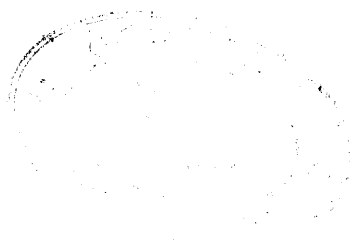


Integral Operators With Rational Kernels



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*In The Name of Allah,
Most Gracious, Most Merciful*

CONTENTS

ABSTRACT	5
DECLARATION	6
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ACKNOWLEDGEMENT	8
CHAPTER 1. Notations and Preliminaries	9
1.1 Background	9
1.2 Integral Operators	13
1.3 G. Little's Theorems	17
1.4 Equivalences	20
CHAPTER 2. $k(s, t) = \frac{1}{a + b(s + t) + cst}$	23
2.1 Introduction	23
2.2 Analysis of $k_0(s, t) = \frac{1}{1 - st}$	25
2.3 Analysis of $k_1(s, t) = \frac{1}{1 + st}$	28
2.4 The General Case $k(s, t) = \frac{1}{a + b(s + t) + cst}$	34

CHAPTER 3.	$k(s, t) = \frac{A + B(s + t) + Cst}{D + E(s + t) + Fst}$	37
3.1	Introduction	37
3.2	The Case $k(s, t) = \frac{a + b(s + t) + cst}{1 - st}$	44
3.3	The Case $k(s, t) = \frac{a + b(s + t) + cst}{1 + st}$	51
CHAPTER 4.	$k(s, t) = \frac{1}{a + b(s + t) + cst + d(s^2 + t^2)}$	55
4.1	Introduction	55
4.2	Preliminaries	56
4.3	Examples	64
4.4	General Remarks	77
4.5	The Case $k(s, t) = \frac{1}{a + b(s + t) + cst + (s^2 + t^2)}$	79
4.6	The Case $k(s, t) = \frac{1}{a + b(s + t) + cst - (s^2 + t^2)}$	86
CHAPTER 5.	Asymptotic behaviour of eigenvalues	95
5.1	Introduction	95
5.2	Factorization Through H^2 and ℓ^2	98
5.3	The E^2 Space of an Elliptic Disc	102
5.4	Tchebyschef Polynomials	106
5.5	Upper Bounds for $\lambda_n(T)$	109
5.6	Lower Bounds for $\lambda_n(T)$	116
REFERENCES		119

ABSTRACT

Let $q(s, t) = a + b(s + t) + cst + d(s^2 + t^2)$, where $a, b, c, d \in \mathbb{R}$, not all zero, and suppose that $k = \frac{1}{q}$. A *closed* (bounded) interval I is called admissible for k if $q(s, t) \neq 0$, for all $s, t \in I$, in which case we obtain a compact symmetric integral operator K_I on $L^2(I)$ with continuous kernel k :

$$K_I f(s) = \int_I \frac{f(t)}{q(s, t)} dt \quad (f \in L^2(I), s \in I).$$

In Chapter 2 we find the number of positive and the number of negative eigenvalues of K_I in the case $d = 0$, and in Chapter 4 we do the same for a general d . In Chapter 3 we find the number of positive and the number of negative eigenvalues of integral operators of kernels of the form

$$\frac{A + B(s + t) + Cst}{D + E(s + t) + Fst}.$$

In Chapter 5 we return to the case $k = \frac{1}{q}$ and $d = 0$ and we find asymptotic estimates for the positive eigenvalues of K_I and the negative eigenvalues.

DECLARATION

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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CHAPTER 1. NOTATIONS AND PRELIMINARIES

1.1 BACKGROUND

1.1.1 DEFINITION

Let T be a bounded symmetric linear operator on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. Then T is called *positive*, written $T \geq 0$, if $\langle Tf, f \rangle \geq 0$ for all $f \in H$. In general, if T and S are two bounded symmetric linear operators on H , then $T \geq S$ is defined to mean that $T - S \geq 0$. Also we say that T is *negative*, written $T \leq 0$, if $-T$ is positive.

Let H and H_1 be Hilbert spaces and $T: H \rightarrow H_1$ be a compact linear operator. Then there is an expansion for T in the form (cf. [2, page 271])

$$Tf = \sum_{n=1}^{\infty} s_n(T) \langle f, \psi_n \rangle \phi_n, \quad f \in H$$

where $(s_n(T))$ a sequence of non-negative real numbers which decreases to 0, called *approximation numbers of T* , (ψ_n) is an orthonormal sequence in H and (ϕ_n) is an orthonormal sequence in H_1 . In the case when $H = H_1$ and T is positive, the s_n 's are the eigenvalues of T .

For any compact symmetric operator T on a Hilbert space H let us denote:

(1) $(\lambda_n^+(T))$ the positive eigenvalues of T in decreasing order

$$\lambda_1^+(T) \geq \lambda_2^+(T) \geq \lambda_3^+(T) \geq \dots$$

with repetitions to account for multiplicities and $(\lambda_n^-(T))$ the negative eigenvalues in increasing order.

(2) $N^+(T), N^-(T)$ the number of positive and the number of negative eigenvalues of T , respectively, $0 \leq N^+(T), N^-(T) \leq +\infty$.

1.1.2 DEFINITION

A symmetric operator T on a Hilbert space H is called *strictly positive* if it is positive and injective.

1.1.3 REMARK

(i) If T is a strictly positive compact linear operator on a Hilbert space H , then

$\overline{\text{ran } T} = H$ so that $N^+(T) = \dim H$, in particular if $H = L^2(I)$ for some interval I

then $N^+(T) = \infty$.

(ii) If T, S are compact positive operators on a Hilbert space H and $T \geq S$ then

S strictly positive implies T strictly positive.

The eigenvalues of any compact symmetric operator T on a Hilbert space H can be determined using the **Courant-Weyl** formulae (see [8, §95]):

$$\lambda_n^+(T) = \sup_{|E|=n} \inf_{f \in E, \|f\|=1} \langle Tf, f \rangle,$$

$$\lambda_n^-(T) = \inf_{|E|=n-1} \sup_{f \in E, \|f\|=1} \langle Tf, f \rangle.$$

In each formula, E denotes a subspace of H and $|E|$ = dimension of E . If T has a finite number N of positive eigenvalues, then each formula on the right evaluates to 0 for $n > N$, and it is conventional to define $\lambda_n^+(T) = 0$ for $n > N$, so that the formulae are valid for all $n \geq 1$. To evaluate $\lambda_n^-(T)$ we can replace T with $-T$ in the formulae. This has the effect of interchanging *sup* and *inf* in both cases. If A is a compact symmetric operator with a finite number $M \geq 0$ of positive eigenvalues, and a finite number $N \geq 0$ of negative eigenvalues, then we have (cf. [8, §95])

$$\lambda_{n+M}^+(T+A) \leq \lambda_n^+(T) \text{ and } \lambda_{n+N}^-(T+A) \geq \lambda_n^-(T) \quad (1.1)$$

respectively, so that

$$N^+(T+A) \leq N^+(T) + N^+(A) \text{ and } N^-(T+A) \leq N^-(T) + N^-(A). \quad (1.2)$$

If $T \geq 0$ and A has N negative eigenvalues then

$$\lambda_{n+N}^+(T) \leq \lambda_n^+(T+A) \leq \lambda_{n-N}^+(T) \quad (1.3)$$

for $n \geq N$. In particular if $T \geq 0$, so that $N^-(T) = 0$ then $N^-(T+A) \leq N$.

If $M : H \rightarrow H_1$ a continuous linear operator, then (cf. [4, page 171])

$$\lambda_n^+(MTM^*) \leq \|M\|^2 \lambda_n^+(T) \quad (1.4)$$

and if M is invertible then

$$\frac{1}{\|M^{-1}\|^2} \lambda_n^+(T) \leq \lambda_n^+(MTM^*) \leq \|M\|^2 \lambda_n^+(T) \quad (1.5)$$

which yields $N^+(MTM^*) = N^+(T)$. Applying (1.5) for a unitary operator $U : H \rightarrow H_1$ we get

$$\lambda_n^+(T) = \lambda_n^+(UTU^*). \quad (1.6)$$

1.1.4 LEMMA [4, Lemma 1.1]

Let T be a compact symmetric operator on a Hilbert spaces H , and let N be a positive integer.

(i) T has at least N positive eigenvalues if and only if there is a subspace $E \subseteq H$ of dimension N and a $\delta > 0$ such that

$$\langle Tf, f \rangle \geq \delta \|f\|^2, \text{ for all } f \in E.$$

(ii) T has at least N positive eigenvalues if and only if there is an N -dimensional subspace $E \subseteq H$ such that

$$\langle Tf, f \rangle > 0, \text{ for all non-zero } f \in E.$$

(iii) If H_1 is a Hilbert space and $M : H_1 \rightarrow H$ a continuous linear operator, then the compact symmetric operator M^*TM on H_1 satisfies

$$N^+(M^*TM) \leq N^+(T).$$

(iv) If also the range of M is dense in H , then

$$N^+(M^*TM) = N^+(T).$$

1.2 INTEGRAL OPERATORS

Let $k : I \times I \rightarrow \mathbb{C}$, $I = [a, b]$, $-\infty < a < b < +\infty$, be a measurable function such that

$$\int_I \int_I |k(s, t)|^2 ds dt < \infty .$$

Then the *integral operator* $K_I : L^2(I) \rightarrow L^2(I)$ with kernel k is defined by

$$K_I f(s) = \int_I k(s, t) f(t) dt .$$

It is well known that K_I is compact, and the adjoint K_I^* of K_I is an integral operator with kernel function k^* , where

$$k^*(t, s) = \overline{k(s, t)} .$$

Hence if $k(t, s) = \overline{k(s, t)}$, then K_I is a self-adjoint operator.

Instead of K_I we also write simply K if it is clear to what interval I we refer in a specific discussion.

We now introduce some well-known results to which we refer throughout this thesis.

1.2.1 MERCER'S THEOREM [8, §98]

Let $k(s, t)$ be the kernel of a positive integral operator on $L^2(I)$, I (bounded) closed interval, and suppose that $k(s, t)$ is continuous on $I \times I$. Then

$$k(s, t) = \sum_{i=1}^{\infty} \lambda_i \phi_i(s) \overline{\phi_i(t)} \quad (s, t \in I)$$

where the series on the right converges uniformly and absolutely to $k(s, t)$ on $I \times I$ and (ϕ_i) denotes an orthonormal sequence of eigenvectors, and (λ_i) the sequence of corresponding eigenvalues.

1.2.2 COROLLARY

Let K be a positive integral operator on $L^2(I)$, (I is closed and bounded) with a kernel $k(s, t)$ that is continuous. Then $k(s, s) \geq 0$ for $s \in I$.

The following theorem will be useful for detecting integral operators T with

$$N^+(T) = N^-(T) = \infty.$$

Let $I = [-\rho, \rho]$, ($\rho > 0$) and let $L_e^2(I)$, $L_o^2(I)$ be the spaces of even, odd functions in $L^2(I)$ respectively. Given $f \in L^2(I)$ write f_e = projection of f on $L_e^2(I)$ and f_o = projection of f on $L_o^2(I)$. That is $f = f_e + f_o$.

1.2.3 THEOREM

Let $I = [-\rho, \rho]$, where $\rho > 0$, and suppose that k is a continuous symmetric kernel on $I \times I$ satisfying $k(s, t) = k(-s, -t)$ ($s, t \in I$). If T is the integral operator on $L^2(I)$ corresponding to k , then

(i) $L_e^2(I)$ and $L_o^2(I)$ are invariant subspaces of T .

(ii) If $T_e : L_e^2(I) \rightarrow L_e^2(I)$ is the restriction of T to $L_e^2(I)$ then T_e is unitarily equivalent to the integral operator T_+ on $L^2(0, \rho)$ given by

$$T_+g(s) = \int_0^\rho (k(s, t) + k(s, -t))g(t) dt \quad (g \in L^2(0, \rho), 0 \leq s \leq \rho).$$

(iii) If $T_o : L_o^2(I) \rightarrow L_o^2(I)$ is the restriction of T to $L_o^2(I)$ then T_o is unitarily equivalent to the integral operator T_- on $L^2(0, \rho)$ given by

$$T_-g(s) = \int_0^\rho (k(s, t) - k(s, -t))g(t) dt \quad (g \in L^2(0, \rho), 0 \leq s \leq \rho).$$

Proof. Write

$$k_e(s, t) = \frac{1}{2}(k(s, t) + k(s, -t)) = \frac{1}{2}(k(-s, -t) + k(-s, t))$$

$$k_o(s, t) = \frac{1}{2}(k(s, t) - k(s, -t)) = \frac{1}{2}(k(-s, -t) - k(-s, t))$$

so that $k = k_e + k_o$ and k_e is an even function of s and an even function of t and k_o

is an odd function of s and an odd function of t . If $f = f_e + f_o \in L^2(I)$ then

$$Tf(s) = \int_{-\rho}^\rho k_e(s, t)f_e(t) dt + \int_{-\rho}^\rho k_o(s, t)f_o(t) dt \quad (s \in I).$$

The first integral here is an even function of s and the second is an odd function of s , and this proves (i). Now suppose $f = f_e \in L_e^2(I)$ then

$$T_e f(s) = \int_0^\rho 2k_e(s, t) f(t) dt \quad (f \in L_e^2(I), s \in I).$$

Define an operator $U: L^2(0, \rho) \rightarrow L_e^2(I)$ by

$$Ug(t) = 2^{-1/2} g(|t|) \quad (g \in L^2(0, \rho), t \in I)$$

then it is easy to see that U is unitary and that

$$U^{-1} f(s) = U^* f(s) = 2^{1/2} f(s) \quad (f \in L_e^2(I), 0 \leq s \leq \rho).$$

Hence for $g \in L^2(0, \rho)$,

$$U^* T_e U g(s) = \int_0^\rho 2k_e(s, t) g(t) dt.$$

This proves (ii). We prove (iii) in a similar way. This time the unitary operator

$V: L^2(0, \rho) \rightarrow L_o^2(I)$ is given by

$$Vg(t) = 2^{-1/2} \operatorname{sgn} t g(|t|) \quad (g \in L^2(0, \rho), t \in I)$$

and we have

$$V^{-1} f(s) = V^* f(s) = 2^{1/2} f(s) \quad (f \in L_o^2(I), 0 \leq s \leq \rho).$$

1.2.4 REMARK

If $T_e \geq 0$ and $T_o \leq 0$ then $N^+(T) = N^+(T_e)$ and $N^-(T) = N^-(T_o)$.

1.3 G. LITTLE'S THEOREMS

In this section we begin by setting up some notation given by G. Little in [4] with some modifications.

An open set $E \subseteq \mathbb{C}^2$ is called *symmetric* if $(z, w) \in E$ implies $(w, z) \in E$. A complex-valued function k on E is called a *symmetric analytic kernel* on E if

- i) k is continuous on E ,
- ii) $k(z, w) = \overline{k(w, z)}$, for all $(z, w) \in E$,
- iii) k is analytic in its first variable at all points of E .

If I is a (bounded) closed real interval such that $I \times I \subseteq E$, then I is called *admissible* for k . In this case the integral operator K on $L^2(I)$ with kernel k is compact and symmetric. We use $Ad(k)$ to denote the set of all admissible intervals for k , $N^+(k, I)$ for the sum of the multiplicities of the (strictly) positive eigenvalues of K , and $N^-(k, I)$ for the sum of the multiplicities of the negative eigenvalues of K . In the previous context $N^+(k, I) = N^+(K)$ and $N^-(k, I) = N^-(K)$. Also we will use $\lambda^+(k, I)$ to denote $\lambda^+(K)$ and $\lambda^-(k, I)$ to denote $\lambda^-(K)$.

1.3.1 EXAMPLE [4, page 169]

Let K be the integral operator with kernel

$$k(s, t) = \frac{1}{1 - st}, \quad s, t \in I.$$

If $I = [a, b]$, with $-1 < a < b < 1$ then $N^+(k, I) = \infty$ and $N^-(k, I) = 0$. This is proved by expanding the kernel in a geometric series.

1.3.2 THEOREM [4, Theorem 1.1]

Let k be a symmetric analytic kernel on a symmetric domain $E \subseteq \mathbb{C}^2$. If I and J are admissible closed intervals for k and $I \cap J \neq \emptyset$, then

$$N^+(k, I) = N^+(k, J) \quad \text{and} \quad N^-(k, I) = N^-(k, J).$$

In other words, if I and J are two such intervals, then K_I and K_J have the same number of positive and the same number of negative eigenvalues.

G. Little [4], extend this theorem to include the case of certain disjoint intervals I, J . He first defined a diagonal point as follows:

Suppose that k is a symmetric analytic kernel on some domain $E \subseteq \mathbb{C}^2$. A point $a \in \mathbb{R}$ is called a *diagonal point* of k if $(a, a) \in E$. For such point he defined $N^+(k, a) := N^+(k, I)$ where I is any closed admissible interval for k which includes a . Analogously he defined $N^-(k, a)$. Because of the theorem, $N^+(k, a)$ and $N^-(k, a)$ are well-defined.

1.3.3 THEOREM [4, Theorem 2.1]

Let a, b be diagonal points of a symmetric analytic kernel k with $a < b$. If every point of the closed interval $[a, b]$ is a diagonal point of k then

$$N^+(k, a) = N^+(k, b) \text{ and } N^-(k, a) = N^-(k, b).$$

Suppose now that k is symmetric analytic kernel on $E = \Delta \times \Delta$, where Δ is the open unit disc in $\mathbb{C}$. Since any point in $[-1, 1]$ is a diagonal point of k , we immediately have

1.3.4 COROLLARY

Let k be a symmetric analytic kernel on $\Delta \times \Delta$. If I and J are admissible closed intervals for k , then

$$N^+(k, I) = N^+(k, J) \text{ and } N^-(k, I) = N^-(k, J).$$

1.4 EQUIVALENCES

Let ψ be a real-valued continuous function on the closed interval $I = [a, b]$, and let $J = [c, d]$ with $c = \psi(a)$ and $d = \psi(b)$. Suppose that ψ has a strictly positive and continuous derivative on the interior I° of I . Then the operator $U : L^2(J) \rightarrow L^2(I)$

$$Ug(s) = g(\psi(s))\psi'(s)^{1/2}, \quad (g \in L^2(J), s \in I)$$

is a unitary operator mapping $L^2(J)$ onto $L^2(I)$ and its inverse $U^{-1} = U^*$ is defined in an analogous way by ψ^{-1} . If K_J is an integral operator on $L^2(J)$ with kernel $k(s, t)$ then, for all $f \in L^2(I)$,

$$\begin{aligned} UK_J U^* f(s) &= \int_J \psi'(s)^{1/2} k(\psi(s), t) (U^* f)(t) dt \\ &= \int_J \psi'(s)^{1/2} k(\psi(s), t) f(\psi^{-1}(t)) \psi^{-1'}(t)^{1/2} dt \\ &= \int_I \psi'(s)^{1/2} k(\psi(s), \psi(t)) \psi'(t)^{1/2} f(t) dt. \end{aligned}$$

Thus $UK_J U^* := \psi K_I$ is an integral operator on $L^2(I)$ with kernel

$$\psi k(s, t) := \psi'(s)^{1/2} k(\psi(s), \psi(t)) \psi'(t)^{1/2} \quad (1.7)$$

so that K_J and ψK_I have exactly the same eigenvalues.

We now introduce some definitions to which we refer throughout this thesis.

1.4.1 DEFINITION

Given two sequences (a_n) , (b_n) of non-negative real numbers we shall write $a_n \approx b_n$ when $a_n = O(b_n)$ and $b_n = O(a_n)$.

Having defined the $\approx$ -notation, we may write (1.5) as $\lambda_n^+(T) \approx \lambda_n^+(MTM^*)$.

1.4.2 DEFINITION

Let K be an integral operator on $L^2(J)$ with kernel k and K' be an integral operator on $L^2(I)$ with kernel k' .

(1) We say that K is *symmetrically equivalent* to K' if

$$K' = MKM^* \quad (1.8)$$

for some continuous and invertible linear operator $M: L^2(J) \rightarrow L^2(I)$.

(2) If M in (1.8) is a unitary operator then we say that K is *unitarily equivalent* to K' .

(3) We say that K is *spatially equivalent* to K' , denoted by $K \equiv K'$, if there exist a real-valued function $\psi: I \rightarrow J$ as above with $\psi(I) = J$ and $\psi' > 0$ on I such that

$$K' = \psi K.$$

That is $k' = \psi k$ as in (1.7).

Observe that spatial equivalence implies unitary equivalence and unitary equivalence implies symmetric equivalence.

In view of the above definitions if K and K' are two integral operators on $L^2(J)$ and $L^2(I)$ respectively then we have the following lemma.

1.4.3 LEMMA

(i) K unitarily equivalent to K' implies $\lambda_n^\pm(K) = \lambda_n^\pm(K')$.

(ii) K symmetrically equivalent to K' implies $\lambda_n^+(K) \approx \lambda_n^+(K')$ and

$|\lambda_n^-(K)| \approx |\lambda_n^-(K')|$, in particular

$$N^\pm(K) = N^\pm(K').$$

Proof. The proof follows from (1.5) and (1.6).

CHAPTER 2. $k(s,t) = \frac{1}{a+b(s+t)+cst}$

2.1 INTRODUCTION

In this chapter we study the kernels

$$k(s,t) = \frac{1}{q(s,t)} = \frac{1}{a+b(s+t)+cst}$$

where $a, b, c \in \mathbb{R}$, not all zero. (If the closed interval $I \in Ad(k)$ then q cannot change sign on $I \times I$ so, where convenient we can always assume $q(s, t) > 0, \forall s, t \in I$.)

We shall use K_I to denote the integral operator on $L^2(I)$ with kernel k . Given $I \in Ad(k)$ we shall determine $N^+(k, I)$ and $N^-(k, I)$. We shall also make progress on the problem of the asymptotic behaviour of $\lambda_n^+(k, I)$ and $\lambda_n^-(k, I)$; a complete answer to this problem will be given at the end of Chapter 5.

Modulo some elementary reductions it will be sufficient for us to consider two canonical special cases; as below.

2.1.1 DEFINITION

(1) Define $k_0(s,t) = \frac{1}{1-st}$. Given $I \in Ad(k_0)$ let K_{0I} to denote the integral operator on $L^2(I)$ with kernel k_0 .

(2) Define $k_1(s, t) = \frac{1}{1 + st}$. Given $I \in Ad(k_1)$ let K_{1I} to denote the integral operator on $L^2(I)$ with kernel k_1 .

As we shall see, it will also be sufficient to assume that I is symmetric about 0:

$I = [-\rho, \rho]$, where $0 < \rho < 1$.

2.2 ANALYSIS OF $k_o(s, t) = \frac{1}{1-st}$

Given two closed intervals I, J . Definition 1.4.2 allows us to transfer a kernel k on $J \times J$ to a kernel ψk on $I \times I$ such that the two operators K_J and ψK_I are spatially equivalent: $K_J \equiv \psi K_I$. Now in general the two functions $k, \psi k$ will be quite different, but in the case of k_o there is a ψ which leaves k_o invariant; more precisely we have

2.2.1 LEMMA

Let $\psi(s) = \frac{s+\lambda}{\lambda s+1}$, where $-1 < \lambda < 1$. Then $\psi k_o(s, t) = k_o(s, t)$, $\forall s, t \in (-1, 1)$.

Proof. Here $\psi'(s) = \frac{1-\lambda^2}{(\lambda s+1)^2}$. Simple manipulation shows that

$$\psi k_o(s, t) = \frac{\psi'(s)^{\frac{1}{2}} \psi'(t)^{\frac{1}{2}}}{1 - \psi(s)\psi(t)} = \frac{1}{1-st} = k_o(s, t).$$

2.2.2 COROLLARY

Let $J \subseteq (-1, 1)$ be a closed interval (so that $J \in Ad(k_o)$). Then there exists ρ , $0 < \rho < 1$, such that $K_{oI} \equiv K_{1J}$, where $I = [-\rho, \rho]$.

Proof. This follows easily from the fact that $\psi(s) = \tanh(\tanh^{-1} s + \tanh^{-1} \lambda)$. We can write $J = [\tanh u, \tanh v]$, where $-\infty < u < v < \infty$. Put $\lambda = \tanh \frac{1}{2}(v + u)$ and $\rho = \tanh \frac{1}{2}(v - u)$. For this choice of λ , ψ maps $I = [-\rho, \rho]$ onto J . Since $\psi k_0 = k_0$, we see by Definition 1.4.2 that $K_{oI} \equiv K_{oJ}$.

If $I \subseteq (-1, 1)$ is a closed interval then $I \in Ad(k_0)$ and it is straight forward to show that K_{oI} is strictly positive so that

$$N^+(k_0, I) = \infty \text{ and } N^-(k_0, I) = 0$$

(see Remark 1.1.3 (i)). For if $f \in L^2(I)$,

$$\begin{aligned} \langle K_{oI} f, f \rangle &= \int_I \int_I \overline{f(s)} f(t) \sum_{n=0}^{\infty} s^n t^n dt ds \\ &= \sum_{n=0}^{\infty} \left| \int_I f(t) t^n dt \right|^2 \\ &\geq 0. \end{aligned}$$

There is equality only if f is orthogonal to all polynomials, i. e. $f = 0$. The geometric series will converge uniformly on $I \times I$: this is enough to justify permuting the summation and the integrations.

More general positive operators can be constructed as follow.

2.2.3 LEMMA

Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and suppose that $\Delta = \alpha\delta - \beta\gamma > 0$. Let $\psi(s) = \frac{\alpha s + \beta}{\gamma s + \delta}$, and

$$k(s, t) = \frac{1}{q(s, t)} = \frac{1}{(\gamma s + \delta)(\gamma t + \delta) - (\alpha s + \beta)(\alpha t + \beta)}.$$

If $I \subseteq \mathbb{R}$ is a closed interval and $|\psi(s)| < 1$, $\forall s \in I$, then $I \in Ad(k)$, $J = \psi(I) \subseteq (-1, 1)$

and $K_I \equiv \Delta^{-1} K_{\omega J}$. In particular

$$(i) \quad N^+(k, I) = \infty \quad \text{and} \quad N^-(k, I) = 0.$$

$$(ii) \quad \lambda_n(k, I) \approx \lambda_n(k_o, J) \quad \text{or equivalently} \quad \lambda_n(K_I) \approx \lambda_n(K_{\omega J}).$$

Proof. Here we have $\psi'(s) = \frac{\Delta}{(\gamma s + \delta)^2}$. So that $\psi'(s)^{\frac{1}{2}} = \Delta^{\frac{1}{2}} |\gamma s + \delta|^{-1}$. Since ψ is

finite in I , $\gamma s + \delta$ does not change sign on I , and hence $|\gamma s + \delta| |\gamma t + \delta| = (\gamma s + \delta)(\gamma t + \delta)$

on $I \times I$. Therefore

$$\psi k_o(s, t) = \frac{\Delta}{(\gamma s + \delta)(\gamma t + \delta)} \cdot \frac{1}{1 - \psi(s)\psi(t)} = \Delta \cdot k(s, t)$$

so that by Definition 1.4.2, $K_I \equiv \Delta^{-1} K_{\omega J}$. Since $J \subseteq (-1, 1)$, (i) and (ii) follows from

Lemma 1.4.3 and our discussion above.

2.3 ANALYSIS OF $k_1(s, t) = \frac{1}{1 + st}$

We follow the same general procedure. But we shall find that, for $I \in Ad(k_1)$, $N^+(K_{1I}) = N^-(K_{1I}) = \infty$. Also, for a suitable closed interval $I' \subseteq (-1, 1)$, we will find a relation between $\lambda_n^+(k_1, I)$ and $\lambda_n(k_0, I')$ and a relation between $|\lambda_n^-(k_1, I)|$ and $\lambda_n(k_0, I')$.

2.3.1 LEMMA

Let $\psi(s) = \frac{s + \mu}{-\mu s + 1}$, where $-\infty < \mu < \infty$.

- (i) $\psi k_1(s, t) = \pm k_1(s, t)$, $s, t \neq 1/\mu$.
- (ii) If $J \in Ad(k_1)$, then there exists ρ , $0 < \rho < 1$, such that $K_{1J} \equiv K_{1I}$, where $I = [-\rho, \rho]$.
- (iii) $\lambda_n^+(k_1, J) = \lambda_n^+(k_1, I)$ and $\lambda_n^-(k_1, J) = \lambda_n^-(k_1, I)$.

Proof. Here $\psi'(s) = \frac{1 + \mu^2}{(1 - \mu s)^2}$, so that

$$\psi k_1(s, t) = \frac{1 + \mu^2}{|-\mu s + 1| |-\mu t + 1|} \cdot \frac{1}{1 + \psi(s)\psi(t)} = \frac{\text{sgn}(-\mu s + 1) \text{sgn}(-\mu t + 1)}{1 + st}.$$

This proves (i). The signs here depend on s and t , and we can say nothing more about them in general. But in proving (ii) we can be more specific.

To prove (ii) suppose $J = [\tan u, \tan v]$, where $-\pi/2 < u < v < \pi/2$. Now if $J \in Ad(k_1)$, we must have $st \neq -1$, for all $s, t \in J$. Since $\tan s \tan(s + \pi/2) = -1$, we must also

have $0 < v - u < \pi/2$. Now put $\mu = \tan \frac{1}{2}(v + u)$ and $\rho = \tan \frac{1}{2}(v - u)$, so that $\mu \in \mathbb{R}$,

$0 < \rho < 1$. Put $I = [-\rho, \rho]$. Since $\psi(s) = \tan\left(\tan^{-1}s + \frac{1}{2}(v+u)\right)$, we see that ψ maps I in a strictly increasing fashion onto J . In particular, the pole $1/\mu$ of ψ cannot be in I . It follows that the sign of $-\mu s + 1$ is constant on I . Therefore $\psi k_1(s, t) = k_1(s, t)$ on $J \times J$. Hence (ii) follows and also (iii) (see Lemma 1.4.3).

Let us now consider the eigenvalues of K_{1I} , $I \in Ad(k)$. Because of Lemma 2.3.1 we can assume that $I = [-\rho, \rho]$, where $0 < \rho < 1$. Let us abbreviate K_{1I} to T , thus

$$Tf(s) = \int_{-\rho}^{\rho} \frac{f(t)}{1+st} dt.$$

Clearly $k_1(s, t) = k_1(-s, -t)$, so by Theorem 1.2.3 (i), $L_e^2(I)$ and $L_o^2(I)$ are invariant subspaces of T . So the eigenvalues of T are the eigenvalues of T_e together with the eigenvalues of T_o . According to Theorem 1.2.3 (ii), $\lambda_n(T_e) = \lambda_n(T_+)$ where $T_+ :$

$L^2(0, \rho) \rightarrow L^2(0, \rho)$ is the integral operator

$$\begin{aligned} T_+g(s) &= \int_0^{\rho} (k_1(s, t) + k_1(s, -t))g(t)dt \\ &= \int_0^{\rho} \frac{2g(t)}{1-s^2t^2} dt. \end{aligned}$$

Since

$$\langle T_+g, g \rangle = 2 \sum_{n=0}^{\infty} \left| \int_0^{\rho} g(t) t^{2n} dt \right|^2,$$

we see that T_+ is strictly positive (even polynomials are dense in $L^2(0, \rho)$). So we

certainly have $N^+(T) = \infty$. Likewise $\lambda_n(T_o) = \lambda_n(T_-)$, where

$$T_-g(s) = -\int_0^\rho \frac{2stg(t)}{1-s^2t^2} dt$$

and T_- is strictly negative. Hence $N^+(T) = N^-(T) = \infty$ and

$$\lambda_n^+(T) = \lambda_n(T_+), \quad \lambda_n^-(T) = \lambda_n^-(T_-) = -\lambda_n^+(-T_-).$$

To elucidate these eigenvalues let $\psi : [0, \rho^2] \rightarrow [0, \rho]$ be the function $\psi(s) = s^{\frac{1}{2}}$ and

let $U : L^2(0, \rho) \rightarrow L^2(0, \rho^2)$ be the corresponding unitary operator:

$$Uf(s) = 2^{-\frac{1}{2}} s^{-\frac{1}{4}} f(s^{\frac{1}{2}}).$$

Then, see (1.7)

$$UT_+U^*f(s) = \int_0^{\rho^2} \frac{f(t)}{s^{\frac{1}{4}}t^{\frac{1}{4}}(1-st)} dt \quad (f \in L^2(0, \rho^2), 0 \leq s \leq \rho^2).$$

Put $J = [0, \rho^2]$ and let $M : L^2(J) \rightarrow L^2(J)$ be the symmetric multiplication operator

$Mf(s) = s^{\frac{1}{4}}f(s)$. Then $\|M\|^2 = \rho$ and the kernel of MUT_+U^*M is $k_0(s, t)$. It

follows from (1.4) that

$$\lambda_n(k_0, J) = \lambda_n(MUT_+U^*M) \leq \rho\lambda_n(UT_+U^*) \leq \rho\lambda_n(T_+) = \rho\lambda_n^+(T). \quad (2.1)$$

To find an upper bound for $\lambda_n^+(T)$, let R be the rank 1 positive operator on $L^2(J)$

with L^2 -kernel $s^{-\frac{1}{4}}t^{-\frac{1}{4}}$. Since

$$\frac{1}{s^{1/4}t^{1/4}} \cdot \frac{1}{1-st} = \frac{s^{3/4}t^{3/4}}{1-st} + \frac{1}{s^{1/4}t^{1/4}}$$

we see that

$$UT_+U^* = M^3K_{0J}M^3 + R.$$

Hence by (1.1) and (1.4) we have

$$\begin{aligned}
\lambda_n^+(T) &= \lambda_n(T_+) = \lambda_n(UT_+U^*) \\
&= \lambda_n(M^3 K_{\omega J} M^3 + R) \\
&\leq \lambda_{n-1}(M^3 K_{\omega J} M^3) \\
&\leq \rho^3 \lambda_{n-1}(K_{\omega J}).
\end{aligned}$$

Therefore, taking into account (2.1), we have

$$\boxed{\frac{1}{\rho} \lambda_n(K_{\omega J}) \leq \lambda_n^+(K_{1I}) \leq \rho^3 \lambda_{n-1}(K_{\omega J}).} \quad (2.2)$$

Now we turn to the negative eigenvalues of $T = K_{1I}$. Let $T^+ = -T_-$, that is

$$T^+ g(s) = \int_0^\rho \frac{2stg(t)}{1-s^2t^2} dt.$$

So we have

$$\lambda_n^+(T^+) = -\lambda_n^-(T) \quad (2.3)$$

and

$$UT^+U^* f(s) = \int_0^{\rho^2} \frac{s^{1/4} t^{1/4} f(t)}{1-st} dt = MK_{\omega J} M.$$

Hence

$$\left| \lambda_n^-(T) \right| = \lambda_n(T^+) = \lambda_n(UT^+U^*) = \lambda_n(MK_{\omega J} M) \leq \rho \lambda_n(K_{\omega J}). \quad (2.4)$$

On the other hand, since

$$\frac{1}{1-st} = s^{3/4} t^{3/4} \cdot \frac{s^{1/4} t^{1/4}}{1-st} + 1$$

we see that

$$K_{\omega J} = M^3 U T^+ U^* M^3 + A$$

where A is a rank 1 positive operator on $L^2(J)$. Hence by (1.1) and (1.4) we have

$$\begin{aligned} \lambda_n(K_{\omega J}) &= \lambda_n(M^3 U T^+ U^* M^3 + A) \\ &\leq \lambda_{n-1}(M^3 U T^+ U^* M^3) \\ &\leq \rho^3 \lambda_{n-1}(U T^+ U^*) \\ &= \rho^3 \lambda_{n-1}(T^+) \\ &= \rho^3 |\lambda_{n-1}^-(T)| \end{aligned}$$

which, together with (2.3) and (2.4), implies

$$\frac{1}{\rho^3} \lambda_{n+1}(K_{\omega J}) \leq |\lambda_n^-(K_{II})| \leq \rho \lambda_n(K_{\omega J}). \quad (2.5)$$

2.3.2 LEMMA (cf. Lemma 2.2.3)

Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and suppose that $\Delta = \alpha\delta - \beta\gamma \neq 0$. Let

$$k(s, t) = \frac{1}{q(s, t)} = \frac{1}{(\gamma s + \delta)(\gamma t + \delta) + (\alpha s + \beta)(\alpha t + \beta)}$$

and $I \in Ad(k)$. Then $N^+(k, I) = N^-(k, I) = \infty$.

Proof. Observe that $q(s, t) > 0$ on $I \times I$ since $q(s, s) > 0$ and I is admissible. Now

either $(\gamma s + \delta)$ does not change sign on I or $(\alpha s + \beta)$ does not change sign on I .

(Otherwise, if $\gamma s_1 + \delta = 0$ and $\alpha s_2 + \beta = 0$ for some $s_1, s_2 \in I$, then $q(s_1, s_2) = 0$).

Assume, and without loss of generality, that $(\gamma s + \delta)$ does not change sign on I . Hence

$$|\gamma s + \delta| |\gamma t + \delta| = (\gamma s + \delta)(\gamma t + \delta) \text{ on } I \times I.$$

Also we may assume that $\Delta = \alpha\delta - \beta\gamma > 0$, since $(\gamma s + \delta)(\gamma t + \delta) = (\gamma's + \delta')(\gamma't + \delta')$

where $\gamma' = -\gamma$, $\delta' = -\delta$ and $\alpha\delta' - \beta\gamma' = -\Delta$. Now let $\psi(s) = \frac{\alpha s + \beta}{\gamma s + \delta}$ and $J = \psi(I)$. Then

we have

$$\psi k_1(s, t) = \frac{\Delta}{(\gamma s + \delta)(\gamma t + \delta)} \cdot \frac{1}{1 + \psi(s)\psi(t)} = \Delta \cdot k(s, t)$$

so that by Definition 1.4.2, $K_I \equiv \Delta^{-1} K_{1J}$. It follows from previous discussion and Lemma 1.4.3 that

$$N^+(k, I) = N^-(k, I) = \infty.$$

2.3.3 REMARK

The proof above not only shows that $N^+(k, I) = N^-(k, I) = \infty$, but it also shows that $K_I \equiv \Delta^{-1} K_{1J}$ and

$$\lambda_n^+(k, I) \approx \lambda_n^+(k_1, J),$$

$$\left| \lambda_n^-(k, I) \right| \approx \left| \lambda_n^-(k_1, J) \right|$$

or equivalently

$$\lambda_n^+(K_I) \approx \lambda_n^+(K_{1J}),$$

$$\left| \lambda_n^-(K_I) \right| \approx \left| \lambda_n^-(K_{1J}) \right|$$

for some closed interval $J \subset \mathbb{R}$.

2.4 THE GENERAL CASE $k(s,t) = \frac{1}{q(s,t)} = \frac{1}{a+b(s+t)+cst}$

Now we can complete our determination of $N^+(k, I), N^-(k, I)$ in the general case. Our results are summarized in Fig. 2.1. First we start with the following remark.

2.4.1 REMARK

Let p be a polynomial of the form

$$p(s,t) = a + b(s+t) + cst \quad (2.6)$$

where $a, b, c \in \mathbb{R}$. Then $p(s,s) = a + 2bs + cs^2$. Hence if $p(s,s) = 0$ on some interval I then $a = b = c = 0$ so that $p(s,t) \equiv 0$ on $\mathbb{R} \times \mathbb{R}$. Therefore, if p and q are any two polynomials of the form (2.6), then

$$p(s,s) = q(s,s) \text{ on } I \times I \Leftrightarrow p(s,t) = q(s,t) \text{ on } \mathbb{R} \times \mathbb{R}. \quad (2.7)$$

Now Let $P(s,t) = A + B(s+t) + Cst$ with $A, B, C \in \mathbb{R}$ and A, B, C not all 0. If $B^2 = AC$ then

$$\begin{aligned} P(s,s) &= A + 2Bs + Cs^2 \\ &= \varepsilon(\gamma s + \delta)^2 \end{aligned}$$

for some $\gamma, \delta \in \mathbb{R}$ and $\varepsilon = \pm 1$. If $p(s,t) = P(s,t) - \varepsilon(\gamma s + \delta)(\gamma t + \delta)$ then p is of the form (2.6). Also $p(s,s) \equiv 0$. Hence by (2.7) when $B^2 = AC$ we can write

$$A + B(s+t) + Cst = \varepsilon(\gamma s + \delta)(\gamma t + \delta).$$

Likewise if $B^2 < AC$ then

$$P(s, s) = \varepsilon \left[(\gamma s + \delta)^2 + (\alpha s + \beta)^2 \right]$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\varepsilon = \pm 1$ and $\alpha\beta - \gamma\delta \neq 0$. Similar argument shows

$$A + B(s+t) + Cst = \varepsilon \left[(\gamma s + \delta)(\gamma t + \delta) + (\alpha s + \beta)(\alpha t + \beta) \right].$$

Similarly if $B^2 > AC$ then

$$A + B(s+t) + Cst = (\gamma s + \delta)(\gamma t + \delta) - (\alpha s + \beta)(\alpha t + \beta)$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha\beta - \gamma\delta \neq 0$. That is

$$A + B(s+t) + Cst = \begin{cases} \varepsilon(\gamma s + \delta)(\gamma t + \delta) & \text{if } B^2 = AC \\ (\gamma s + \delta)(\gamma t + \delta) - (\alpha s + \beta)(\alpha t + \beta) & \text{if } B^2 > AC \\ \varepsilon[(\gamma s + \delta)(\gamma t + \delta) + (\alpha s + \beta)(\alpha t + \beta)] & \text{if } B^2 < AC \end{cases}$$

Observe that if for some closed interval I , $A + B(s+t) + Cst > 0$ on $I \times I$, then $\varepsilon = +1$.

Now let K be the integral operator on $L^2(I)$ with kernel k :

$$k(s, t) = \frac{1}{q(s, t)} = \frac{1}{a + b(s+t) + cst}$$

where $a, b, c \in \mathbb{R}$, not all zero and let $I \in \text{Ad}(k)$ be a closed interval. Since $q(s, t) \neq 0$

on $I \times I$, it is sufficient to assume that $q(s, t) > 0$ on $I \times I$.

(I) If $b^2 = ac$, then by Remark 2.4.1, $k(s, t)$ can be written as follows

$$\begin{aligned} k(s, t) &= \frac{1}{(\gamma s + \delta)(\gamma t + \delta)} \\ &= \varphi(s)\varphi(t) \end{aligned}$$

where $\varphi(s) \in L^2(I)$. This is a rank 1 positive kernel. That is

$$N^+(K) = 1 \text{ and } N^-(K) = 0.$$

(II) $b^2 > ac$. In this case

$$k(s, t) = \frac{1}{(\gamma s + \delta)(\gamma t + \delta) - (\alpha s + \beta)(\alpha t + \beta)},$$

by Remark 2.4.1, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. By replacing $\alpha s + \beta$ with $-\alpha s - \beta$ if necessary

we can assume $\Delta = \alpha\delta - \beta\gamma > 0$. Let $\psi(s) = \frac{\alpha s + \beta}{\gamma s + \delta}$. Since $q(s, s) > 0$ on $I \times I$ we have

$|\gamma s + \delta| > |\alpha s + \beta|$ and hence $J = \psi(I) \subseteq (-1, 1)$, which implies by Lemma 2.2.3,

$$N^+(K) = \infty \text{ and } N^-(K) = 0.$$

(III) For $b^2 < ac$, we have, by Remark 2.4.1,

$$k(s, t) = \frac{1}{(\gamma s + \delta)(\gamma t + \delta) + (\alpha s + \beta)(\alpha t + \beta)}$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Hence by Lemma 2.3.2, we have

$$N^+(K) = N^-(K) = \infty.$$

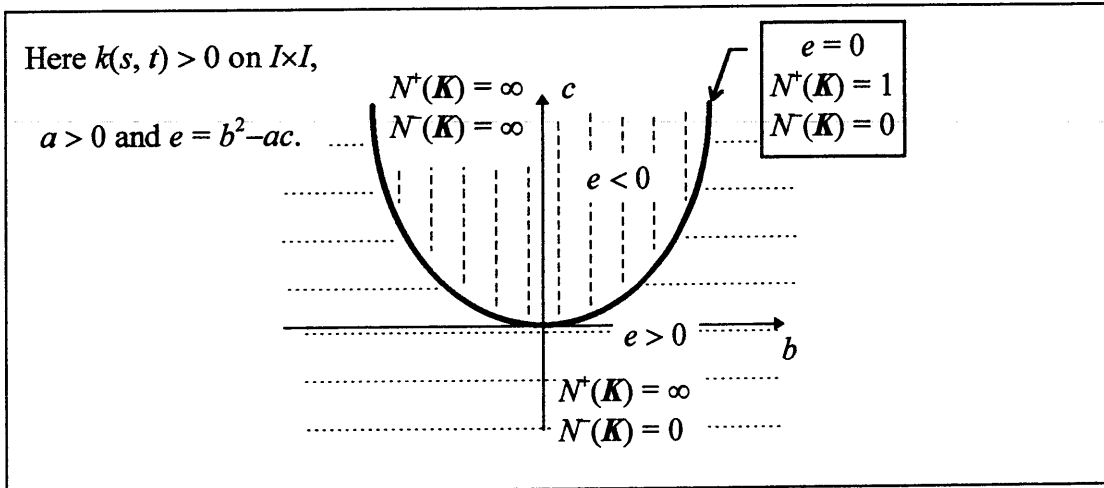


Fig. 2.1

CHAPTER 3. $k(s,t) = \frac{A + B(s+t) + Cst}{D + E(s+t) + Fst}$

3.1 INTRODUCTION

In this chapter we study the kernels

$$k(s,t) = \frac{A + B(s+t) + Cst}{D + E(s+t) + Fst} = \frac{P(s,t)}{Q(s,t)}$$

where $A, B, C, D, E, F \in \mathbb{R}$ (D, E, F not all zero). Let $\mathbf{K}$ be the integral operator on $L^2(I)$ with kernel k . Given $I \in Ad(k)$ we shall determine $N^+(k, I)$ and $N^-(k, I)$.

3.1.1 THEOREM

Let $k(s,t) = \frac{A + B(s+t) + Cst}{D + E(s+t) + Fst} = \frac{P(s,t)}{Q(s,t)}$, where $A, B, C, D, E, F \in \mathbb{R}$ and

$I \in Ad(k)$. Let $\mathbf{K}$ be the integral operator on $L^2(I)$ with kernel $k(s, t)$. Then

(i) If $E^2 = DF$ then $\mathbf{K}$ is a finite rank operator.

(ii) If $E^2 > DF$ then $\mathbf{K}$ is symmetrically equivalent to an integral operator $\mathbf{K}'$ on

$L^2(J)$, $J = [-\rho, \rho]$, $0 < \rho < 1$, with kernel of the form

$$k'(s,t) = \frac{a + b(s+t) + cst}{1 - st} \quad (3.1)$$

where $a, b, c \in \mathbb{R}$.

(iii) If $E^2 < DF$ then $\mathbf{K}$ is symmetrically equivalent to an integral operator $\mathbf{K}'$

on $L^2(J)$, $J = [-\rho, \rho]$, $0 < \rho < 1$, with kernel of the form

$$k'(s,t) = \frac{a + b(s+t) + cst}{1 + st} \quad (3.2)$$

where $a, b, c \in \mathbb{R}$.

Proof. Since $I \in Ad(k)$ then Q cannot change sign on $I \times I$, so we can always assume

$$Q(s, t) > 0, \forall s, t \in I.$$

(i) If $E^2 = DF$ then, by Remark 2.4.1, k can be reformulated as follows

$$k(s, t) = \frac{A + B(s + t) + Cst}{(\gamma s + \delta)(\gamma t + \delta)}$$

and this proves (i).

(ii) We will prove (ii) in two steps. First we will show that K is symmetrically equivalent to an integral operator on $L^2(I_1)$, for some closed interval $I_1 \subset (-1, 1)$ with kernel of the form (3.1). Then we will symmetrize I_1 .

Step 1. Using Remark 2.4.1 let

$$Q(s, t) = (\gamma s + \delta)(\gamma t + \delta) - (\alpha s + \beta)(\alpha t + \beta)$$

with $\Delta = \alpha\delta - \beta\gamma > 0$. Let

$$\varphi(s) = \frac{\alpha s + \beta}{\gamma s + \delta}$$

so that

$$\varphi'(s) = \frac{\alpha\delta - \beta\gamma}{(\gamma s + \delta)^2} > 0.$$

Since $Q(s, s) > 0$ on I we have $|\gamma s + \delta| > |\alpha s + \beta|$ and hence $I_1 = \varphi(I) \subset (-1, 1)$. Let

$\chi = \varphi^{-1}$. That is

$$\chi(s) = \frac{\delta s - \beta}{-\gamma s + \alpha} \text{ and } \chi'(s) = \frac{\alpha\delta - \beta\gamma}{(-\gamma s + \alpha)^2} > 0.$$

Now

$$k(s, t) = \frac{P(s, t)}{(\gamma s + \delta)(\gamma t + \delta)[1 - \varphi(s)\varphi(t)]}$$

and hence

$$\begin{aligned} \chi k(s, t) &= \frac{\Delta}{(-\gamma s + \alpha)(-\gamma t + \alpha)} \cdot \frac{1}{[\gamma \chi(s) + \delta][\gamma \chi(t) + \delta]} \cdot \frac{P(\chi(s), \chi(t))}{1 - st} \\ &= \frac{1}{\Delta} \cdot \frac{A + B[\chi(s) + \chi(t)] + C\chi(s)\chi(t)}{1 - st} \\ &= \frac{1}{(-\gamma s + \alpha)(-\gamma t + \alpha)} \cdot \frac{\tilde{A} + \tilde{B}(s + t) + \tilde{C}st}{1 - st} \end{aligned}$$

for some $\tilde{A}, \tilde{B}, \tilde{C} \in \mathbb{R}$. That is $\chi K = M\tilde{K}M^*$ where $\tilde{K}$ is the integral operator on

$L^2(I_1)$ with kernel

$$\tilde{k}(s, t) = \frac{\tilde{A} + \tilde{B}(s + t) + \tilde{C}st}{1 - st}$$

and $Mg(s) = m(s)g(s)$, $m(s) = \frac{1}{-\gamma s + \alpha}$. Necessarily $-\gamma s + \alpha \neq 0$ on I . Hence K is

symmetrically equivalent to $\tilde{K}$.

Step 2. Since $I_1 \subset (-1, 1)$, we know from Corollary 2.2.2 and its proof that there exist

$\rho, 0 < \rho < 1$ and $\psi(s) = \frac{s + \lambda}{\lambda s + 1}$, $\psi: J = [-\rho, \rho] \rightarrow I_1$ such that $\psi k_{oI_1} = k_{oJ}$. Hence

$$\begin{aligned} \psi \tilde{k}(s, t) &= \frac{1}{1 - st} \left\{ \tilde{A} + \tilde{B}[\psi(s) + \psi(t)] + \tilde{C}\psi(s)\psi(t) \right\} \\ &= \frac{1}{1 - st} \cdot \frac{a + b(s + t) + cst}{(\lambda s + 1)(\lambda t + 1)} \end{aligned}$$

for some $a, b, c \in \mathbb{R}$. Again this shows that $\tilde{K}$ is symmetrically equivalent to an integral operator say K' on $L^2(J)$, $J = [-\rho, \rho]$ with kernel of the form (3.1) which in turns shows that K itself is symmetrically equivalent to K' .

(iii) Again we will prove (iii) in two steps. First we will show that K is symmetrically equivalent to an integral operator on $L^2(I_1)$, $I_1 \subset \mathbb{R}$ with kernel of the form (3.2). Then we will symmetrize I_1 inside $(-1, 1)$.

Step 1. Using Remark 2.4.1, let

$$Q(s, t) = (\gamma s + \delta)(\gamma t + \delta) + (\alpha s + \beta)(\alpha t + \beta).$$

Now either $(\gamma s + \delta)$ or $(\alpha s + \beta)$ does not change sign on I . So we may assume, and without loss of generality, that $(\gamma s + \delta)$ does not change sign on I and that

$\Delta = \alpha\delta - \beta\gamma > 0$. Let $\varphi(s) = \frac{\alpha s + \beta}{\gamma s + \delta}$ and $I_1 = \varphi(I)$. Let $\chi = \varphi^{-1}$. Following exactly

the same technique used in Step 1 of (ii) above we see that K is symmetrically equivalent to an integral operator $\tilde{K}$ (on $L^2(I_1)$) with kernel

$$\tilde{k}(s, t) = \frac{\tilde{A} + \tilde{B}(s+t) + \tilde{C}st}{1+st}$$

with $\tilde{A}, \tilde{B}, \tilde{C} \in \mathbb{R}$.

Step 2. This step is very similar to Step 2 of (ii) above, using in this case Lemma 2.3.1 and its proof.

We complete Section 3.1 by exhibiting two more canonical examples.

3.1.2 THEOREM

Let $\mathbf{K}$ be the integral operator on $L^2(I)$ with kernel k . If $I = [-\rho, \rho]$, $\rho > 0$ and $k(-s, -t) = -k(s, t)$, $\forall s, t \in I$, then

$$(i) \mathbf{K}(L_e^2(I)) \subseteq L_o^2(I) \text{ and } \mathbf{K}(L_o^2(I)) \subseteq L_e^2(I);$$

$$(ii) \lambda_n^+(\mathbf{K}) = -\lambda_n^-(\mathbf{K}).$$

Proof. Proof of (i) is easy so we leave it to the reader. To prove (ii) let $U: L^2(I) \rightarrow L^2(I)$ be defined by $Uf(s) = f(-s)$. Then U is unitary operator with $U = U^* = U^{-1}$. Now it is easy to see that $UKU = -\mathbf{K}$ and hence by (1.6) the proof is complete.

3.1.3 EXAMPLE

Let $\mathbf{K}$ be the integral operator on $L^2(I)$, $I = [-\rho, \rho]$, $0 < \rho < 1$, with kernel

$$k(s, t) = \frac{s+t}{1-st}.$$

Then $\lambda_n^+(\mathbf{K}) = |\lambda_n^-(\mathbf{K})|$ and $N^+(\mathbf{K}) = N^-(\mathbf{K}) = \infty$.

Proof. Clearly $k(-s, -t) = -k(s, t)$. So by Theorem 3.1.2 we have $\lambda_n^+(\mathbf{K}) = |\lambda_n^-(\mathbf{K})|$ and hence $N^+(\mathbf{K}) = N^-(\mathbf{K})$. In order to show $N^+(\mathbf{K}) = \infty$ it is sufficient to show

$\mathbf{K}$ has infinite rank. To do this reformulate $k(s, t)$ as follows

$$\begin{aligned} k(s, t) &= \frac{s+t}{1-st} \\ &= \frac{(1+s^2)t}{1-s^2t^2} + \frac{(1+t^2)s}{1-s^2t^2} \\ &= k'(s, t) + k''(s, t) \end{aligned}$$

where $k'(s, t) = \frac{(1+s^2)t}{1-s^2t^2}$ and $k''(s, t) = \frac{(1+t^2)s}{1-s^2t^2}$. Let K' and K'' be the integral operators on $L^2(I)$ corresponding to k' and k'' respectively. So $K = K' + K''$. Now it is easy to see that

$$K(L_o^2(I)) = K'(L_o^2(I)) \subseteq L_e^2(I)$$

and

$$K(L_e^2(I)) = K''(L_e^2(I)) \subseteq L_o^2(I).$$

So in order to prove that $N^+(K) = \infty$ it suffices to show that $\dim K''(L_e^2(I)) = \infty$.

So let $M: L_e^2(I) \rightarrow L_e^2(I)$ be defined by $Mf(s) = (1+s^2)f(s)$, $T: L_e^2(I) \rightarrow L_e^2(I)$ be defined by $Tf(s) = \int_I \frac{f(t)}{1-s^2t^2} dt$ and $M': L_e^2(I) \rightarrow L_o^2(I)$ be defined by

$M'f(s) = sf(s)$, so that $K'' = M'TM$. Now M , M' are obviously injective, and T is injective as we saw on page 29. So K'' is injective on the even functions in $L^2(I)$.

Therefore $\dim K''(L_e^2(I)) = \infty$.

Finally, we close this section with the following example.

3.1.4 EXAMPLE

Let K be the integral operator on $L^2(I)$, $I = [-\rho, \rho]$, $0 < \rho < 1$, with kernel

$$k(s, t) = \frac{st}{1-st}.$$

Then $K \geq 0$ and $N^+(K) = \infty$.

Proof.

$$\begin{aligned}\langle Kf, f \rangle &= \int_I \int_I \frac{st}{1-st} \overline{f(s)} f(t) dt ds \\ &= \sum_{n=1}^{\infty} \left| \int_I f(t) t^n dt \right|^2 \geq 0.\end{aligned}$$

This shows that $K \geq 0$. Now $N^+(K) = \infty$ will follow, since

$$\begin{aligned}k(s, t) &= \frac{1}{1-st} - 1 \\ &= k_0(s, t) - 1\end{aligned}$$

and since $N^+(K_0) = \infty$ (see Section 2.2 and (1.2)).

3.2 THE CASE $k(s,t) = \frac{a+b(s+t)+cst}{1-st}$

Let K be the integral operator on $L^2(I)$, $I = [-\rho, \rho]$, $0 < \rho < 1$, with kernel

$$k(s,t) = \frac{a+b(s+t)+cst}{1-st}$$

where $a, b, c \in \mathbb{R}$, not all 0. In this section we shall determine $N^+(K)$ and $N^-(K)$.

Our results are summarized in Figures 3.1 and 3.3.

Case (I) $b = 0$: In this case and by considering $-k$ instead of k if necessary we can assume (without loss of generality) $a \geq 0$. Now for $b = 0$ we have

$$\begin{aligned} k(s,t) &= \frac{a+cst}{1-st} \\ &= a + \frac{(a+c)st}{1-st} \\ &= a + (a+c)k'(s,t) \text{ (say).} \end{aligned}$$

It is clear that if $a+c = 0$ ($a > 0$) we have

$$N^+(K) = 1 \text{ and } N^-(K) = 0.$$

For $a+c \neq 0$ let K' be the integral operator on $L^2(I)$ corresponding to k' . By

Example 3.1.4 we have $N^+(K') = \infty$ and $N^-(K') = 0$ so for $a+c > 0$ we have

$$N^+(K) = \infty \text{ and } N^-(K) = 0.$$

For $a + c < 0$, either $a = 0$, and hence

$$N^+(K) = 0 \text{ and } N^-(K) = \infty,$$

or $a > 0$ and hence, by (1.2), $N^+(K) \leq 1$ and $N^-(K) = \infty$. We claim that K has exactly one positive eigenvalue, that is $N^+(K) = 1$. Since if $K \leq 0$, then $k(s, s) \leq 0$ for all $s \in I$ (see Corollary 1.2.2). But $k(0, 0) = a > 0$, hence K is not negative and therefore K has one positive eigenvalue. That is

$$N^+(K) = 1 \text{ and } N^-(K) = \infty.$$

Case(II) $b \neq 0$: In this case, and without loss of generality, we may assume that $|b| = 1$. That is by replacing k with $\frac{1}{|b|}k$. Also, we can assume that $a + c \geq 0$ (again by considering $-k$ instead of k if necessary). Now for $ac \geq 1$ (see Fig. 3.2), so in fact $a > 0$, $k(s, t)$ can be reformulated as follows

$$\begin{aligned} k(s, t) &= \frac{l(s)l(t) + \lambda st}{1 - st} \\ &= k'(s, t) + k''(s, t) \end{aligned}$$

where $l(u) = a^{\frac{1}{2}} + (b / a^{\frac{1}{2}})u$, $\lambda = (c - 1/a) \geq 0$ and

$$k'(s, t) = \frac{l(s)l(t)}{1 - st}, \quad k''(s, t) = \frac{\lambda st}{1 - st}.$$

Let K' and K'' be the integral operators on $L^2(I)$ corresponding to k' and k'' respectively, so we have $K = K' + K''$. Now $K'' \geq 0$ (see Example 3.1.4) and hence $K \geq K'$. By the invariance theorem (Theorem 1.3.2) we can assume that ρ is so small that $l(s) > 0$ on I . This implies that K' is symmetrically equivalent to K_{0I} . Since we

know that $K_{oI} \geq 0$ and $N^+(K_{oI}) = \infty$ (see Section 2.2), we have, by Remark 1.1.3

(ii),

$$N^+(K) = \infty \text{ and } N^-(K) = 0 \text{ if } a > 0 \text{ and } ac \geq 1. \quad (3.3)$$

If $ac < 1$, then we have the following three cases (see Fig. 3.2):

- (1) $a + c \geq 2$ and $a > 1$. In this case it is easy to see the existence of a real number $r > 0$ such that $(a - r)(c + r) = 1$ and $(a - r) > 0$ (See Fig. 3.2). So, for this r , let $\tilde{k}(s, t) = k(s, t) - r$. That is

$$\begin{aligned} \tilde{k}(s, t) &= \frac{(a - r) + b(s + t) + (c + r)st}{1 - st} \\ &= \frac{\tilde{a} + b(s + t) + \tilde{c}st}{1 - st} \end{aligned}$$

where $\tilde{a} = a - r$ and $\tilde{c} = c + r$. Now $\tilde{a} > 0$ and $\tilde{a}\tilde{c} = 1$, which implies, by (3.3),

$N^+(\tilde{K}) = \infty$ and $N^-(\tilde{K}) = 0$ where $\tilde{K}$ is the integral operator on $L^2(I)$

corresponding to $\tilde{k}$. Here $K = \tilde{K} + R$, where R is a rank 1 operator and $R \geq 0$.

Hence

$$N^+(K) = \infty \text{ and } N^-(K) = 0 \text{ if } a + c \geq 2 \text{ and } a > 1.$$

- (2) $a + c \geq 2$ and $c > 1$. Here $1 > 1/c > 0$, so we can assume, by the invariance theorem (Theorem 1.3.2), that $1/c < \rho < 1$. In this case there exist a real number $r > 0$ such that $(a + r)(c - r) = 1$ and $(a + r) > 0$ (see Fig. 3.2). So, for this r , let $k'(s, t) = k(s, t) + r$. That is

$$\begin{aligned}
k'(s, t) &= \frac{(a+r) + b(s+t) + (c-r)st}{1-st} \\
&= \frac{a' + b(s+t) + c'st}{1-st}
\end{aligned}$$

where $a' = a + r$ and $c' = c - r$. Now $a' > 0$ and $a'c' = 1$. So, again by (3.3), we have $N^+(K') = \infty$ and $N^-(K') = 0$ where K' is the integral operator on $L^2(I)$ corresponding to k' . Here $K' = K + R$, where R is a rank 1 operator and $R \geq 0$. Hence, by (1.2) we have $N^+(K) = \infty$ and $N^-(K) \leq 1$. Since for $s = -b/c \in I$ we have

$$k(s, s) = \frac{a - 1/c}{1 - 1/c^2} < 0$$

so we conclude that K is not positive (see Corollary 1.2.2) and hence $N^-(K) = 1$.

That is

$$N^+(K) = \infty \text{ and } N^-(K) = 1 \text{ if } a + c \geq 2 \text{ and } c > 1.$$

(3) $0 \leq a + c < 2$. Let us start with $a = c$ (so that $0 \leq a < 1$). If $a = 0$, then

$$k(s, t) = \frac{b(s+t)}{1-st}.$$

Hence, by Example 3.1.3 we have $N^+(K) = N^-(K) = \infty$. Next we consider the case

$a = c$ and $0 < a < 1$ (see Fig 3.2). Let

$$k'(s, t) = \frac{b(s+t)}{1-st}$$

and let K' be the integral operator on $L^2(I)$ corresponding to k' . We know that

$$N^+(K') = N^-(K') = \infty. \text{ Now for}$$

$$\psi(s) = \frac{s + \lambda}{\lambda s + 1}, \lambda \in \mathbb{R},$$

consider $\psi k'$ as in (1.7)

$$\begin{aligned} \psi k'(s, t) &= \psi'(s)^{\frac{1}{2}} k'(\psi(s), \psi(t)) \psi'(t)^{\frac{1}{2}} \\ &= \frac{b(1 - \lambda^2)}{(1 + \lambda s)(1 + \lambda t)} \cdot \frac{(s + \lambda)(1 + \lambda t) + (t + \lambda)(1 + \lambda s)}{(1 + \lambda s)(1 + \lambda t) - (s + \lambda)(t + \lambda)} \\ &= \frac{b}{(1 + \lambda s)(1 + \lambda t)} \cdot \frac{2\lambda + (1 + \lambda^2)(s + t) + 2\lambda st}{1 - st}. \end{aligned}$$

By putting $\lambda = \frac{1 - \sqrt{1 - a^2}}{ab}$ we have

$$\begin{aligned} \psi k'(s, t) &= \frac{1 + \lambda^2}{(1 + \lambda s)(1 + \lambda t)} \cdot \frac{a + b(s + t) + ast}{1 - st} \\ &= \frac{1 + \lambda^2}{(1 + \lambda s)(1 + \lambda t)} k(s, t) \\ &= m(s) k(s, t) m(t) \end{aligned}$$

where $m(u) = \frac{\sqrt{1 + \lambda^2}}{1 + \lambda u}$. Since $|b| = 1$, $0 < a < 1$ we see $|\lambda| < 1$ and hence $J = \psi^{-1}(I)$

$\subseteq (-1, 1)$. So

$$\begin{aligned} N^\pm(k', I) &= N^\pm(\psi k', J) \text{ (by spatial equivalence)} \\ &= N^\pm(\psi k', I) \text{ (by invariance)} \\ &= N^\pm(k, I) \text{ (by symmetric equivalence)} \end{aligned}$$

and hence $N^+(K) = N^-(K) = \infty$. That is

$N^+(K) = N^-(K) = \infty \text{ if } a = c \text{ and } 0 \leq a < 1$	(3.4)
--	---------

Now for any a and c with $a \neq c$ and $0 \leq a + c < 2$, let $A = \frac{c-a}{2}$ and

$\tilde{k}(s, t) = k(s, t) + A$. Then

$$\begin{aligned}\tilde{k}(s, t) &= \frac{a + b(s+t) + cst}{1-st} + A \\ &= \frac{(a+A) + b(s+t) + (c-A)st}{1-st} \\ &= \frac{\tilde{a} + b(s+t) + \tilde{a}st}{1-st}\end{aligned}$$

where $\tilde{a} = \frac{1}{2}(a+c)$. Hence, by (3.4), $N^+(\tilde{K}) = N^-(\tilde{K}) = \infty$ where $\tilde{K}$ is the integral

operator on $L^2(I)$ corresponding to $\tilde{k}$. Here $\tilde{K} = K + A$, where A is a rank 1

operator. Therefore $N^+(K) = N^-(K) = \infty$.

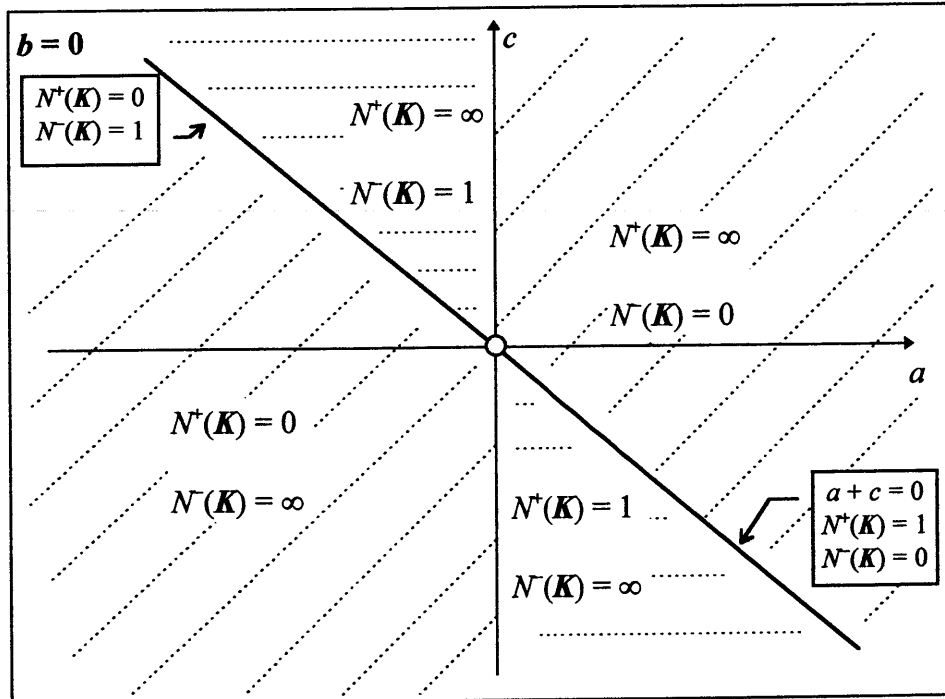


Fig. 3.1

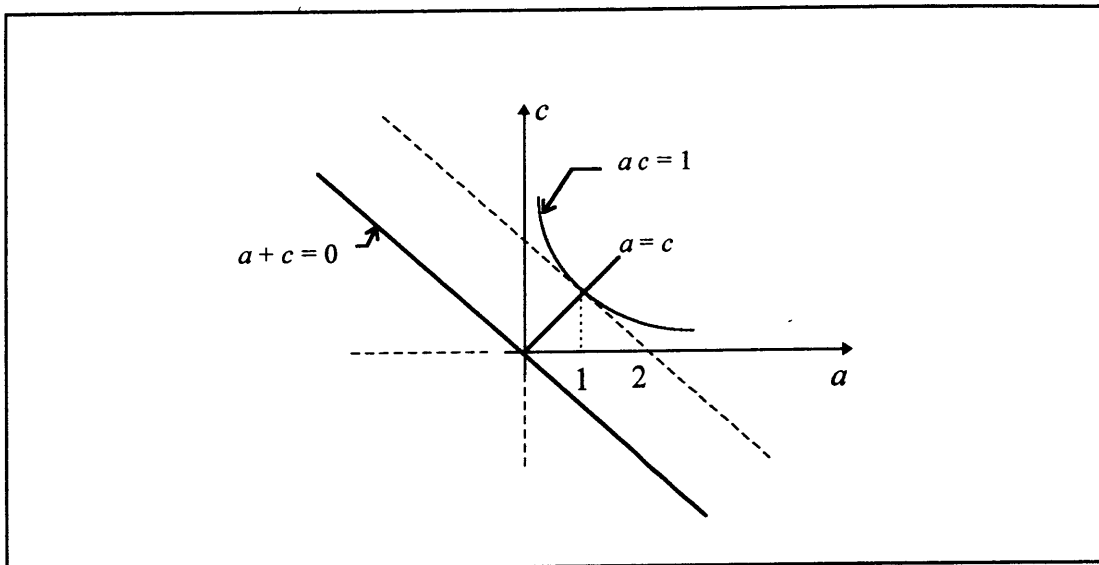


Fig. 3.2

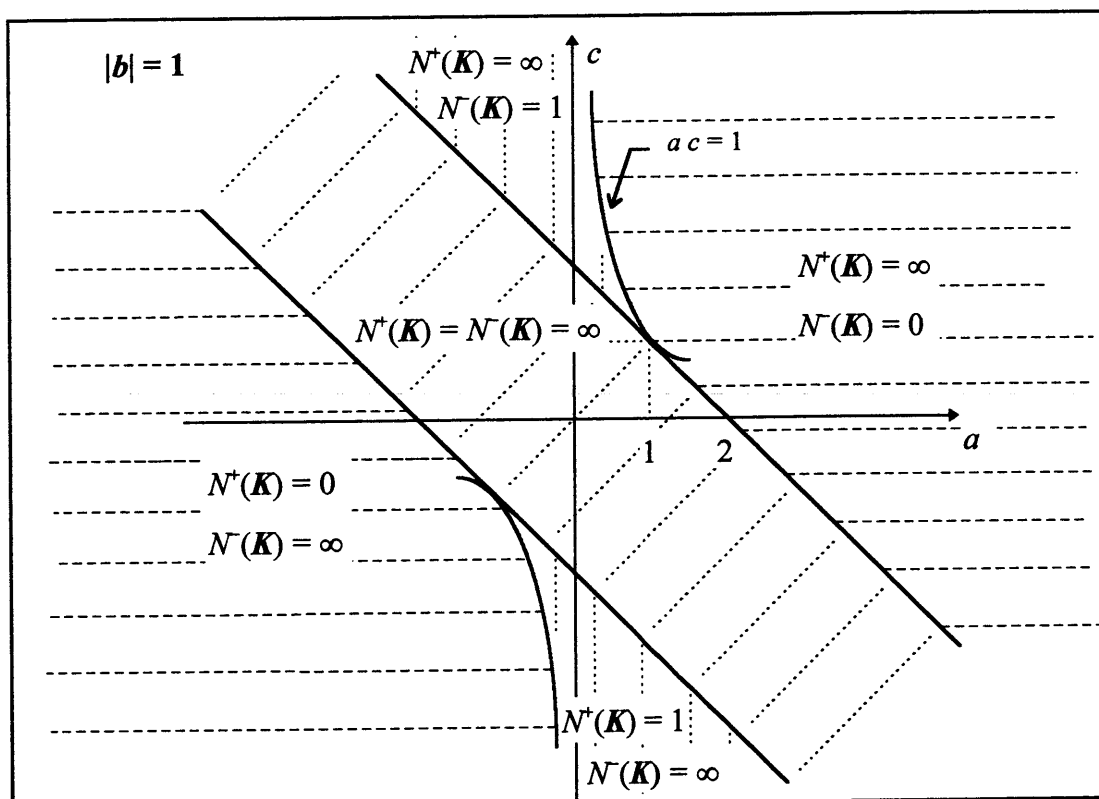


Fig. 3.3

3.3 THE CASE $k(s,t) = \frac{a+b(s+t)+cst}{1+st}$

Let $\mathbf{K}$ be the integral operator on $L^2(I)$, $I = [-\rho, \rho]$, $0 < \rho < 1$, with kernel

$$k(s,t) = \frac{a+b(s+t)+cst}{1+st}$$

where $a, b, c \in \mathbb{R}$, not all 0. In this section we shall determine $N^+(\mathbf{K})$ and $N^-(\mathbf{K})$.

This case is actually rather simpler. Our results are summarized in Fig. 3.4 and 3.6.

First, and without loss of generality, we assume that $a \geq 0$.

Case (I) $b = 0$: If $b = 0$ then

$$\begin{aligned} k(s,t) &= \frac{a+cst}{1+st} \\ &= c + \frac{a-c}{1+st}. \end{aligned}$$

Now if $a = c > 0$ then

$$N^+(\mathbf{K}) = 1 \text{ and } N^-(\mathbf{K}) = 0.$$

For $a \neq c$, let $k'(s,t) = k(s,t) - c$, and let $\mathbf{K}'$ be the integral operator on $L^2(I)$

corresponding to k' . Now $k'(s,t) = (a-c)k_1(s,t)$ where

$$k_1(s,t) = \frac{1}{1+st}$$

and we know from Section 2.3 that $N^+(\mathbf{K}_1) = N^-(\mathbf{K}_1) = \infty$ where $\mathbf{K}_1$ be the

integral operator on $L^2(I)$ corresponding to k_1 . So for $a \neq c$, we have

$N^+(\mathbf{K}') = N^-(\mathbf{K}') = \infty$ and hence

$$N^+(\mathbf{K}) = N^-(\mathbf{K}) = \infty.$$

Case(II) $b \neq 0$: Here we may again assume that $|b| = 1$. The existence of a real number r such that $(a+r)(c+r) = 1$ and $(a+r) > 0$ is obvious (see Fig. 3.5). So, for this r , let $\tilde{k}(s, t) = k(s, t) + r$. That is

$$\tilde{k}(s, t) = \frac{\tilde{a} + b(s+t) + \tilde{c}st}{1+st},$$

where $\tilde{a} = a+r$ and $\tilde{c} = c+r$. Now $\tilde{k}$ can be reformulated as follows

$$\tilde{k}(s, t) = \frac{m(s)m(t)}{1+st}; \quad m(u) = \tilde{a}^{1/2} + \frac{b}{\tilde{a}^{1/2}}u$$

or

$$\tilde{k}(s, t) = m(s)k_1(s, t)m(t).$$

Now let $\tilde{\mathbf{K}}$ be the integral operator on $L^2(I)$ corresponding to $\tilde{k}$ and let $Mg(s) = m(s)g(s)$. Here $M = M^*$. Because of the invariance theorem (Theorem 1.3.2) we can assume ρ is so small that $m(s) > 0$ on I , so that M is invertible. This implies that $\tilde{\mathbf{K}}$ is symmetrically equivalent to $\mathbf{K}_1$. We know from Section 2.3 that $N^+(\mathbf{K}_1) = N^-(\mathbf{K}_1) = \infty$. Hence $N^+(\tilde{\mathbf{K}}) = N^-(\tilde{\mathbf{K}}) = \infty$ and therefore

$$N^+(\mathbf{K}) = N^-(\mathbf{K}) = \infty.$$

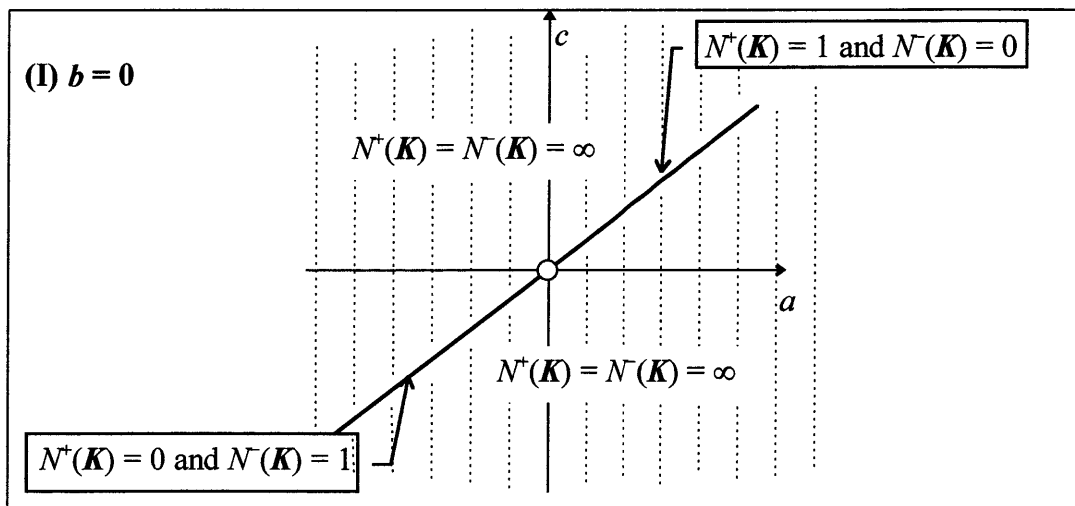


Fig. 3.4

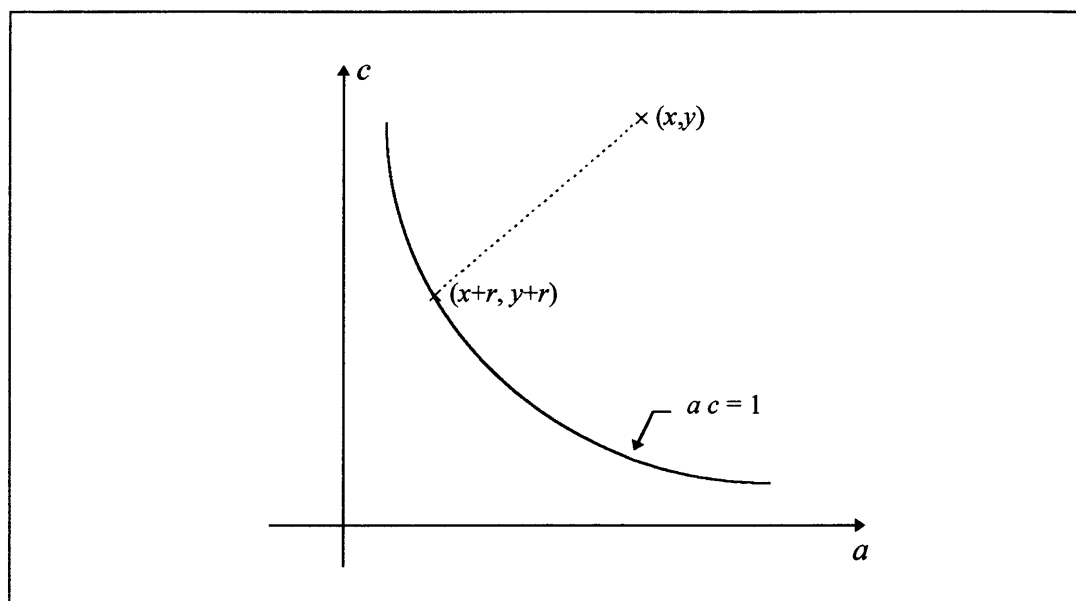


Fig. 3.5

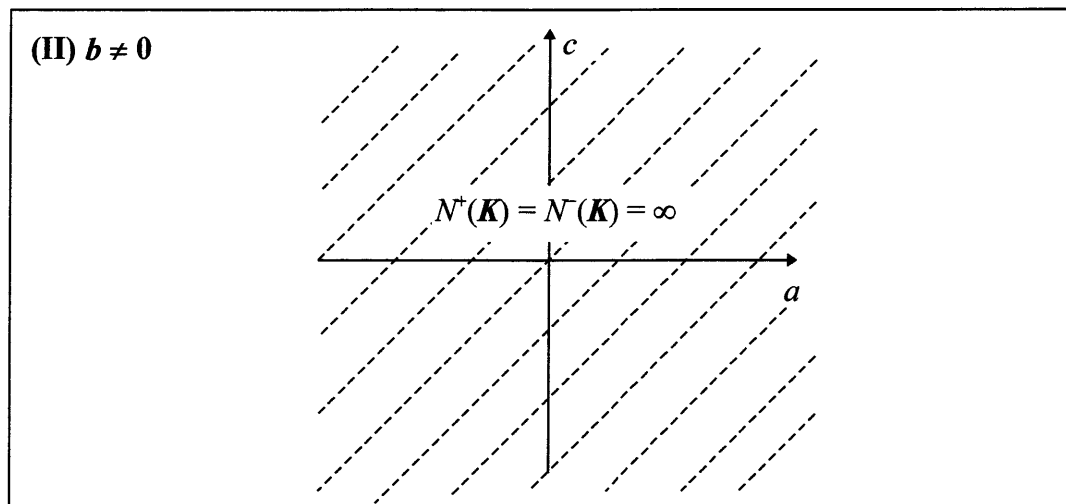


Fig. 3.6

CHAPTER 4. $k(s,t) = \frac{1}{a+b(s+t)+cst+d(s^2+t^2)}$

4.1 INTRODUCTION

In this chapter we study the kernel

$$k(s,t) = \frac{1}{q(s,t)} = \frac{1}{a+b(s+t)+cst+d(s^2+t^2)}.$$

where a, b, c and d are real numbers and $d \neq 0$. Given $I \in Ad(k)$ we shall determine $N^+(k, I)$ and $N^-(k, I)$. Since $d \neq 0$, it will be sufficient for us to consider $d = \pm 1$.

In the following section some further techniques are given, and in Section 4.3 we give some examples. These examples are in fact a part of studying the general case. In Section 4.4 we give some results which are true for all values of d . Finally in Sections 4.5 and 4.6 we specialize to the cases $d = +1$ and $d = -1$ respectively.

4.2 PRELIMINARIES

Let K and L be two integral operators on $L^2(I)$ with kernels k and l on $I \times I$ respectively. In the sequel we shall say that k is *positive definite* when $K \geq 0$ and we shall use the notation $k(s, t) \overset{\circ}{\geq} 0$ and we shall always understand that $s, t \in I$. In general we will write $k(s, t) \overset{\circ}{\geq} l(s, t)$ when $K \geq L$.

4.2.1 LEMMA

Let T_n be a sequence of compact symmetric operators on a Hilbert space H . Let T be a bounded symmetric operator on H and suppose $T_n \rightarrow T$ weakly (i.e. $\langle T_n f, g \rangle \rightarrow \langle T f, g \rangle$ for all $f, g \in H$). Suppose there exist an N -dimensional (sub-) space $E \subseteq H$ such that

$$\langle T f, f \rangle > 0 \text{ for all } f \in E, f \neq 0.$$

Then

$$\underline{\lim} N^+(T_n) \geq N.$$

Proof. We shall show that if n is sufficiently large then $\langle T_n f, f \rangle > 0$ for all $f \in E, f \neq 0$ (see Lemma 1.1.4 (i) and (ii)). Now since the map $A : E \rightarrow \mathbb{R}$ defined by $Af = \langle T f, f \rangle$ is continuous, and since the unit sphere of E is compact, we have

$$\inf_{\|f\|=1, f \in E} \langle T f, f \rangle = \delta \text{ for some } \delta > 0.$$

So

$$\langle Tf, f \rangle \geq \delta \text{ for all } f \in E, \|f\| = 1$$

and hence

$$\langle Tf, f \rangle \geq \delta \|f\|^2 \text{ for all } f \in E.$$

Now let $e_1, e_2, \dots, e_N$ be an orthonormal basis for E and let $B(E)$ be the set of all bounded operators on E . If

$$f = \sum_{j=1}^N b_j e_j \text{ and } g = \sum_{i=1}^N a_i e_i$$

then

$$\begin{aligned} \langle Tf, g \rangle &= \sum_{i=1}^N \sum_{j=1}^N \bar{a}_i b_j \langle Te_j, e_i \rangle \\ &= \langle \mathbf{A} \mathbf{b}, \mathbf{a} \rangle \end{aligned}$$

where $\mathbf{A}$ is the $N \times N$ matrix $(\langle Te_j, e_i \rangle)$, $\mathbf{b} = (b_j) \in \mathbb{C}^N$ and $\mathbf{a} = (a_i) \in \mathbb{C}^N$. Similarly

$\langle T_n f, g \rangle = \langle \mathbf{A}_n \mathbf{b}, \mathbf{a} \rangle$ where $\mathbf{A}_n$ is the matrix $(\langle T_n e_j, e_i \rangle)$. Now

$$\langle T_n e_j, e_i \rangle \rightarrow \langle Te_j, e_i \rangle \quad \forall i, j.$$

Since $B(E)$ is finite dimensional, $\mathbf{A}_n \rightarrow \mathbf{A}$ in norm. Now for all $f \in E, \|f\| = 1$

$$\begin{aligned} \langle T_n f, f \rangle &= \langle Tf, f \rangle - \langle (T - T_n)f, f \rangle \\ &= \langle Tf, f \rangle - \langle (\mathbf{A} - \mathbf{A}_n)\mathbf{a}, \mathbf{a} \rangle \\ &\geq \delta - \|\mathbf{A} - \mathbf{A}_n\|. \end{aligned}$$

So if n_0 is such that $\|\mathbf{A} - \mathbf{A}_n\| \leq \delta/2$ for all $n \geq n_0$ we have for $n \geq n_0$

$$\langle T_n f, f \rangle \geq \delta/2 \text{ for all } f \in E, \|f\| = 1.$$

Also $\langle T_n f, f \rangle > 0$ for all $f \in E, f \neq 0$. So, by Lemma 1.1.4 (ii), we have

$$N^+(T_n) \geq N \text{ for all } n \geq n_0.$$

4.2.2 COROLLARY

Let T and T_n be as above. If also T is compact then

$$\lim N^+(T_n) \geq N^+(T).$$

Proof. By Lemma 1.1.4 (ii), for any finite $N \leq N^+(T)$ there is a subspace E of dimension N satisfying hypotheses of Lemma 4.2.1.

4.2.3 LEMMA [5, Lemma 2]

Let S, T be the symmetric integral operators on $L^2(I)$ with kernels

$$\frac{1}{q(s,t)}, \frac{1}{q(s,t) - r(s,t)}$$

respectively, and suppose that

- (i) q and r are continuous on $I \times I$,
- (ii) $q(s,t) = \overline{q(t,s)}$, $r(s,t) = \overline{r(t,s)}$ on $I \times I$,
- (iii) $|r(s,t)| < q(s,t)$ on $I \times I$.

If S is positive and r is a positive definite kernel on I , then T is positive and $S \leq T$; in particular $\text{rank } T \geq \text{rank } S$ and $0 \leq \lambda_n(S) \leq \lambda_n(T)$ for all $n \geq 0$.

There is an obvious corollary that we shall find useful.

4.2.4 LEMMA

Let S, T be the symmetric integral operators on $L^2(I)$ with kernels

$$\frac{1}{r(s,t)}, \frac{1}{r(s,t) - r_1(s,t) - r_2(s,t)}$$

respectively, and suppose that

- (i) r, r_1 and r_2 are continuous on $I \times I$,
- (ii) $r(s, t) = \overline{r(t, s)}$, $r_i(s, t) = \overline{r_i(t, s)}$ ($i = 1, 2$) on $I \times I$,
- (iii) $|r_1(s, t)| < r(s, t)$ and $|r_2(s, t)| < r(s, t) - r_1(s, t)$ on $I \times I$.

If S is positive and r_1, r_2 are positive definite kernels on I , then $S \leq T$ and $0 \leq \lambda_n(S) \leq \lambda_n(T)$ for all $n \geq 0$.

Proof. By Lemma 4.2.3

$$\frac{1}{(r(s, t) - r_1(s, t)) - r_2(s, t)} \stackrel{\circ}{\geq} \frac{1}{r(s, t) - r_1(s, t)} \stackrel{\circ}{\geq} \frac{1}{r(s, t)} \stackrel{\circ}{\geq} 0.$$

4.2.5 LEMMA

Let ψ be a continuous real valued function on $\mathbb{R}$ such that $|\psi(u)| = O(e^{-\beta|u|})$ as $|u| \rightarrow \infty$, where $\beta > 0$. Let $I \subseteq \mathbb{R}$ be a closed interval and let T be the integral operator on $L^2(I)$:

$$Tf(s) = \int_I \hat{\psi}(s-t) f(t) dt$$

where $\hat{\psi}(\omega) = \int_{\mathbb{R}} \psi(u) e^{-iu\omega} du$. Then

- (i) T is compact and symmetric.
- (ii) $\psi(u) \geq 0$ for all $u \in \mathbb{R}$ implies $T \geq 0$.
- (iii) If $\psi(a) > 0$ for some $a \in \mathbb{R}$ then $N^+(T) = \infty$.
- (iv) If $\psi(b) < 0$ for some $b \in \mathbb{R}$ then $N^-(T) = \infty$.

Proof. (i) $\psi \in L^1(\mathbb{R})$ implies $\hat{\psi} \in C_0(\mathbb{R})$ and hence $k(s, t) = \hat{\psi}(s-t)$ is continuous on $I \times I$. So T is compact. Also $\hat{\psi}(s-t) = \overline{\hat{\psi}(t-s)}$, since ψ is real, so T is symmetric.

(ii) For all $f \in L^2(I)$, $Tf(s) = \int_I \hat{\psi}(s-t) f(t) dt$ and

$$\begin{aligned}
\langle Tf, f \rangle &= \int_I \int_I \hat{\psi}(s-t) f(t) \overline{f(s)} dt ds \\
&= \int_I \int_I \int_{\mathbb{R}} \psi(u) e^{-iut} f(t) e^{iut} \overline{f(s)} du dt ds \\
&= \int_{\mathbb{R}} \psi(u) \int_I f(t) e^{-iut} dt \int_I \overline{f(s)} e^{ius} ds du.
\end{aligned} \tag{4.1}$$

Permuting the order of the integrals is legitimate for if l is the length of I then, by Schwarz's inequality

$$\int_{\mathbb{R}} \psi(u) \int_I |f(t)| dt \int_I |f(s)| ds du \leq \int_{\mathbb{R}} \psi(u) \|f\|^2 l du < \infty.$$

So in all cases

$$\langle Tf, f \rangle = \int_{\mathbb{R}} \psi(u) \left| \int_I f(t) e^{-iut} dt \right|^2 du \geq 0 \tag{4.2}$$

with equality if and only if $f = 0$. Hence $T \geq 0$.

(iii) In order to prove (iii) we need to modify (4.2) so that the inner integral defines an operator $L^2(I) \rightarrow L^2(\mathbb{R})$ with a dense range. Write $\beta = 4\delta$ and put

$$\varphi(u) = e^{2\delta|u|} \psi(u).$$

Then (4.2) becomes

$$\begin{aligned}
\langle Tf, f \rangle &= \int_{\mathbb{R}} \varphi(u) \left| \int_I f(t) e^{-\delta|u|} e^{-iut} dt \right|^2 du \\
&= \int_{\mathbb{R}} \varphi(u) |Af(u)|^2 du
\end{aligned} \tag{4.3}$$

where $A: L^2(I) \rightarrow L^2(\mathbb{R})$ is the integral operator

$$Af(u) = \int_I f(t) e^{-\delta|u|} e^{-iut} dt \quad (f \in L^2(I), u \in \mathbb{R}).$$

The kernel of A is square summable on $\mathbb{R} \times I$. Hence $A^*: L^2(\mathbb{R}) \rightarrow L^2(I)$ is given by

$$A^*g(s) = \int_{\mathbb{R}} g(u) e^{-\delta|u|} e^{ius} du \quad (g \in L^2(\mathbb{R}), s \in I).$$

Because of the factor $e^{-\delta|u|}$ each A^*g is analytic in the strip $|\operatorname{Im} z| < \delta$. So if $A^*g \equiv 0$ on I then $A^*g \equiv 0$ on $\mathbb{R}$. Since the Fourier transform is injective we have $g(u) \equiv 0$ on $\mathbb{R}$. So $g(u) \equiv 0$. Hence A^* is injective so that the range of A is dense. Now for $g \in L^2(\mathbb{R})$ define $B(g) = \int_{\mathbb{R}} \varphi(u) |g(u)|^2 du$ so that, for $g \in L^2(I)$,

$$\langle Tf, f \rangle = B(Af). \quad (4.4)$$

We shall show that $N^+(T) = \infty$ by appealing to Lemma 1.1.4. Let $n \geq 0$ be fixed integer. We shall find an $n + 1$ -dimensional subspace $E \subseteq L^2(I)$ such that

$$\langle Tf, f \rangle = B(Af) > 0 \text{ when } f \in E, f \neq 0. \quad (4.5)$$

To do this choose a closed interval J , with $a \in J$, such that

$$\inf_{u \in J} \varphi(u) = \rho > 0.$$

Let P be the set of all functions in $L^2(\mathbb{R})$ of the form

$$g(u) = \begin{cases} p(u), & u \in J \\ 0, & u \notin J \end{cases}$$

where p is a polynomial of degree $\leq n$. Then P is an $n + 1$ -dimensional subspace of $L^2(\mathbb{R})$. Since every $g \in P$ is supported on J it is obvious that

$$B(g) \geq \rho \|g\|^2 \quad (g \in P) \quad (4.6)$$

Since the range of A is dense in $L^2(\mathbb{R})$ we shall be able to construct E so that $A(E)$ is in some sense “close” to P . Let $p_0, p_1, \dots, p_n$ be an orthonormal basis for P . Now for $\varepsilon > 0$ choose $f_0, f_1, \dots, f_n \in L^2(I)$ such that

$$\|Af_i - p_i\|_{L^2(\mathbb{R})} < \varepsilon \quad (0 \leq i \leq n) \quad (4.7)$$

and let $E \subseteq L^2(I)$ be the $(n + 1)$ -dimensional space spanned by $f_0, f_1, \dots, f_n$. Suppose

$f = \sum_{i=0}^n \alpha_i f_i$ and $p = \sum_{i=0}^n \alpha_i p_i$, where $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{C}$. Then by (4.4)

$$\left. \begin{aligned} \langle Tf, f \rangle &= B(Af) = \int_{\mathbb{R}} \varphi(u) |p(u) + (Af(u) - p(u))|^2 du \\ &= \int_{\mathbb{R}} \varphi(u) |p(u)|^2 du + \int_{\mathbb{R}} \varphi(u) |Af(u) - p(u)|^2 du \\ &\quad + 2 \operatorname{Re} \int_{\mathbb{R}} \varphi(u) p(u) \overline{(Af(u) - p(u))} du. \end{aligned} \right\} \quad (4.8)$$

For the first integral we have

$$\int_{\mathbb{R}} \varphi(u) |p(u)|^2 du \geq \rho \|p\|^2 = \rho \sum_{i=0}^n |\alpha_i|^2. \quad (4.9)$$

For the second integral we have

$$\left| \int_{\mathbb{R}} \varphi(u) |Af(u) - p(u)|^2 du \right| \leq \|\varphi\|_{\infty} \|Af - p\|^2.$$

Now

$$\|Af - p\|^2 = \int_{\mathbb{R}} \sum_i \sum_j \alpha_i \bar{\alpha}_j (Af_i(u) - p_i(u)) \overline{(Af_j(u) - p_j(u))} du.$$

So, by Schwarz’s inequality and (4.7) we get

$$\begin{aligned}
\|Af - p\|^2 &\leq \sum_i \sum_j |\alpha_i| |\alpha_j| \varepsilon^2 \\
&\leq (n+1) \varepsilon^2 \sum_{i=0}^n |\alpha_i|^2
\end{aligned} \tag{4.10}$$

and so

$$\left| \int_{\mathbb{R}} \varphi(u) |Af(u) - p(u)|^2 du \right| \leq \|\varphi\|_{\infty} (n+1) \varepsilon^2 \sum_{i=0}^n |\alpha_i|^2. \tag{4.11}$$

Finally, by Schwarz's inequality and (4.10) we have

$$\begin{aligned}
\left| \int_{\mathbb{R}} \varphi(u) p(u) \overline{(Af(u) - p(u))} du \right| &\leq \left(\int_{\mathbb{R}} |\varphi(u)|^2 |p(u)|^2 du \right)^{\frac{1}{2}} \|Af - p\| \\
&\leq \|\varphi\|_{\infty} \|p\| \sqrt{n+1} \varepsilon \left(\sum_{i=0}^n |\alpha_i|^2 \right)^{\frac{1}{2}} \\
&= \sqrt{n+1} \varepsilon \|\varphi\|_{\infty} \sum_{i=0}^n |\alpha_i|^2.
\end{aligned} \tag{4.12}$$

So by (4.8), (4.9), (4.11) and (4.12) we get

$$\langle Tf, f \rangle \geq \left(\sum_{i=0}^n |\alpha_i|^2 \right) \left(\rho - (n+1) \varepsilon^2 \|\varphi\|_{\infty} - 2\sqrt{n+1} \varepsilon \|\varphi\|_{\infty} \right).$$

Since ρ , $\|\varphi\|_{\infty}$ and n are fixed we can choose $\varepsilon > 0$ such that last factor is positive. So

for such an ε

$$\langle Tf, f \rangle > 0 \text{ for } f \in E, f \neq 0.$$

Since n was arbitrary it follows from Lemma 1.1.4 (ii) that $N^+(T) = \infty$.

(iv) Replace, in (iii), ψ with $-\psi$.

4.3 EXAMPLES

4.3.1 EXAMPLE

Let $k(s, t) = \frac{1}{q(s, t)} = \frac{1}{(\varphi(s) + \varphi(t))^\gamma}$, where φ is non constant (real)

polynomial, γ is positive integer. Let K be the integral operator on $L^2(I)$ corresponding to k , where I is a closed interval such that

$$\inf_{s \in I} \varphi(s) = \delta > 0.$$

Then $I \in Ad(k)$ and $K \geq 0$ with $N^+(K) = \infty$.

Proof. Obviously $\varphi(s) + \varphi(t) > 0$ on $I \times I$, so

$$k(s, t) = \int_0^\infty e^{-u[\varphi(s) + \varphi(t)]} \frac{u^{\gamma-1}}{(\gamma-1)!} du$$

and hence, for all $f \in L^2(I)$,

$$\begin{aligned} (\gamma-1)! \langle Kf, f \rangle &= \int_I \int_I f(t) \overline{f(s)} \left(\int_0^\infty u^{\gamma-1} e^{-\varphi(s)u} e^{-\varphi(t)u} du \right) dt ds \\ &= \int_0^\infty u^{\gamma-1} \left(\int_I f(t) e^{-\varphi(t)u} dt \right) \left(\int_I \overline{f(s)} e^{-\varphi(s)u} ds \right) du \end{aligned}$$

provided we can show that one of these two repeated integrals is absolutely summable. But this is easy for the second, for

$$\begin{aligned}
& \int_0^\infty u^{\gamma-1} \left(\int_I |f(t)| e^{-\varphi(t)u} dt \right) \left(\int_I |f(s)| e^{-\varphi(s)u} ds \right) du \\
& \leq \int_0^\infty u^{\gamma-1} \left(\int_I |f(t)| e^{-\delta u} dt \right) \left(\int_I |f(s)| e^{-\delta u} ds \right) du \\
& = \int_0^\infty u^{\gamma-1} e^{-2\delta u} \left(\int_I |f(t)| dt \right)^2 du \\
& < \infty,
\end{aligned}$$

by Schwarz's inequality. Therefore

$$\begin{aligned}
\langle Kf, f \rangle &= \frac{1}{(\gamma-1)!} \int_0^\infty u^{\gamma-1} \left(\int_I f(t) e^{-\varphi(t)u} dt \right) \left(\int_I \overline{f(s)} e^{-\varphi(s)u} ds \right) du \\
&= \frac{1}{(\gamma-1)!} \int_0^\infty u^{\gamma-1} \left| \int_I f(t) e^{-\varphi(t)u} dt \right|^2 du \geq 0.
\end{aligned}$$

Hence $K \geq 0$. Now by the invariance theorem (Theorem 1.3.2) we may assume that $\varphi'(s) \neq 0$ on I . (replace, if necessary, I with subinterval $J \subseteq I$ such that $\varphi'(s) \neq 0$ on J). So if $\langle Kf, f \rangle = 0$ then

$$\int_I f(t) e^{-\varphi(t)u} dt = 0$$

so that

$$\int_{\varphi(I)} f(\varphi^{-1}(t)) |\varphi^{-1}'(t)| e^{-tu} dt = 0$$

so, since the Laplace transform is injective, $f(\varphi^{-1}(t)) |\varphi^{-1}'(t)| \equiv 0$ on $\varphi(I)$ so that

$f \equiv 0$ on I . Hence, by Remark 1.1.3 (i), $N^+(K) = \infty$.

4.3.2 EXAMPLE

Let $l(s, t) = \frac{a + d(s^2 + t^2)}{\left[a + d(s^2 + t^2)\right]^2 - c^2 s^2 t^2}$, where $a, c, d \in \mathbb{R}$, $a > 0$ and

suppose I is a symmetric interval $I = [-\rho, \rho]$ where $\rho > 0$ is small enough to ensure that $I \in Ad(k)$. Let L be the integral operator on $L^2(I)$ corresponding to l . Then $L \geq 0$ and $N^+(L) = \infty$.

Proof. For $d = 0$, L is injective on the even functions and $L \geq 0$ with $N^+(L) = \infty$ (see Section 2.3, page 29). Now suppose $d \neq 0$. By using Lemma 4.2.3 we have

$$\begin{aligned} l(s, t) &= \frac{a + d(s^2 + t^2)}{\left[a + d(s^2 + t^2)\right]^2 - c^2 s^2 t^2} \\ &= \frac{1}{\left[a + d(s^2 + t^2)\right] - \left(c^2 s^2 t^2\right) / \left[a + d(s^2 + t^2)\right]} \\ &\stackrel{o}{\geq} \frac{1}{a + d(s^2 + t^2)} \end{aligned}$$

provided the last kernel here is positive definite. But this follows easily from Example 4.3.1 with $\varphi(u) = a/2 + du^2$. Hence, by Remark 1.1.3 (ii), we have $L \geq 0$ with $N^+(L) = \infty$.

4.3.3 EXAMPLE

Let $k(s, t) = \frac{1}{a + cst + d(s^2 + t^2)}$, where $a, c, d \in \mathbb{R}$, $a > 0$, $c > 0$ and suppose

I is a symmetric interval $I = [-\rho, \rho]$ where $\rho > 0$ is small enough to ensure that $I \in Ad(k)$. Let $\mathbf{K}$ be the integral operator on $L^2(I)$ corresponding to k . Then $N^+(\mathbf{K}) = N^-(\mathbf{K}) = \infty$.

Proof. For $d = 0$, we have $N^+(\mathbf{K}) = N^-(\mathbf{K}) = \infty$ (see Section 2.4, page 36).

Suppose $d \neq 0$. Clearly $k(s, t) = k(-s, -t)$, so by Theorem 1.2.3, $L_e^2(I)$ and $L_o^2(I)$ are invariant subspaces of $T = \mathbf{K}$ and we can write $T = T_e + T_o$ where $T_e: L_e^2(I) \rightarrow L_e^2(I)$ with kernel

$$k_e(s, t) = \frac{1}{2}[k(s, t) + k(s, -t)]$$

and $T_o: L_o^2(I) \rightarrow L_o^2(I)$ with kernel

$$k_o(s, t) = \frac{1}{2}[k(s, t) - k(s, -t)].$$

We show that $T_e \geq 0$ with $N^+(T_e) = \infty$ and $T_o \leq 0$ with $N^-(T_o) = \infty$, so that

$N^+(\mathbf{K}) = N^-(\mathbf{K}) = \infty$ (see Remark 1.2.4). Now

$$\begin{aligned} k_e(s, t) &= \frac{1}{2}[k(s, t) + k(s, -t)] \\ &= \frac{1}{2} \left[\frac{1}{a + cst + d(s^2 + t^2)} + \frac{1}{a - cst + d(s^2 + t^2)} \right] \\ &= \frac{a + d(s^2 + t^2)}{\left[a + d(s^2 + t^2) \right]^2 - c^2 s^2 t^2}. \end{aligned}$$

Hence, by Example 4.3.2, $T_e \geq 0$ with $N^+(T_e) = \infty$. To show $T_o \leq 0$ and

$N^-(T_o) = \infty$, let $T' = -T_o$ and $k' = -k_o$. Then

$$\begin{aligned} k'(s, t) &= \frac{1}{2} [k(s, -t) - k(s, t)] \\ &= \frac{1}{2} \left[\frac{1}{a - cst + d(s^2 + t^2)} - \frac{1}{a + cst + d(s^2 + t^2)} \right] \\ &= \frac{cst}{\left[a + d(s^2 + t^2) \right]^2 - c^2 s^2 t^2}. \end{aligned}$$

Let $l(s, t) = \frac{1}{p(s, t)}$ where $p(s, t) = \left[a + d(s^2 + t^2) \right]^2 - c^2 s^2 t^2$ and let L be the

integral operator on $L^2(I)$ corresponding to l . By Lemma 4.2.3 we have

$$l(s, t) = \frac{1}{\left[a + d(s^2 + t^2) \right]^2 - c^2 s^2 t^2} \stackrel{o}{\geq} \frac{1}{\left[a + d(s^2 + t^2) \right]^2}.$$

provided that the last kernel here is positive definite. But this follows from Example

4.3.1 with $\varphi(u) = a/2 + du^2$ and $\gamma = 2$. Hence, by Remark 1.1.3 (ii), we have $L \geq 0$

with $N^+(L) = \infty$. Now let M be the multiplication operator defined by

$Mf(s) = \sqrt{c} s f(s)$, so $MLM = T'$. Hence $T' \geq 0$ and by Lemma 1.1.4 (iv),

$N^+(T') = N^+(L) = \infty$. Since $T' = -T_o$ we have $T_o \leq 0$, $N^-(T_o) = \infty$ and this

completes the proof.

4.3.4 EXAMPLE

Let $k(s, t) = \frac{1}{(s+t)^2}$, $I \in Ad(k)$ and let K be the integral operator on $L^2(I)$

corresponding to k . Then $K \geq 0$ and $N^+(K) = \infty$.

Proof. Clearly $0 \notin I$, so either $I \subseteq (0, \infty)$ or $I \subseteq (-\infty, 0)$. In the first case we can choose $\delta > 0$ so that $s \geq \delta$ for all $s \in I$. Hence, by Example 4.3.1 with $\varphi(s) = s$ and $\gamma = 2$, we have

$$K \geq 0 \text{ and } N^+(K) = \infty.$$

When $I \subseteq (-\infty, 0)$, let $U : L^2(-I) \rightarrow L^2(I)$ such that $Uf(s) = f(-s)$. Then $U = U^* = U^{-1}$ and $UKU = K'$ where K' is the integral operator on $L^2(J)$, $J = -I \subseteq (0, \infty)$, with the same kernel k . Hence, by above, $K' \geq 0$ and $N^+(K') = \infty$. On the other hand, U is a unitary operator and by (1.6) we have $\lambda^+(K) = \lambda^+(K')$. Therefore $K \geq 0$ and $N^+(K) = \infty$.

4.3.5 EXAMPLE

Let $k(s, t) = \frac{1}{\gamma + \cosh(s-t)}$, where $\gamma > 1$ and I be symmetric interval

$I = [-\rho, \rho]$ where $\rho > 0$. Necessarily $I \in Ad(k)$. Let K be the integral operator on $L^2(I)$ corresponding to k . Then $N^+(K) = N^-(K) = \infty$.

Proof. Let $\theta = \cosh^{-1} \gamma$, so that $k(s, t) = \frac{1}{\cosh(s-t) + \cosh \theta}$. We will show that

$$k(s, t) = \hat{\psi}(s-t), \quad (4.13)$$

where

$$\psi(u) = \frac{\sin \theta u}{\sinh \theta \sinh \pi u} \quad (4.14)$$

and then the result will follow from Lemma 4.2.5 and the existence of $a, b \in \mathbb{R}$ with

$\psi(a) > 0 > \psi(b)$. Now $\hat{\psi}(\omega) = \int_{\mathbb{R}} \psi(u) e^{-iu\omega} du$ so (4.13) and (4.14) are equivalent to

$$\frac{\sin \theta u}{\sinh \theta \sinh \pi u} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iux}}{\cosh x + \cosh \theta} dx.$$

Observe that the function $f(z) = \frac{e^{iuz}}{\cosh z + \cosh \theta}$, $z \in \mathbb{C}$, has an infinite number of

singularities in both the upper and lower half-planes; these occur at the points

$$z = \pm\theta + (2n+1)\pi i \quad (n = 0, \pm 1, \pm 2, \dots)$$

Let Γ_ρ be the rectangular contour as in Fig. 4.1 below

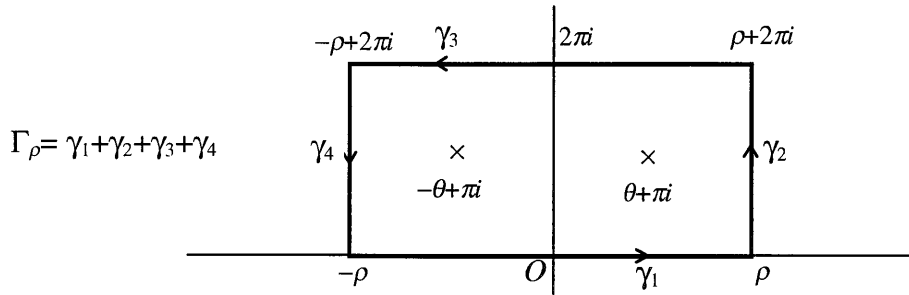


Fig. 4.1

It is not difficult to see that the integrals over the vertical segments γ_2 and γ_4 both approach zero as $\rho \rightarrow \infty$ and that the integral over the horizontal segments γ_3 is just $-e^{-2\pi u}$ times the integral over γ_1 . In other words, on taking the limit as $\rho \rightarrow \infty$ we have

$$\lim_{\rho \rightarrow \infty} \int_{\Gamma_\rho} \frac{e^{iuz}}{\cosh z + \cosh \theta} dz = (1 - e^{-2\pi u}) \int_{-\infty}^{\infty} \frac{e^{iux}}{\cosh x + \cosh \theta} dx. \quad (4.15)$$

Now we use residue theory to evaluate the contour integral in (4.15). For each $\rho > \theta$,

the function $f(z) = \frac{e^{iuz}}{\cosh z + \cosh \theta}$ is analytic inside and on Γ_ρ except for two

simple poles at $z_1 = -\theta + \pi i$ and $z_2 = \theta + \pi i$. Thus,

$$\begin{aligned} \int_{\Gamma_\rho} \frac{e^{iuz}}{\cosh z + \cosh \theta} dz &= 2\pi i [\text{Res}(z_1) + \text{Res}(z_2)] \\ &= 2\pi i \left[\frac{e^{i(-\theta+\pi i)u}}{\sinh(-\theta + \pi i)} + \frac{e^{i(\theta+\pi i)u}}{\sinh(\theta + \pi i)} \right] \\ &= \frac{2\pi i}{\sinh \theta} [e^{-(\theta i + \pi)u} - e^{(\theta i - \pi)u}]. \end{aligned}$$

Consequently,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{iux}}{\cosh x + \cosh \theta} dx &= \frac{2\pi i}{\sinh \theta} \cdot \frac{e^{-(\theta i + \pi)u} - e^{(\theta i - \pi)u}}{1 - e^{-2\pi u}} \\ &= \frac{2\pi}{\sinh \theta} \cdot \frac{\sin \theta u}{\sinh \pi u}, \end{aligned}$$

and hence the proof is complete.

4.3.6 EXAMPLE

Let $k_\gamma(s, t) = \frac{1}{(A - st)^\gamma}$ and let $I = [\alpha, \beta]$ with $A > \max\{\alpha^2, \beta^2\}$. Let K_γ be

the integral operator on $L^2(I)$ corresponding to k_γ . Then $K_\gamma \geq 0$ and is injective for

$\gamma > 0$, in particular $N^+(K_\gamma) = \infty$.

Proof. For $\gamma > 0$ we have

$$k_\gamma(s, t) = \sum_{n=0}^{\infty} a_{n\gamma} \frac{s^n t^n}{A^{n+\gamma}},$$

where $a_{n\gamma} = \left| \binom{-\gamma}{n} \right| > 0$ for all n . Now

$$\langle K_\gamma f, f \rangle = \frac{1}{A^\gamma} \sum_{n=0}^{\infty} \left| \int_I a_{n\gamma}^{\frac{1}{2}} f(t) \frac{t^n}{A^{n/2}} dt \right|^2 \geq 0$$

with equality if and only if $f = 0$. Hence, by Remark 1.1.3 (i), $K_\gamma \geq 0$ and

$$N^+(K_\gamma) = \infty.$$

4.3.7 EXAMPLE

Let $k(s, t) = \frac{1}{\gamma - \cosh(s-t)}$, where $\gamma > 1$ and suppose I is a symmetric

interval $I = [-\rho, \rho]$ where $\rho > 0$ is small enough to ensure that $I \in \text{Ad}(k)$. Let K be the integral operator on $L^2(I)$ corresponding to k . Then $N^+(K) = N^-(K) = \infty$.

Proof. Let $\gamma = \cosh \theta$, so that

$$k(s, t) = \frac{1}{\cosh \theta - \cosh(s-t)}.$$

Clearly $k(s, t) = k(-s, -t)$. Referring to Theorem 1.2.3, $L_e^2(I)$ and $L_o^2(I)$ are invariant subspaces of $T = K$ and we can write $T = T_e + T_o$ where $T_e: L_e^2(I) \rightarrow L_e^2(I)$ with kernel

$$k_e(s, t) = \frac{1}{2}[k(s, t) + k(s, -t)]$$

and $T_o: L_o^2(I) \rightarrow L_o^2(I)$ with kernel

$$k_o(s, t) = \frac{1}{2}[k(s, t) - k(s, -t)].$$

We show that $T_e \geq 0$ with $N^+(T_e) = \infty$ and $T_o \leq 0$ with $N^-(T_o) = \infty$, so that

$N^+(K) = N^-(K) = \infty$ (see Remark 1.2.4). Now

$$\begin{aligned} k_e(s, t) &= \frac{1}{2} \left[\frac{1}{\cosh \theta - \cosh s \cosh t + \sinh s \sinh t} + \frac{1}{\cosh \theta - \cosh s \cosh t - \sinh s \sinh t} \right] \\ &= \frac{\cosh \theta - \cosh s \cosh t}{(\cosh \theta - \cosh s \cosh t)^2 - \sinh^2 s \sinh^2 t} \\ &= \frac{1}{(\cosh \theta - \cosh s \cosh t) - (\sinh^2 s \sinh^2 t) / (\cosh \theta - \cosh s \cosh t)} \\ &\stackrel{o}{\geq} \frac{1}{\cosh \theta - \cosh s \cosh t} \\ &= l(s, t) \quad (\text{say}) \end{aligned}$$

provided the last kernel here is positive definite. But this follows from Lemma 4.2.3.

Necessarily that $\cosh^2 \rho < \cosh \theta$. Let L be the integral operator on $L^2(I)$ corresponding to l , so we have $T_e \geq L$. Hence, by Remark 1.1.3 (ii), it is enough now to show that $L \geq 0$ and $N^+(L) = \infty$. Now choose δ , $0 < \delta < \rho$ and let $J = [\delta, \rho]$. By the invariance theorem (Theorem 1.3.2)

$$N^\pm(l, I) = N^\pm(l, J). \quad (4.16)$$

Let $\psi(s) = \cosh^{-1} s$. Here $\psi'(s) = \frac{1}{(s^2 - 1)^{1/2}}$ which is bounded and bounded away

from 0 on $J' = [\cosh \delta, \cosh \rho]$. Now

$$\psi l(s, t) = \frac{1}{(s^2 - 1)^{1/4} (t^2 - 1)^{1/4}} \cdot \frac{1}{\cosh \theta - st}.$$

This shows that the integral operator L_J on $L^2(J)$ with kernel l is symmetrically equivalent to an integral operator on $L^2(J')$ with kernel $m(s, t) = \frac{1}{\cosh \theta - st}$. That is

$$N^\pm(l, J) = N^\pm(\psi l, J') = N^\pm(m, J').$$

Now by Example 4.3.6 (with $\gamma = 1$) we have $m \stackrel{\circ}{\geq} 0$ and $N^+(m, J') = \infty$ which implies $L_J \geq 0$ and $N^+(l, J) = \infty$. Hence, by (4.16), $L \geq 0$ and $N^+(L) = \infty$ which shows that $T_e \geq 0$ with $N^+(T_e) = \infty$.

Now we turn to T_o and we shall find that $T_o \leq 0$ and $N^-(T_o) = \infty$. Let $T' = -T_o$ and $k' = -k_o$. Then

$$\begin{aligned} k'(s, t) &= \frac{1}{2} [k(s, -t) - k(s, t)] \\ &= \frac{\sinh s \sinh t}{(\cosh \theta - \cosh s \cosh t)^2 - \sinh^2 s \sinh^2 t} \\ &\stackrel{\circ}{\geq} \frac{\sinh s \sinh t}{(\cosh \theta - \cosh s \cosh t)^2} \quad (\text{by Lemma 4.2.3}) \\ &= l'(s, t) \quad (\text{say}). \end{aligned}$$

Let L' be the integral operator on $L^2(I)$ corresponding to l' . So we have $T' \geq L'$ and hence, by Remark 1.1.3 (ii), it is enough to show that $L' \geq 0$ and $N^+(L') = \infty$.

As above by letting $J = [\delta, \rho]$, $0 < \delta < \rho$ and $\psi(s) = \cosh^{-1} s$ we have

$$\psi l'(s, t) = \frac{(s^2 - 1)^{1/4} (t^2 - 1)^{1/4}}{(\cosh \theta - st)^2}.$$

Again this shows that the integral operator L_J on $L^2(J)$ with kernel l' is symmetrically equivalent to an integral operator on $L^2(J')$, $J' = [\cosh \delta, \cosh \rho]$, with

kernel $m'(s, t) = \frac{1}{(\cosh \theta - st)^2}$. Hence

$$N^\pm(l', I) = N^\pm(l', J) \quad (\text{by invariance})$$

$$= N^\pm(\psi l', J') \quad (\text{by spatial equivalence})$$

$$= N^\pm(m', J') \quad (\text{by symmetric equivalence}).$$

Now by Example 4.3.6 (with $\gamma = 2$) we have $m' \geq 0$ and $N^+(m', J') = \infty$ which implies that $L' \geq 0$ and $N^+(L') = \infty$. Hence $T' \geq 0$ and $N^+(T') = \infty$. Since $T' = -T_o$ we have $T_o \leq 0$, $N^-(T_o) = \infty$ and this completes the proof.

4.3.8 EXAMPLE

Let $k(s, t) = \frac{1}{A - (s - t)^2}$, where $A > 0$ and suppose I is a symmetric interval I

$= [-\rho, \rho]$ where $\rho > 0$ is small enough to ensure that $I \in Ad(k)$. Let K be the integral operator on $L^2(I)$ corresponding to k . Then $N^+(K) = N^-(K) = \infty$.

Proof. Here $k(s, t) = k(-s, -t)$ and by Theorem 1.2.3, $L_e^2(I)$ and $L_o^2(I)$ are invariant subspaces of $T = K$ and $T = T_e + T_o$. As in Example 4.3.7 we will show that $T_e \geq 0$ with $N^+(T_e) = \infty$ and that $T_o \leq 0$ with $N^-(T_o) = \infty$. Now

$$\begin{aligned}
k_e(s, t) &= \frac{1}{2} [k(s, t) + k(s, -t)] \\
&= \frac{1}{2} \left[\frac{1}{(A - s^2 - t^2) - 2st} + \frac{1}{(A - s^2 - t^2) + 2st} \right] \\
&= \frac{A - s^2 - t^2}{(A - s^2 - t^2)^2 - 4s^2 t^2}.
\end{aligned}$$

Hence, by Example 4.3.2 (with $A = a$, $d = -1$ and $c = 2$) we have $T_e \geq 0$ with

$N^+(T_e) = \infty$. To complete the proof let $T' = -T_o$ and $k' = -k_o$. Then

$$\begin{aligned}
k'(s, t) &= \frac{1}{2} [k(s, -t) - k(s, t)] \\
&= \frac{2st}{(A - s^2 - t^2)^2 - 4s^2 t^2} \\
&= \frac{2st}{\varphi(s) + \varphi(t) - 2s^2 t^2},
\end{aligned}$$

where $\varphi(u) = A/2 - 2Au^2 + u^4$. Now let L be the integral operator on $L^2(I)$ with kernel

$$l(s, t) = \frac{1}{\varphi(s) + \varphi(t) - 2s^2 t^2}.$$

Then, by Lemma 4.2.3, we have

$$l(s, t) \geq \frac{1}{\varphi(s) + \varphi(t)}$$

and hence, by Example 4.3.1 and Remark 1.1.3 (ii), we have $L \geq 0$ and $N^+(L) = \infty$.

Now let M be the multiplication operator defined by $Mf(s) = \sqrt{2}sf(s)$, so

$MLM = T'$. Hence $T' \geq 0$ and by Lemma 1.1.4 (iv), $N^+(T') = \infty$ and this

completes the proof.

4.4 GENERAL REMARKS

Let q be a polynomial of the form

$$q(s, t) = a + b(s + t) + cst + d(s^2 + t^2)$$

where $a, b, c, d \in \mathbb{R}$. Let λ be any real number. Define q_λ —translation of q by λ —to be

$$q_\lambda(s, t) = q(s - \lambda, t - \lambda).$$

Hence

$$q_\lambda(s, t) = A + B(s + t) + cst + d(s^2 + t^2)$$

for some $A, B \in \mathbb{R}$. Now let K be the integral operator on $L^2(I)$ with kernel $k(s, t) = \frac{1}{q(s, t)}$ and $I \in \text{Ad}(k)$. Choose $\lambda \in \mathbb{R}$ such that $0 \in I_\lambda$ where $I_\lambda = I + \lambda$ and

let $\psi : I_\lambda \rightarrow I$ be defined by $\psi(s) = s - \lambda$. Then $\psi k(s, t) = \frac{1}{q_\lambda(s, t)}$ and by Lemma

1.3.6 we have

$$\lambda_n^\pm(\psi K) = \lambda_n^\pm(K).$$

Here $q_\lambda(s, t) \neq 0$ on $I_\lambda \times I_\lambda$ because $I \in \text{Ad}(k) \Leftrightarrow (I + \lambda) \in \text{Ad}(\psi k)$. Therefore, we may assume and without loss of generality that $q_\lambda(s, t) > 0$ on $I_\lambda \times I_\lambda$ and this is equivalent to $A > 0$. Hence we conclude, from this discussion, that to study the kernel

$$k(s, t) = \frac{1}{a + b(s + t) + cst + d(s^2 + t^2)}$$

we can assume, and without loss of generality, that $0 \in I$, $a > 0$. We may farther assume $d = \pm 1$. Now we have two cases

Case (I) $c \leq 0$. In this case $k(s, t)$ can be reformulated as follows

$$k(s, t) = \frac{1}{\varphi(s) + \varphi(t) + cst}$$

where $\varphi(u) = a/2 + bu + du^2$. Let $p(s, t) = \varphi(s) + \varphi(t)$ and $r(s, t) = |c|st$. Then

$$k(s, t) = \frac{1}{p(s, t) - r(s, t)}.$$

Now for any interval J with $0 \in J \subset I$ we have, by Theorem 1.3.2,

$$N^+(k, I) = N^+(k, J) \text{ and } N^-(k, I) = N^-(k, J).$$

Because $0 = r(0, 0) < a = p(0, 0)$ then, by continuity, we can assume that J is small enough so that

$$|r(s, t)| < p(s, t)$$

on $J \times J$. Hence, by using Lemma 4.2.3, we have

$$k(s, t) = \frac{1}{p(s, t) - r(s, t)} \stackrel{o}{\geq} \frac{1}{p(s, t)}.$$

Hence, by Example 4.3.1 and Remark 1.1.3 (ii), K is positive with $N^+(K) = \infty$.

Case(II) $c > 0$. Define $U : L^2(-I) \rightarrow L^2(I)$ by $Uf(s) = f(-s)$. Then UKU^* has kernel

$$k'(s, t) = k(-s, -t)$$

$$k'(s, t) = \frac{1}{a - b(s + t) + cst + d(s^2 + t^2)}.$$

and K is unitarily equivalent to UKU^* . This will allow us to assume, and without loss of generality, that $b \geq 0$. In next section we will study the case $c > 0$ and $d = +1$ and in Section 4.6 we will study the case $c > 0$ and $d = -1$.

4.5 THE CASE $k(s, t) = \frac{1}{q(s, t)} = \frac{1}{a + b(s + t) + cst + (s^2 + t^2)}$

Let

$$k(s, t) = \frac{1}{q(s, t)} = \frac{1}{a + b(s + t) + cst + (s^2 + t^2)}$$

and $I \in Ad(k)$, $0 \in I$, $a > 0$ and $b \geq 0$. Let K be the integral operator on $L^2(I)$ corresponding to k . We already know that $K \geq 0$ and $N^+(K) = \infty$ when $c \leq 0$. So suppose $c > 0$. In order to simplify our manipulations it is convenient to write $c = 2\gamma$ so that $\gamma > 0$. So our kernel becomes

$$k(s, t) = \frac{1}{q(s, t)} = \frac{1}{a + b(s + t) + s^2 + t^2 + 2\gamma st}.$$

Suppose $\alpha \in \mathbb{R}$ is fixed and that $\psi: I + \alpha \rightarrow I$ is defined by $\psi(s) = s - \alpha$. Let $l(s, t) = \psi k(s, t)$ (see (1.7)) so that

$$N^+(k, I) = N^+(l, I_\alpha) \text{ and } N^-(k, I) = N^-(l, I_\alpha)$$

where $I_\alpha = I + \alpha$. Then

$$\begin{aligned} l(s, t) &= \psi'(s)^{\frac{1}{2}} k(\psi(s), \psi(t)) \psi'(t)^{\frac{1}{2}} \\ &= \frac{1}{a + b(s + t - 2\alpha) + (s - \alpha)^2 + (t - \alpha)^2 + 2\gamma(s - \alpha)(t - \alpha)} \\ &= \frac{1}{[a - 2\alpha b + 2\alpha^2(1 + \gamma)] + [b - 2\alpha(1 + \gamma)](s + t) + s^2 + t^2 + 2\gamma st}. \end{aligned}$$

Notice the new denominator is strictly positive on $I_\alpha \times I_\alpha$. Put

$$\alpha = \frac{b}{2(1+\gamma)},$$

then for this choice of α the denominator has no term in $s + t$, thus

$$l(s,t) = \frac{1}{e + s^2 + t^2 + 2\gamma st} = \frac{1}{p(s,t)} \text{ (say)}$$

where

$$e = a - \frac{b^2}{2(1+\gamma)}.$$

Let L be the integral operator on $L^2(I_\alpha)$ corresponding to l . Now we have

$$N^+(L) = N^+(K) \text{ and } N^-(L) = N^-(K).$$

As in Fig. 4.2 below we have four possibilities:

- | | |
|--|---|
| (1) Area A: $e > 0$ and $\gamma > 0$ | (2) Area B: $e \leq 0$ and $0 < \gamma \leq 1$ |
| (3) Curve C: $e = 0$ and $\gamma > 1$ | (4) Area D: $e < 0$ and $\gamma > 1$. |

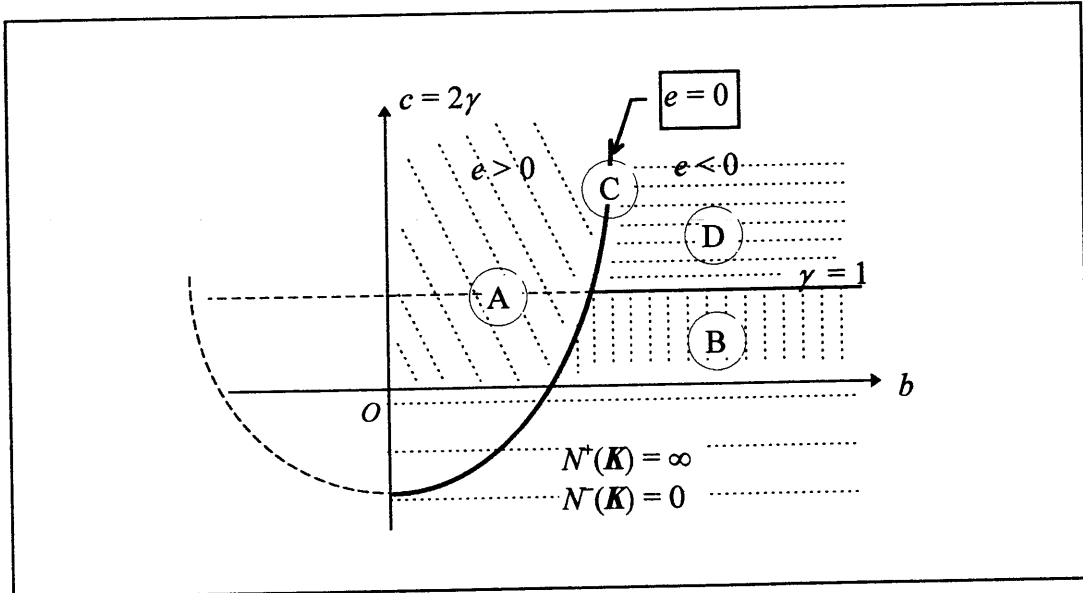


Fig. 4.2

$$(1) \text{ Area A: } e > 0, \gamma > 0 \text{ and } l(s, t) = \frac{1}{e + s^2 + t^2 + 2\gamma st} = \frac{1}{p(s, t)} \text{ on } I_\alpha \times I_\alpha,$$

$I_\alpha = I + \alpha$. Now $p(s, s) = e + 2s^2(1 + \gamma) > 0$ for all $s \in \mathbb{R}$. Hence every $s \in \mathbb{R}$ is a diagonal point of l . So by Theorem 1.3.3 we have

$$\begin{aligned} N^+(k, 0) &= N^+(l, \alpha) \\ &= N^+(l, 0) \\ &= \infty \quad (\text{by Example 4.3.3}). \end{aligned}$$

Similarly, $N^-(k, 0) = \infty$ and hence $N^+(K) = N^-(K) = \infty$.

$$(2) \text{ Area B: } e \leq 0 \text{ and } 0 < \gamma \leq 1. \quad l(s, t) = \frac{1}{e + s^2 + t^2 + 2\gamma st} = \frac{1}{p(s, t)} \text{ on } I_\alpha.$$

$$\left. \begin{aligned} l(s, t) &= \frac{1}{e + s^2 + t^2 + 2\gamma st} \\ &= \frac{1}{(s + t)^2 - 2(1 - \gamma)st - |e|} \\ &= \frac{1}{r(s, t) - r_1(s, t) - r_2(s, t)} \end{aligned} \right\} \quad (4.17)$$

where $r(s, t) = (s + t)^2$, $r_1(s, t) = 2(1 - \gamma)st$ and $r_2(s, t) = |e|$. Since $p(s, t) > 0$ on $I_\alpha \times I_\alpha$

we see by (4.17) that $0 \notin I_\alpha$. Hence by Example 4.3.4

$$N^+(r, I_\alpha) = \infty \text{ and } N^-(r, I_\alpha) = 0.$$

We can prove that

$$l(s, t) = \frac{1}{p(s, t)} \stackrel{0}{\geq} \frac{1}{r(s, t)} \stackrel{0}{\geq} 0 \quad (s, t \in I_\alpha)$$

by showing that r , r_1 , r_2 satisfy the hypotheses of Lemma 4.2.4, at least on some suitably small subinterval J of I_α . Now

$$0 < p(\alpha, \alpha) = r(\alpha, \alpha) - r_1(\alpha, \alpha) - r_2(\alpha, \alpha)$$

so that

$$|r_2(s, t)| = |e| < r(\alpha, \alpha) - r_1(\alpha, \alpha).$$

Hence

$$|r_2(s, t)| < r(s, t) - r_1(s, t)$$

in a neighborhood of (α, α) in $\mathbb{R}^2$. We also have

$$|r_1(\alpha, \alpha)| = r_1(\alpha, \alpha) = 2(1 - \gamma)\alpha^2 < 2\alpha^2 = r(\alpha, \alpha)$$

so on some neighborhood of (α, α) in $\mathbb{R}^2$ we have

$$|r_1(s, t)| < r(s, t).$$

So on some suitably small subinterval $J \subseteq I_\alpha$ we have

$$l(s, t) = \frac{1}{r(s, t) - r_1(s, t) - r_2(s, t)} \stackrel{\circ}{\geq} \frac{1}{r(s, t)} \stackrel{\circ}{\geq} 0.$$

Hence, by Remark 1.1.3 (ii),

$$N^+(k, I) = N^+(l, I_\alpha) = N^+(l, J) = \infty$$

$$N^-(k, I) = N^-(l, I_\alpha) = N^-(l, J) = 0.$$

(3) Curve C: $e = 0$ and $\gamma > 1$. Here

$$l(s, t) = \frac{1}{s^2 + t^2 + 2\gamma st} = \frac{1}{p(s, t)} \text{ on } I_\alpha \times I_\alpha.$$

In this case we must have $b > 0$ so $\alpha > 0$. Also $0 \notin I_\alpha$ and $\alpha \in I_\alpha$, so $I_\alpha \subset (0, \infty)$. Now

for any $\delta > 0$

$$\inf_{s,t \geq \delta} p(s,t) = p(\delta, \delta) = 2\delta^2(1+\gamma) > 0$$

hence any closed interval $J \subseteq (0, \infty)$ is admissible for l and

$$N^+(l, I_\alpha) = N^+(l, J) \text{ and } N^-(l, I_\alpha) = N^-(l, J)$$

for any interval $J \subseteq (0, \infty)$. Choose $J = [e^{-\beta}, e^\beta]$, $\beta > 0$, and let $\eta(s) = e^s$ then

$$\begin{aligned} \eta l(s,t) &= \frac{e^{s/2} e^{t/2}}{e^{2s} + e^{2t} + 2\gamma e^{(s+t)}} \\ &= \frac{e^{-s/2} e^{-t/2}}{2[\cosh(s-t) + \gamma]} \end{aligned}$$

acting on $L^2(J)$, $J' = [-\beta, \beta]$. Hence $\eta L = M L' M^*$ where M is the multiplication

operator on $L^2(J')$ defined by $Mf(s) = \frac{1}{\sqrt{2}} e^{-s/2} f(s)$ and L' is the integral operator

on $L^2(J')$, as defined in Example 4.3.5, with kernel

$$l'(s,t) = \frac{1}{\cosh(s-t) + \gamma}.$$

Thus ηL is symmetrically equivalent to L' . Hence

$$\begin{aligned} N^+(k, I) &= N^+(l, J) \\ &= N^+(\eta l, J') \quad (\text{by spatial equivalence}) \\ &= N^+(l', J') \quad (\text{by symmetric equivalence}) \\ &= \infty \quad (\text{by Example 4.3.5}). \end{aligned}$$

Similarly $N^-(k, I) = \infty$, and hence $N^+(K) = N^-(K) = \infty$.

(4) Area D: $e < 0$ and $\gamma > 1$.

$$l(s, t) = \frac{1}{e + s^2 + t^2 + 2\gamma st} = \frac{1}{p(s, t)} \text{ on } I_\alpha \times I_\alpha.$$

Here again we must have $b > 0$, so $\alpha > 0$. Let $m(s, t) = \frac{1}{s^2 + t^2 + 2\gamma st}$. Let $\delta > 0$, then

$$\inf_{s, t \geq \delta} p(s, t) = p(\delta, \delta) = -|e| + 2\delta^2(1 + \gamma). \text{ So fix } \delta \text{ so that } \delta^2 = \frac{|e|}{2(1 + \gamma)} \text{ (i. e.}$$

$\inf_{s, t \geq \delta} p(s, t) = 0$). Then if J is any closed interval in (δ, ∞) , J is admissible for l and

$N^+(l, J)$, $N^-(l, J)$ are independent of the choice of $J \subset (\delta, \infty)$. Now

$$0 < a = p(\alpha, \alpha) = -|e| + 2\alpha^2(1 + \gamma)$$

so $\alpha^2 > \delta^2$, i. e. $\alpha \in (\delta, \infty)$. It follows that for all $n \geq 1$,

$$J_n = [n\alpha, 2n\alpha] \subseteq (\delta, \infty)$$

and that

$$N^\pm(l, I_\alpha) = N^\pm(l, J_n)$$

for all $n \geq 1$. Let $\psi_n(s) = ns$, then

$$\psi_n l(s, t) = \frac{n}{e + n^2(s^2 + t^2 + 2\gamma st)},$$

(acting on $L^2(J_1)$). Now

$$n\psi_n l(s, t) = \frac{1}{\left(\frac{e}{n^2}\right) + s^2 + t^2 + 2\gamma st} \rightarrow \frac{1}{s^2 + t^2 + 2\gamma st} = m(s, t) \text{ as } n \rightarrow \infty$$

(pointwise on $J_1 \times J_1$). Also

$$N^+(l, I_\alpha) = N^+(\psi_n l, J_1)$$

$$= N^+(n\psi_n l, J_n)$$

$n = 1, 2, 3, \dots$. But since $e < 0$, $n\psi_n l$ decreases to m , hence, by Dini's theorem

$n\psi_n l$ tends to m uniformly on $I_1 \times I_1$. So, by Lemma 4.2.1, we have

$$N^+(l, I_\alpha) = \lim_{n \rightarrow \infty} N^+(n\psi_n l, J_1)$$

$$\geq N^+(m, J_1).$$

Since $N^+(m, J_1) = \infty$, (see the previous case) we get $N^+(l, I_\alpha) = \infty$. Similarly one

can show $N^-(l, I_\alpha) = \infty$. Hence $N^+(\mathbf{K}) = N^-(\mathbf{K}) = \infty$.

4.6 THE CASE $k(s,t) = \frac{1}{a+b(s+t)+cst-(s^2+t^2)}$

In this section, we will complete the case $d = -1$. That is

$$k(s,t) = \frac{1}{q(s,t)} = \frac{1}{a+b(s+t)-s^2-t^2+2\gamma st}$$

with $a > 0$, $b \geq 0$ and $\gamma > 0$. As before let $\psi: I + \alpha \rightarrow I$ with $\psi(s) = s - \alpha$. Then

$$\begin{aligned} l(s,t) &= \psi k(s,t) = \psi'(s)^{\frac{1}{2}} k(\psi(s), \psi(t)) \psi'(t)^{\frac{1}{2}} \\ &= \frac{1}{a+b(s+t-2\alpha)-(s-\alpha)^2-(t-\alpha)^2+2\gamma(s-\alpha)(t-\alpha)} \\ &= \frac{1}{[a-2\alpha b-2\alpha^2(1-\gamma)]+[b+2\alpha(1-\gamma)](s+t)-s^2-t^2+2\gamma st}. \end{aligned}$$

In the case $\gamma \neq 1$, put

$$\alpha = \frac{b}{2(\gamma-1)}$$

to get

$$l(s,t) = \frac{1}{e-(s^2+t^2)+2\gamma st} = \frac{1}{p(s,t)} \quad (\text{say})$$

where

$$e = a - \frac{b^2}{2(\gamma-1)}.$$

Let L be the integral operator on $L^2(I_\alpha)$ corresponding to l where $I_\alpha = I + \alpha$. Now we have, at least for $\gamma \neq 1$,

$$N^+(L) = N^+(K) \text{ and } N^-(L) = N^-(K).$$

As in Fig. 4.3 below we have five possibilities:

- | | |
|--|---|
| (1) Area A: $e > 0$ and $\gamma > 1$ | (2) Area B: $e > 0$ and $0 < \gamma < 1$ |
| (3) Curve $\mathcal{L}$: $e = 0$ | (4) Area D: $e < 0$ |
| (5) Line $\mathcal{L}$: $\gamma = 1$. | |

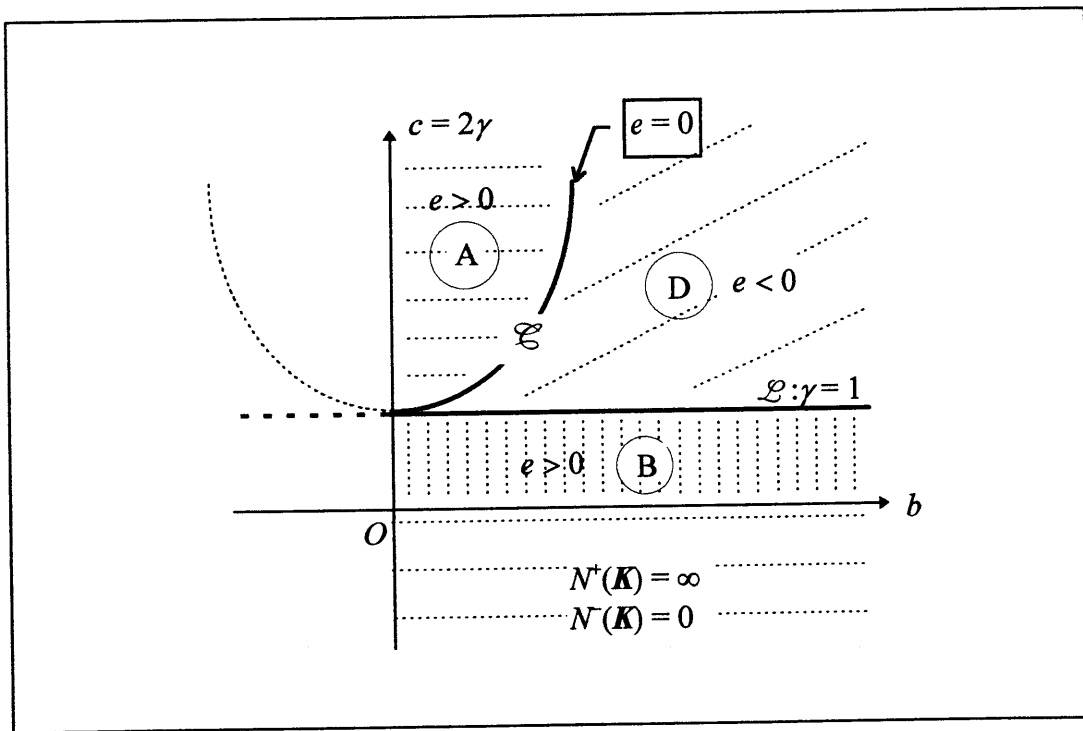


Fig. 4.3

(1) **Area A:** $e > 0, \gamma > 1$ and $l(s, t) = \frac{1}{e - s^2 - t^2 + 2\gamma st} = \frac{1}{p(s, t)}$ on $I_\alpha \times I_\alpha$,

$I_\alpha = I + \alpha$. Here $p(s, s) = e + 2s^2(\gamma - 1) > 0$, for all $s \in \mathbb{R}$, this means that any point in $\mathbb{R}$ is a diagonal point of l . Hence by Theorem 1.3.3 we have

$$N^+(k, 0) = N^+(l, \alpha)$$

$$= N^+(l, 0)$$

$$= \infty$$

(see Example 4.3.3). Similarly $N^-(l, 0) = \infty$. Hence $N^+(\mathbf{K}) = N^-(\mathbf{K}) = \infty$.

(2) **Area B:** $e > 0$ and $0 < \gamma < 1$. Here $\alpha \leq 0$ and $p(s, s) = e - 2s^2(1 - \gamma)$ so

if $\tau = \sqrt{\frac{e}{2(1 - \gamma)}}$ each point in $(-\tau, \tau)$ is a diagonal point of l . Now

$$\tau^2 = \frac{e}{2(1 - \gamma)} = \frac{a}{2(1 - \gamma)} + \frac{b^2}{4(1 - \gamma)} > \frac{b^2}{4(1 - \gamma)} = \alpha^2.$$

Hence $\alpha \in (-\tau, \tau)$, so that

$$N^+(k, 0) = N^+(l, \alpha)$$

$$= N^+(l, 0)$$

$$= \infty.$$

As above, this will implies $N^+(\mathbf{K}) = N^-(\mathbf{K}) = \infty$.

(3) **Curve $\mathcal{E}$:** $e = 0$ (and hence $\gamma > 1$). In this case we must have $b > 0$, so

$\alpha > 0$. Here

$$l(s, t) = \frac{1}{2\gamma st - s^2 - t^2} = \frac{1}{p(s, t)} \text{ on } I_\alpha \times I_\alpha.$$

Now $0 \notin I_\alpha$ and $\alpha \in I_\alpha$, so $I_\alpha \subset (0, \infty)$. Now $p(s, s) = 2(\gamma - 1)s^2 > 0$ for all s and hence

$$N^+(l, \zeta) \text{ independent of } \zeta > 0.$$

Since $\alpha > 0$

$$N^+(k, I) = N^+(l, \alpha) = N^+(l, 1).$$

Choose $\beta > 0$ such that $p(s, t) > 0$ on $J \times J$, where $J = [e^{-\beta}, e^\beta]$. Then

$$N^+(k, I) = N^+(l, J).$$

Let $\eta(s) = e^s$ then

$$\begin{aligned} \eta l(s, t) &= \frac{e^{s/2} e^{t/2}}{2\gamma e^{(s+t)} - e^{2s} - e^{2t}} \\ &= \frac{e^{-s/2} e^{-t/2}}{2[\gamma - \cosh(s-t)]} \end{aligned}$$

acting on $L^2(J')$, $J' = [-\beta, \beta]$. Hence $\eta L = M L' M^*$ where M is the multiplication

operator on $L^2(J')$ defined by $Mf(s) = \frac{1}{\sqrt{2}} e^{-s/2} f(s)$ and L' is the integral

operator on $L^2(J')$, as defined in Example 4.3.7, with kernel

$$l'(s, t) = \frac{1}{\gamma - \cosh(s-t)}.$$

Thus ηL is symmetrically equivalent to L' . Hence we have

$$N^+(k, I) = N^+(l, J)$$

$$= N^+(\eta l, J') \quad (\text{by spatial equivalence})$$

$$= N^+(l', J') \quad (\text{by symmetric equivalence})$$

$$= \infty \quad (\text{by Example 4.3.7}).$$

Similarly $N^-(k, I) = \infty$, and hence $N^+(K) = N^-(K) = \infty$.

(4) Area D: $e < 0$, $\gamma > 1$, hence also $\alpha \geq 0$. Here

$$l(s, t) = \frac{1}{e - (s^2 + t^2) + 2\gamma st} = \frac{1}{p(s, t)} \text{ on } I_\alpha \times I_\alpha.$$

Now $p(s, s) = 2(\gamma - 1)s^2 - |e|$, hence

$$p(s, s) > 0 \Leftrightarrow |s| > \delta = \left(\frac{|e|}{2(\gamma - 1)} \right)^{\frac{1}{2}}.$$

Since $p(\alpha, \alpha) = q(0, 0) = a > 0$ and $\alpha \geq 0$ we must have $\alpha > \delta$. So for any closed admissible interval $J \subseteq (\delta, \infty)$ we have

$$N^\pm(k, I) = N^\pm(l, \alpha) = N^\pm(l, J).$$

Suppose that $r > 1$ and that $I_n = [\frac{n}{r}, nr]$, $n = 1, 2, 3, \dots$. We shall choose r such that,

for large n , $I_n \in Ad(I)$, we will automatically have $I_n \subseteq (\delta, \infty)$, so that

$$N^+(k, I) = N^+(l, I_n).$$

To choose r , write

$$p(s, t) = e - (s - t)^2 + 2(\gamma - 1)st.$$

Now $\forall s, t \in I_n$ we have

$$\begin{aligned}
p(s, t) &\geq e - n^2 \left(r - \frac{1}{r} \right)^2 + 2(\gamma - 1) \left(\frac{n}{r} \right)^2 \\
&= e + n^2 \left[\frac{2(\gamma - 1)}{r^2} - \left(r - \frac{1}{r} \right)^2 \right] \\
&= e + n^2 \sigma(r) \text{ (say).}
\end{aligned}$$

As $r \downarrow 1$; $\sigma(r) \rightarrow 2(\gamma - 1)$ and $\left(\frac{\gamma}{r^2} - r^2 \right) \rightarrow (\gamma - 1)$. So choose $r > 1$ so that $\sigma(r) > 0$

and

$$\left(\frac{\gamma}{r^2} - r^2 \right) > 0 \quad (4.18)$$

Then for this r , there exist N such that

$$e + n^2 \sigma(r) > 0 \quad \forall n \geq N.$$

Hence $I_n = [\frac{n}{r}, nr]$ is admissible for l for all $n \geq N$, and by invariance

$$\begin{aligned}
N^+(l, I_n) &= N^+(l, I_\alpha) \\
&= N^+(k, I) \quad \forall n \geq N.
\end{aligned}$$

Now, for $n \geq N$, let $\psi_n(s) = ns$, so that

$$\psi_n l(s, t) = \frac{n}{p(ns, nt)}$$

and hence

$$n\psi_n l(s,t) = \frac{n^2}{p(ns,nt)}$$

$$= \frac{1}{\left(\frac{e}{n^2}\right) + 2\gamma st - s^2 - t^2}.$$

We shall consider these kernels on $I_1 \times I_1$. Since $p(ns,nt) > 0$ on $I_n \times I_n$, $n\psi_n l(s,t) > 0$ on $I_1 \times I_1$. Since $e < 0$ we see that $n\psi_n l(s,t)$ decreases to $\frac{1}{2\gamma st - s^2 - t^2} = m(s,t)$ pointwise on $I_1 \times I_1$ as $n \rightarrow \infty$. Because of (4.18), m is continuous on $I_1 \times I_1$. Hence, by Dini's theorem $n\psi_n l \rightarrow m$ uniformly on $I_1 \times I_1$. Now for all $n \geq N$,

$$N^+(l, I_\alpha) = N^+(l, I_n)$$

$$= N^+(\psi_n l, I_1)$$

$$= N^+(n\psi_n l, I_1).$$

Hence, by Lemma 4.2.1, we have

$$N^+(l, I_\alpha) = \underline{\lim} N^+(n\psi_n l, I_n) \geq N^+(m, I_1).$$

But $N^+(m, I_1) = \infty$ (see the previous case). Hence $N^+(l, I_\alpha) = \infty$ and, by similar way, we can show $N^-(l, I_\alpha) = \infty$. So $N^+(\mathbf{K}) = N^-(\mathbf{K}) = \infty$.

(5) Line $\mathcal{L}$: $\gamma = 1$. On the line $\gamma = 1$ the original kernel $k(s, t)$ can be written

as

$$k(s,t) = \frac{1}{q(s,t)} = \frac{1}{a + b(s+t) - (s-t)^2}$$

with $a > 0$, $b \geq 0$ and $0 \in I$. The case $b = 0$ has already been done in Example 4.3.8

where we saw that $N^+(K) = N^-(K) = \infty$. So suppose that $b > 0$. Now

$$q(s, s) = a + 2bs > 0 \text{ when } s > -\delta = -\frac{a}{2b}.$$

So for any closed admissible interval $J \subseteq (-\delta, \infty)$ we have

$$N^+(k, I) = N^+(k, J) \text{ and } N^-(k, I) = N^-(k, J).$$

We shall find an r ($0 < r < 1$) such that for large n the interval $I_n = [n^2 - nr, n^2 + nr]$

is an admissible closed subinterval of $(-\delta, \infty)$. For $s, t \in I_n$ we have

$$\begin{aligned} q(s, t) &\geq a + 2b(n^2 - nr) - 4n^2r^2 \\ &= a + 2n[n(b - 2r^2) - br] \\ &= a + 2n\sigma(n, r) \text{ (say).} \end{aligned}$$

So choose r small such that $(b - 2r^2) > 0$ then choose n large such that $\sigma(n, r) > 0$.

That is

$$q(s, t) > 0 \text{ on } I_n \quad \forall n \geq N \text{ for some } N.$$

We now have

$$N^\pm(k, I) = N^\pm(k, I_n), \quad n \geq N.$$

Let $\psi_n(s) = n^2 + ns$, then for $n \geq N$,

$$N^\pm(k, I) = N^\pm(k, I_n) = N^\pm(\psi_n k, I') = N^\pm(n\psi_n k, I')$$

where $I' = [-r, r]$. For $s, t \in I'$ we have

$$\psi_n k(s, t) = \frac{n}{a + 2bn^2 + bn(s+t) - n^2(s-t)^2}.$$

So $n\psi_n k(s,t) \downarrow \frac{1}{2b-(s-t)^2} = m(s,t)$ (say) uniformly on $I' \times I'$ as $n \rightarrow +\infty$. (Note

we already have $b-2r^2 > 0$ and hence $2b-(s-t)^2 > 0$ on $I' \times I'$). Now, by

Lemma 4.2.1, we have

$$N^+(k, I) = \underline{\lim} N^+(n\psi_n k, I') \geq N^+(m, I').$$

But $N^+(m, I') = \infty$, by Example 4.3.8. Hence $N^+(K) = \infty$. Similarly $N^-(K) = \infty$.

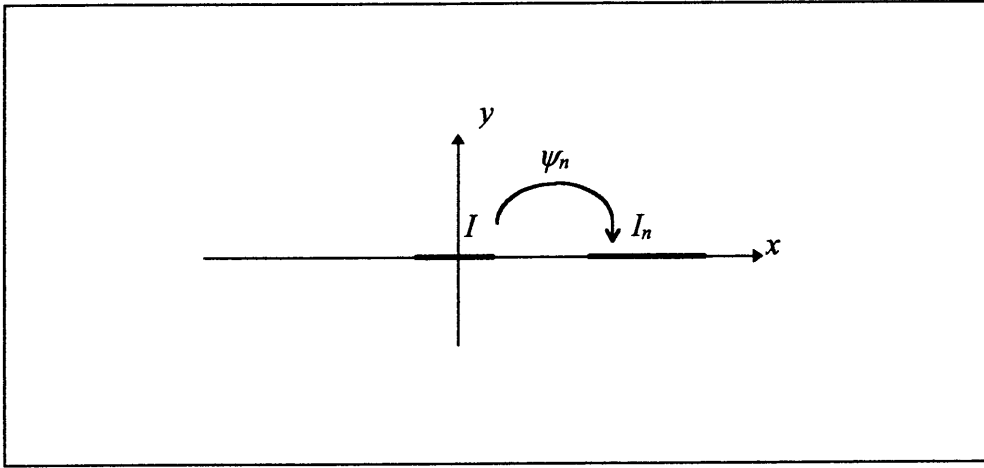


Fig. 4.4

CHAPTER 5. ASYMPTOTIC BEHAVIOUR OF EIGENVALUES

5.1 INTRODUCTION

In this chapter we resume our study of kernels of the form

$$k(s,t) = \frac{1}{q(s,t)} = \frac{1}{a + b(s+t) + st}$$

where $a, b, c \neq 0$. Our aim is to find asymptotic estimates of $\lambda_n^+(k, I)$, $\lambda_n^-(k, I)$ when $I \in Ad(k)$. We use the same notation and conventions as in Chapter 2. Thus if $I \in Ad(k)$ we write K_I for the integral operator on $L^2(I)$ with kernel k . We always assume $q(s,t) > 0$ on $I \times I$ and we reject straight away the trivial finite rank case $b^2 = ac$.

5.1.1 Reductions in the case $b^2 > ac$

Suppose k is as above with $b^2 > ac$ and that $I \in Ad(k)$. This is the same case in which K_I is strictly positive. Using Lemma 2.2.3 we can first construct a closed interval $J \subset (-1,1)$ such that K_I is unitarily equivalent to a positive multiple of K_{oJ} , where K_{oJ} is the integral operator on $L^2(J)$ with kernel

$$k_o(s,t) = \frac{1}{1-st}.$$

Next, using Corollary 2.2.2 we can find a ρ , $0 < \rho < 1$, such that K_{oJ} is unitarily equivalent to $K_{oI'}$ where $I' = [-\rho, \rho]$. Thus the problem of finding asymptotic estimates for the eigenvalues of K_I can be reduced to that of finding asymptotic estimates for the eigenvalues of positive integral operators of the form

$$Tf(s) = \int_{-\rho}^{\rho} \frac{f(t)}{1-st} dt \quad (f \in L^2(-\rho, \rho), -\rho \leq s \leq \rho) \quad (5.1)$$

where $0 < \rho < 1$. This is the main problem we shall solve in this chapter. We shall show that $\lambda_n(T) \approx e^{-2n\alpha}$, for suitable $\alpha > 0$ depending on ρ . Essentially the same problem was solved in [2]. As a bonus we shall be able to estimate the eigenvalues of the two kernels

$$\frac{\cosh 2\alpha - st}{\sinh^2 2\alpha + s^2 + t^2 - (2 \cosh 2\alpha)st},$$

$$\frac{1}{\sinh^2 2\alpha + s^2 + t^2 - (2 \cosh 2\alpha)st}$$

acting on $L^2(-1,1)$.

5.1.2 Reductions in the case $b^2 < ac$

Suppose k is as above with $b^2 < ac$ and that $I \in Ad(k)$. This is the case in which $N^+(K_I) = N^-(K_I) = \infty$. Let

$$k_1(s, t) = \frac{1}{1+st}.$$

Using Lemma 2.3.2 and Remark 2.3.3 we can construct a closed interval $J \in Ad(k_1)$ such that K_I is unitarily equivalent to a positive multiple of K_{1J} . Next using Lemma

2.3.1 we can find a $\rho', 0 < \rho' < 1$, such that $\mathbf{K}_{IJ}$ is unitarily equivalent to $\mathbf{K}_{IJ'}$ where $I' = [-\rho', \rho']$. So for this case we need only to find estimates for the eigenvalues of integral operators of the form

$$T'f(s) = \int_{-\rho'}^{\rho'} \frac{f(t)}{1+st} dt \quad (f \in L^2(-\rho', \rho'), -\rho' \leq s \leq \rho').$$

But we are able to reduce this problem still further. For if we put $J' = [0, \rho'^2]$ we have, by (2.2) and (2.5)

$$\frac{1}{\rho'} \lambda_n(\mathbf{K}_{0J'}) \leq \lambda_n^+(\mathbf{K}_{IJ'}) \leq \rho'^3 \lambda_{n-1}(\mathbf{K}_{0J'})$$

and

$$\frac{1}{\rho'^3} \lambda_{n+1}(\mathbf{K}_{0J'}) \leq |\lambda_n^-(\mathbf{K}_{IJ'})| \leq \rho' \lambda_n(\mathbf{K}_{0J'}).$$

Here $\mathbf{K}_{0J'}$ is the operator

$$T''f(s) = \int_0^{\rho'^2} \frac{f(t)}{1-st} dt \quad (f \in L^2(0, \rho'), 0 \leq s \leq \rho').$$

By Corollary 2.2.2 this operator is unitarily equivalent to an operator of the form T in

(5.1) where $\rho = \tanh \frac{1}{2} (\tanh^{-1} \rho'^2)$. Given that we shall prove that, for this T ,

$\lambda_n(T) \approx e^{-2n\beta}$, for some $\beta > 0$, we will obviously have $\lambda_{n-1}(T) \approx \lambda_n(T) \approx \lambda_{n+1}(T)$

so that

$$\lambda_n(\mathbf{K}_I) \approx |\lambda_n^-(\mathbf{K}_I)| \approx e^{-2n\beta}.$$

5.2 FACTORIZATION THROUGH H^2 AND l^2

From now on suppose ρ is fixed, $0 < \rho < 1$, that $I = [-\rho, \rho]$ and that $T = K_{oI}$, i. e.

$$Tf(s) = \int_{-\rho}^{\rho} \frac{f(t)}{1-st} dt \quad (f \in L^2(-\rho, \rho), -\rho \leq s \leq \rho). \quad (5.2)$$

This operator is strictly positive and we seek estimates of $\lambda_n(T)$. In this section we recall G. Little's factorizations of T through H^2 and l^2 (see [2, Section 5]) and let $\Delta =$ open unit disc, and let $\mathbb{T}$ be its bounding circle. The space H^2 can be thought of as either a space $H^2(\Delta)$ of analytic functions on Δ or as a closed subspace $H^2(\mathbb{T})$ of $L^2(\mathbb{T})$. We summarize here what we need to know about H^2 . For a full discussion see [1]. The space $H^2(\Delta)$ is the set of all analytic functions f on Δ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ where } a = (a_n) \in l^2. \quad (5.3)$$

This space can be made into a Hilbert space by defining $\langle f, g \rangle_{H^2(\Delta)} = \sum_{n=0}^{\infty} a_n \bar{b}_n$ and

so $\|f\|_{H^2(\Delta)}^2 = \sum_{n=0}^{\infty} |a_n|^2$. Thus, by construction, the map $W_{\Delta} : l^2 \rightarrow H^2(\Delta)$ given by

$$W_{\Delta}a(z) = \sum_{n=0}^{\infty} a_n z^n \quad (a = (a_n) \in l^2, z \in \Delta) \quad (5.4)$$

is a unitary operator. Clearly the functions $e_n(z) = z^n$ ($n \geq 0$) form an orthonormal basis for $H^2(\Delta)$. Moreover

polynomials are dense in $H^2(\Delta)$.

We use $L^2(\mathbb{T})$ to denote the L^2 -space of arc-length measure on $\mathbb{T}$ normalized by the factor $\frac{1}{2\pi}$. Thus for $f \in L^2(\mathbb{T})$,

$$\|f\|_{L^2(\mathbb{T})}^2 = \frac{1}{2\pi} \int_{\mathbb{T}} |f(z)|^2 |dz| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta. \quad (5.5)$$

Here and throughout we shall always use $|dz|$ to denote integration with respect to arc-length measure. The functions $\tilde{e}_n(e^{i\theta}) = e^{ni\theta}$ ($n \in \mathbb{Z}$) form an orthonormal basis for $L^2(\mathbb{T})$ and if $f \in L^2(\mathbb{T})$ has Fourier coefficients $(a_n)_{n \in \mathbb{Z}}$ then

$$\|f\|_{L^2(\mathbb{T})}^2 = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} |a_n|^2. \quad (5.6)$$

The space $H^2(\mathbb{T})$ is the closed subspace of $L^2(\mathbb{T})$ spanned by the functions $\tilde{e}_n$ with $n \geq 0$.

Given an $f \in H^2(\Delta)$ with Taylor coefficients $a = (a_n) \in \ell^2$ as in (5.3) let $\tilde{f} \in H^2(\mathbb{T})$ be the function

$$\tilde{f}(z) = \sum_{n=0}^{\infty} a_n z^n \quad (z \in \mathbb{T}),$$

the convergence being in mean-square on $\mathbb{T}$. By (5.5) and (5.6) it is clear that the map

$f \mapsto \tilde{f}$ ($H^2(\Delta) \rightarrow H^2(\mathbb{T})$) is unitary. This map can be realized by taking radial limits.

Thus

$$\tilde{f}(z) = \lim_{r \uparrow 1} f(rz)$$

for almost all $z \in \mathbb{T}$. It is customary to identify f with $\tilde{f}$ and hence $H^2(\Delta)$ with $H^2(\mathbb{T})$, and we shall do so from now on. In what follows we shall be able to avoid subtle problems of integration because, for the most part, we will be considering

functions f which are analytic in a neighbourhood of $\overline{\Delta}$. For such a function $\tilde{f}$ is just the restriction of f to $\mathbb{T}$.

We can now exhibit our factorization of T as in (5.2). Because $I = [-\rho, \rho] \subset \Delta$ we can restrict any element of $H^2(\Delta)$ to I . Thus we define $R_\Delta: H^2(\Delta) \rightarrow L^2(I)$ by

$$R_\Delta f(s) = f(s) \quad (f \in H^2(\Delta), s \in I). \quad (5.7)$$

5.2.1 LEMMA ([2])

(i) R_Δ is compact,

(ii) $T = R_\Delta R_\Delta^*$.

Proof. Let $W_\Delta: l^2 \rightarrow H^2(\Delta)$ be the unitary operator defined in (5.4). To prove (i) it is sufficient to show that $S_\Delta = R_\Delta W_\Delta$ is compact. Now $S_\Delta = R_\Delta W_\Delta: l^2 \rightarrow L^2(I)$ is given by

$$S_\Delta a(s) = \sum_{n=0}^{\infty} a_n s^n \quad (a = (a_n) \in l^2, s \in I).$$

Hence S_Δ is an “integral” operator with an L^2 -kernel:

$$\sum_{n=0}^{\infty} \int_I s^{2n} ds = \sum_{r=0}^{\infty} \frac{2}{2r+1} \rho^{2r+1} < \infty.$$

Therefore S_Δ is compact and so is R_Δ .

To prove (ii) we note first that $S_\Delta S_\Delta^* = R_\Delta W_\Delta W_\Delta^* R_\Delta^* = R_\Delta R_\Delta^*$. Now $S_\Delta^*: L^2(I) \rightarrow l^2$ is given by

$$\left(S_{\Delta}^* f\right)_n = \int_I f(t) t^n dt \quad (f \in L^2(I), n \geq 0).$$

Therefore for any $f \in L^2(I)$ and $s \in I$

$$\begin{aligned} S_{\Delta} S_{\Delta}^* f(s) &= \sum_{n=0}^{\infty} s^n \int_I f(t) t^n dt \\ &= \int_I \frac{f(t)}{1-st} dt. \end{aligned}$$

Therefore $R_{\Delta} R_{\Delta}^* = S_{\Delta} S_{\Delta}^* = T$.

5.3 THE E^2 SPACE OF AN ELLIPTIC DISC

For $\alpha > 0$ let

$$D_\alpha = \left\{ z \in \mathbb{C} : \frac{x^2}{\cosh^2 \alpha} + \frac{y^2}{\sinh^2 \alpha} < 1 \right\},$$

$$\gamma_\alpha = \partial D_\alpha = \left\{ z \in \mathbb{C} : \frac{x^2}{\cosh^2 \alpha} + \frac{y^2}{\sinh^2 \alpha} = 1 \right\}$$

and let us write $J = [-1, 1]$. For our fixed ρ there is a unique $\alpha(\rho) > 0$ and a conformal map φ of $D_{\alpha(\rho)}$ onto Δ which is real and increasing on $D_{\alpha(\rho)} \cap \mathbb{R}$ and satisfies $\varphi(J) = I$ (see [6, Chapter 6] and [7, Section 3.5]).

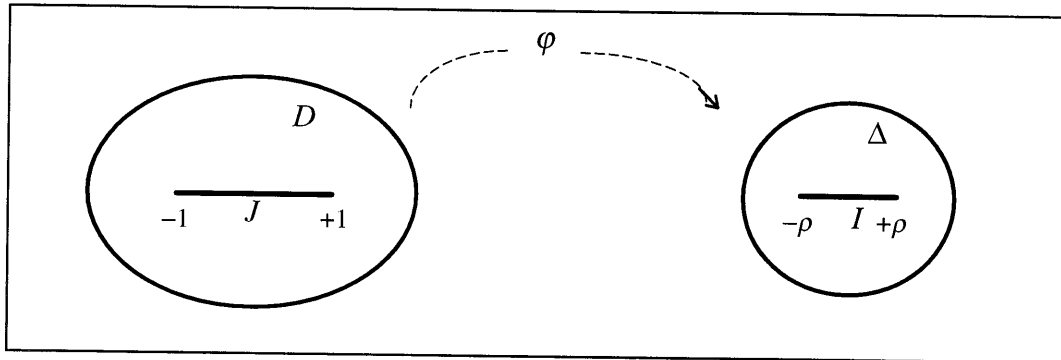


Fig. 5.1

Let us assume that $\alpha(\rho)$ has now been chosen so that we can abbreviate $\alpha(\rho)$ to α , $D_{\alpha(\rho)}$ to D and $\gamma_{\alpha(\rho)}$ to γ .

5.3.1 DEFINITION

An analytic function g on D is said to be of class $E^2(D)$ if there exists a function $f \in H^2(\Delta)$ such that

$$g(z) \equiv f(\varphi(z))\varphi'(z)^{\frac{1}{2}} \quad (z \in D)$$

where $\varphi'^{\frac{1}{2}}$ is the branch of the square root of φ' which is real and positive on $D \cap \mathbb{R}$. We define

$$\|g\|_{E^2(D)} = \|f\|_{H^2(\Delta)}.$$

Thus, by construction, $E^2(D)$ is a Hilbert space with

$$\langle g_1, g_2 \rangle_{E^2(D)} = \langle f_1, f_2 \rangle_{H^2(\Delta)}$$

where $g_i(z) = f_i(\varphi(z))\varphi'(z)^{\frac{1}{2}}$, ($i = 1, 2$) and the operator $V_\varphi : H^2(\Delta) \rightarrow E^2(D)$ given by

$$V_\varphi f = (f \circ \varphi)\varphi'^{\frac{1}{2}}$$

is unitary. By restricting φ to J we obtain a unitary operator $U_\varphi : L^2(I) \rightarrow L^2(J)$ defined by

$$U_\varphi f(s) = f(\varphi(s))\varphi'(s)^{\frac{1}{2}} \quad (f \in L^2(I), s \in J).$$

Now, by analogy with (5.7) we define $R_D : E^2(D) \rightarrow L^2(J)$ by

$$R_D f(s) = f(s) \quad (f \in E^2(D), s \in J).$$

Clearly, $R_D V_\varphi = U_\varphi R_\Delta$

$$\begin{array}{ccc}
 H^2(\Delta) & \xrightleftharpoons[V_\varphi^* = V_\varphi^{-1}]{V_\varphi} & E^2(D) \\
 \downarrow R_\Delta & & \downarrow R_D \\
 L^2(I) & \xrightarrow{U_\varphi} & L^2(J)
 \end{array}$$

so $R_D = U_\varphi R_\Delta V_\varphi^*$ is compact. The above diagram commutes. Since V_φ is unitary,

$$R_D R_D^* = U_\varphi R_\Delta R_\Delta^* U_\varphi^* = U_\varphi T U_\varphi^*. \quad (5.8)$$

If we put $T_D = R_D R_D^*$, then by (5.8)

$$\lambda_n(T) = \lambda_n(T_D) \quad (n \geq 0) \quad (5.9)$$

and, by Lemma 1.4.3, T_D is the positive integral operator on $L^2(J)$ with kernel

$$\begin{aligned}
 \varphi k_o(s, t) &= \frac{\varphi'(s)^{\frac{1}{2}} \varphi'(t)^{\frac{1}{2}}}{1 - \varphi(s)\varphi(t)} \\
 &= \sum_{n=0}^{\infty} \varphi(s)^n \varphi'(s)^{\frac{1}{2}} \varphi(t)^n \varphi'(t)^{\frac{1}{2}},
 \end{aligned}$$

the series being uniformly convergent on $J \times J$.

Since the functions $e_n(z) = z^n$ ($n \geq 0$) form an orthonormal basis for $H^2(\Delta)$, the functions $V_\varphi e_n = \varphi^n \varphi'^{\frac{1}{2}}$ ($n \geq 0$) form an orthonormal basis for $E^2(D)$. By analogy with our construction for H^2 let us define a unitary operator $W_D : l^2 \rightarrow E^2(D)$ and a compact $S_D : l^2 \rightarrow L^2(J)$ by

$$S_D a(s) = \sum_{n=0}^{\infty} a_n \varphi(s)^n \varphi'(s)^{\frac{1}{2}}, \quad (a = (a_n) \in l^2, s \in J).$$

$$\begin{array}{ccccc}
& & H^2(\Delta) & \xrightarrow{V_\varphi} & E^2(D) \\
R_\Delta^* \nearrow & & \downarrow R_\Delta & \swarrow W_\Delta \quad \searrow W_D & \nwarrow R_D^* \\
L^2(I) & & & \ell^2 & L^2(J) \\
T \searrow & & \downarrow S_\Delta \quad \uparrow S_D & & \swarrow T_D \\
& L^2(I) & \xrightarrow{U_\varphi} & L^2(J)
\end{array}$$

105

5.4 TCHEBYSCHEF POLYNOMIALS

Because of (5.9) we need to find estimates of the eigenvalues of $T_D = S_D S_D^* = R_D R_D^*$. Every function in $E^2(D)$ can be expanded in Tchebyschef polynomials of both first and second kind. Such expansions will be used to find separate upper and lower bounds for $\lambda_n(T_D)$ and hence $\lambda_n(T)$.

The Tchebyschef polynomials (T_n) of first kind are defined by

$$T_n(s) = \cos(n \cos^{-1} s), \quad n \geq 0, \quad -1 \leq s \leq 1.$$

It is convenient to normalize these polynomials and define

$$\tilde{T}_0 = \frac{1}{\sqrt{\pi}} T_0 \quad \text{and} \quad \tilde{T}_n = \sqrt{\frac{2}{\pi}} T_n \quad (n \geq 1).$$

It is well known that

$$\int_{-1}^1 \frac{\tilde{T}_n(s) \tilde{T}_m(s)}{(1-s^2)^{1/2}} ds = \delta_{mn}.$$

Let us write

$$w(s) = (1-s^2)^{\frac{1}{4}}, \quad s \in J = [-1, 1]$$

and let $V_c: L^2(J) \rightarrow L^2(0, \pi)$ be the unitary operator

$$V_c f(\theta) = f(\cos \theta) (\sin \theta)^{\frac{1}{2}} \quad (f \in L^2(J), 0 \leq \theta \leq \pi).$$

Then we have

$$V_c \left(\frac{\tilde{T}_n}{w} \right) = c_n,$$

where $c_0(\theta) = 1/\sqrt{\pi}$ and $c_n(\theta) = \sqrt{\frac{2}{\pi}} \cos(n\theta)$, $n \geq 1$. Since $(c_n)_{n \geq 0}$ is an

orthonormal basis of $L^2(0, \pi)$ we see that

$$(\tilde{T}_n/w)_{n \geq 0} \text{ is an orthonormal bases for } L^2(J). \quad (5.10)$$

Likewise the Tchebyschef polynomials of the second kind are defined by

$$U_n(s) = \frac{\sin((n+1)\cos^{-1}s)}{\sin(\cos^{-1}s)}, \quad n \geq 0, \quad -1 \leq s \leq 1$$

and if we define $\tilde{U}_n = \sqrt{\frac{2}{\pi}} U_n$ ($n \geq 0$) we have

$$\int_{-1}^1 \tilde{U}_n(s) \tilde{U}_m(s) (1-s^2)^{\frac{1}{2}} ds = \delta_{mn}.$$

A similar argument to the one above shows that

$$(\tilde{U}_n w)_{n \geq 0} \text{ is an orthonormal bases for } L^2(J). \quad (5.11)$$

Using these orthonormal bases it is easy to define two positive integral operators

T'_1, T'_2 on $L^2(J)$ with eigenvalues $e^{-2n\alpha}$. Define $S'_1, S'_2 : l^2 \rightarrow L^2(J)$ by

$$S'_1 a(s) = \sum_{n=0}^{\infty} a_n e^{-n\alpha} \frac{\tilde{T}_n(s)}{w(s)}, \quad S'_2 a(s) = \sum_{n=0}^{\infty} a_n e^{-n\alpha} \tilde{U}_n(s) w(s) \quad (a = (a_n) \in l^2, s \in J).$$

Since

$$\sup_{s \in J} |T_n(s)| = 1, \quad \sup_{s \in J} |U_n(s)| = n+1$$

it is easy to see that S'_1, S'_2 have L^2 -kernels. Now define $T'_1 = S'_1 S'^*_1$, $T'_2 = S'_2 S'^*_2$.

These are the positive integral operators on $L^2(J)$ with respective kernels

$$\left. \begin{aligned} t'_1(s, t) &= \sum_{n=0}^{\infty} e^{-2n\alpha} \frac{\tilde{T}_n(s)}{w(s)} \cdot \frac{\tilde{T}_n(t)}{w(t)}, & s, t \in J, \\ t'_2(s, t) &= \sum_{n=0}^{\infty} e^{-2n\alpha} \left(\tilde{U}_n(s) w(s) \right) \left(\tilde{U}_n(t) w(t) \right) & s, t \in J. \end{aligned} \right\} (5.12)$$

Because of (5.10), (5.11) it follows that

$$\lambda_n(T'_1) = \lambda_n(T'_2) = e^{-2n\alpha}, \quad n \geq 0. \quad (5.13)$$

We will ultimately show that $\lambda_n(T_D) = O(\lambda_n(T'_1))$ and $\lambda_n(T'_2) = O(\lambda_n(T_D))$. From this it will follow immediately that

$$\lambda_n(T) = \lambda_n(T_D) \approx e^{-2n\alpha}.$$

5.5 UPPER BOUNDS FOR $\lambda_n(T)$

Consider again the conformal map φ of D onto Δ . By using Schwarz's reflection principle it can be shown that φ can be continued analytically to define a function which is meromorphic on the whole plane, the nearest poles to $\mathbb{R}$ being at $\pm i \sinh 2\alpha$. If

$$D' = \left\{ z \in \mathbb{C} : \frac{x^2}{\cosh^2 2\alpha} + \frac{y^2}{\sinh^2 2\alpha} < 1 \right\} \text{ and } \Delta' = \mathbb{C} - \{(-\infty, -1/\rho] \cup [1/\rho, \infty)\}$$

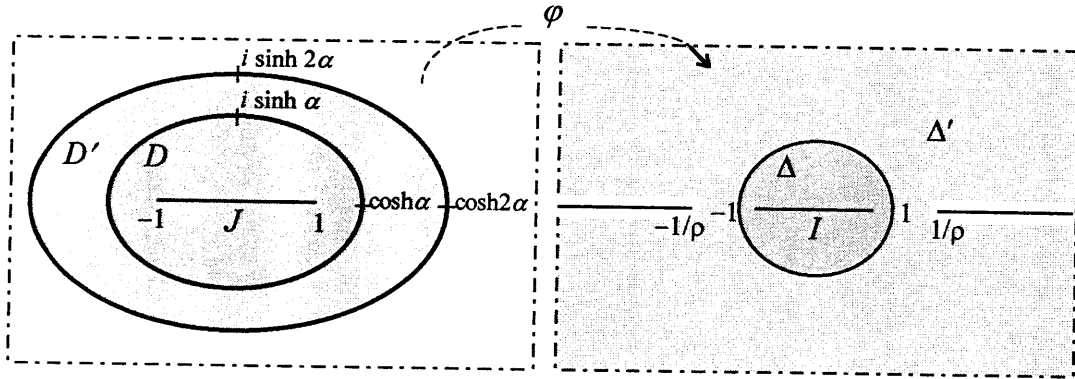


Fig. 5.2

then φ maps D' conformally onto Δ' (see Section 4 of [3]). We can also continue analytically our branch of $\varphi^{1/2}$ to D' and we can define a branch of $\varphi^{-1/2}$ on Δ' using the formula

$$\varphi^{-1/2}(w)^{1/2} = \frac{1}{\varphi'(\varphi^{-1}(w))^{1/2}}.$$

Next let us recall the formula defining the unitary operator $V_\varphi: H^2(\Delta) \rightarrow E^2(D)$, namely $V_\varphi f = (f \circ \varphi) \varphi'^{\frac{1}{2}}$. Its inverse $V_\varphi^{-1}: E^2(D) \rightarrow H^2(\Delta)$, is given by $V_\varphi^{-1} g = (g \circ \varphi^{-1}) \varphi'^{-\frac{1}{2}}$. We see from this that if F is the set of all analytic functions on Δ' and G is the set of all analytic functions on D' then $V_\varphi F = G$. Since F is dense in $H^2(\Delta)$ we see that G is dense in $E^2(D)$.

The following lemma can be deduced from sophisticated results in Chapter 10 of [1]. But we prefer the following elementary proof.

5.5.1 LEMMA

(i) If $g \in G$ then

$$\|g\|_{E^2(D)}^2 = \frac{1}{2\pi} \int_{\gamma} |g(z)|^2 |dz|. \quad (5.14)$$

(ii) Polynomials are dense in $E^2(D)$.

Proof. Given $g \in G$ let $f \in F$ be the function $V_\varphi^{-1} g$. Then

$$\|g\|_{E^2(D)}^2 = \|V_\varphi^{-1} g\|_{H^2(\Delta)}^2 = \frac{1}{2\pi} \int_{\Gamma} |g(\varphi^{-1}(w))|^2 |\varphi'^{-1}(w)| |dw|.$$

Substituting $w = \varphi(z)$ we obtain (5.14).

To prove (ii) it is sufficient to show that if $g \in G$ then g is a norm limit of polynomials. But since $\overline{D}$ is compact and $C - \overline{D}$ is connected it follows from Runge's theorem [9, Sec. 13.7] that there is a sequence (p_n) of polynomials which converges to g uniformly on $\overline{D}$ hence on $\gamma = \partial D$. Now it follows easily from (5.14) that $p_n \rightarrow g$ in norm. Hence polynomials are dense in $E^2(D)$.

5.5.2 THEOREM

(i) There exist $\beta, \delta, 0 < \beta < \delta$ depending only on α , such that, for all complex sequences (a_r) and all $n \geq m \geq 0$,

$$\beta \sum_{r=m}^n |a_r|^2 \leq \left\| \sum_{r=m}^n a_r e^{-r\alpha} \tilde{T}_r \right\|_{E^2(D)}^2 \leq \delta \sum_{r=m}^n |a_r|^2. \quad (5.15)$$

(ii) The formula

$$W_1 a(z) = \sum_{n=0}^{\infty} a_n e^{-n\alpha} \tilde{T}_n(z), \quad (a = (a_n) \in l^2, z \in D)$$

defines a continuous invertible operator mapping l^2 onto $E^2(D)$.

(iii) Every $g \in E^2(D)$ has a unique Tchebyschef expansion

$$g = \sum_{n=0}^{\infty} b_n e^{-n\alpha} \tilde{T}_n, \quad (5.16)$$

where $(b_n) \in l^2$. The series converges in the norm of $E^2(D)$ and uniformly on compact subsets of D .

Proof. It is convenient to define

$$\varepsilon_n = \begin{cases} \pi, & n = 0 \\ \pi/2, & n \geq 1 \end{cases}$$

so that $\tilde{T}_n = \varepsilon_n^{-1/2} T_n$. Notice also that $\tilde{T}_n(\cos z) = \varepsilon_n^{-1/2} \cos(nz)$ for all $z \in \mathbb{C}$.

To prove (i) let g be the function inside the norm signs in (5.15). Then by Lemma 5.5.1 (i)

$$\|g\|_{E^2(D)}^2 = \frac{1}{2\pi} \int_{\gamma} |g(w)|^2 |dw|.$$

Now the function $\cos$ maps the line segment $\Gamma = [i\alpha, 2\pi+i\alpha)$ one to one, onto $\gamma = \partial D$.

Putting $w = \cos z$ we see that

$$\|g\|_{E^2(D)}^2 = \frac{1}{2\pi} \int_{\Gamma} |g(\cos z)|^2 |\sin z| |dz|.$$

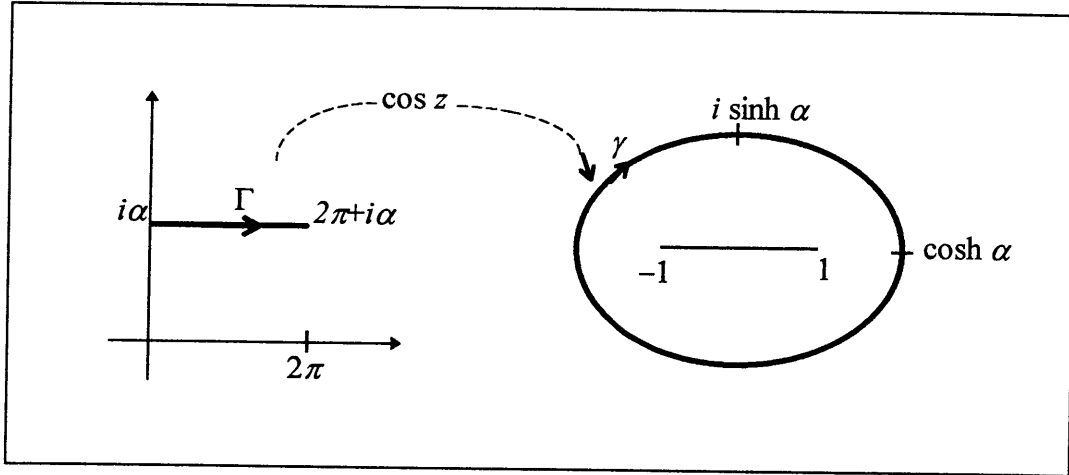


Fig. 5.3

Now let

$$A = \frac{1}{2\pi} \int_{\Gamma} |g(\cos z)|^2 |dz|. \quad (5.17)$$

On Γ we have $\sinh \alpha \leq |\sin z| \leq \cosh \alpha$. Therefore

$$(\sinh \alpha) A \leq \|g\|_{E^2(D)}^2 \leq (\cosh \alpha) A. \quad (5.18)$$

Substituting for g and parametrizing we have

$$\begin{aligned} 2\pi A &= \int_{\Gamma} g(\cos z) \overline{g(\cos z)} |dz| \\ &= \int_0^{2\pi} \sum_{k=-n}^n \sum_{l=-n}^n a_k \bar{a}_l e^{-(k+l)\alpha} \varepsilon_k^{-\frac{1}{2}} \varepsilon_l^{-\frac{1}{2}} \cos k(t+i\alpha) \cos l(t-i\alpha) dt. \end{aligned}$$

Now

$$\begin{aligned}
\int_0^{2\pi} \cos k(t+i\alpha) \cos l(t-i\alpha) dt &= \int_0^{2\pi} (\cos kt \cosh k\alpha - i \sin kt \sinh k\alpha) \\
&\quad \times (\cos lt \cosh l\alpha + i \sin lt \sinh l\alpha) dt \\
&= \delta_{kl} \int_0^{2\pi} (\cosh^2 k\alpha \cos^2 kt + \sinh^2 k\alpha \sin^2 kt) dt \\
&= 2\delta_{kl} \varepsilon_k (\cosh^2 k\alpha + \sinh^2 k\alpha) \\
&= 2\delta_{kl} \varepsilon_k \cosh 2k\alpha.
\end{aligned}$$

Therefore

$$A = \frac{1}{\pi} \sum_{k=m}^n |a_k|^2 e^{-2k\alpha} \cosh 2k\alpha.$$

So by (5.17) and (5.18) we can put

$$\begin{aligned}
\beta &= \frac{\sinh \alpha}{\pi} \inf_{k \geq 0} e^{-2k\alpha} \cosh 2k\alpha = \frac{\sinh \alpha}{2\pi}, \\
\delta &= \frac{\cosh \alpha}{\pi} \sup_{k \geq 0} e^{-2k\alpha} \cosh 2k\alpha = \frac{\cosh \alpha}{\pi}.
\end{aligned}$$

So (i) is proved.

(ii) The right-hand inequality in (5.15) shows that W_1 is a continuous operator mapping l^2 into $E^2(D)$. And left-hand inequality shows that W_1 is injective and has a closed range. Since polynomials are dense in $E^2(D)$ we see that W_1 is continuous and invertible.

To prove (iii) let $g \in E^2(D)$, By (ii) there is a unique $b = (b_n) \in l^2$ such that $g = W_1 b$. Hence g has a unique norm convergent expansion as in (5.16). To show

that this expansion is uniformly convergent on compact sets it is sufficient to show that it converges uniformly on any $\overline{D}_{\alpha'}$, where $0 < \alpha' < \alpha$. But it is well known that

$$\sup_{z \in \overline{D}_{\alpha'}} |T_n(z)| = O(e^{n\alpha'}).$$

So the theorem is now proved.

We now have a compact operator $R_D W_1: l^2 \rightarrow L^2(J)$, so that $T^{(1)} = R_D W_1 W_1^* R_D^*$ is a compact positive operator on $L^2(J)$.

5.5.3 THEOREM

(i) $T^{(1)}$ is the positive integral operator on $L^2(J)$ with kernel

$$t_1(s, t) = \sum_{n=0}^{\infty} e^{-2n\alpha} \tilde{T}_n(s) \tilde{T}_n(t). \quad (5.19)$$

(ii) $\lambda_n(T_D) \approx \lambda_n(T^{(1)}) \leq e^{-2n\alpha}$.

Proof. If $a = (a_n) \in l^2$ then

$$R_D W_1 a(s) = \sum_{n=0}^{\infty} a_n e^{-n\alpha} \tilde{T}_n(s).$$

This is an integral operator with an L^2 -kernel. Just as in the proof of Lemma 5.2.1 (ii)

we see that the kernel of $T^{(1)}$ is

$$\sum_{n=0}^{\infty} e^{-n\alpha} \tilde{T}_n(s) e^{-n\alpha} \tilde{T}_n(t).$$

To prove the first relation in (ii) we note that if $f \in L^2(J)$ then

$$\langle T_D f, f \rangle = \langle R_D R_D^* f, f \rangle = \|R_D^* f\|^2,$$

$$\langle T^{(1)} f, f \rangle = \langle R_D W_1 W_1^* R_D^* f, f \rangle = \|W_1^* R_D^* f\|^2.$$

Since W_1 is invertible and $\|W_1^*\| = \|W_1\|$

$$\frac{1}{\|W^{-1}\|^2} \|R_D^* f\|^2 \leq \|W_1^* R_D^* f\|^2 \leq \|W_1\|^2 \|R_D^* f\|^2.$$

Hence

$$\frac{1}{\|W^{-1}\|^2} T_D \leq T^{(1)} \leq \|W_1\|^2 T_D,$$

so that $\lambda_n(T_D) \approx \lambda_n(T^{(1)})$.

To prove the second relation let T_1' be the integral operator on $L^2(J)$ with kernel t_1' as in (5.12)

$$t_1'(s, t) = \sum_{n=0}^{\infty} e^{-2n\alpha} \frac{\tilde{T}_n(s)}{w(s)} \frac{\tilde{T}_n(t)}{w(t)},$$

where $w(s) = (1 - s^2)^{\frac{1}{4}}$. If M is the multiplication operator on $L^2(J)$: $Mf = wf$ then M

is symmetric and $\|M\| = 1$. Clearly $T^{(1)} = MT_1' M$ so that $\lambda_n(T^{(1)}) \leq \lambda_n(T_1') = e^{-2n\alpha}$,

by (1.4) and (5.13).

5.6 LOWER BOUNDS FOR $\lambda_n(T)$

We have now established that $\lambda_n(T) = \lambda_n(T_D) = O(e^{-2n\alpha})$. To find lower bounds we proceed analogously with Tchebyschef polynomials of second kind. We give sketch proofs, pointing out the essential differences.

5.6.1 THEOREM

(i) There exist $\beta', \delta', 0 < \beta' < \delta'$ depending only on α , such that for all complex sequences (a_r) and all $n \geq m \geq 0$

$$\beta' \sum_{r=m}^n |a_r|^2 \leq \left\| \sum_{r=m}^n a_r e^{-r\alpha} \tilde{U}_r \right\|_{E^2(D)}^2 \leq \delta' \sum_{r=m}^n |a_r|^2.$$

(ii) The formula

$$W_2 a(z) = \sum_{n=0}^{\infty} a_n e^{-n\alpha} \tilde{U}_n(z), \quad (a = (a_n) \in l^2, z \in D)$$

define a continuous invertible operator mapping l^2 onto $E^2(D)$.

(iii) Every $g \in E^2(D)$ has a unique Tchebyschef expansion

$$g = \sum_{n=0}^{\infty} b_n e^{-n\alpha} \tilde{U}_n,$$

where $(b_n) \in l^2$. The series converges in the norm of $E^2(D)$ and uniformly on compact subsets of D .

Proof. First of all (ii) and (iii) follow from (i) just as they did in Theorem 5.5.2. Note that if $0 < \alpha' < \alpha$ then we do have

$$\sup_{z \in \overline{D}_{\alpha'}} |U_n(z)| = O(e^{n\alpha'}).$$

To prove (i) let g be the function inside the norm signs. As before we have

$$\|g\|_{E^2(D)}^2 = \frac{1}{2\pi} \int_{\Gamma} |g(\cos z)|^2 |\sin z| |dz|.$$

This time we put

$$B = \frac{1}{2\pi} \int_{\Gamma} |g(\cos z)|^2 |\sin z|^2 |dz|$$

so that

$$(\operatorname{sech} \alpha) B \leq \|g\|_{E^2(D)}^2 \leq (\operatorname{csch} \alpha) B.$$

Now

$$B = \frac{1}{\pi^2} \int_{\Gamma} \sum_{k=0}^n \sum_{l=m}^n a_k \bar{a}_l e^{-(k+l)\alpha} \sin(k+1)z \sin(l+1)\bar{z} |dz|.$$

The proof is completed as before. The new constants are

$$\beta' = \frac{\operatorname{sech} \alpha}{\pi} \inf_{k \geq 0} e^{-2k\alpha} \cosh 2(k+1)\alpha = \frac{\operatorname{sech} \alpha}{2\pi} e^{2\alpha},$$

$$\delta' = \frac{\operatorname{csch} \alpha}{\pi} \sup_{k \geq 0} e^{-2k\alpha} \cosh 2(k+1)\alpha = \frac{\operatorname{csch} \alpha}{\pi} \cosh 2\alpha.$$

Having proved this theorem we now have a positive integral operator

$T^{(2)} = R_D W_2 W_2^* R_D^*$ on $L^2(J)$ with kernel

$$t_2(s, t) = \sum_{n=0}^{\infty} e^{-2n\alpha} \tilde{U}_n(s) \tilde{U}_n(t) \quad (s, t \in J). \quad (5.20)$$

Its eigenvalues satisfy $\lambda_n(T^{(2)}) \approx \lambda_n(T_D)$.

If T'_2 is the integral operator on $L^2(J)$ with kernel t'_2 as in (5.12)

$$t'_2(s, t) = \sum_{n=0}^{\infty} e^{-2n\alpha} \tilde{U}_n(s) w(s) \tilde{U}_n(t) w(t)$$

then by (5.13), $\lambda_n(T'_2) = e^{-2n\alpha}$. This time we have $T'_2 = MT^{(2)}M$, so that

$$e^{-2n\alpha} = \lambda_n(T'_2) \leq \lambda_n(T^{(2)}).$$

Combining this with our previous result we have, finally,

$$\lambda_n(T) = \lambda_n(T_D) \approx e^{-2n\alpha}. \quad (5.21)$$

5.6.2 Two Further Examples

In proving our main result (5.21) we have also shown that $\lambda_n(T^{(1)}) \approx \lambda_n(T^{(2)}) \approx e^{-2n\alpha}$. It is not difficult to compute their kernels by summing the series in (5.19) and (5.20). Thus

$$t_1(s, t) = \frac{\sinh 2\alpha}{\pi} \cdot \frac{\cosh 2\alpha - st}{\sinh^2 2\alpha + s^2 + t^2 - (2 \cosh 2\alpha)st},$$

$$t_2(s, t) = \frac{\sinh 2\alpha}{\pi} \cdot \frac{e^{2\alpha}}{\sinh^2 2\alpha + s^2 + t^2 - (2 \cosh 2\alpha)st}.$$

So both of these kernels are positive definite and $\lambda_n(t_1, J) \approx \lambda_n(t_2, J) \approx e^{-2n\alpha}$.

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