrary parameter $s \neq 0$. Letting for example $s = 1$, we obtain the following eigenvector corresponding to the second

eigenvalue $\lambda_2 = 3$ is $\boldsymbol{\nu}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

For $\lambda_3 = 1$, the system $(A - \lambda I)\nu = 0$ is written

$$\begin{pmatrix} 3 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -0 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} 3\nu_1 + 2\nu_2 + 2\nu_3 \\ 2\nu_2 + \nu_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The solution to the equations $3\nu_1 + 2\nu_2 + 2\nu_3 = 0$, $2\nu_2 + \nu_3 = 0$ are $\nu_1 = 2s, \nu_2 = 3s, \nu_3 = -6s$, With an arbitrary parameter $s \neq 0$. Letting for example $s = 1$, we obtain the following eigenvector corresponding to the third

eigenvalue $\lambda_3 = 1$ is $\boldsymbol{\nu}_3 = \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$.

3.3 Solution of the Homogeneous System

Theorem 3.3.1: (Distinct real eigenvalues)

Let the matrix A have n distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Then, solving equation (3.2.3) for each eigenvalue, one obtains precisely n linearly independent eigenvectors $v_1, \dots, v_n$. consequently (3.2.2) provides the following fundamental system of solutions: $y_1 = e^{\lambda_1 x} v_1, \dots, y_n = e^{\lambda_n x} v_n$. (3.3.1)

The general solution to the system (3.2.1) is given by

$$Y = c_1 e^{\lambda_1 x} v_1 + \dots + c_n e^{\lambda_n x} v_n. \quad (3.3.2)$$

Example 3.3.1:

According to theorem 3.3.1, the general solution for the system $Y' = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} Y$

from example 3.2.1 is given by $Y = c_1 e^{5x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2x} \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

hence, the solution has the form $y_1 = c_1 e^{5x} - 4c_2 e^{-2x}$, $y_2 = c_1 e^{5x} + 3c_2 e^{-2x}$

Example 3.3.2:

According to theorem 3.3.1, the general solution for the system $Y' = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} Y$

from example 3.2.2 is given by $Y = c_1 e^{4x} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{3x} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + c_3 e^x \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$.

hence, the solution has the form

$$y_1 = c_1 e^{4x} + 2c_2 e^{3x} + 2c_3 e^x, \quad y_2 = -c_2 e^{3x} + 3c_3 e^x, \quad y_3 = -6c_3 e^x.$$

Theorem 3.3.2: (repeated eigenvalues)

If the equation (3.3.4) has a complex root λ , then one can proceed as in the similar situation for single higher-order equations. Therefore, assume that the matrix A has repeated roots $\lambda_1 = \lambda_2$ and the eigenvectors with $\nu_1 = \nu_2$. Hence one solution $y_1 = e^{\lambda x} \nu_1$ and second solution is obtained by letting $y_2 = e^{\lambda x} \nu_1 x$

The general solution to the equation $Y = c_1 y_1 + c_2 y_2$.

Example 3.3.3:

Consider as an example the system $Y' = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} Y$, the matrix $A = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$ has repeated roots $\lambda_1 = \lambda_2 = 1$, and the eigenvector with $\nu_1 = \nu_2$.

Hence, one solution $y_1 = e^x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, a second solution is obtained by letting

$y_2 = e^x \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} x + \begin{pmatrix} a \\ b \end{pmatrix} \right]$, substitution in the equation $Y' = AY$ yields $b = 1 + a$,

So $Y = c_1 y_1 + c_2 y_2 = c_1 e^x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \left[x e^x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^x \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$

Hence, the solution has the form

$$y_1 = c_1 e^x + c_2 x e^x, \quad y_2 = c_1 e^x + c_2 x e^x + c_2 e^x$$

Theorem 3.3.3: (complex eigenvalues)

If the equation (3.4.4) has a complex root λ , then it has also the complex conjugate root $\bar{\lambda}$. Therefore, assume that the matrix A has complex conjugate eigenvalues $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$. Then equation (3.3.3) provides the

complex conjugate eigenvectors $v = p + iq$ and $\bar{v} = p - iq$. The corresponding complex solutions obtained by (3.1.2) gives two real solutions:

$$\begin{aligned} y_1 &= e^{\alpha x} (p \cos \beta x - q \sin \beta x) \\ y_2 &= e^{\alpha x} (p \sin \beta x + q \cos \beta x) \end{aligned} \quad (3.3.3)$$

It remains to substitute (3.3.3) in the general solution (3.3.2).

Example 3.3.4:

Consider as an example the system $Y' = \begin{pmatrix} -2 & 5 \\ -1 & 2 \end{pmatrix} Y$ with $A = \begin{pmatrix} -2 & 5 \\ -1 & 2 \end{pmatrix}$, the matrix A has complex eigenvalues $\lambda = 2i$, $\bar{\lambda} = -2i$.

equation $(A - \lambda I)v = 0$ yields the complex conjugate eigenvectors are

$$v = \begin{pmatrix} 5s \\ (2 - i)s \end{pmatrix} \quad \text{and}$$

$$\bar{v} = \begin{pmatrix} 5s \\ (2 + i)s \end{pmatrix}.$$

Let us set $s = 1$ and write the eigenvector v in the form $v = p + iq$ as

$$v = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \text{where} \quad p = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad q = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\text{equations} \quad y_1 = e^{\alpha x} (p \cos \beta x - q \sin \beta x)$$

$$y_2 = e^{\alpha x} (p \sin \beta x + q \cos \beta x)$$

Provide the following fundamental system of solutions:

$$y_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \cos(2x) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2x)$$

$$y_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \sin(2x) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2x)$$

Thus, the general solution of our system is $\mathbf{Y} = c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2$ or

$$\mathbf{Y} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} [c_1 \cos(2x) + c_2 \sin(2x)] + \begin{pmatrix} 0 \\ 1 \end{pmatrix} [c_1 \sin(2x) - c_2 \cos(2x)]$$

Chapter 4

Nonhomogeneous Linear Systems

4.1 Extension to the homogeneous system

The extension to the homogeneous system is the nonhomogeneous system which define by :

$$Y'(x) = A(x)Y(x) + b(x) , \quad (4.1.1)$$

where $A(x) = (a_{ij})_{n \times n}$ is a $n \times n$ matrix, $b(x) = \begin{pmatrix} b_{11} \\ \vdots \\ b_{n1} \end{pmatrix}$ is a vector.

Theorem 4.1.1:

Let $\varphi_j(x)$ ($1 \leq j \leq n$) form a fundamental set for

$$Y'(x) = A(x)Y(x) \quad (4.1.2)$$

and let Ψ and Ψ_0 be solutions of (4.1.1) then there are a unique constants

$$c_1, c_2, \dots, c_n \text{ such that } \Psi = c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n + \Psi_0 . \quad (4.1.3)$$

proof:

Let Ψ and Ψ_0 be solutions of (4.1.1) , so

$$\Psi' = A\Psi + b \quad (1)$$

$$\Psi_0' = A\Psi_0 + b \quad (2)$$

Subtract (2) from (1) we get

$$(\Psi' - \Psi_0') = A(\Psi - \Psi_0)$$

$\therefore (\Psi - \Psi_0)$ satisfies homogenous system (4.1.2)

Thus by theorem 3.1.2

there exist a unique constants $c_1, c_2, \dots, c_n$ such that

$$\Psi - \Psi_0 = c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n$$

then

$$\Psi = c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n + \Psi_0 \blacksquare$$

4.2 Variation of Parameters:

Consider the linear nonhomogeneous system:

$$Y'(x) = A(x)Y(x) + b(x), \quad (4.1.1)$$

where A is a $n \times n$ constant matrix.

Theorem 4.2.1:

Let $\Phi(x)$ be a fundamental matrix for (4.1.2) then the function vector

$$\Psi_0 = \Phi(x) \int_a^x \Phi^{-1}(t)b(t)dt \quad (4.2.1)$$

is also solution for (4.1.1)

proof:

by differentiate (4.2.1) we get

$$\Psi_0'(x) = \Phi'(x) \int_a^x \Phi^{-1}(t)b(t)dt + \Phi(x)\Phi^{-1}(x)b(x)$$

$$\Psi_0'(x) = A\Phi(x) \int_a^x \Phi^{-1}(t)b(t)dt + b(x)$$

$$\Psi_0'(x) = A\Psi_0(x) + b(x)$$

So it's a solution for (4.1.1) ■

Theorem 4.2.2:

Let $\Phi_1(x)$ and $\Phi_2(x)$ be solutions of

$$\Phi'(x) = A\Phi(x) \quad (4.2.2)$$

Let

$$\Phi(x) = \Phi_1(x) + \Phi_2(x) \quad (4.2.3)$$

and let $\Phi(x)$ be non-singular, Let a and b be fixed points. Then the functions

$$\Psi_0(x) = \Phi_1(x) \int_a^x \Phi^{-1}(t)b(t)dt - \Phi_2(x) \int_x^b \Phi^{-1}(t)b(t)dt \quad (4.2.4)$$

is a solution for (4.1.1).

proof:

by differentiate (4.2.4) we get

$$\Psi_0'(x) = \Phi_1'(x) \int_a^x \Phi^{-1}(t)b(t)dt + \Phi_1(x)\Phi^{-1}(x)b(x) - \Phi_2'(x) \int_x^b \Phi^{-1}(t)b(t)dt + \Phi_2(x)\Phi^{-1}(x)b(x)$$

$$\Psi_0'(x) = A\Phi_1(x) \int_a^x \Phi^{-1}(t)b(t)dt - A\Phi_2(x) \int_x^b \Phi^{-1}(t)b(t)dt + [\Phi_1(x) + \Phi_2(x)]\Phi^{-1}(x)b(x)$$

$$\Psi_0'(x) = A[\Phi_1(x) \int_a^x \Phi^{-1}(t)b(t)dt - \Phi_2(x) \int_x^b \Phi^{-1}(t)b(t)dt] + \Phi(x)\Phi^{-1}(x)b(x)$$

$$\Psi_0'(x) = A\Psi_0(x) + b(x)$$

$\therefore \Psi_0$ is a solution for (4.1.1) ■

Example 4.2.1:

To solve the following non-homogeneous system:

$$Y' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 3 & 2 & 4 \end{pmatrix} Y + \begin{pmatrix} e^{-x} \\ 0 \\ -e^{-x} \end{pmatrix}$$

the fundamental system of solutions for the homogeneous system is given by

$$\Phi(x) = \begin{pmatrix} -6e^x & 0 & 0 \\ 3e^x & e^{3x} & 0 \\ 4e^x & -2e^{3x} & e^{4x} \end{pmatrix}$$

The inverse of this matrix is

$$\Phi^{-1}(x) = \begin{pmatrix} \frac{-1}{6}e^{-x} & 0 & 0 \\ \frac{1}{2}e^{-3x} & e^{-3x} & 0 \\ \frac{5}{3}e^{-4x} & 2e^{-4x} & e^{-4x} \end{pmatrix}$$

We have

$$\Phi^{-1}(x)b(x) = \begin{pmatrix} \frac{-1}{6}e^{-x} & 0 & 0 \\ \frac{1}{2}e^{-3x} & e^{-3x} & 0 \\ \frac{5}{3}e^{-4x} & 2e^{-4x} & e^{-4x} \end{pmatrix} \begin{pmatrix} e^{-x} \\ 0 \\ -e^{-x} \end{pmatrix} = \begin{pmatrix} \frac{-1}{6}e^{-2x} \\ \frac{1}{2}e^{-4x} \\ \frac{2}{3}e^{-5x} \end{pmatrix}$$

$$\text{Therefore } \int \Phi^{-1}(x)b(x)dx = \int \begin{pmatrix} \frac{-1}{6}e^{-2x} \\ \frac{1}{2}e^{-4x} \\ \frac{2}{3}e^{-5x} \end{pmatrix} dx = \begin{pmatrix} \frac{1}{12}e^{-2x} \\ -\frac{1}{8}e^{-4x} \\ -\frac{2}{15}e^{-5x} \end{pmatrix}$$

$$\text{and hence } \Phi(x) \int \Phi^{-1}(x)b(x)dx = \begin{pmatrix} -6e^x & 0 & 0 \\ 3e^x & e^{3x} & 0 \\ 4e^x & -2e^{3x} & e^{4x} \end{pmatrix} \begin{pmatrix} \frac{1}{12}e^{-2x} \\ -\frac{1}{8}e^{-4x} \\ -\frac{2}{15}e^{-5x} \end{pmatrix}$$

$$\therefore \Phi(x) \int \Phi^{-1}(x)b(x)dx = \frac{1}{2}e^{-x} \begin{pmatrix} -1 \\ \frac{1}{4} \\ \frac{9}{10} \end{pmatrix}$$

So the general solution to the nonhomogeneous system is

$$\mathbf{Y} = c_1 e^x \begin{pmatrix} -6 \\ 3 \\ 4 \end{pmatrix} + c_2 e^{3x} \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + c_3 e^{4x} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2} e^{-x} \begin{pmatrix} -1 \\ \frac{1}{4} \\ \frac{9}{10} \end{pmatrix}$$

4.3 Extension for the nonhomogeneous system:

Theorem 4.3.1:

Let $\boldsymbol{\varphi}_j(x)$ form a fundamental set for

$$\mathbf{Y}'(x) = A(x)\mathbf{Y}(x)$$

and let $\Psi(x)$ be a solution of

$$\mathbf{Y}'(x) = \{A(x) + B(x)\}\mathbf{Y}(x) + \mathbf{b}(x), \quad (4.3.1)$$

Let $\Psi_0(x)$ be a solution of

$$\mathbf{Y}'(x) = A(x)\mathbf{Y}(x) + B(x)\Psi(x) + \mathbf{b}(x) \quad (4.3.2)$$

Then there exist a unique constants $c_1, c_2, \dots, c_n$, such that

$$\Psi(x) = c_1\boldsymbol{\varphi}_1 + c_2\boldsymbol{\varphi}_2 + \dots + c_n\boldsymbol{\varphi}_n + \Psi_0(x).$$

Proof:

Since $\Psi(x)$ be a solution for (4.3.1), then

$$\Psi'(x) = \{A(x) + B(x)\}\Psi(x) + \mathbf{b}(x) \quad (1)$$

and $\Psi_0(x)$ be a solution of (4.3.2), then

$$\Psi_0'(x) = A(x)\Psi_0(x) + B(x)\Psi(x) + \mathbf{b}(x) \quad (2)$$

(1) – (2) we get $(\Psi(x) - \Psi_0(x))' = A(x)(\Psi(x) - \Psi_0(x))$

i.e. $\Psi(x) - \Psi_0(x)$ satisfies the homogeneous equation $\mathbf{Y}' = A(x)\mathbf{Y}$,

and $\boldsymbol{\varphi}_j(x)$ fundamental set ,

$\therefore$ by Theorem 3.1.2 $\exists$ a unique constant $c_1, c_2, \dots, c_n$, such that

$$\Psi(x) - \Psi_0(x) = c_1\boldsymbol{\varphi}_1 + c_2\boldsymbol{\varphi}_2 + \dots + c_n\boldsymbol{\varphi}_n.$$

i.e. $\Psi(x) = c_1\boldsymbol{\varphi}_1 + c_2\boldsymbol{\varphi}_2 + \dots + c_n\boldsymbol{\varphi}_n + \Psi_0(x).$

Example 4.3.1:

We consider the second order equation

$$a_0 y'' + a_1 y' + (a_2 + \alpha)y = \beta \quad (4.3.2)$$

with corresponding homogeneous equation $a_0 y'' + a_1 y' + a_2 y = 0$ (4.3.3)

rewrite (4.3.2) in form

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix} \quad (4.3.4)$$

Now let φ_1 and φ_2 be linearly independent solutions of (4.3.3) then,

$\Phi = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{pmatrix}$ is a fundamental matrix for $Y'(x) = A(x)Y(x)$.

If we replace $\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} \psi \\ \psi' \end{pmatrix}$ in the second term of (4.3.4), and

$$b(x) = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha \psi + \beta \end{pmatrix}$$

Then (4.3.4) can be written in form

$$\begin{aligned} \begin{pmatrix} \psi_0 \\ \psi_0' \end{pmatrix} &= \begin{pmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{pmatrix} \int_a^x \frac{1}{W(t)} \begin{pmatrix} \varphi_2'(t) & -\varphi_2(t) \\ -\varphi_1'(t) & \varphi_1(t) \end{pmatrix} \begin{pmatrix} 0 \\ -\alpha \psi + \beta \end{pmatrix} dt \\ \begin{pmatrix} \psi_0 \\ \psi_0' \end{pmatrix} &= \begin{pmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{pmatrix} \int_a^x \frac{1}{W(t)} \left(\frac{\beta}{a_0} - \frac{\alpha}{a_0} \psi \right) \begin{pmatrix} -\varphi_2(t) \\ \varphi_1(t) \end{pmatrix} dt \\ \therefore \psi_0 &= \int_a^x \left\{ \frac{\varphi_2(x)\varphi_1(t) - \varphi_1(x)\varphi_2(t)}{W(t)} \right\} \left(\frac{\beta - \alpha \psi}{a_0} \right) dt \end{aligned} \quad (3.4.5)$$

$$\text{Also } \psi_0' = \int_a^x \left\{ \frac{\varphi_2'(x)\varphi_1(t) - \varphi_1'(x)\varphi_2(t)}{W(t)} \right\} \left(\frac{\beta - \alpha \psi}{a_0} \right) dt$$

(3.4.5) is called the variation of parameter for equation (4.3.2).

4.4 The Adjoint System:

Theorem 4.4.1:

Let $\Phi(x)$ be a fundamental matrix for $Y'(x) = A(x)Y(x)$ then $\Theta^* = \Phi^{*-1} = \Psi$ is a fundamental matrix for the system

$$Y'(x) = -A^*(x)Y(x) \quad (4.4.1)$$

where $*$ denote adjoint matrix define for $A = (a_{ij})$; $A^* = (a_{ij})^* = (\bar{a}_{ij})$.

Proof:

Let $\Theta(x) = \Phi^{-1}(x)$

Multiply both side by $\Phi(x)$ we get

$$\Theta(x)\Phi(x) = \Phi^{-1}(x)\Phi(x)$$

then

$$\Theta(x)\Phi(x) = I ,$$

where I is identity matrix.

By differentiate both side

$$\Theta'(x)\Phi(x) + \Theta(x)\Phi'(x) = 0$$

Substitute by $\Phi'(x) = A(x)\Phi(x)$

$$\Theta'(x)\Phi(x) + \Theta(x)A(x)\Phi(x) = 0$$

multiply both side by $\Phi^{-1}(x)$

$$\Theta'(x)\Phi(x)\Phi^{-1}(x) + \Theta(x)A(x)\Phi(x)\Phi^{-1}(x) = 0$$

$$\Theta'(x) + \Theta(x)A(x) = 0$$

$$\Theta'(x) = -\Theta(x)A(x)$$

$$\Theta'^*(x) = -A^*(x)\Theta^*(x)$$

$$\Theta'^*(x) = -A^*(x)\Theta^*(x)$$

$$\because \Theta = \Phi^{-1}, \Theta^* = \Phi^{*-1}$$

$$\therefore Y'(x) = -A^*(x)Y(x) \quad \blacksquare$$

Remark : Now $\Psi_0 = \Phi \int_a^x \Phi^{-1}(t)b(t)dt$

Put $\Psi = \Phi^{*-1}$

$$\Rightarrow \Psi^* = \Phi^{-1}$$

$$\therefore \Psi_0 = \Phi \int_a^x \Psi^*(t)b(t)dt \quad (1)$$

Now we consider the n^{th} - order differential equation.

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0 \quad (2)$$

Then

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ \frac{-a_n}{a_0} & \frac{-a_{n-1}}{a_0} & \frac{-a_{n-2}}{a_0} & \dots & \frac{-a_1}{a_0} \end{pmatrix}$$

We need to show that $Y' = A(x)Y$ is equivalent to an n^{th} order differential equation i.e. the adjoint equation of (2). Now the system $Y' = A(x)Y$ take the form

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & \frac{\bar{a}_n}{\bar{a}_0} \\ -1 & 0 & \dots & \frac{\bar{a}_{n-1}}{\bar{a}_0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & -1 & \frac{\bar{a}_1}{\bar{a}_0} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad (3)$$

$$\left. \begin{aligned} y_1' &= \frac{\bar{a}_n}{\bar{a}_0} y_n \\ y_2' &= \frac{\bar{a}_{n-1}}{\bar{a}_0} y_n - y_1 \\ y_3' &= \frac{\bar{a}_{n-2}}{\bar{a}_0} y_n - y_2 \\ &\vdots \\ y_n' &= \frac{\bar{a}_1}{\bar{a}_0} y_n - y_{n-1} \end{aligned} \right\} (4)$$

Now

$$y_1' = \frac{\bar{a}_n}{\bar{a}_0} y_n$$

$$y_2'' = -y_1' + \left(\frac{\bar{a}_{n-1}}{\bar{a}_0} y_n \right)' \times (-1)$$

$$y_3^{(3)} = -y_2'' + \left(\frac{\bar{a}_{n-2}}{\bar{a}_0} y_n \right)'' \times (-1)^2$$

$\vdots$

$$y_n^{(n)} = -y_{n-1}^{(n-1)} + \left(\frac{\bar{a}_1}{\bar{a}_0} y_n \right)^{(n-1)} \times (-1)^{(n-1)}$$

and adding the equations

$$(-1)^{n-1} y_n^{(n)} = (-1) \left(\frac{\bar{a}_{n-1}}{\bar{a}_0} y_n \right)' + (-1)^2 \left(\frac{\bar{a}_{n-2}}{\bar{a}_0} y_n \right)'' + \dots + (-1)^{n-1} \left(\frac{\bar{a}_1}{\bar{a}_0} y_n \right)^{(n-1)} + \frac{\bar{a}_n}{\bar{a}_0} y_n \quad (5)$$

Define $z = \frac{y_n}{\bar{a}_0}$, then $y_n = \bar{a}_0 z$

So (5) gives

$$(-1)^n (\bar{a}_0 z)^{(n)} + (-1)^{n-1} (\bar{a}_1 z)^{(n-1)} + \dots + (-1)^2 (\bar{a}_{n-2} z)'' + (-1) (\bar{a}_{n-1} z)' + \bar{a}_n z = 0 \blacksquare$$

4.5 periodic system:

Definition 4.5.1:

The system $Y'(x) = A(x)Y(x)$ (4.1.1)

is periodic if the system $A(x + \omega) = A(x)$, (4.5.1)

where ω nono-zero constant.

Theorem 4.5.1:

Let $y(x)$ be solution for (4.5.1), then $y(x + \omega)$ is also solution.

Proof:

Let $t = x + \omega$ and $dx = dt$

Thus $Y'(x) = A(x)Y(x)$,

so $Y'(x) - A(x)Y(x) = 0$

$$Y'(x + \omega) - A(x + \omega)Y(x + \omega) = \frac{d}{dt}(Y(t)) - A(t)Y(t) = 0$$

$$Y'(t) - A(t)Y(t) = 0 \quad \blacksquare$$

Theorem 4.5.2:

There are a non-zero constant and a non-trivial solution $\psi(x)$ of (4.5.1) such that

$$\psi(x + \omega) = \delta\psi(x) \quad (4.5.2)$$

Proof:

Let $\Phi(x)$ be a fundamental matrix for (4.5.1) with $\Phi(0) = I$,
then by theorem (4.5.1) $\Phi(x + \omega)$ also a fundamental matrix for (4.5.1).

$$\Phi(x + \omega) = \Phi(x)C$$

Put $x = 0$, then

$$\Phi(\omega) = \Phi(0)C$$

$$\therefore C = \Phi(\omega)$$

Now any solution for (4.5.1) takes the form $\psi(x) = \Phi(x)b$

$$\therefore \psi(x + \omega) = \Phi(x + \omega)b = \Phi(x)Cb$$

Now (4.5.2) holds if

$$\delta\psi(x) = \Phi(x)Cb$$

$$\delta\Phi(x)b = \Phi(x)Cb$$

$$\delta b = Cb$$

i.e. δ eigenvalue for the matrix C, b eigenvector for C . So if

$$\det(C - \delta I) = 0 \quad (4.5.3)$$

has at least one root (equation of degree n). ■

Now (4.5.3) has N distinct roots $\delta_1, \dots, \delta_N$ and linearly independent solutions

$$\psi_1, \dots, \psi_N$$

$$C = J \wedge J^{-1} \quad (4.5.4)$$

Where J is the Jordan canonical form and let the fundamental matrix ψ define to

$$\text{be } \psi = \Phi J$$

Then by (4.5.2)

$$\psi(x + \omega) = \Phi(x + \omega)J = \Phi(x)CJ$$

$$\psi(x + \omega) = \Phi(x)J \wedge$$

By (4.5.4)

$$\text{i.e. } \psi(x + \omega) = \psi(x) \wedge \quad (4.5.5)$$

Example 4.5.1:

The second order differential equation $a_0 y'' + a_1 y' + a_2 y = 0$ we have two cases for the matrix Λ

Case1: $\Lambda = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$, then (4.5.5) gives

$$(\psi_1(x + \omega) \quad \psi_2(x + \omega)) = (\psi_1(x) \quad \psi_2(x)) \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$$

$$\psi_1(x + \omega) = \delta_1 \psi_1(x), \quad \psi_2(x + \omega) = \delta_2 \psi_2(x)$$

Define

$$\delta_j = e^{m_j \omega}, \quad P_j(x) = e^{-m_j x} \psi_j(x)$$

We use $\psi_j(x + \omega) = \delta_j \psi_j(x)$

Then $P_j(x + \omega) = e^{-m_j(x + \omega)} \psi_j(x + \omega)$

$$P_j(x + \omega) = e^{-m_j(x + \omega)} \delta_j \psi_j(x)$$

$$P_j(x + \omega) = e^{-m_j x} e^{-m_j \omega} \delta_j \psi_j(x)$$

$$P_j(x + \omega) = e^{-m_j x} \psi_j(x) = P_j(x)$$

with $\delta_j = e^{m_j \omega}$,

$$\text{So } \psi_j(x) = e^{m_j x} P_j(x) \quad (j = 1, 2) \quad (4.5.6)$$

where $P_j(x)$ is periodic.

Case2: $\Lambda = \begin{pmatrix} \delta & 1 \\ 0 & \delta \end{pmatrix}$, then (4.5.5) gives ,

$$(\psi_1(x + \omega) \quad \psi_2(x + \omega)) = (\psi_1(x) \quad \psi_2(x)) \begin{pmatrix} \delta & 1 \\ 0 & \delta \end{pmatrix}$$

So

$$\psi_1(x + \omega) = \delta\psi_1 \quad , \quad \psi_2(x + \omega) = \psi_1 + \delta\psi_2$$

$$\text{Now } \psi_1 = e^{mx} p_1 \quad , \quad \delta = e^{m\omega}$$

$$\text{Define } \psi_2(x) = e^{mx} \{kxp_1(x) + p_2(x)\}$$

$$\text{then } \psi_2(x + \omega) = \psi_1 + \delta\psi_2$$

$$\text{gives } e^{m(x+\omega)} \{k(x+\omega)p_1(x) + p_2(x+\omega)\} = e^{mx} p_1(x) + e^{m(x+\omega)} \{kxp_1(x) + p_2(x)\}$$

$$e^{m(x+\omega)} \{k\omega p_1(x) + p_2(x+\omega)\} = e^{mx} p_1(x) + e^{m(x+\omega)} p_2(x)$$

$$e^{m\omega} e^{mx} \{k\omega p_1(x) + p_2(x+\omega)\} = e^{mx} p_1(x) + e^{m(x+\omega)} p_2(x)$$

$$\text{Choose } e^{m\omega} k\omega = 1 \quad , \quad \text{then}$$

$$e^{mx} p_1(x) + e^{m(x+\omega)} p_2(x+\omega) = e^{mx} p_1(x) + e^{m(x+\omega)} p_2(x)$$

$$\text{then } p_2(x + \omega) = p_2(x)$$

$$\text{Now define } \psi_3(x) = \frac{\psi_2(x)}{k}$$

$$\text{i.e. } \psi_3(x) = e^{mx} \left\{ xp_1(x) + \frac{p_2(x)}{k} \right\} = e^{mx} \{xp_1(x) + p_3\}$$

$$\text{where } p_3 = \frac{p_2(x)}{k} \text{ periodic because } p_2 \text{ is periodic } \blacksquare$$

4.6 Extensions:

Case I: the diagonal system

$$Y' = \Lambda(x)Y, \quad \Lambda = dg(\lambda_1(x), \dots, \lambda_n(x))$$

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$y_1' = \lambda_1 y_1, \dots, y_n' = \lambda_n y_n$$

$$y_j' = \lambda_j y_j \quad (1 \leq j \leq n)$$

$$\text{so } \frac{dy_j'}{y_j} = \lambda_j dx, \text{ then } \ln y_j = \int \lambda_j(t) dt + c_j$$

$$y_j = c_j \exp\left(\int_a^x \lambda_j(t) dt\right), (1 \leq j \leq n) \quad (4.6.1)$$

are solutions for $Y'(x) = \Lambda Y(x)$, where c_j are arbitrary constants.

If we now take any integer $(1 \leq j \leq n)$ and $c_k = 1$ and other c_j zero, we obtain $y_k = e_k \exp\left(\int_a^x \lambda_k(t) dt\right)$ explicit solutions of n linearly independent of the system

$$Y' = \Lambda Y,$$

where e_k is the coordinate vector whose k^{th} component is unity and other components zero.

A special case for $Y'(x) = \Lambda Y(x)$ when $\Lambda = dg(\lambda_1, \dots, \lambda_n) = \text{constant}$ then by (4.6.1) $y_j = c_j e^{\lambda_j x}$, $(1 \leq j \leq n)$ and the explicit solutions

$y_k = e_k e^{\lambda_k x}$ are solutions for $Y'(x) = \Lambda Y(x)$

Case II :The transformation $Y = PZ$

If we consider the system $Y'(x) = AY(x)$, where A is constant. (4.6.2)

Now $A_{n \times n}$ matrix is diagonalizable if and only if A has n linearly independent eigenvectors $P_1, P_2, \dots, P_n$ such that $P^{-1}AP = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, where λ_j is the eigenvalue associated with the vector P_j .

By using the transformation $Y = PZ$, where P is constant, such that $P^{-1}AP = \Lambda$ differentiate both side we get $Y' = PZ' + P'Z$, A has n linearly independent eigenvectors.

so $P' = 0$, thus $Y' = PZ'$

so (4.6.2) gives $PZ' = APZ$

multiplying both side by P^{-1}

$$P^{-1}PZ' = P^{-1}APZ$$

$$Z' = P^{-1}APZ$$

$$\therefore P^{-1}AP = \Lambda$$

$$\text{Thus } Z' = \Lambda Z \quad (4.6.3)$$

$$z_j = c_j e^{\lambda_j x}, \quad (1 \leq j \leq n) \quad (4.6.4)$$

Solution for (4.6.3).

to find the solution for (4.6.2) we transform back by using $Y = PZ$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = P \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = P \begin{pmatrix} c_1 e^{\lambda_1 x} \\ \vdots \\ c_n e^{\lambda_n x} \end{pmatrix} \quad (4.6.5)$$

is the solution for (4.6.2)

Example 4.5.1:

By solving the system

$$\mathbf{Y}' = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} \mathbf{Y}$$

the eigenvalue are $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 2$, and the eigenvector are

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \text{ and } v_3 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \text{ respectively.}$$

$$\text{Thus } p = \begin{pmatrix} -1 & -1 & -2 \\ 1 & 1 & 1 \\ 4 & 2 & 4 \end{pmatrix} \text{ and the inverse is } p^{-1} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 2 & -\frac{1}{2} \\ -1 & -1 & 0 \end{pmatrix}$$

$$\text{Then } p^{-1}Ap = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 2 & -\frac{1}{2} \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} \begin{pmatrix} -1 & -1 & -2 \\ 1 & 1 & 1 \\ 4 & 2 & 4 \end{pmatrix}$$

$$p^{-1}Ap = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \Lambda$$

Now by (4.6.4)

$$\text{So } \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 x} \\ c_2 e^{\lambda_2 x} \\ c_3 e^{\lambda_3 x} \end{pmatrix} = \begin{pmatrix} c_1 e^{3x} \\ c_2 e^x \\ c_3 e^{2x} \end{pmatrix}$$

And by (4.6.5)

$$\therefore \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = P \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = P \begin{pmatrix} c_1 e^{\lambda_1 x} \\ \vdots \\ c_n e^{\lambda_n x} \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -2 \\ 1 & 1 & 1 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} c_1 e^{3x} \\ c_2 e^x \\ c_3 e^{2x} \end{pmatrix} = \begin{pmatrix} -c_1 e^{3x} - c_2 e^x - 2c_3 e^{2x} \\ c_1 e^{3x} + c_2 e^x + c_3 e^{2x} \\ 4c_1 e^{3x} + 2c_2 e^x + 4c_3 e^{2x} \end{pmatrix}$$

So $y_1 = -c_1 e^{3x} - c_2 e^x - 2c_3 e^{2x}$, $y_2 = c_1 e^{3x} + c_2 e^x + c_3 e^{2x}$ and
 $y_3 = 4c_1 e^{3x} + 2c_2 e^x + 4c_3 e^{2x}$

Or $Y = \nu_1 c_1 e^{3x} + \nu_2 c_2 e^x + \nu_3 c_3 e^{2x}$

Case III : The transformation $Y = P(x)Z$

More extension than $Y' = AY$ (4.6.3)

is the $Y' = A(x)Y$, (4.6.6)

now $A_{n \times n}$ matrix is diagonalizable if and only if $A=A(x)$ has n linearly independent eigenvectors $P_1, P_2, \dots, P_n$ such that

$P^{-1}AP = \Lambda(x) = \text{diag}(\lambda_1(x), \dots, \lambda_n(x))$, where λ_j is the eigenvalue associated with the vector P_j .

when we use the transformation $Y = PZ$, where $P = P(x)$ with $P^{-1}AP = \Lambda(x)$.

differentiate both side we get $Y' = PZ' + P'Z$

substitute in (4.6.2) we get $PZ' + P'Z = APZ$

$$PZ' = APZ - P'Z$$

$$PZ' = (AP - P')Z$$

Multiply both side by P^{-1} we get

$$P^{-1}PZ' = P^{-1}(AP - P')Z$$

$$Z' = (P^{-1}AP - P^{-1}P')Z$$

now $P^{-1}AP = \Lambda$, then $Z' = (\Lambda - P^{-1}P')Z$ (4.6.7)

to solve (4.6.7) we need the following theorem of (Eastham theorem 1985,1989)

for the system regarding the general system $Y'(x) = (\Lambda(x) + R(x))Y(x)$.

Theorem (Eastham theorem 1985, 1989)

Let $\Lambda(x)$ be $n \times n$ diagonal matrix,

$$\Lambda(x) = \text{diag}(\lambda_1(x), \dots, \lambda_n(x)) \quad ,$$

in which $\lambda_i(x) - \lambda_j(x)$ is nowhere zero in some interval $[a, \infty)$ for each pair of unequal integers i and j in $[1, n]$. Let the $n \times n$ matrix $R(x)$ satisfy

- i. $\{\lambda_i(x) - \lambda_j(x)\}^{-1} R(x) \rightarrow 0$ as $x \rightarrow \infty$;
- ii. $(\{\lambda_i(x) - \lambda_j(x)\}^{-1} R(x))' \in L(a, \infty)$

for each unequal i and j in $[1, n]$.

Also, let the eigenvalues $\mu_k(x)$ ($1 \leq k \leq n$) of $\Lambda(x) + R(x)$ satisfy the dichotomy condition L .

Then, as $x \rightarrow \infty$, the solutions $Y_k(x)$ with the asymptotic form

$$Y_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t) dt\right).$$

e_k is the coordinate vector whose k^{th} component is unity and other components zero.

Now we return back to (4.6.7) to apply Eastham theorem so if $\lambda_i - \lambda_j \neq 0$ for

($i \neq j, 1 \leq i, j \leq n$) and the eigenvalues $\mu_j(x)$ of the matrix $\Lambda - P^{-1}P'$

satisfies the dichotomy condition and $\{\lambda_i(x) - \lambda_j(x)\}^{-1} P^{-1}P' \rightarrow 0$ as $x \rightarrow \infty$ with

$$[\{\lambda_i(x) - \lambda_j(x)\}^{-1} P^{-1}P']' \in L(a, \infty) \text{ for } (i \neq j, 1 \leq i, j \leq n),$$

then Eastham theorems gives as $x \rightarrow \infty$ $Z_k(x)$ solutions for (4.6.7) with asymptotic form

$$Z_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t) dt\right) \quad , \quad (1 \leq k \leq n) \quad (4.6.8)$$

To find the asymptotic form of solutions of the system (4.6.3) we transform back by using $\mathbf{Y} = P\mathbf{Z}$ and (4.6.8) i.e.

$$Y_k(x) = PZ_k = P\{e_k + \mathbf{o}(\mathbf{1})\}\exp\left(\int_a^x \mu_k(t) dt\right)$$

as solution for $\mathbf{Y}' = A(x)\mathbf{Y}$.

DISCUSSIONS AND CONCLUSION

- i. As we see from our research that we consider several cases for systems of equation as $\mathbf{Y}' = A\mathbf{Y}$ (1)
and $\mathbf{Y}' = A\mathbf{Y} + \mathbf{b}$ (2)
we obtained the general solutions for (1) and (2) when $A = \text{constant}$ in chapters three and four.
- ii. In chapter 4 we give some more extended results for systems as when $A = A(x)$ variable matrix.
- iii. As we see in 4.6 we give the diagonal system $\mathbf{Y}'(x) = \Lambda \mathbf{Y}(x)$ when $\Lambda = dg(\lambda_1(x), \dots, \lambda_n(x))$ and we obtained (4.6.1).
A special case when Λ diagonal matrix we obtained explicit solution.
- iv. In equation (4.6.2) we consider $A = \text{constant}$ matrix we start to diagonalize the matrix A by using the transformation $\mathbf{Y} = P\mathbf{Z}$, where P constant matrix, we get (4.6.3).
- v. The importance of the system $\mathbf{Z}' = (\Lambda - P^{-1}P')\mathbf{Z}$ (1)
When we transform $\mathbf{Y}' = A(x)\mathbf{Y}$ (2)
by mean's of the transformation $\mathbf{Y} = P(x)\mathbf{Z}(x)$ (3)
to (1), this suggest that any Linear differential equation of any order can be transform to (1) after writing it in the form (2) and using the transformation (3) to get the system (1) to obtain the fundamental set of solutions by applying Eastham theorem to (1).

For example the fourth-order differential equation

$$(p_0 y'')'' + (p_1 y')' + p_2 y = 0 \quad (4)$$

is the case $m = 2$ of

$$(p_0 y^{(m)})^{(m)} + (p_1 y^{(m-1)})^{(m-1)} + \dots + (p_{m-1} y')' + p_m y = 0$$

with $p_i (i = 0, 1, 2)$ nowhere zero in $(0, \infty)$ can be written as first-order system $Y' = A(x)Y$ (2)

$$\text{with } A(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & p_0^{-1} & 0 \\ 0 & -p_1 & 0 & 1 \\ -p_2 & 0 & 0 & 0 \end{bmatrix} \quad (5)$$

then to diagonalize A we need the eigenvalues λ_j of A i.e. to solve $\det(A - \lambda I) = 0$ gives the characteristic equation

$$p_0 \lambda^4 + p_1 \lambda^2 + p_2 = 0 \quad (6)$$

In this case as open problems, we need to discuss some cases of the eigenvalues as $x \rightarrow \infty$ to obtain the asymptotic form of fundamental solutions by applying Eastham to (1).

Bibliography

Alexander, Graham (1979) . Matrix Theory And Applications For Engineers And Mathematicians. Sydney: Halsted Press.

AL-Hammadi,A.S.A., Ibragimov,N.H.(2006) . Introduction to Differential Equations.Karlskrona: Alga Publications.

AL-Hammadi,A.S.A. Asymptotic theory for a class of fourth-order differential equation. *Mathematika* 43(1996), 198-208.

AL-Hammadi,A.S.A. Asymptotic Formulae of Liouville-Green type for Rocky Mountain Journal of Mathematics. Vol.28, N.3, 1998 , 801-811.

Arnold,V. I. (1989).Mathematical Methods of Classical Mechanics. Berlin: Springer-Verlag.

Arnold,V. I. (2006).Ordinary Differential Equations, Berlin :Springer-Verlag.

Ayaz,Fatma (2004).Solution of the System Of Differential Equations By Differential Transform Method. *Applied Mathematics and Computation* , 12 January, Vol. 147(2).

Bhatia, Rajendra (1997). Matrix Analysis. Berlin: springer.

Chen,Chi-Tsong (1995).Linear System Theory and Design. USA: Oxford University Press.

Eastham, M.S.P.(1989).The Asymptotic Solution of linear Differential systems Applications of the Levinson Theorem. Oxford: Clarendon press.

Finkbeiner,Daniel T.(1978). Introduction to Matrices And Linear Transformations. California:W H Freeman &Co.

Giaquinta, Mariano (2007). Mathematical Analysis. Boston: Birkhauser.

Hirsch, Morris W.(2013). Differential Equations, Dynamical Systems, and an Introduction to Chaos, Oxford: Elsevier Inc.

Jang,Bongsoo (2010). Solving Linear and Nonlinear Initial Value Problems By the Projected Differential Transform Method. Computer Physics Communications, May, Vol.181(5).

Levi,M. (2014).Classical Mechanics with Calculus of Variations and Optimal Control, *AnIntuitive Introduction*.USA: American Mathematical Society.

Ovsiannikov, L.V.(1982). Group Analysis of Differential Equations. Newyork: Academic Press Inc.

Perko, Lawrence (2001). Differential Equations and Dynamical Systems. USA : Springer.

Watkins, David S. (2007). The Matrix Eigenvalue Problem. USA: The Society For Industrial And Applied Mathematics.

Wilkinson, J.H. (1965). The Algebraic Eigenvalue Problem. Oxford: Clarendon Press.

الخلاصة

نعرض في هذه الأطروحة إلى دراسة حلول المنظومة الخطية للمصفوفات لنظم المعادلات التفاضلية

$$Y' = AY$$

كما نستعرض خلالها إلى دراسة دقيقة للحلول في حالات متعددة، عندما تكون المصفوفة A ذات عناصر ثابتة، وكذلك عندما تكون ذات عناصر متغيرة، ونستخدم كذلك نظريات عديدة للحصول على الحلول في كل حالة، كما نطبق نظرية حديثة إيستام (١٩٨٥) للحصول على تعميمات دقيقة في حالات خاصة لهذه المعادلات.

جامعة البحرين

قسم: الرياضيات

كلية العلوم



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أطروحة مقدمة كجزء من متطلبات الحصول على درجة الماجستير في الرياضيات

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