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STABILITY OF SOME OPERATOR IDEALS

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PREFACE

An operator ideal O is said to be tensor stable with respect to a tensor norm $\|\cdot\|_\alpha$ if for all $T_1 \in O(X_1, Y_1)$ and $T_2 \in O(X_2, Y_2)$,

$$T_1 \overset{\alpha}{\otimes} T_2 \in O(X_1 \overset{\alpha}{\otimes} X_2, Y_1 \overset{\alpha}{\otimes} Y_2).$$

The earliest examples of tensor stable ideals with respect to $\|\cdot\|_\varepsilon$ are those of compact operators and absolutely p -summing operators. These results are due to Vala [14] and Holub [5], respectively. In 1988, Carl, Defant and Ramanujan, [2], identified ε - or π -stability of some operator ideals including those discussed in our thesis.

The primary goal of this thesis is to give simple proofs for ε -stability of the ideals of p -summing operators, p -integral operators and p -nuclear operators using factorization techniques. The thesis also provides a proof for the ε -stability of the injective hull of an ε -stable operator ideal as well as for the π -stability of the ideal of compact operators.

The thesis consists of three chapters. In Chapter I, basic definitions and results about tensor product spaces are given. In particular, it is shown that $C(I) \overset{\varepsilon}{\otimes} X = C(I, X)$ and $L^1(I) \overset{\pi}{\otimes} X = L^1(I, X)$. In these theorems the unit interval equipped with the Lebesgue measure can be replaced with any compact Hausdorff space and any measure space, respectively.

In Chapter II, the most important examples of operator ideals are considered. In fact, factorization theorems about the p -summing operators, the p -nuclear operators and the p -integral operators are presented in sections 2, 3, and 4.

Finally, Chapter III presents our work on stability of operator ideals. These results are not new but the methods for the proof are new and simpler than those given by Carl, Defant and Ramanujan, [2].

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CHAPTER I

TENSOR PRODUCT SPACES

§1. Tensor Product – The Definition.

This chapter introduces some of the basic material on the tensor product of two Banach spaces.

Throughout, X and Y will denote real Banach spaces.

Definition 1.1.1. *Let $x \in X$ and $y \in Y$. Define*

$$\begin{aligned} x \otimes y: X^* &\longrightarrow Y \\ x^* &\longmapsto \langle x^*, x \rangle y. \end{aligned}$$

Then $x \otimes y$ is a bounded linear operator from X^* into Y . Define

$$X \otimes Y = \mathbf{Span} \{ x \otimes y : x \in X, y \in Y \} \quad \text{in } L(X^*, Y).$$

Then every $z \in X \otimes Y$ has a representation

$$z = \sum_{i=1}^n x_i \otimes y_i$$

for some $n \in \mathbb{N}$, $\{x_1, \dots, x_n\} \subseteq X$ and $\{y_1, \dots, y_n\} \subseteq Y$. $X \otimes Y$ is called the tensor product of X and Y .

Basic Properties 1.1.2.

1. $X \otimes Y$ is a subspace of $L(X^*, Y)$.
Every element of $X \otimes Y$ is a continuous linear operator of finite rank from X^* into Y . (Refer to 2.1.4.)
2. $\lambda(x \otimes y) = \lambda x \otimes y = x \otimes \lambda y$ for all $\lambda \in \mathbb{R}$, $x \in X$, $y \in Y$.
3. $x \otimes 0 = 0 \otimes y = 0 \otimes 0$.
4. Every $z \in X \otimes Y$ can have either the representation

$$z = 0 \otimes 0 \quad \text{or} \quad z = \sum_{i=1}^n x_i \otimes y_i$$

where $n \in \mathbb{N}$, $\{x_1, \dots, x_n\} \subseteq X$ and $\{y_1, \dots, y_n\} \subseteq Y$ are linearly independent sets.

5. Every $z \in X \otimes Y$ can be represented as

$$z = \sum_{i=1}^n \lambda_i \hat{x}_i \otimes \hat{y}_i$$

for some $n \in \mathbb{N}$, where $\lambda_i \in \mathbb{R}$, $\hat{x}_i \in X$, $\hat{y}_i \in Y$, $\|\hat{x}_i\| = \|\hat{y}_i\| = 1$ for all $i = 1, \dots, n$.

The proof of such properties is elementary and will be omitted.

Definition 1.1.3. A norm $\|\cdot\|$ is called a crossnorm on $X \otimes Y$ if

$$\|x \otimes y\| = \|x\| \|y\| \quad \text{for all } x \in X, y \in Y.$$

Definition 1.1.4. Let $x^* \in X^*$, $y^* \in Y^*$. Define

$$(x^* \otimes y^*)(x \otimes y) = \langle x^*, x \rangle \langle y^*, y \rangle.$$

REMARK 1.1.5. $x^* \otimes y^*$ is a well-defined linear functional on $X \otimes Y$.

Definition 1.1.6. A norm $\|\cdot\|$ on $X \otimes Y$ is called a reasonable norm if, for all $x^* \in X^*$ and $y^* \in Y^*$,

$$x^* \otimes y^* \in (X \otimes Y, \|\cdot\|)^* \quad \text{and} \quad \|x^* \otimes y^*\|_d = \|x^*\| \|y^*\|$$

where $\|\cdot\|_d$ is the operator-norm defined on the dual of $(X \otimes Y, \|\cdot\|)$.

REMARK 1.1.7.

1. If a norm $\|\cdot\|$ is a reasonable norm on $X \otimes Y$, then $X^* \otimes Y^*$ is a subspace of $(X \otimes Y, \|\cdot\|)^*$.
2. A norm $\|\cdot\|$ is a reasonable crossnorm on $X \otimes Y$ if and only if

$$\|x \otimes y\| = \|x\| \|y\| \quad \text{for all } x \in X, y \in Y$$

and

$$\|x^* \otimes y^*\|_d = \|x^*\| \|y^*\| \quad \text{for all } x^* \in X^*, y^* \in Y^*.$$

Theorem 1.1.8. *A norm $\|\cdot\|$ on $X \otimes Y$ is a reasonable crossnorm if and only if*

$$\|x \otimes y\| \leq \|x\| \|y\| \quad \text{for all } x \in X, y \in Y \quad (1)$$

and

$$\|x^* \otimes y^*\|_d \leq \|x^*\| \|y^*\| \quad \text{for all } x^* \in X^*, y^* \in Y^*. \quad (2)$$

Proof. Obviously, if a $\|\cdot\|$ is a reasonable crossnorm on $X \otimes Y$ then both inequalities are immediate. For the converse, we need to show that

$$\|x \otimes y\| \geq \|x\| \|y\| \quad \text{for all } x \in X, y \in Y$$

and

$$\|x^* \otimes y^*\|_d \geq \|x^*\| \|y^*\| \quad \text{for all } x^* \in X^*, y^* \in Y^*.$$

Then Definitions 1.1.3 and 1.1.6 ensure that $\|\cdot\|$ is a reasonable crossnorm on $X \otimes Y$.

Now, let $x \in X, y \in Y, x^* \in X^*$ and $y^* \in Y^*$. Then

$$|\langle x^*, x \rangle \langle y^*, y \rangle| = |(x^* \otimes y^*)(x \otimes y)| \leq \|x^* \otimes y^*\|_d \|x \otimes y\|.$$

Using inequality (1) we obtain

$$|\langle x^*, x \rangle| |\langle y^*, y \rangle| \leq \|x^* \otimes y^*\|_d \|x\| \|y\|.$$

If we take a supremum over all $x \in B_1(X)$ and $y \in B_1(Y)$, we get

$$\|x^*\| \|y^*\| \leq \|x^* \otimes y^*\|_d.$$

Hence

$$\|x^* \otimes y^*\|_d = \|x^*\| \|y^*\|. \quad (*)$$

On the other hand, if we use inequality (2) we get

$$|\langle x^*, x \rangle| |\langle y^*, y \rangle| \leq \|x^*\| \|y^*\| \|x \otimes y\|.$$

Taking the supremum over all $x^* \in B_1(X^*)$ and $y^* \in B_1(Y^*)$, we get

$$\|x\| \|y\| \leq \|x \otimes y\|.$$

So

$$\|x \otimes y\| = \|x\| \|y\|. \quad (**)$$

Then it follows from $(*)$, $(**)$, and Definitions 1.1.3, 1.1.6 that $\| \cdot \|$ is a reasonable crossnorm on $X \otimes Y$. $\square$

Theorem 1.1.9. *If a norm $\| \cdot \|$ is a reasonable crossnorm on $X \otimes Y$, then its dual norm $\| \cdot \|_d$ is a reasonable crossnorm on $X^* \otimes Y^*$.*

Proof. We claim that

$$\|x^* \otimes y^*\|_d \leq \|x^*\| \|y^*\| \quad \text{for all } x^* \in X^*, y^* \in Y^* \quad (1)$$

and

$$\|x^{**} \otimes y^{**}\|_{dd} \leq \|x^{**}\| \|y^{**}\| \quad \text{for all } x^{**} \in X^{**}, y^{**} \in Y^{**}. \quad (2)$$

Indeed, the first inequality follows from Theorem 1.1.8. To prove the second inequality, let $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$. Then, by Goldstine's Theorem [3, p.424], X and Y are weak* dense in X^{**} and Y^{**} respectively. Hence there exist nets

$$(x_t)_t \subset X \quad \text{and} \quad (y_t)_t \subset Y$$

such that

$$x_t \xrightarrow{w^*} x^{**} \quad \text{and} \quad y_t \xrightarrow{w^*} y^{**}.$$

Further,

$$\|x_t\| \leq \|x^{**}\|, \quad \|y_t\| \leq \|y^{**}\| \quad \text{for all } t.$$

Let

$$f = \sum_{i=1}^n x_i^* \otimes y_i^* \in X^* \otimes Y^*.$$

Then

$$\begin{aligned} |(x^{**} \otimes y^{**})(f)| &= \left| \sum_{i=1}^n \langle x^{**}, x_i^* \rangle \langle y^{**}, y_i^* \rangle \right| \\ &= \left| \lim_t \sum_{i=1}^n \langle x_t, x_i^* \rangle \langle y_t, y_i^* \rangle \right| \\ &= \left| \lim_t \sum_{i=1}^n \langle x_i^*, x_t \rangle \langle y_i^*, y_t \rangle \right| \\ &= \left| \lim_t \left(\sum_{i=1}^n x_i^* \otimes y_i^* \right) (x_t \otimes y_t) \right| \\ &= \lim_t |f(x_t \otimes y_t)|. \end{aligned}$$

So

$$\begin{aligned} |(x^{**} \otimes y^{**})(f)| &\leq \varliminf_t \|f\|_d \|x_t \otimes y_t\| \\ &= \varliminf_t \|f\|_d \|x_t\| \|y_t\| \\ &\leq \varliminf_t \|f\|_d \|x^{**}\| \|y^{**}\| \\ &= \|x^{**}\| \|y^{**}\| \|f\|_d. \end{aligned}$$

Hence

$$\|x^{**} \otimes y^{**}\|_{dd} \leq \|x^{**}\| \|y^{**}\|. \quad \square$$

§2. The Injective Tensor Product.

One can put different norms on $X \otimes Y$. In this section we will study the first of three interesting norms that we need in our thesis.

Definition 1.2.1. For $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$, define

$$\|z\|_\varepsilon = \sup_{\substack{x^* \in X^*, \|x^*\| \leq 1 \\ y^* \in Y^*, \|y^*\| \leq 1}} \left| \sum_{i=1}^n \langle x^*, x_i \rangle \langle y^*, y_i \rangle \right|.$$

This is nothing but the operator norm on $X \otimes Y$ as a subspace of $L(X^*, Y)$.

Theorem 1.2.2. $\|\cdot\|_\varepsilon$ is a reasonable crossnorm on $X \otimes Y$. Further, if $\|\cdot\|_\alpha$ is any reasonable crossnorm on $X \otimes Y$, then

$$\|z\|_\varepsilon \leq \|z\|_\alpha \quad \text{for all } z \in X \otimes Y.$$

Proof. Let $x \in X$ and $y \in Y$. Then

$$\begin{aligned} \|x \otimes y\|_\varepsilon &= \sup_{\substack{x^* \in B_1(X^*) \\ y^* \in B_1(Y^*)}} |\langle x^*, x \rangle \langle y^*, y \rangle| \\ &= \sup_{x^* \in B_1(X^*)} |\langle x^*, x \rangle| \sup_{y^* \in B_1(Y^*)} |\langle y^*, y \rangle| \\ &= \|x\| \|y\|. \end{aligned}$$

Now, if $x^* \in X^*$, $y^* \in Y^*$, $x^* \neq 0$, $y^* \neq 0$, and $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$, then

$$\begin{aligned} |(x^* \otimes y^*)(z)| &= \left| \sum_{i=1}^n \langle x^*, x_i \rangle \langle y^*, y_i \rangle \right| \\ &= \|x^*\| \|y^*\| \left| \sum_{i=1}^n \left\langle \frac{x^*}{\|x^*\|}, x_i \right\rangle \left\langle \frac{y^*}{\|y^*\|}, y_i \right\rangle \right| \\ &\leq \|x^*\| \|y^*\| \|z\|_\varepsilon. \end{aligned}$$

Therefore $x^* \otimes y^* \in (X \otimes Y, \|\cdot\|_\varepsilon)^*$ and $\|x^* \otimes y^*\|_d \leq \|x^*\| \|y^*\|$. By Theorem 1.1.8 $\|\cdot\|_\varepsilon$ is a reasonable crossnorm on $X \otimes Y$.

To show that $\|\cdot\|_\varepsilon$ is the least reasonable crossnorm on $X \otimes Y$, let $\|\cdot\|_\alpha$ be any other reasonable crossnorm on $X \otimes Y$. Then for any $x^* \in X^*$, $y^* \in Y^*$ and $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$, we have

$$\left| \sum_{i=1}^n \langle x^*, x_i \rangle \langle y^*, y_i \rangle \right| = |(x^* \otimes y^*)(z)| \leq \|x^* \otimes y^*\|_d \|z\|_\alpha \leq \|x^*\| \|y^*\| \|z\|_\alpha.$$

Taking the supremum over all $x^* \in B_1(X^*)$ and $y^* \in B_1(Y^*)$, we get

$$\|z\|_\varepsilon \leq \|z\|_\alpha. \quad \square$$

The normed space $(X \otimes Y, \|\cdot\|_\varepsilon)$ need not be complete. Let $X \overset{\varepsilon}{\otimes} Y$ denote its completion with respect to $\|\cdot\|_\varepsilon$. $(X \overset{\varepsilon}{\otimes} Y, \|\cdot\|_\varepsilon)$ is called the completed injective tensor product of X and Y .

Theorem 1.2.3. *If G is a closed subspace of X , then $G \overset{\varepsilon}{\otimes} Y$ is a closed subspace of $X \overset{\varepsilon}{\otimes} Y$, for any Banach space Y .*

Proof. It is enough to show that the inclusion map

$$I: G \overset{\varepsilon}{\otimes} Y \longrightarrow X \overset{\varepsilon}{\otimes} Y$$

is an isometry.

Let

$$\|\cdot\|_1 \quad \text{denote the} \quad \|\cdot\|_\varepsilon \quad \text{on} \quad G \overset{\varepsilon}{\otimes} Y$$

and

$$\|\cdot\|_2 \quad \text{denote the} \quad \|\cdot\|_\varepsilon \quad \text{on} \quad X \overset{\varepsilon}{\otimes} Y.$$

Let $z = \sum_{i=1}^n g_i \otimes y_i \in G \overset{\varepsilon}{\otimes} Y$, then

$$\begin{aligned}
\|z\|_1 &= \sup_{\substack{g^* \in B_1(G^*) \\ y^* \in B_1(Y^*)}} \left| \sum_{i=1}^n \langle g_i, g^* \rangle \langle y_i, y^* \rangle \right| \\
&= \sup_{\substack{g^* \in B_1(G^*) \\ y^* \in B_1(Y^*)}} \left| \left\langle \sum_{i=1}^n \langle y_i, y^* \rangle g_i, g^* \right\rangle \right| \\
&= \sup_{y^* \in B_1(Y^*)} \left\| \sum_{i=1}^n \langle y_i, y^* \rangle g_i \right\|_G \\
&= \sup_{y^* \in B_1(Y^*)} \left\| \sum_{i=1}^n \langle y_i, y^* \rangle g_i \right\|_X \\
&= \sup_{\substack{x^* \in B_1(X^*) \\ y^* \in B_1(Y^*)}} \left| \sum_{i=1}^n \langle g_i, x^* \rangle \langle y_i, y^* \rangle \right| \\
&= \|z\|_2. \quad \square
\end{aligned}$$

Let $I = [0, 1]$ and $C(I, X)$ be the space of real-valued continuous functions from I into X . On $C(I, X)$, there is the natural norm

$$\|f\|_\infty = \sup_{t \in I} \|f(t)\|.$$

$(C(I, X), \|\cdot\|_\infty)$ is a Banach space, [10, p.9].

Theorem 1.2.4. $C(I) \overset{\varepsilon}{\otimes} X = C(I, X)$.

Proof. On the dense subspace $C(I) \otimes X$ of $C(I) \overset{\varepsilon}{\otimes} X$, we define

$$J: C(I) \otimes X \longrightarrow C(I, X)$$

by

$$J\left(\sum_{i=1}^n f_i \otimes x_i\right) = g$$

where

$$g(t) = \sum_{i=1}^n f_i(t)x_i.$$

Then,

$$\begin{aligned}
\left\| J\left(\sum_{i=1}^n f_i \otimes x_i\right) \right\|_{\infty} &= \sup_t \left\| \sum_{i=1}^n f_i(t)x_i \right\| \\
&= \sup_t \sup_{x^* \in B_1(X^*)} \left| \sum_{i=1}^n f_i(t) \langle x_i, x^* \rangle \right| \\
&= \sup_{x^* \in B_1(X^*)} \sup_t \left| \left(\sum_{i=1}^n \langle x_i, x^* \rangle f_i \right)(t) \right| \\
&= \sup_{x^* \in B_1(X^*)} \left\| \sum_{i=1}^n \langle x_i, x^* \rangle f_i \right\| \\
&= \sup_{x^* \in B_1(X^*)} \sup_{\mu \in B_1(M(I))} \left| \sum_{i=1}^n \langle x_i, x^* \rangle \langle f_i, \mu \rangle \right| \\
&= \left\| \sum_{i=1}^n f_i \otimes x_i \right\|_{\epsilon}.
\end{aligned}$$

Thus J is an isometric isomorphism of $C(I) \otimes X$ onto $J(C(I) \otimes X)$.

It remains to show that $J(C(I) \otimes X)$ is a dense subspace of $C(I, X)$. Let $f \in C(I, X)$ and $\epsilon > 0$. Since f is continuous and I is compact, then $f(I)$ is compact and hence totally bounded. Therefore there exist points $f(t_1), \dots, f(t_n)$ in $f(I)$ such that the open balls $B(f(t_1), \epsilon), \dots, B(f(t_n), \epsilon)$ cover $f(I)$.

Let

$$U_i = f^{-1}(B(f(t_i), \epsilon)), \quad \text{for all } i = 1, \dots, n.$$

Then $(U_i)_{i=1}^n$ is an open finite cover of I . Since I is paracompact, there exists a partition of unity subordinated to $(U_i)_{i=1}^n$. That is, there exist functions $f_1, \dots, f_n$ in $C(I)$ such that

$$\begin{aligned}
\text{Supp}(f_i) &\subseteq U_i & \text{for all } i = 1, \dots, n, \\
0 \leq f_i(t) &\leq 1 & \text{for all } t \in I
\end{aligned}$$

and

$$\sum_{i=1}^n f_i(t) = 1 \quad \text{for all } t \in I.$$

Now, define

$$\hat{f} = \sum_{i=1}^n f(t_i) \cdot f_i.$$

Then $\hat{f} \in J(C(I) \otimes X)$. Moreover, for any $t \in I$, we have

$$\begin{aligned} \|\hat{f}(t) - f(t)\| &= \left\| \sum_{i=1}^n f(t_i) f_i(t) - f(t) \right\| \\ &= \left\| \sum_{i=1}^n f(t_i) f_i(t) - \sum_{i=1}^n f(t) f_i(t) \right\| \\ &= \left\| \sum_{i=1}^n f_i(t) (f(t_i) - f(t)) \right\| \\ &\leq \sum_{i=1}^n f_i(t) \|f(t_i) - f(t)\| \leq \epsilon \end{aligned}$$

because $f_i(t) = 0$ if $\|f(t_i) - f(t)\| > \epsilon$. Taking the supremum over all $t \in I$, we get $\|f - \hat{f}\|_\infty \leq \epsilon$. $\square$

Corollary 1.2.5. *Let S and T be compact Hausdorff spaces. Then*

$$C(S) \overset{\varepsilon}{\otimes} C(T) = C(S \times T).$$

Proof. By Theorem 1.2.4, we have

$$C(S) \overset{\varepsilon}{\otimes} C(T) = C(S, C(T)).$$

Therefore, we need only to show that the mapping

$$J: C(S \times T) \longrightarrow C(S, C(T))$$

defined by

$$J(f) = \hat{f}$$

where

$$\hat{f}(s) = f_s \quad \text{and} \quad f_s(t) = f(s, t)$$

is an isometric isomorphism of $C(S \times T)$ onto $C(S, C(T))$.

Let $f \in C(S \times T)$, then

$$\begin{aligned}
\|J(f)\|_\infty &= \|\hat{f}\|_\infty \\
&= \sup_{s \in S} \|\hat{f}(s)\| \\
&= \sup_{s \in S} \|f_s\|_\infty \\
&= \sup_{s \in S} \sup_{t \in T} |f_s(t)| \\
&= \sup_{\substack{s \in S \\ t \in T}} |f(s, t)| \\
&= \|f\|_\infty.
\end{aligned}$$

J is clearly onto, since given $\hat{f} \in C(S, C(T))$ there exists $f \in C(S \times T)$ such that

$$f(s, t) = (\hat{f}(s))(t). \quad \square$$

§3. The Projective Tensor Product.

In this section we study the second interesting norm on $X \otimes Y$.

Definition 1.3.1. For $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$, define

$$\|z\|_\pi = \inf \sum_{i=1}^n \|x_i\| \|y_i\|$$

where the infimum is taken over all representations of z in $X \otimes Y$.

Theorem 1.3.2. $\|\cdot\|_\pi$ is a reasonable crossnorm on $X \otimes Y$. Further, if $\|\cdot\|_\alpha$ is any reasonable crossnorm on $X \otimes Y$, then

$$\|z\|_\alpha \leq \|z\|_\pi \quad \text{for all } z \in X \otimes Y.$$

Proof. The first step is to show that $\|\cdot\|_\pi$ is indeed a norm on $X \otimes Y$. Let $0 \neq z \in X \otimes Y$, then our claim is that $\|z\|_\pi > 0$. But $z: X^* \rightarrow Y$ is not the zero

operator means there exists $x^* \in X^*$ such that $\|x^*\| = 1$ and $z(x^*) \neq 0$. It follows that

$$\begin{aligned} 0 < \|z(x^*)\| &= \left\| \left(\sum_{i=1}^n x_i \otimes y_i \right) (x^*) \right\| \\ &= \left\| \sum_{i=1}^n \langle x^*, x_i \rangle y_i \right\| \\ &\leq \sum_{i=1}^n |\langle x^*, x_i \rangle| \|y_i\| \\ &\leq \sum_{i=1}^n \|x_i\| \|y_i\|. \end{aligned}$$

Since the last inequality is true for all representations of $z \in X \otimes Y$, we get $\|z\|_\pi > 0$. The other conditions for $\|\cdot\|_\pi$ to be a norm are easily verified and their proofs will be omitted.

Next, we show that $\|\cdot\|_\pi$ is a crossnorm on $X \otimes Y$. Let $0 \neq x \in X$ and $0 \neq y \in Y$, and assume that

$$x \otimes y = \sum_{i=1}^n x_i \otimes y_i.$$

Then, since $0 \neq x \in X$, there exists $x^* \in X^*$ such that $\|x^*\| = 1$ and $\langle x^*, x \rangle = \|x\|$. Applying $x \otimes y$ to x^* , we get

$$\langle x^*, x \rangle y = \sum_{i=1}^n \langle x^*, x_i \rangle y_i$$

thus

$$\|x\| y = \sum_{i=1}^n \langle x^*, x_i \rangle y_i$$

and hence

$$\begin{aligned} \|x\| \|y\| &= \left\| \sum_{i=1}^n \langle x^*, x_i \rangle y_i \right\| \\ &\leq \sum_{i=1}^n \|x_i\| \|y_i\|. \end{aligned}$$

Therefore,

$$\|x\| \|y\| \leq \left\| \sum_{i=1}^n x_i \otimes y_i \right\|_{\pi} \leq \|x\| \|y\|.$$

So we obtain

$$\|x \otimes y\|_{\pi} = \|x\| \|y\|.$$

To show that $\|\cdot\|_{\pi}$ is a reasonable norm, let $x^* \in X^*$ and $y^* \in Y^*$. Then for any $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ we have

$$\begin{aligned} |(x^* \otimes y^*)(z)| &= \left| \sum_{i=1}^n \langle x^*, x_i \rangle \langle y^*, y_i \rangle \right| \\ &\leq \|x^*\| \|y^*\| \sum_{i=1}^n \|x_i\| \|y_i\|. \end{aligned}$$

Since the last inequality is true for all representations of z in $X \otimes Y$, we get

$$|(x^* \otimes y^*)(z)| \leq \|x^*\| \|y^*\| \|z\|_{\pi}.$$

Thus

$$\|x^* \otimes y^*\|_d \leq \|x^*\| \|y^*\|$$

and hence, by Theorem 1.1.8, $\|\cdot\|_{\pi}$ is a reasonable crossnorm on $X \otimes Y$.

Now we show that $\|\cdot\|_{\pi}$ is the greatest reasonable crossnorm on $X \otimes Y$. Let $\|\cdot\|_{\alpha}$ be any reasonable crossnorm on $X \otimes Y$ and $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$. Then

$$\begin{aligned} \|z\|_{\alpha} &= \left\| \sum_{i=1}^n x_i \otimes y_i \right\|_{\alpha} \\ &\leq \sum_{i=1}^n \|x_i \otimes y_i\|_{\alpha} \\ &= \sum_{i=1}^n \|x_i\| \|y_i\|. \end{aligned}$$

Taking infimum over all representations of z in $X \otimes Y$, we get

$$\|z\|_{\alpha} \leq \|z\|_{\pi}. \quad \square$$

The normed linear space $(X \otimes Y, \|\cdot\|_\pi)$ need not be complete. Its completion with respect to $\|\cdot\|_\pi$ is denoted by $X \overset{\pi}{\otimes} Y$. $(X \overset{\pi}{\otimes} Y, \|\cdot\|_\pi)$ is called the completed projective tensor product of X and Y .

Theorem 1.3.3. *Let $z \in X \overset{\pi}{\otimes} Y$. Then*

$\forall \epsilon > 0, \exists (\lambda_n) \in c_0, (x_n) \subset X, (y_n) \subset Y$, with $\|x_n\| = \|y_n\| = 1$ for all $n = 1, 2, \dots$

such that

$$z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n$$

and

$$\|z\|_\pi \leq \sum_{n=1}^{\infty} \lambda_n \leq \|z\|_\pi + \epsilon.$$

Proof. Since $X \otimes Y$ is dense in $X \overset{\pi}{\otimes} Y$, there exists a sequence $(z_n) \subset X \otimes Y$ such that $\|z_n - z\|_\pi \rightarrow 0$. Let $\epsilon > 0$. We choose (z_n) such that

$$\|z_n - z\|_\pi < \frac{\epsilon}{2^{n+2}} \quad \text{for } n = 1, 2, \dots$$

From the definition of $\|\cdot\|_\pi$, there exists a representation for z_1 in $X \otimes Y$, say

$$z_1 = \sum_{i=1}^{k(1)} \hat{x}_i \otimes \hat{y}_i \quad \text{such that} \quad \sum_{i=1}^{k(1)} \|\hat{x}_i\| \|\hat{y}_i\| < \|z_1\|_\pi + \frac{\epsilon}{4} < \|z\|_\pi + \frac{\epsilon}{2}.$$

For each n , $z_{n+1} - z_n$ in $X \otimes Y$ satisfies

$$\begin{aligned} \|z_{n+1} - z_n\|_\pi &\leq \|z_{n+1} - z\|_\pi + \|z - z_n\|_\pi \\ &< \frac{\epsilon}{2^{n+3}} + \frac{\epsilon}{2^{n+2}} \\ &< \frac{\epsilon}{2^{n+1}}. \end{aligned}$$

Therefore, there exists a representation for $z_{n+1} - z_n$ in $X \otimes Y$, say

$$z_{n+1} - z_n = \sum_{i=k(n)+1}^{k(n+1)} \hat{x}_i \otimes \hat{y}_i \quad \text{such that} \quad \sum_{i=k(n)+1}^{k(n+1)} \|\hat{x}_i\| \|\hat{y}_i\| < \frac{\epsilon}{2^{n+1}}.$$

Now we claim that the series

$$z_1 + \sum_{n=1}^{\infty} (z_{n+1} - z_n)$$

converges to z in $X \overset{\pi}{\otimes} Y$. The series $z_1 + \sum_{n=1}^{\infty} (z_{n+1} - z_n)$ is absolutely convergent since

$$\begin{aligned} \|z_1\|_{\pi} + \sum_{n=1}^{\infty} \|z_{n+1} - z_n\|_{\pi} &= \left\| \sum_{i=1}^{k(1)} \hat{x}_i \otimes \hat{y}_i \right\|_{\pi} + \sum_{n=1}^{\infty} \left\| \sum_{i=k(n)+1}^{k(n+1)} \hat{x}_i \otimes \hat{y}_i \right\|_{\pi} \\ &\leq \sum_{i=1}^{k(1)} \|\hat{x}_i \otimes \hat{y}_i\|_{\pi} + \sum_{n=1}^{\infty} \sum_{i=k(n)+1}^{k(n+1)} \|\hat{x}_i \otimes \hat{y}_i\|_{\pi} \\ &= \sum_{i=1}^{k(1)} \|\hat{x}_i\| \|\hat{y}_i\| + \sum_{n=1}^{\infty} \sum_{i=k(n)+1}^{k(n+1)} \|\hat{x}_i\| \|\hat{y}_i\| \\ &< \|z\|_{\pi} + \frac{\epsilon}{2} + \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} \\ &= \|z\|_{\pi} + \epsilon < \infty. \end{aligned}$$

Therefore the series $z_1 + \sum_{n=1}^{\infty} (z_{n+1} - z_n)$ is convergent by the completeness of $X \overset{\pi}{\otimes} Y$. Its limit is z since the sequence of partial sums

$$S_1 = z_1, S_2 = z_2, \dots, S_n = z_n, \dots$$

satisfies

$$\|S_n - z\|_{\pi} = \|z_n - z\|_{\pi} \longrightarrow 0 \quad \text{in } X \overset{\pi}{\otimes} Y.$$

Thus

$$\begin{aligned} z &= z_1 + \sum_{n=1}^{\infty} (z_{n+1} - z_n) \\ &= \sum_{i=1}^{k(1)} \hat{x}_i \otimes \hat{y}_i + \sum_{n=1}^{\infty} \sum_{i=k(n)+1}^{k(n+1)} \hat{x}_i \otimes \hat{y}_i \\ &= \sum_{i=1}^{\infty} \hat{x}_i \otimes \hat{y}_i \\ &= \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i \end{aligned}$$

where

$$\lambda_i = \|\hat{x}_i\| \|\hat{y}_i\|, \quad \|x_i\| = \|y_i\| = 1 \quad \text{for all } i=1,2, \dots$$

and

$$\sum_{i=1}^{\infty} \lambda_i = \sum_{i=1}^{\infty} \|\hat{x}_i\| \|\hat{y}_i\| < \|z\|_{\pi} + \epsilon.$$

Finally, since $\sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i$ is a representation of z in $X \otimes Y$, it follows that

$$\|z\|_{\pi} \leq \sum_{i=1}^{\infty} \lambda_i < \|z\|_{\pi} + \epsilon. \quad \square$$

Let $I = [0, 1]$ and for $1 \leq p < \infty$ let $L^p(I, X)$ be the Banach space of all (equivalence classes of) p -Bochner integrable functions from I into X . If $X = \mathbb{R}$, we simply write $L^p(I)$. The norm in $L^p(I, X)$ is defined by

$$\|f\|_p = \left(\int_I \|f(t)\|^p d\mu(t) \right)^{\frac{1}{p}}.$$

Let $L^{\infty}(I, X)$ be the Banach space of all (equivalence classes of) essentially bounded strongly measurable functions on I with norm

$$\|f\|_{\infty} = \operatorname{ess\,sup}_t \|f(t)\|.$$

We refer to [10, pp.113,121] for the basic properties of such spaces.

If E is any set, we define the characteristic function χ_E of the set E to be the function given by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

A function $s: I \longrightarrow X$ is called simple if

$$s = \sum_{i=1}^n x_i \chi_{E_i}$$

where $E_1, \dots, E_n$ are disjoint sets of I of finite measure and $x_1, \dots, x_n \in X$. Let $S(I, X)$ denote the set of all simple functions from I into X . Then $S(I, X)$ is a subspace of $L^p(I, X)$.

Theorem 1.3.4. $L^1(I) \overset{\pi}{\otimes} X = L^1(I, X)$.

Proof. Define

$$J: L^1(I) \overset{\pi}{\otimes} X \longrightarrow L^1(I, X)$$

by

$$J\left(\sum_{i=1}^n f_i \otimes x_i\right) = g$$

where

$$g(t) = \sum_{i=1}^n f_i(t)x_i.$$

Clearly J is linear. Further, J is bounded. To prove the boundedness of J , we have

$$\begin{aligned} \left\| J\left(\sum_{i=1}^n f_i \otimes x_i\right) \right\|_{L^1(I, X)} &= \int_I \left\| J\left(\sum_{i=1}^n f_i \otimes x_i\right)(t) \right\| d\mu(t) \\ &= \int_I \left\| \sum_{i=1}^n f_i(t)x_i \right\| d\mu(t) \\ &\leq \int_I \sum_{i=1}^n |f_i(t)| \|x_i\| d\mu(t) \\ &= \sum_{i=1}^n \int_I |f_i(t)| \|x_i\| d\mu(t) \\ &= \sum_{i=1}^n \|f_i\| \|x_i\|. \end{aligned}$$

Taking the infimum over all representations of $\sum_{i=1}^n f_i \otimes x_i$ in $L^1(I) \overset{\pi}{\otimes} X$, we get

$$\left\| J\left(\sum_{i=1}^n f_i \otimes x_i\right) \right\|_{L^1(I, X)} \leq \left\| \sum_{i=1}^n f_i \otimes x_i \right\|_{L^1(I) \overset{\pi}{\otimes} X}.$$

Therefore, J is a bounded linear operator and $\|J\| \leq 1$.

Let $S(I) \otimes X$ denote the subspace of $L^1(I) \otimes X$ consisting of elements of the form

$$s = \sum_{i=1}^n \chi_{E_i} \otimes x_i$$

where $E_1, \dots, E_n$ are disjoint sets of I of finite measure and $x_1, \dots, x_n \in X$.

To show the isometric isomorphism of $L^1(I) \overset{\pi}{\otimes} X$ onto $L^1(I, X)$ it is enough to prove that J maps the dense subspace $S(I) \otimes X$ of $L^1(I) \otimes X$ isometrically onto the dense subspace $S(I, X)$ of $L^1(I, X)$.

Finally to establish the isometric isomorphism of $S(I) \otimes X$ onto $S(I, X)$ all we need to show is that

$$\left\| J\left(\sum_{i=1}^n \chi_{E_i} \otimes x_i\right) \right\|_{L^1(I, X)} \geq \left\| \sum_{i=1}^n \chi_{E_i} \otimes x_i \right\|_{L^1(I) \overset{\pi}{\otimes} X}.$$

For this, we have

$$\begin{aligned} \left\| J\left(\sum_{i=1}^n \chi_{E_i} \otimes x_i\right) \right\|_{L^1(I, X)} &= \int_I \left\| J\left(\sum_{i=1}^n \chi_{E_i} \otimes x_i\right)(t) \right\| d\mu(t) \\ &= \int_I \left\| \sum_{i=1}^n \chi_{E_i}(t) x_i \right\| d\mu(t) \\ &= \sum_{i=1}^n \int_{E_i} \|x_i\| d\mu(t) \\ &= \sum_{i=1}^n \|x_i\| \mu(E_i) \\ &= \sum_{i=1}^n \|x_i\| \|\chi_{E_i}\| \\ &\geq \left\| \sum_{i=1}^n \chi_{E_i} \otimes x_i \right\|_{L^1(I) \overset{\pi}{\otimes} X}. \quad \square \end{aligned}$$

Corollary 1.3.5. *If S and T are σ -finite measure spaces, then*

$$L^1(S) \overset{\pi}{\otimes} L^1(T) = L^1(S \times T).$$

Proof. Similar to the proof of Corollary 1.2.5. $\square$

Definition 1.3.6. Let $T = \sum_{i=1}^n x_i^* \otimes x_i \in (X^* \otimes X, \|\cdot\|_\pi)$. Define

$$\text{trace}(T) = \sum_{i=1}^n \langle x_i^*, x_i \rangle.$$

Theorem 1.3.7. The trace can be extended to be a well-defined linear bounded functional on $X^* \overset{\pi}{\otimes} X$.

Proof. We need to show that trace of a given T in $X^* \overset{\pi}{\otimes} X$ is independent of the representations of T , that is

$$\text{if } T = \sum_{i=1}^n x_i^* \otimes x_i \sim 0 \otimes 0 \quad \text{then} \quad \text{trace}(T) = \sum_{i=1}^n \langle x_i^*, x_i \rangle = 0. \quad (*)$$

Our proof is built on an induction argument.

First, suppose that

$$T = x^* \otimes x \sim 0 \otimes 0.$$

Then either $x^* = 0$ or $x = 0$. Hence $\text{trace}(T) = \langle x^*, x \rangle = 0$.

Next, assume that

$$\text{if } T = \sum_{i=1}^{n-1} x_i^* \otimes x_i \sim 0 \otimes 0 \quad \text{then} \quad \text{trace}(T) = \sum_{i=1}^{n-1} \langle x_i^*, x_i \rangle = 0.$$

We claim that $(*)$ is true. There are two cases to be considered:

Case I : $\{x_1, \dots, x_n\}$ is linearly independent. For each $x \in X$ we have

$$Tx = \sum_{i=1}^n \langle x_i^*, x \rangle x_i = 0.$$

Because of the linear independence of $\{x_1, \dots, x_n\}$ we have $\langle x_i^*, x \rangle = 0$ for all $i = 1, \dots, n$ and for all $x \in X$. Therefore, $\langle x_i^*, x_i \rangle = 0$ for all $i = 1, \dots, n$. Hence $\text{trace}(T) = \sum_{i=1}^n \langle x_i^*, x_i \rangle = 0$.

Case II : $\{x_1, \dots, x_n\}$ is linearly dependent. Without loss of generality assume that $x_n \in \text{Span}\{x_1, \dots, x_{n-1}\}$. In this case x_n can be written as

$$x_n = \sum_{i=1}^{n-1} \lambda_i x_i.$$

Hence, we have for T

$$\begin{aligned}
T &= \sum_{i=1}^n x_i^* \otimes x_i \\
&= \sum_{i=1}^{n-1} x_i^* \otimes x_i + x_n^* \otimes x_n \\
&= \sum_{i=1}^{n-1} x_i^* \otimes x_i + x_n^* \otimes \sum_{i=1}^{n-1} \lambda_i x_i \\
&= \sum_{i=1}^{n-1} x_i^* \otimes x_i + \sum_{i=1}^{n-1} x_n^* \otimes \lambda_i x_i \\
&= \sum_{i=1}^{n-1} (x_i^* + \lambda_i x_n^*) \otimes x_i.
\end{aligned}$$

Now, it follows from our assumption that

$$\text{trace}(T) = \sum_{i=1}^n \langle x_i^*, x_i \rangle = \sum_{i=1}^{n-1} \langle (x_i^* + \lambda_i x_n^*), x_i \rangle = 0$$

which completes the induction.

Finally, it is clear that trace is a linear operator on $X^* \otimes X$. To show the boundedness of trace, we have

$$\begin{aligned}
\left| \text{trace} \left(\sum_{i=1}^n x_i^* \otimes x_i \right) \right| &= \left| \sum_{i=1}^n \langle x_i^*, x_i \rangle \right| \\
&\leq \sum_{i=1}^n |\langle x_i^*, x_i \rangle| \\
&\leq \sum_{i=1}^n \|x_i^*\| \|x_i\|.
\end{aligned}$$

Taking the infimum over all representations of $\sum_{i=1}^n x_i^* \otimes x_i$ in $X^* \otimes X$ gives

$$\left| \text{trace} \left(\sum_{i=1}^n x_i^* \otimes x_i \right) \right| \leq \left\| \sum_{i=1}^n x_i^* \otimes x_i \right\|_{\pi}.$$

Consequently, trace is a well-defined linear bounded functional on $(X^* \otimes X, \|\cdot\|_{\pi})$. Therefore, trace can be extended continuously to $X^* \overset{\pi}{\otimes} X$. $\square$

Theorem 1.3.8. $(X \overset{\pi}{\otimes} Y)^* = L(X, Y^*)$.

Proof. Let $A \in L(X, Y^*)$. Define

$$F_A: X \otimes Y \longrightarrow \mathbb{R}$$

by

$$F_A\left(\sum_{i=1}^n x_i \otimes y_i\right) = \sum_{i=1}^n \langle Ax_i, y_i \rangle = \text{trace}\left(\sum_{i=1}^n Ax_i \otimes y_i\right).$$

Then F_A is a well-defined linear functional on $X \otimes Y$.

Further,

$$\begin{aligned} \left|F_A\left(\sum_{i=1}^n x_i \otimes y_i\right)\right| &= \left|\sum_{i=1}^n \langle Ax_i, y_i \rangle\right| \\ &\leq \sum_{i=1}^n |\langle Ax_i, y_i \rangle| \\ &\leq \sum_{i=1}^n \|Ax_i\| \|y_i\| \\ &\leq \|A\| \sum_{i=1}^n \|x_i\| \|y_i\|. \end{aligned}$$

Taking the infimum over all representations of $\sum_{i=1}^n x_i \otimes y_i$ in $X \otimes Y$, we get

$$\left|F_A\left(\sum_{i=1}^n x_i \otimes y_i\right)\right| \leq \|A\| \left\|\sum_{i=1}^n x_i \otimes y_i\right\|_{\pi}$$

and hence $\|F_A\| \leq \|A\|$. As $X \otimes Y$ is dense in $X \overset{\pi}{\otimes} Y$ and F_A is bounded, we can extend F_A uniquely to $X \overset{\pi}{\otimes} Y$. Let $\tilde{F}_A$ denote this extension.

Now let

$$F: X \overset{\pi}{\otimes} Y \longrightarrow \mathbb{R}$$

be a bounded linear functional on $X \overset{\pi}{\otimes} Y$. Define

$$A: X \longrightarrow Y^*$$

by

$$\langle Ax, y \rangle = F(x \otimes y).$$

Then A is a bounded linear operator from X into Y^* since

$$\begin{aligned} |\langle Ax, y \rangle| &= |F(x \otimes y)| \\ &\leq \|F\| \|x \otimes y\|_\pi \\ &\leq \|F\| \|x\| \|y\|. \end{aligned}$$

Taking the supremum over all $x \in B_1(X)$ and $y \in B_1(Y)$, we get $\|A\| \leq \|F\|$. Further, $F = F_A$ on $X \otimes Y$ since

$$F\left(\sum_{i=1}^n x_i \otimes y_i\right) = \sum_{i=1}^n F(x_i \otimes y_i) = \sum_{i=1}^n \langle Ax_i, y_i \rangle = F_A\left(\sum_{i=1}^n x_i \otimes y_i\right).$$

Thus the linear map $A \mapsto \tilde{F}_A$ is an isometric isomorphism of $L(X, Y^*)$ onto $(X \otimes^\pi Y)^*$. $\square$

§4. The Nuclear Tensor Product.

We study in this section the third interesting norm on $X \otimes Y$.

Definition 1.4.1. Let $(x_i)_{i=1}^n \subseteq X$ and $1 \leq p < \infty$. Define

$$\begin{aligned} 1. \quad \|(x_i)_{i=1}^n\|_{\alpha(p)} &= \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}}. \\ 2. \quad \|(x_i)_{i=1}^n\|_{\varepsilon(p)} &= \sup_{\substack{x^* \in X^* \\ \|x^*\| \leq 1}} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Definition 1.4.2. Let $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$. For $1 \leq p, p^* \leq \infty$, $\frac{1}{p} + \frac{1}{p^*} = 1$, define

$$\|z\|_{\nu_p} = \inf \|(x_i)_{i=1}^n\|_{\alpha(p)} \|(y_i)_{i=1}^n\|_{\varepsilon(p^*)}$$

where the infimum is taken over all representations of z in $X \otimes Y$.

Theorem 1.4.3. $\|\cdot\|_{\nu_p}$ is a reasonable crossnorm on $X \otimes Y$.

Proof. First of all we show that $\|\cdot\|_{\nu_p}$ is indeed a norm on $X \otimes Y$. Let $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$. Define

$$A: Y^* \longrightarrow X$$

by

$$Ay^* = \sum_{i=1}^n \langle y^*, y_i \rangle x_i.$$

Then $A \in L(Y^*, X)$ since for all $y^* \in B_1(Y^*)$

$$\begin{aligned} \|Ay^*\| &= \left\| \sum_{i=1}^n \langle y^*, y_i \rangle x_i \right\| \\ &\leq \sum_{i=1}^n |\langle y^*, y_i \rangle| \|x_i\| \\ &\leq \left(\sum_{i=1}^n |\langle y^*, y_i \rangle|^{p^*} \right)^{\frac{1}{p^*}} \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \\ &\leq \|(y_i)_{i=1}^n\|_{\varepsilon(p^*)} \|(x_i)_{i=1}^n\|_{\alpha(p)}. \end{aligned}$$

Taking the infimum over all representations of $z \in X \otimes Y$, we get

$$\|Ay^*\| \leq \|z\|_{\nu_p}.$$

Taking now the supremum over all $y^* \in B_1(Y^*)$, we obtain

$$\|A\| \leq \|z\|_{\nu_p}.$$

It follows from the definition of A that if $z \neq 0$ then $A \neq 0$. But the last inequality shows that $A \neq 0$ implies $\|z\|_{\nu_p} > 0$. It is clear that if $z = 0$ then $\|z\|_{\nu_p} = 0$. Hence $z = 0 \iff \|z\|_{\nu_p} = 0$. It is obvious that $\|az\|_{\nu_p} = |a| \|z\|_{\nu_p}$ for all $a \in \mathbb{R}$ and $z \in X \otimes Y$. It remains to show that $\|z + w\|_{\nu_p} \leq \|z\|_{\nu_p} + \|w\|_{\nu_p}$ for all z and w in $X \otimes Y$. Let $\epsilon > 0$ be given. Then, by the definition of $\|\cdot\|_{\nu_p}$ there exists a representation for z in $X \otimes Y$, say $z = \sum_{i=1}^n x_i \otimes y_i$, such that

$$\|(x_i)_{i=1}^n\|_{\alpha(p)} \|(y_i)_{i=1}^n\|_{\varepsilon(p^*)} \leq \|z\|_{\nu_p} + \epsilon.$$

It follows that

$$\left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \sup_{y^* \in B_1(Y^*)} \left(\sum_{i=1}^n |\langle y^*, y_i \rangle|^{p^*} \right)^{\frac{1}{p^*}} \leq \|z\|_{\nu_p} + \epsilon.$$

Since the factors on the left of the last inequality are homogeneous, we can assume that

$$\left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \leq [\|z\|_{\nu_p} + \epsilon]^{\frac{1}{p}}$$

and that

$$\sup_{y^* \in B_1(Y^*)} \left(\sum_{i=1}^n |\langle y^*, y_i \rangle|^{p^*} \right)^{\frac{1}{p^*}} \leq [\|z\|_{\nu_p} + \epsilon]^{\frac{1}{p^*}}.$$

Hence we have

$$\sum_{i=1}^n \|x_i\|^p \leq \|z\|_{\nu_p}^p + \epsilon^p$$

and

$$\sum_{i=1}^n |\langle y^*, y_i \rangle|^{p^*} \leq \|z\|_{\nu_p}^{p^*} + \epsilon^{p^*} \quad \text{for all } y^* \in B_1(Y^*).$$

For $w \in X \otimes Y$, we can select a representation $w = \sum_{i=n+1}^m x_i \otimes y_i$ such that

$$\|(x_i)_{i=n+1}^m\|_{\alpha(p)} \|(y_i)_{i=n+1}^m\|_{\epsilon(p^*)} \leq \|w\|_{\nu_p} + \epsilon.$$

By the same argument we used for z , we obtain

$$\sum_{i=n+1}^m \|x_i\|^p \leq \|w\|_{\nu_p}^p + \epsilon^p$$

and

$$\sum_{i=n+1}^m |\langle y^*, y_i \rangle|^{p^*} \leq \|w\|_{\nu_p}^{p^*} + \epsilon^{p^*} \quad \text{for all } y^* \in B_1(Y^*).$$

Thus,

$$\sum_{i=1}^m \|x_i\|^p \leq \|z\|_{\nu_p}^p + \|w\|_{\nu_p}^p + 2\epsilon^p$$

and

$$\sum_{i=1}^m |\langle y^*, y_i \rangle|^{p^*} \leq \|z\|_{\nu_p} + \|w\|_{\nu_p} + 2\epsilon \quad \text{for all } y^* \in B_1(Y^*).$$

It follows that

$$\left(\sum_{i=1}^m \|x_i\|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^m |\langle y^*, y_i \rangle|^{p^*} \right)^{\frac{1}{p^*}} \leq \|z\|_{\nu_p} + \|w\|_{\nu_p} + 2\epsilon.$$

Taking the supremum over all $y^* \in B_1(Y^*)$, we get

$$\|(x_i)_{i=1}^m\|_{\alpha(p)} \|(y_i)_{i=1}^m\|_{\varepsilon(p^*)} \leq \|z\|_{\nu_p} + \|w\|_{\nu_p} + 2\epsilon.$$

Taking the infimum over all representations of $z + w$ in $X \otimes Y$, yeilds

$$\|z + w\|_{\nu_p} \leq \|z\|_{\nu_p} + \|w\|_{\nu_p} + 2\epsilon.$$

Since ϵ was arbitrary, we have $\|z + w\|_{\nu_p} \leq \|z\|_{\nu_p} + \|w\|_{\nu_p}$.

Next, we show that $\|\cdot\|_{\nu_p}$ is a reasonable crossnorm on $X \otimes Y$. Let $x \in X$ and $y \in Y$. Then by the definition of $\|\cdot\|_{\nu_p}$, we have

$$\begin{aligned} \|x \otimes y\|_{\nu_p} &\leq \|(x)\|_{\alpha(p)} \|(y)\|_{\varepsilon(p^*)} \\ &= \|x\| \|y\|. \end{aligned} \tag{1}$$

For $x^* \in X^*$ and $y^* \in Y^*$ we have

$$\begin{aligned} \left| (x^* \otimes y^*) \left(\sum_{i=1}^n x_i \otimes y_i \right) \right| &= \left| \sum_{i=1}^n \langle x^*, x_i \rangle \langle y^*, y_i \rangle \right| \\ &\leq \sum_{i=1}^n |\langle x^*, x_i \rangle| |\langle y^*, y_i \rangle| \\ &\leq \|x^*\| \sum_{i=1}^n \|x_i\| |\langle y^*, y_i \rangle| \\ &= \|x^*\| \|y^*\| \sum_{i=1}^n \|x_i\| \left| \left\langle \frac{y^*}{\|y^*\|}, y_i \right\rangle \right| \\ &\leq \|x^*\| \|y^*\| \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n \left| \left\langle \frac{y^*}{\|y^*\|}, y_i \right\rangle \right|^{p^*} \right)^{\frac{1}{p^*}} \\ &\leq \|x^*\| \|y^*\| \|(x_i)_{i=1}^n\|_{\alpha(p)} \|(y_i)_{i=1}^n\|_{\varepsilon(p^*)}. \end{aligned}$$

Taking the infimum over all representations of z in $X \otimes Y$, gives

$$\left| (x^* \otimes y^*) \left(\sum_{i=1}^n x_i \otimes y_i \right) \right| \leq \|x^*\| \|y^*\| \left\| \sum_{i=1}^n x_i \otimes y_i \right\|_{\nu_p}.$$

Hence

$$\|x^* \otimes y^*\|_d \leq \|x^*\| \|y^*\|. \quad (2)$$

By (1), (2) and Theorem 1.1.8, $\|\cdot\|_{\nu_p}$ is a reasonable crossnorm on $X \otimes Y$. $\square$

Corollary 1.4.4. $\|\cdot\|_{\varepsilon} \leq \|\cdot\|_{\nu_p} \leq \|\cdot\|_{\pi}$.

Proof. Follows from Theorems 1.2.2 and 1.3.2. $\square$

$(X \otimes Y, \|\cdot\|_{\nu_p})$ is a normed linear space that need not be complete. Its completion with respect to $\|\cdot\|_{\nu_p}$ is denoted by $X \overset{\nu_p}{\otimes} Y$. The Banach space $(X \overset{\nu_p}{\otimes} Y, \|\cdot\|_{\nu_p})$ is called the completed p -nuclear tensor product of X and Y .

Theorem 1.4.5. Let S and T be finite measure spaces. For $1 \leq p < \infty$

$$L^p(S) \overset{\nu_p}{\otimes} L^p(T) = L^p(S \times T).$$

Proof. For the proof of this theorem we refer to [10, p.33]. $\square$

Theorem 1.4.6. Let S and T be σ -finite measure spaces. Then $L^\infty(S) \overset{\varepsilon}{\otimes} L^\infty(T)$ is a subspace of $L^\infty(S \times T)$.

Proof. For any Banach spaces X and Y we have

$$X^* \overset{\varepsilon}{\otimes} Y^* = \overline{\mathcal{F}(X, Y^*)}$$

where $\overline{\mathcal{F}(X, Y^*)}$ denotes the closure in the norm topology of the set of finite rank operators from X into Y^* [10, p.18].

By Theorem 1.3.8 we know that

$$(X \overset{\pi}{\otimes} Y)^* = L(X, Y^*)$$

Hence we get

$$X^* \overset{\varepsilon}{\otimes} Y^* = \overline{\mathcal{F}(X, Y^*)} \subset L(X, Y^*) = (X \overset{\pi}{\otimes} Y)^*,$$

since $\overline{\mathcal{F}(X, Y^*)}$ is a subspace of $L(X, Y^*)$.

Now, with $X = L^1(S)$ and $Y = L^1(T)$, using Corollary 1.3.5 we obtain

$$(L^1(S))^* \overset{\varepsilon}{\otimes} (L^1(T))^* \subset (L^1(S) \overset{\pi}{\otimes} L^1(T))^* = (L^1(S \times T))^* = L^\infty(S \times T).$$

Hence,

$$L^\infty(S) \overset{\varepsilon}{\otimes} L^\infty(T) \subset L^\infty(S \times T). \quad \square$$

REMARK 1.4.7. In Theorem 1.4.6 the inclusion in general is proper.

§5. Tensor Product Mapping.

Throughout this section let X_1, X_2, Y_1 and Y_2 denote Banach spaces.

Definition 1.5.1. Let $A \in L(X_1, X_2)$ and $B \in L(Y_1, Y_2)$. Define

$$A \otimes B: X_1 \otimes Y_1 \longrightarrow X_2 \otimes Y_2$$

by

$$(A \otimes B) \left(\sum_{i=1}^n x_i \otimes y_i \right) = \sum_{i=1}^n Ax_i \otimes By_i.$$

Lemma 1.5.2. $A \otimes B$ is a well-defined linear operator.

Proof. The definition of $(A \otimes B)(z)$ is independent of the representation of z . To see this, let $z = \sum_{i=1}^n x_i \otimes y_i \in X_1 \otimes Y_1$ such that $z \sim 0 \otimes 0$. We claim that $(A \otimes B)(z) = 0$. If $A = 0$ or $B = 0$, then it is clear that $(A \otimes B)(z) = 0$. Let $A \neq 0$,

$B \neq 0$ and on the contrary suppose that $(A \otimes B)(z) \neq 0$. Then

$$\begin{aligned}
0 < \|(A \otimes B)(z)\|_\pi &= \left\| \sum_{i=1}^n Ax_i \otimes By_i \right\|_\pi \\
&\leq \sum_{i=1}^n \|Ax_i \otimes By_i\|_\pi \\
&= \sum_{i=1}^n \|Ax_i\| \|By_i\| \\
&\leq \|A\| \|B\| \sum_{i=1}^n \|x_i\| \|y_i\|.
\end{aligned}$$

Taking infimum over all representations of z in $X_1 \otimes Y_1$, we get

$$\|(A \otimes B)(z)\|_\pi \leq \|A\| \|B\| \|z\|_\pi.$$

This implies that $\|z\|_\pi > 0$, but this is a contradiction since $z \sim 0 \otimes 0$. $\square$

Definition 1.5.3. A norm $\|\cdot\|_\alpha$, $\|\cdot\|_\varepsilon \leq \|\cdot\|_\alpha \leq \|\cdot\|_\pi$, on the tensor product $X \otimes Y$ is called a tensor norm if for every pair of operators $A \in L(X_1, X_2)$ and $B \in L(Y_1, Y_2)$ we have

$$\left\| \sum_{i=1}^n Ax_i \otimes By_i \right\|_\alpha \leq \|A\| \|B\| \left\| \sum_{i=1}^n x_i \otimes y_i \right\|_\alpha \quad \text{for all } \sum_{i=1}^n x_i \otimes y_i \in X_1 \otimes Y_1.$$

Let $X \overset{\alpha}{\otimes} Y$ denote the completion in the $\|\cdot\|_\alpha$ -norm of $X \otimes Y$. Then, the continuous extension $A \overset{\alpha}{\otimes} B$ of $A \otimes B$ to $X_1 \overset{\alpha}{\otimes} Y_1$ is an element of $L(X_1 \overset{\alpha}{\otimes} Y_1, X_2 \overset{\alpha}{\otimes} Y_2)$. It is called the α -tensor product of the operators A and B .

Theorem 1.5.4. Let $A \in L(X_1, X_2)$ and $B \in L(Y_1, Y_2)$. Then

$$A \overset{\varepsilon}{\otimes} B \in L(X_1 \overset{\varepsilon}{\otimes} Y_1, X_2 \overset{\varepsilon}{\otimes} Y_2)$$

and

$$A \overset{\pi}{\otimes} B \in L(X_1 \overset{\pi}{\otimes} Y_1, X_2 \overset{\pi}{\otimes} Y_2).$$

Proof. In the proof of Lemma 1.5.2 we have shown that

$$\|(A \otimes B)(z)\|_\pi \leq \|A\| \|B\| \|z\|_\pi \quad \text{for all } z \in X_1 \otimes Y_1.$$

It follows that $A \otimes B \in L((X_1 \otimes Y_1, \|\cdot\|_\pi), (X_2 \otimes Y_2, \|\cdot\|_\pi))$ and $\|A \otimes B\| \leq \|A\| \|B\|$. Consequently, $A \otimes B$ has the continuous extension $A \overset{\pi}{\otimes} B$ from $X_1 \overset{\pi}{\otimes} Y_1$ into $X_2 \overset{\pi}{\otimes} Y_2$.

To show that $A \overset{\varepsilon}{\otimes} B \in L(X_1 \overset{\varepsilon}{\otimes} Y_1, X_2 \overset{\varepsilon}{\otimes} Y_2)$, let $\sum_{i=1}^n x_i \otimes y_i \in X_1 \otimes Y_1$. Then $(A \otimes B)(\sum_{i=1}^n x_i \otimes y_i) = \sum_{i=1}^n Ax_i \otimes By_i \in X_2 \otimes Y_2$. Therefore on $X_2 \overset{\varepsilon}{\otimes} Y_2$, we have

$$\begin{aligned} \left\| (A \otimes B) \left(\sum_{i=1}^n x_i \otimes y_i \right) \right\|_\varepsilon &= \left\| \sum_{i=1}^n Ax_i \otimes By_i \right\|_\varepsilon \\ &= \sup_{\substack{x_2^* \in B_1(X_2^*) \\ y_2^* \in B_1(Y_2^*)}} \left| \sum_{i=1}^n \langle Ax_i, x_2^* \rangle \langle By_i, y_2^* \rangle \right|. \end{aligned}$$

But for all $x_2^* \in B_1(X_2^*)$ and $y_2^* \in B_1(Y_2^*)$ we have

$$\begin{aligned} \left| \sum_{i=1}^n \langle Ax_i, x_2^* \rangle \langle By_i, y_2^* \rangle \right| &= \left| \sum_{i=1}^n \langle x_i, A^* x_2^* \rangle \langle y_i, B^* y_2^* \rangle \right| \\ &= \|A^*\| \|B^*\| \left| \sum_{i=1}^n \langle x_i, \frac{A^* x_2^*}{\|A^*\|} \rangle \langle y_i, \frac{B^* y_2^*}{\|B^*\|} \rangle \right| \\ &\leq \|A\| \|B\| \sup_{\substack{x_1^* \in B_1(X_1^*) \\ y_1^* \in B_1(Y_1^*)}} \left| \sum_{i=1}^n \langle x_i, x_1^* \rangle \langle y_i, y_1^* \rangle \right| \\ &= \|A\| \|B\| \left\| \sum_{i=1}^n x_i \otimes y_i \right\|_\varepsilon. \end{aligned}$$

Taking the supremum over all $x_2^* \in B_1(X_2^*)$ and $y_2^* \in B_1(Y_2^*)$, we get

$$\left\| (A \otimes B) \left(\sum_{i=1}^n x_i \otimes y_i \right) \right\|_\varepsilon \leq \|A\| \|B\| \left\| \sum_{i=1}^n x_i \otimes y_i \right\|_\varepsilon.$$

Hence $A \otimes B \in L((X_1 \otimes Y_1, \|\cdot\|_\varepsilon), (X_2 \otimes Y_2, \|\cdot\|_\varepsilon))$ and $\|A \otimes B\| \leq \|A\| \|B\|$. Therefore, $A \otimes B$ has the continuous extension $A \overset{\varepsilon}{\otimes} B$ from $X_1 \overset{\varepsilon}{\otimes} Y_1$ into $X_2 \overset{\varepsilon}{\otimes} Y_2$. $\square$

Corollary 1.5.5. $\|\cdot\|_\varepsilon$ and $\|\cdot\|_\pi$ are tensor norms on $X \otimes Y$.

CHAPTER II

OPERATOR IDEALS

§1. Operator Ideal – The Definition.

This chapter studies three well-known operator ideals: the p -summing operators, the p -nuclear operators and the p -integral operators. The injective hull of an operator ideal is also presented.

Throughout, X and Y will denote real Banach spaces.

Definition 2.1.1. Let $L = \cup L(X, Y)$, where the union is taken over all Banach spaces X and Y . A subclass $O \subseteq L$ is called an operator ideal if the components

$$O(X, Y) = O \cap L(X, Y)$$

satisfy the following conditions:

- O1. $O(X, Y)$ is a subspace of $L(X, Y)$ for all X and Y .
- O2. $X^* \otimes Y \subseteq O(X, Y)$ for all X and Y .
- O3. If $A \in L(W, X)$, $T \in O(X, Y)$ and $B \in L(Y, Z)$ then $BT A \in O(W, Z)$.

Definition 2.1.2. O is called small if $O(X, Y) = L(X, Y)$ implies either X or Y is finite dimensional.

Definition 2.1.3. O is called closed if $\overline{O(X, Y)}$ in $(L(X, Y), \| \cdot \|)$ is $O(X, Y)$.

Now we give two simple examples of operator ideals.

Finite Rank Operators 2.1.4.

Definition 2.1.4.1. An operator $T \in L(X, Y)$ is called a *finite rank operator* if the range $T(X)$ is finite dimensional. Then $\text{rank}(T)$ is defined to be the dimension of $T(X)$.

Notation 2.1.4.2. The class of all finite rank operators is denoted by $\mathcal{F}$.

REMARKS 2.1.4.3.

1. $\mathcal{F}$ is a small operator ideal.
2. Let $x^* \in X^*$ and $y \in Y$. Obviously

$$x^* \otimes y: x \longmapsto \langle x^*, x \rangle y$$

is a finite rank operator, and we have $\|x^* \otimes y\| = \|x^*\| \|y\|$.

3. An operator $T \in L(X, Y)$ is a finite rank operator if and only if it admits a so-called finite representation

$$T = \sum_{i=1}^n x_i^* \otimes y_i$$

where $x_1^*, \dots, x_n^* \in X^*$ and $y_1, \dots, y_n \in Y$.

4. If X is not a Banach space, then a linear operator $T: X \longrightarrow Y$ such that $\dim T(X) < \infty$ need not be bounded. To see this, let $\{e_1, e_2, \dots\}$ be any orthonormal sequence in a Hilbert space H . Define

$$T: \text{Span}\{e_1, e_2, \dots\} \longrightarrow H$$

by

$$Te_n = ne_1.$$

Then $(e_n) \subset H$ satisfies that $\|e_n\| = 1$ for all n , however, $(\|Te_n\|)$ is not bounded. Thus T is not bounded although $\dim T(\text{Span}\{e_1, e_2, \dots\}) = 1$.

Compact Operators 2.1.5.

Definition 2.1.5.1. An operator $T: X \longrightarrow Y$ is called a compact operator if for every bounded subset M of X , the image $T(M)$ is relatively compact in the norm topology.

Lemma 2.1.5.2. Every compact operator $T: X \longrightarrow Y$ is bounded.

Proof. The unit sphere $S_1(X) = \{x \in X \mid \|x\| = 1\}$ is bounded. Since T is compact, $\overline{T(S_1(X))}$ is compact. Therefore $\overline{T(S_1(X))}$ is bounded. So there exists a number $c > 0$ such that

$$\|Tx\| \leq c \quad \text{for all } x \in S_1(X).$$

Hence T is bounded. $\square$

Notation 2.1.5.3. The class of all compact operators is denoted by $\mathcal{K}$.

Theorem 2.1.5.4. An operator $T: X \longrightarrow Y$ is compact if and only if it maps every bounded sequence (x_n) in X into a sequence (Tx_n) in Y which has a convergent subsequence.

Proof. Suppose that T is compact. Let (x_n) be a bounded sequence in X . Then (Tx_n) is relatively compact. Consequently, (Tx_n) has a convergent subsequence.

Conversely, suppose that if (x_n) is a bounded sequence in X then it has a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that $(Tx_{n_k})_{k \in \mathbb{N}}$ converges in Y . Let $B \subset X$ be a bounded set. We show that $T(B)$ is relatively compact. Let (y_n) be an arbitrary sequence in $T(B)$. For each n we have $y_n = Tx_n$ for some x_n in B . The sequence $(x_n) \subset X$ is bounded because B is a bounded subset of X . Hence by assumption there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of (x_n) such that $(Tx_{n_k})_{k \in \mathbb{N}}$ converges. This shows that $T(B)$ is relatively compact. Therefore T is compact. $\square$

Theorem 2.1.5.5. $\mathcal{K}$ is an operator ideal.

Proof. It is elementary to show that $\mathcal{K}(X, Y)$ is a subspace of $L(X, Y)$, therefore we will only verify the conditions O2 and O3 of Definition 2.1.1.

First, let T be a finite rank operator from X into Y . We show T belongs to $\mathcal{K}(X, Y)$. Let (x_n) be a bounded sequence in X , then (Tx_n) is bounded. Hence (Tx_n) is relatively compact since $\dim T(X) < \infty$. It follows that (Tx_n) has a

convergent subsequence. Since (x_n) was an arbitrary bounded sequence in X , the operator T is compact.

Next, let $R \in L(W, X)$, $T \in \mathcal{K}(X, Y)$ and $S \in L(Y, Z)$. We show that $TR \in \mathcal{K}(W, Y)$ and $ST \in \mathcal{K}(X, Z)$. Let $B \subset W$ be any bounded set. Then $R(B)$ is a bounded set, and the set $T(R(B)) = TR(B)$ is relatively compact because T is compact. Hence TR is a compact operator. Now let $(x_n) \subset X$ be a bounded sequence. Then (Tx_n) has a convergent subsequence $(Tx_{n_k})_{k \in \mathbb{N}}$ by the compactness of T . Hence $(STx_{n_k})_{k \in \mathbb{N}}$ converges. Hence ST is also compact. $\square$

Theorem 2.1.5.6. $\mathcal{K}$ is a closed operator ideal.

Proof. Let (T_n) be a sequence of compact operators in $L(X, Y)$. Further, we suppose that there exists an operator $T: X \rightarrow Y$ such that $\|T_n - T\| \rightarrow 0$ as $n \rightarrow \infty$, then we claim that T is compact. We show that for every bounded sequence (x_m) in X , the image (Tx_m) has a convergent subsequence.

Now, since T_1 is compact (x_m) has a subsequence $(x_{1,m})$ such that $(T_1 x_{1,m})$ converges. Similarly, $(x_{1,m})$ has a subsequence $(x_{2,m})$ such that $(T_2 x_{2,m})$ converges. Continuing in this way we obtain the diagonal sequence $(z_m) = (x_{m,m})$, which is a subsequence of (x_m) such that $(T_n z_m)_m$ converges for every n .

The sequence (x_m) is bounded, so there exists $c > 0$ such that $\|x_m\| \leq c$ for all $m \in \mathbb{N}$. Hence $\|z_m\| \leq c$ for all $m \in \mathbb{N}$. Let $\epsilon > 0$. Since $T_n \rightarrow T$, there exists an $n_0 \in \mathbb{N}$ such that $\|T - T_{n_0}\| < \epsilon/3c$.

The sequence $(T_{n_0} z_m)$ converges, hence is Cauchy. Therefore, there exists $N \in \mathbb{N}$ such that

$$\|T_{n_0} z_j - T_{n_0} z_k\| < \frac{\epsilon}{3} \quad \text{for all } j, k > N.$$

Hence for $j, k > N$, we have

$$\begin{aligned} \|Tz_j - Tz_k\| &\leq \|Tz_j - T_{n_0} z_j\| + \|T_{n_0} z_j - T_{n_0} z_k\| + \|T_{n_0} z_k - Tz_k\| \\ &\leq \|T - T_{n_0}\| \|z_j\| + \frac{\epsilon}{3} + \|T_{n_0} - T\| \|z_k\| \\ &< \frac{\epsilon}{3c} \cdot c + \frac{\epsilon}{3} + \frac{\epsilon}{3c} \cdot c = \epsilon. \end{aligned}$$

Thus (Tz_m) is Cauchy and converges by the completeness of Y . Since the arbitrary bounded sequence $(x_m) \subset X$ has a subsequence (z_m) whose image $(Tz_m) \subset Y$ converges; T is compact. $\square$

Corollary 2.1.5.7. $(\mathcal{K}(X, Y), \|\cdot\|)$ is a Banach space.

On each component $O(X, Y)$ of an operator ideal O one can define norms. Then $O(X, Y)$ has its own norm together with the norm it inherits from being a subspace of $L(X, Y)$. In the following sections the normed operator ideals of p -summing operators, p -nuclear operators, p -integral operators and the injective hull of an operator ideal are studied.

§2. Absolutely p -Summing Operators.

The notion of absolutely p -summing operators has been introduced by Pietsch [11, p.228] in 1966. At the beginning of this section, we need to study some sequence spaces with values in a Banach space X .

Definition 2.2.1. Let X be a Banach space and $1 \leq p \leq \infty$. Define

1. $\ell^p(X) = \{ (x_i) \subset X : \sum_{i=1}^{\infty} \|x_i\|^p < \infty \}$

$$\|(x_i)\|_{\alpha(p)} = \left(\sum_{i=1}^{\infty} \|x_i\|^p \right)^{\frac{1}{p}}$$

$$\|(x_i)\|_{\alpha(\infty)} = \max_{1 \leq i \leq n} \|x_i\|.$$
2. $\ell^p\langle X \rangle = \{ (x_i) \subset X : \sup_{\|x^*\| \leq 1} \sum_{i=1}^{\infty} |\langle x^*, x_i \rangle|^p < \infty \}$

$$\|(x_i)\|_{\varepsilon(p)} = \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^{\infty} |\langle x^*, x_i \rangle|^p \right)^{\frac{1}{p}}$$

$$\|(x_i)\|_{\varepsilon(\infty)} = \sup_{\|x^*\| \leq 1} \max_{1 \leq i \leq n} |\langle x^*, x_i \rangle|.$$

Theorem 2.2.2. Let $1 \leq p < \infty$. Then $\ell^p(X)$ and $\ell^p\langle X \rangle$ are Banach spaces. Moreover,

$$\ell^p(X) \subseteq \ell^p\langle X \rangle \quad \text{and} \quad \| \cdot \|_{\varepsilon(p)} \leq \| \cdot \|_{\alpha(p)}.$$

Proof. See [1, p.15].

Definition 2.2.3. A bounded operator $T: X \longrightarrow Y$ is called absolutely p -summing, $1 \leq p \leq \infty$, if there exists a constant $\lambda \geq 0$ such that the inequality

$$\|(Tx_i)_{i=1}^n\|_{\alpha(p)} \leq \lambda \|(x_i)_{i=1}^n\|_{\varepsilon(p^*)} \quad (*)$$

holds for all finite sets $\{x_1, \dots, x_n\}$ in X .

Notation 2.2.4. The class of all absolutely p -summing operators is denoted by Π_p .

Theorem 2.2.5. For $1 \leq p \leq \infty$, $\Pi_p(X, Y)$ is a subspace of $L(X, Y)$.

Proof. Let $T, S \in \Pi_p(X, Y)$. Then there exist $\lambda, \gamma \geq 0$ such that for all finite sets $\{x_1, \dots, x_n\} \subseteq X$, we have

$$\left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} \leq \lambda \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^p \right)^{\frac{1}{p}}$$

and

$$\left(\sum_{i=1}^n \|Sx_i\|^p \right)^{\frac{1}{p}} \leq \gamma \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^p \right)^{\frac{1}{p}}.$$

Using Minkowski's Inequality, we get

$$\begin{aligned} \left(\sum_{i=1}^n \|(T+S)(x_i)\|^p \right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n \|Tx_i + Sx_i\|^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n \|Sx_i\|^p \right)^{\frac{1}{p}} \\ &\leq (\lambda + \gamma) \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^p \right)^{\frac{1}{p}}. \end{aligned} \quad (1)$$

Thus $T + S \in \Pi_p(X, Y)$. Since for all scalars k we have

$$\begin{aligned} \left(\sum_{i=1}^n \|(kT)(x_i)\|^p \right)^{\frac{1}{p}} &= |k| \left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} \\ &\leq |k| \cdot \lambda \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^p \right)^{\frac{1}{p}}. \end{aligned} \quad (2)$$

It follows that $kT \in \Pi_p(X, Y)$. Hence $\Pi_p(X, Y)$ is a subspace of $L(X, Y)$. $\square$

Definition 2.2.6. For $T \in \Pi_p(X, Y)$, $1 \leq p \leq \infty$, define

$$\|T\|_{\pi_p} = \inf \{ \lambda \mid * \text{ holds} \}.$$

Theorem 2.2.7. For $1 \leq p \leq \infty$, $\|\cdot\|_{\pi_p}$ is a norm on $\Pi_p(X, Y)$. Moreover

$$\|T\| \leq \|T\|_{\pi_p} \quad \text{for all } T \in \Pi_p(X, Y).$$

Proof. Let $T \in \Pi_p(X, Y)$. We first show that $\|T\| \leq \|T\|_{\pi_p}$. By Definitions 2.2.3 and 2.2.6 we have

$$\left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} \leq \|T\|_{\pi_p} \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^p \right)^{\frac{1}{p}}$$

for all finite sets $\{x_1, \dots, x_n\} \subseteq X$. In particular, for the singletons $\{x\} \subset X$, we have

$$\|Tx\| \leq \|T\|_{\pi_p} \sup_{\|x^*\| \leq 1} |\langle x^*, x \rangle|.$$

Hence

$$\|Tx\| \leq \|T\|_{\pi_p} \|x\| \quad \text{for all } x \in X$$

and so we get $\|T\| \leq \|T\|_{\pi_p}$. Now if $T \neq 0$ then $\|T\| > 0$ and therefore $\|T\|_{\pi_p} > 0$. If $T = 0$ then obviously $\|T\|_{\pi_p} = 0$. Thus $T = 0$ if and only if $\|T\|_{\pi_p} = 0$. From the inequalities (1) and (2) one can easily see that

$$\|T + S\|_{\pi_p} \leq \|T\|_{\pi_p} + \|S\|_{\pi_p} \quad \text{for all } T, S \in \Pi_p(X, Y)$$

and

$$\|kT\|_{\pi_p} = |k| \|T\|_{\pi_p} \quad \text{for all } k \in \mathbb{R} \text{ and } T \in \Pi_p(X, Y).$$

Hence $\|\cdot\|_{\pi_p}$ is a norm on $\Pi_p(X, Y)$. $\square$

Theorem 2.2.8. Π_p is an operator ideal.

Proof. We need to verify the conditions O2 and O3 of Definition 2.1.1.

First we show that $\mathcal{F}(X, Y) \subseteq \Pi_p(X, Y)$. Let $0 \neq x^* \in X^*$ and $0 \neq y \in Y$. We claim that $x^* \otimes y \in \Pi_p(X, Y)$. For an arbitrary finite set $\{x_1, \dots, x_n\} \subseteq X$, we have

$$\begin{aligned} \left(\sum_{i=1}^n \|(x^* \otimes y)(x_i)\|^p \right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n \|\langle x^*, x_i \rangle y\|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^p \|y\|^p \right)^{\frac{1}{p}} \\ &= \|y\| \|x^*\| \left(\sum_{i=1}^n \left| \left\langle \frac{x^*}{\|x^*\|}, x_i \right\rangle \right|^p \right)^{\frac{1}{p}} \\ &\leq \|y\| \|x^*\| \sup_{\substack{z^* \in X^* \\ \|z^*\| \leq 1}} \left(\sum_{i=1}^n |\langle z^*, x_i \rangle|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Hence $x^* \otimes y \in \Pi_p(X, Y)$ and $\|x^* \otimes y\|_{\pi_p} \leq \|x^*\| \|y\|$. By Theorem 2.2.7 we have

$$\|x^*\| \|y\| = \|x^* \otimes y\| \leq \|x^* \otimes y\|_{\pi_p} \leq \|x^*\| \|y\|.$$

So we get $\|x^* \otimes y\|_{\pi_p} = \|x^*\| \|y\|$.

Now let $R \in L(W, X)$, $T \in \Pi_p(X, Y)$ and $S \in L(Y, Z)$. We show that $STR \in \Pi_p(W, Z)$. Since T is absolutely p -summing we have, for any finite set $\{w_1, \dots, w_n\} \subseteq W$,

$$\begin{aligned} \left(\sum_{i=1}^n \|(STR)(w_i)\|^p \right)^{\frac{1}{p}} &\leq \|S\| \left(\sum_{i=1}^n \|(TR)(w_i)\|^p \right)^{\frac{1}{p}} \\ &= \|S\| \left(\sum_{i=1}^n \|T(R(w_i))\|^p \right)^{\frac{1}{p}} \\ &\leq \|S\| \|T\|_{\pi_p} \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |\langle R w_i, x^* \rangle|^p \right)^{\frac{1}{p}} \\ &= \|S\| \|T\|_{\pi_p} \|R^*\| \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n \left| \left\langle w_i, \frac{R^* x^*}{\|R^*\|} \right\rangle \right|^p \right)^{\frac{1}{p}} \\ &\leq \|S\| \|T\|_{\pi_p} \|R\| \sup_{\|w^*\| \leq 1} \left(\sum_{i=1}^n |\langle w_i, w^* \rangle|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Thus $STR \in \Pi_p(W, Z)$ and $\|STR\|_{\pi_p} \leq \|S\| \|T\|_{\pi_p} \|R\|$. $\square$

Theorem 2.2.9. $(\Pi_p(X, Y), \| \cdot \|_{\pi_p})$ is a Banach space.

Proof. Let (T_n) be a Cauchy sequence in $\Pi_p(X, Y)$. Then, given $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$\|T_n - T_m\|_{\pi_p} < \epsilon \quad \text{for all } n, m > N.$$

By Theorem 2.2.7, (T_n) is a Cauchy sequence in $(L(X, Y), \| \cdot \|)$. Since $(L(X, Y), \| \cdot \|)$ is complete, the sequence (T_n) converges in the operator norm, say

$$T_n \longrightarrow T \quad \text{as } n \longrightarrow \infty$$

for some $T \in L(X, Y)$.

For all finite sets $\{x_1, \dots, x_n\} \subseteq X$ we have

$$\begin{aligned} \left(\sum_{i=1}^n \|(T_n - T_m)(x_i)\|^p \right)^{\frac{1}{p}} &\leq \|T_n - T_m\|_{\pi_p} \|(x_i)_{i=1}^n\|_{\varepsilon(p)} \\ &< \epsilon \|(x_i)_{i=1}^n\|_{\varepsilon(p)} \quad \text{for all } n, m > N. \end{aligned}$$

Letting $n \longrightarrow \infty$, we obtain for $m > N$

$$\left(\sum_{i=1}^n \|(T - T_m)(x_i)\|^p \right)^{\frac{1}{p}} < \epsilon \|(x_i)_{i=1}^n\|_{\varepsilon(p)}.$$

Therefore,

$$\|T - T_m\|_{\pi_p} < \epsilon \quad \text{for all } m > N.$$

It follows that (T_n) converges to T in $(\Pi_p(X, Y), \| \cdot \|_{\pi_p})$. Since (T_n) was an arbitrary Cauchy sequence in $\Pi_p(X, Y)$, this proves the completeness of $\Pi_p(X, Y)$. $\square$

REMARK 2.2.10. $(\Pi_\infty(X, Y), \| \cdot \|_{\pi_\infty}) = (L(X, Y), \| \cdot \|)$ for all X and Y .

Theorem 2.2.11. If $p \leq q$, then $\Pi_p(X, Y) \subseteq \Pi_q(X, Y)$ and $\| \cdot \|_{\pi_q} \leq \| \cdot \|_{\pi_p}$.

Proof. Let $T \in \Pi_p(X, Y)$. If $p \leq q$ then $q = p + pq/r$ for some $r > 0$. Given

$\{x_1, \dots, x_n\} \subseteq X$ we have

$$\begin{aligned}
\left(\sum_{i=1}^n \|Tx_i\|^q \right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n \left[\|Tx_i\| \|Tx_i\|^{\frac{q}{r}} \right]^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^n [a_i \|Tx_i\|]^p \right)^{\frac{1}{p}} \quad \text{where } a_i = \|Tx_i\|^{\frac{q}{r}} \\
&= \left(\sum_{i=1}^n \|T(a_i x_i)\|^p \right)^{\frac{1}{p}} \\
&\leq \|T\|_{\pi_p} \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |\langle x^*, a_i x_i \rangle|^p \right)^{\frac{1}{p}} \\
&= \|T\|_{\pi_p} \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |a_i|^p |\langle x^*, x_i \rangle|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Using the Generalized Hölder's Inequality, we get

$$\begin{aligned}
\left(\sum_{i=1}^n \|Tx_i\|^q \right)^{\frac{1}{p}} &\leq \|T\|_{\pi_p} \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |a_i|^r \right)^{\frac{1}{r}} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^q \right)^{\frac{1}{q}} \\
&= \|T\|_{\pi_p} \left(\sum_{i=1}^n \|Tx_i\|^q \right)^{\frac{1}{r}} \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^q \right)^{\frac{1}{q}}.
\end{aligned}$$

Upon division by $(\sum_{i=1}^n \|Tx_i\|^q)^{\frac{1}{r}}$ and by noting that $\frac{1}{p} - \frac{1}{r} = \frac{1}{q}$, we obtain

$$\left(\sum_{i=1}^n \|Tx_i\|^q \right)^{\frac{1}{q}} \leq \|T\|_{\pi_p} \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^q \right)^{\frac{1}{q}}.$$

Hence $T \in \Pi_q(X, Y)$ and $\|T\|_{\pi_q} \leq \|T\|_{\pi_p}$. $\square$

Theorem 2.2.12. Π_p is not small.

Proof. The proof follows from the result of Kwapien [8] that

$$\Pi_1(\ell^1, \ell^2) = L(\ell^1, \ell^2).$$

However, neither ℓ^1 nor ℓ^2 is a finite dimensional space. $\square$

Theorem 2.2.13. For $1 \leq p < \infty$, the inclusion map

$$J_p: C(I) \longrightarrow L^p(I, \mu)$$

is absolutely p -summing.

Proof. Let $f_1, f_2, \dots, f_n \in C(I)$. Then

$$\begin{aligned} \sum_{i=1}^n \|J_p f_i\|^p &= \sum_{i=1}^n \|f_i\|^p \\ &= \sum_{i=1}^n \int_I |f_i(x)|^p d\mu(x) \\ &= \int_I \sum_{i=1}^n |f_i(x)|^p d\mu(x). \end{aligned}$$

If δ_x denotes the Dirac measure at the point x , then we have

$$\begin{aligned} \sum_{i=1}^n \|J_p f_i\|^p &= \int_I \sum_{i=1}^n |\langle f_i, \delta_x \rangle|^p d\mu(x) \\ &\leq \int_I \sup_{\|\nu\| \leq 1} \sum_{i=1}^n |\langle f_i, \nu \rangle|^p d\mu(x) \\ &= \|(f_i)_{i=1}^n\|_{\varepsilon(p)}^p \int_I d\mu(x) \\ &= \|(f_i)_{i=1}^n\|_{\varepsilon(p)}^p \mu(I). \end{aligned}$$

Finally, we get

$$\|(J_p f_i)_{i=1}^n\|_{\alpha(p)} \leq (\mu(I))^{\frac{1}{p}} \|(f_i)_{i=1}^n\|_{\varepsilon(p)}.$$

This proves that $J_p \in \Pi_p(C(I), L^p(I, \mu))$ and moreover $\|J_p\|_{\pi_p} \leq (\mu(I))^{\frac{1}{p}}$. $\square$

REMARK 2.2.14. $\Pi_p \not\subset \mathcal{K}$. To see this, consider $J_p: C(I) \longrightarrow L^p(I, \mu)$. $J_p \in \Pi_p(C(I), L^p(I, \mu))$. However, $J_p \notin \mathcal{K}(C(I), L^p(I, \mu))$.

The following theorem gives the fundamental characterization of absolutely p -summing operators.

Theorem (Pietsch Integral Theorem) 2.2.15. *Let $1 \leq p < \infty$. An operator $T \in L(X, Y)$ is absolutely p -summing if and only if there exists a constant $\lambda \geq 0$ and a measure $\mu \in B_1(M(B_1(X^*)))$ such that*

$$\|Tx\| \leq \lambda \left(\int_{B_1(X^*)} |\langle x^*, x \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \quad \text{for all } x \in X.$$

In this case,

$$\|T\|_{\pi_p} = \inf \lambda.$$

Proof. See [11, p.232].

Theorem (Factorization Theorem) 2.2.16. *Let $1 \leq p < \infty$. An operator $T \in L(X, Y)$ is absolutely p -summing if and only if it admits the factorization*

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \xrightarrow{J_Y} & \ell^\infty(B_1(Y^*)) \\ A \downarrow & & & & \uparrow B \\ C(B_1(X^*)) & \xrightarrow{J_p} & L^p(B_1(X^*), \mu) & & \end{array}$$

where $\mu \in B_1(M(B_1(X^*)))$, $J_Y: Y \rightarrow \ell^\infty(B_1(Y^*))$ is the natural isometric embedding, $J_p: C(B_1(X^*)) \rightarrow L^p(B_1(X^*), \mu)$ is the natural inclusion and $A: X \rightarrow C(B_1(X^*))$ and $B: L^p(B_1(X^*), \mu) \rightarrow \ell^\infty(B_1(Y^*))$ are bounded linear operators.

Proof. First suppose that T admits the factorization claimed. We show that T is absolutely p -summing. By Theorem 2.2.13, $J_p \in \Pi_p(C(B_1(X^*)), L^p(B_1(X^*), \mu))$. Hence $J_Y T = B J_p A \in \Pi_p(X, \ell^\infty(B_1(Y^*)))$. Since J_Y is an isometry, we have

$$\begin{aligned} \left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n \|J_Y Tx_i\|^p \right)^{\frac{1}{p}} \\ &\leq \lambda \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^p \right)^{\frac{1}{p}} \quad \text{for some } \lambda \geq 0. \end{aligned}$$

Consequently $T \in \Pi_p(X, Y)$.

Conversely, let $T \in \Pi_p(X, Y)$. Define

$$\begin{aligned} A: X &\longrightarrow C(B_1(X^*)) \\ x &\longmapsto f_x \end{aligned}$$

where

$$f_x(x^*) = \langle x^*, x \rangle.$$

Then $\|Ax\| = \|f_x\| = \sup_{x^* \in B_1(X^*)} |f_x(x^*)| = \sup_{x^* \in B_1(X^*)} |\langle x^*, x \rangle| = \|x\|$. Hence we obtain an operator $A \in L(X, C(B_1(X^*)))$.

Next, define

$$B_0: \text{Range}(J_p A) \longrightarrow Y$$

by setting

$$B_0(J_p Ax) = Tx.$$

B_0 is a continuous linear operator from $\text{Range}(J_p A)$ into Y since for all $x \in X$ we have, by Theorem 2.2.15,

$$\begin{aligned} \|B_0(J_p Ax)\| &= \|Tx\| \leq \lambda \left(\int_{B_1(X^*)} |\langle x^*, x \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \\ &= \lambda \left(\int_{B_1(X^*)} |f_x(x^*)|^p d\mu(x^*) \right)^{\frac{1}{p}} \\ &= \lambda \|f_x\|_p, \end{aligned}$$

for some $\lambda > 0$. Consequently $\|B_0(f_x)\| \leq \lambda \|f_x\|_p$, which proves the continuity of B_0 on $\text{Range}(J_p A)$. Let $\hat{B}_0$ be the extension of B_0 to $\overline{\text{Range}(J_p A)}$.

Now consider the composition

$$J_Y \hat{B}_0: \overline{\text{Range}(J_p A)} \subseteq L^p(B_1(X^*), \mu) \longrightarrow \ell^\infty(B_1(Y^*)).$$

Because of the metric extension property of $\ell^\infty(B_1(Y^*))$, Theorem 2.5.4, we can find a continuous extension of $J_Y \hat{B}_0$, say

$$B: L^p(B_1(X^*), \mu) \longrightarrow \ell^\infty(B_1(Y^*)).$$

Moreover, for all $x \in X$ we have

$$\begin{aligned}
(BJ_p A)(x) &= (BJ_p)(f_x) \\
&= B(f_x) \\
&= (\langle y^*, Tx \rangle)_{y^* \in B_1(Y^*)} \\
&= (J_Y T)(x).
\end{aligned}$$

Thus $BJ_p A = J_Y T$. $\square$

Theorem 2.2.17. *Let $g \in L^p(I, \mu)$, $1 \leq p < \infty$. Define*

$$T: L^\infty(I, \mu) \longrightarrow L^p(I, \mu)$$

by

$$Tf = g \cdot f.$$

Then T is an absolutely p -summing operator. (T is called a diagonal or a multiplication operator.)

Proof. Let $f_1, \dots, f_n \in L^\infty(I, \mu)$. Our claim is that there exists $\lambda \geq 0$ so that

$$\left(\sum_{i=1}^n \|Tf_i\|^p \right)^{\frac{1}{p}} \leq \sup_{\substack{\nu \in (L^\infty(I, \mu))^* \\ \|\nu\| \leq 1}} \left(\sum_{i=1}^n |\langle f_i, \nu \rangle|^p \right)^{\frac{1}{p}}.$$

But we have

$$\begin{aligned}
\left(\sum_{i=1}^n \|Tf_i\|^p \right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n \int_I |(Tf_i)(t)|^p d\mu(t) \right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^n \int_I |g(t)f_i(t)|^p d\mu(t) \right)^{\frac{1}{p}} \\
&= \left(\int_I |g(t)|^p \sum_{i=1}^n |f_i(t)|^p d\mu(t) \right)^{\frac{1}{p}} \\
&\leq \|g\|_p \operatorname{ess\,sup}_t \left(\sum_{i=1}^n |f_i(t)|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Consequently, all what we need to show is that

$$\operatorname{ess\,sup}_t \left(\sum_{i=1}^n |f_i(t)|^p \right)^{\frac{1}{p}} \leq \sup_{\substack{\nu \in (L^\infty(I, \mu))^* \\ \|\nu\| \leq 1}} \left(\sum_{i=1}^n |\langle f_i, \nu \rangle|^p \right)^{\frac{1}{p}}.$$

Let $t_0 \in I$. Then, by the Lebesgue Convergence Theorem [13, p.91], we have

$$f_i(t_0) = \lim_{h \rightarrow 0} \frac{1}{2h} \int_{t_0-h}^{t_0+h} f_i(t) dt \quad \text{a.e.}$$

Take $\nu_j = \frac{dt}{2/j}$ on $\left[t_0 - \frac{1}{j}, t_0 + \frac{1}{j}\right]$ and zero elsewhere. Then $\nu_j \in B_1((L^\infty(I, \mu))^*)$ for all j . Hence, for all j ,

$$\begin{aligned} \sup_{\substack{\nu \in (L^\infty(I, \mu))^* \\ \|\nu\| \leq 1}} \left(\sum_{i=1}^n |\langle f_i, \nu \rangle|^p \right)^{\frac{1}{p}} &\geq \left(\sum_{i=1}^n |\langle f_i, \nu_j \rangle|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^n \left| \int_I f_i(t) d\nu_j(t) \right|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^n \left| \frac{1}{2/j} \int_{t_0 - \frac{1}{j}}^{t_0 + \frac{1}{j}} f_i(t) dt \right|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Letting $j \rightarrow \infty$, we get

$$\sup_{\substack{\nu \in (L^\infty(I, \mu))^* \\ \|\nu\| \leq 1}} \left(\sum_{i=1}^n |\langle f_i, \nu \rangle|^p \right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^n |f_i(t_0)|^p \right)^{\frac{1}{p}} \quad \text{a.e.}$$

Since the last inequality is true almost everywhere in I , we have

$$\sup_{\substack{\nu \in (L^\infty(I, \mu))^* \\ \|\nu\| \leq 1}} \left(\sum_{i=1}^n |\langle f_i, \nu \rangle|^p \right)^{\frac{1}{p}} \geq \operatorname{ess\,sup}_t \left(\sum_{i=1}^n |f_i(t)|^p \right)^{\frac{1}{p}}.$$

Hence, we get

$$\left(\sum_{i=1}^n \|Tf_i\|^p \right)^{\frac{1}{p}} \leq \|g\|_p \sup_{\substack{\nu \in (L^\infty(I, \mu))^* \\ \|\nu\| \leq 1}} \left(\sum_{i=1}^n |\langle f_i, \nu \rangle|^p \right)^{\frac{1}{p}}.$$

Therefore, $T \in \Pi_p(L^\infty(I, \mu), L^p(I, \mu))$ and $\|T\|_{\pi_p} \leq \|g\|_p$. $\square$

§3. p -Nuclear Operators.

The concept of p -nuclear operators was introduced during the study of operators on Hilbert spaces that have a well-defined trace. The material of this section includes those nice operators which possess trace.

Definition 2.3.1. Let $1 \leq p, p^* < \infty$, $\frac{1}{p} + \frac{1}{p^*} = 1$. An operator $T \in L(X, Y)$ is called p -nuclear if

$$T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n \quad (*)$$

where

1. $(\lambda_n) \in \ell^p$
2. $(x_n^*) \in \ell^\infty(X^*)$, $\|x_n^*\| = 1$ for all n
3. $(y_n) \in \ell^{p^*} \langle Y \rangle$.

Notation 2.3.2. The class of all p -nuclear operators is denoted by $\mathcal{N}_p$.

Lemma 2.3.3. $\mathcal{N}_p(X, Y)$ is a subspace of $L(X, Y)$.

Proof. Given $T_1, T_2 \in \mathcal{N}_p(X, Y)$, we claim that $T_1 + T_2 \in \mathcal{N}_p(X, Y)$. Suppose that

$$T_1 = \lambda_1 x_1^* \otimes y_1 + \lambda_2 x_2^* \otimes y_2 + \lambda_3 x_3^* \otimes y_3 + \dots$$

and that

$$T_2 = \xi_1 z_1^* \otimes w_1 + \xi_2 z_2^* \otimes w_2 + \xi_3 z_3^* \otimes w_3 + \dots$$

where

$(\lambda_n), (\xi_n) \in \ell^p$
 $(x_n^*), (z_n^*) \in \ell^\infty(X^*)$ with $\|x_n^*\| = 1$ and $\|z_n^*\| = 1$ for all n
and $(y_n), (w_n) \in \ell^{p^*} \langle Y \rangle$.

For $T_1 + T_2$ we choose the representation

$$T_1 + T_2 = \sum_{n=1}^{\infty} \gamma_n u_n^* \otimes v_n$$

where

$(\gamma_n) = (\lambda_1, \lambda_2, \xi_1, \lambda_3, \xi_2, \dots)$
 $(u_n^*) = (x_1^*, x_2^*, z_1^*, x_3^*, z_2^*, \dots)$
and $(v_n) = (y_1, y_2, w_1, y_3, w_2, \dots)$.

Consequently, $(\gamma_n) \in \ell^p$, $(u_n^*) \in \ell^\infty(X^*)$ with $\|u_n^*\| = 1$ for all n , and $(v_n) \in \ell^{p^*} \langle Y \rangle$. Hence $T_1 + T_2 \in \mathcal{N}_p(X, Y)$. It is obvious that $\alpha T \in \mathcal{N}_p(X, Y)$ for all $\alpha \in \mathbb{R}$ and $T \in \mathcal{N}_p(X, Y)$. Therefore, $\mathcal{N}_p(X, Y)$ is a subspace of $L(X, Y)$. $\square$

Definition 2.3.4. Let $T \in \mathcal{N}_p(X, Y)$. For $1 \leq p, p^* < \infty$, $\frac{1}{p} + \frac{1}{p^*} = 1$, define

$$\|T\|_{\mathcal{N}_p} = \inf \|(\lambda_n)\|_{\alpha(p)} \| (y_n) \|_{\varepsilon(p^*)}$$

where the infimum is taken over all p -nuclear representations of T given in (*).

Lemma 2.3.5. For $1 \leq p < \infty$, $\| \cdot \|_{\mathcal{N}_p}$ is a norm on $\mathcal{N}_p(X, Y)$. Moreover,

$$\|T\| \leq \|T\|_{\mathcal{N}_p} \quad \text{for all } T \in \mathcal{N}_p(X, Y).$$

Proof. We shall first show that $\|T\| \leq \|T\|_{\mathcal{N}_p}$ for all $T \in \mathcal{N}_p(X, Y)$. Let $x \in X$. Then, we have

$$\begin{aligned} \|Tx\| &= \left\| \sum_{n=1}^{\infty} \lambda_n \langle x_n^*, x \rangle y_n \right\| \\ &= \sup_{y^* \in B_1(Y^*)} \left| \sum_{n=1}^{\infty} \lambda_n \langle x_n^*, x \rangle \langle y^*, y_n \rangle \right| \\ &\leq \sup_{y^* \in B_1(Y^*)} \sum_{n=1}^{\infty} |\lambda_n| |\langle x_n^*, x \rangle| |\langle y^*, y_n \rangle| \\ &\leq \|x\| \sup_{y^* \in B_1(Y^*)} \left(\sum_{n=1}^{\infty} |\lambda_n|^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |\langle y^*, y_n \rangle|^{p^*} \right)^{\frac{1}{p^*}} \\ &= \|x\| \|(\lambda_n)\|_{\alpha(p)} \| (y_n) \|_{\varepsilon(p^*)}. \end{aligned}$$

Taking the infimum over all p -nuclear representations of T , gives

$$\|Tx\| \leq \|T\|_{\mathcal{N}_p} \|x\|.$$

Hence, $\|T\| \leq \|T\|_{\mathcal{N}_p}$.

Next we show that $\| \cdot \|_{\mathcal{N}_p}$ is indeed a norm on $\mathcal{N}_p(X, Y)$. Since $\|T\| \leq \|T\|_{\mathcal{N}_p}$, it follows that if $T \neq 0$ then $\|T\|_{\mathcal{N}_p} > 0$. Also, it is easy to see that if $T = 0$ then $\|T\|_{\mathcal{N}_p} = 0$. Thus, $T = 0$ if and only if $\|T\|_{\mathcal{N}_p} = 0$. Further, it is clear

that $\|\alpha T\|_{\mathcal{N}_p} = |\alpha| \|T\|_{\mathcal{N}_p}$ for all $\alpha \in \mathbb{R}$ and $T \in \mathcal{N}_p(X, Y)$. Now it remains to prove the triangle inequality. Let $S, T \in \mathcal{N}_p(X, Y)$. Then, there exist p -nuclear representations for S and T , say

$$S = \sum_{i=1}^{\infty} \lambda_i x_i^* \otimes y_i$$

and

$$T = \sum_{i=1}^{\infty} \xi_i z_i^* \otimes w_i$$

where $(\lambda_i), (\xi_i) \in \ell^p$, $(x_i^*), (z_i^*) \in \ell^\infty(X^*)$ with $\|x_i^*\| = \|z_i^*\| = 1$ for all i , and $(y_i), (w_i) \in \ell^{p^*} \langle Y \rangle$.

By noting that $(X^* \overset{\nu_p}{\otimes} Y, \|\cdot\|_{\nu_p}) \subseteq (\mathcal{N}_p(X, Y), \|\cdot\|_{\mathcal{N}_p})$, we claim that there exists $(S_n) \subset X^* \overset{\nu_p}{\otimes} Y$ such that $\|S - S_n\|_{\mathcal{N}_p} \longrightarrow 0$ as $n \longrightarrow \infty$. Consider

$$S_n = \sum_{i=1}^n \hat{x}_i^* \otimes y_i$$

where $\hat{x}_i^* = \lambda_i x_i^*$ for all $i = 1, \dots, n$. Then $(S_n) \subset X^* \overset{\nu_p}{\otimes} Y$. Moreover,

$$\begin{aligned} \|S - S_n\|_{\mathcal{N}_p} &= \left\| \sum_{i=n+1}^{\infty} \hat{x}_i^* \otimes y_i \right\|_{\mathcal{N}_p} \\ &= \left\| \sum_{i=n+1}^{\infty} \lambda_i x_i^* \otimes y_i \right\|_{\mathcal{N}_p} \\ &\leq \|(\lambda_i)_{i=n+1}^{\infty}\|_{\alpha(p)} \| (y_i)_{i=n+1}^{\infty} \|_{\varepsilon(p^*)} \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty \end{aligned}$$

since $S \in \mathcal{N}_p(X, Y)$. Similarly, for $T \in \mathcal{N}_p(X, Y)$ consider the sequence

$$T_n = \sum_{i=1}^n \hat{z}_i^* \otimes w_i$$

where $\hat{z}_i^* = \xi_i z_i^*$ for all $i = 1, \dots, n$. As $T \in \mathcal{N}_p(X, Y)$, the sequence $(T_n) \subset X^* \overset{\nu_p}{\otimes} Y$ satisfies $\|T - T_n\|_{\mathcal{N}_p} \longrightarrow 0$ as $n \longrightarrow \infty$.

Furthermore, as in the proof of Lemma 2.3.3, $S + T$ has the representation

$$S + T = \sum_{i=1}^{\infty} \gamma_i u_i^* \otimes v_i$$

where $(\gamma_i) \in \ell^p$, $(u_i^*) \in \ell^\infty(X^*)$ with $\|u_i^*\| = 1$ for all i , and $(v_i) \in \ell^{p^*} \langle Y \rangle$. Then, one can easily see that

$$S_n + T_n = \sum_{i=1}^n \hat{u}_i^* \otimes v_i$$

where $\hat{u}_i^* = \gamma_i u_i^*$ for all $i = 1, \dots, n$, satisfies that $(S_n + T_n) \subset X^* \otimes^{\nu_p} Y$ and $\|(S + T) - (S_n + T_n)\|_{\mathcal{N}_p} \longrightarrow 0$ as $n \longrightarrow \infty$.

Now for each n , the triangle inequality

$$\|S_n + T_n\|_{\nu_p} \leq \|S_n\|_{\nu_p} + \|T_n\|_{\nu_p}$$

holds in $X^* \otimes^{\nu_p} Y$ by Theorem 1.4.3. Equivalently, the inequality

$$\|S_n + T_n\|_{\mathcal{N}_p} \leq \|S_n\|_{\mathcal{N}_p} + \|T_n\|_{\mathcal{N}_p}$$

holds in $\mathcal{N}_p(X, Y)$. Letting $n \longrightarrow \infty$, we get

$$\|S + T\|_{\mathcal{N}_p} \leq \|S\|_{\mathcal{N}_p} + \|T\|_{\mathcal{N}_p}. \quad \square$$

Lemma 2.3.6. $\mathcal{N}_p$ is an operator ideal.

Proof. By Lemma 2.3.3, $\mathcal{N}_p(X, Y)$ is a subspace of $L(X, Y)$. So it remains to verify the conditions O2 and O3 of Definition 2.1.1.

First we show $\mathcal{F}(X, Y) \subseteq \mathcal{N}_p(X, Y)$. Let $x^* \in X^*$ and $y \in Y$. Then

$$x^* \otimes y = \|x^*\| \cdot \left(\frac{x^*}{\|x^*\|} \right) \otimes y.$$

Consequently $x^* \otimes y \in \mathcal{N}_p(X, Y)$. Moreover

$$\begin{aligned} \|x^* \otimes y\|_{\mathcal{N}_p} &\leq \|(\|x^*\|)\|_{\alpha(p)} \|y\|_{\varepsilon(p^*)} \\ &= \|x^*\| \|y\|. \end{aligned}$$

By the previous lemma, we have

$$\|x^*\| \|y\| = \|x^* \otimes y\| \leq \|x^* \otimes y\|_{\mathcal{N}_p} \leq \|x^*\| \|y\|.$$

So we obtain $\|x^* \otimes y\|_{\mathcal{N}_p} = \|x^*\| \|y\|$.

Next we show that $STR \in \mathcal{N}_p(W, Z)$ for all $R \in L(W, X)$, $T \in \mathcal{N}_p(X, Y)$ and $S \in L(Y, Z)$. Since $T \in \mathcal{N}_p(X, Y)$ we have

$$T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n$$

where $(\lambda_n) \in \ell^p$, $(x_n^*) \in \ell^\infty(X^*)$ with $\|x_n^*\| = 1$ for all n , and $(y_n) \in \ell^{p^*} \langle Y \rangle$. It follows that for all $w \in W$, we have

$$\begin{aligned} (STR)(w) &= S(T(Rw)) \\ &= S\left(\sum_{n=1}^{\infty} \lambda_n \langle x_n^*, Rw \rangle y_n\right) \\ &= \sum_{n=1}^{\infty} \lambda_n \langle x_n^*, Rw \rangle Sy_n \\ &= \sum_{n=1}^{\infty} \lambda_n \langle R^* x_n^*, w \rangle Sy_n \\ &= \left(\sum_{n=1}^{\infty} \lambda_n R^* x_n^* \otimes Sy_n\right)(w). \end{aligned}$$

Hence, we get

$$\begin{aligned} STR &= \sum_{n=1}^{\infty} \lambda_n R^* x_n^* \otimes Sy_n \\ &= \sum_{n=1}^{\infty} \gamma_n w_n^* \otimes z_n \end{aligned}$$

where $(\gamma_n) = (\|R^* x_n^*\| \lambda_n)$, $(w_n^*) = \left(\frac{R^* x_n^*}{\|R^* x_n^*\|}\right)$ and $(z_n) = (Sy_n)$. It is clear that $(\gamma_n) \in \ell^p$, $(w_n^*) \in \ell^\infty(W^*)$ with $\|w_n^*\| = 1$ for all n . It remains to show that

$(z_n) \in \ell^{p^*} \langle Z \rangle$. For this let $z^* \in B_1(Z^*)$, then we have

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} |\langle z_n, z^* \rangle|^{p^*} \right)^{\frac{1}{p^*}} &= \left(\sum_{n=1}^{\infty} |\langle Sy_n, z^* \rangle|^{p^*} \right)^{\frac{1}{p^*}} \\
&= \left(\sum_{n=1}^{\infty} |\langle y_n, S^* z^* \rangle|^{p^*} \right)^{\frac{1}{p^*}} \\
&= \|S^* z^*\| \left(\sum_{n=1}^{\infty} \left| \langle y_n, \frac{S^* z^*}{\|S^* z^*\|} \rangle \right|^{p^*} \right)^{\frac{1}{p^*}} \\
&\leq \|S\| \sup_{y^* \in B_1(Y^*)} \left(\sum_{n=1}^{\infty} |\langle y_n, y^* \rangle|^{p^*} \right)^{\frac{1}{p^*}}.
\end{aligned}$$

Hence

$$\|(z_n)\|_{\varepsilon(p^*)} \leq \|S\| \|(y_n)\|_{\varepsilon(p^*)} < \infty.$$

Therefore $(z_n) \in \ell^{p^*} \langle Z \rangle$. Consequently, $STR \in \mathcal{N}_p(W, Z)$. Moreover,

$$\begin{aligned}
\|STR\|_{\mathcal{N}_p} &\leq \|(\gamma_n)\|_{\alpha(p)} \|(z_n)\|_{\varepsilon(p^*)} \\
&\leq \|R\| \|(\lambda_n)\|_{\alpha(p)} \|S\| \|(y_n)\|_{\varepsilon(p^*)} \\
&= \|S\| \|(\lambda_n)\|_{\alpha(p)} \|(y_n)\|_{\varepsilon(p^*)} \|R\|.
\end{aligned}$$

Taking infimum over all p -nuclear representations of T in $\mathcal{N}_p(X, Y)$, we get

$$\|STR\|_{\mathcal{N}_p} \leq \|S\| \|T\|_{\mathcal{N}_p} \|R\|. \quad \square$$

Lemma 2.3.7. For $1 \leq p < \infty$, $(\mathcal{N}_p(X, Y), \|\cdot\|_{\mathcal{N}_p})$ is a Banach space.

Proof. Let $T_1, T_2, \dots \in \mathcal{N}_p(X, Y)$ and $\sum_{j=1}^{\infty} \|T_j\|_{\mathcal{N}_p} < \infty$. Our claim is that $\sum_{j=1}^{\infty} T_j \in \mathcal{N}_p(X, Y)$. By Lemma 2.3.5, we have $\sum_{j=1}^{\infty} \|T_j\| < \infty$. Let $T = \sum_{j=1}^{\infty} T_j$ in $L(X, Y)$. We choose the p -nuclear representations:

$$\begin{aligned}
T_1 &= \lambda_{11} x_{11}^* \otimes y_{11} + \lambda_{12} x_{12}^* \otimes y_{12} + \lambda_{13} x_{13}^* \otimes y_{13} + \lambda_{14} x_{14}^* \otimes y_{14} + \dots \\
T_2 &= \lambda_{21} x_{21}^* \otimes y_{21} + \lambda_{22} x_{22}^* \otimes y_{22} + \lambda_{23} x_{23}^* \otimes y_{23} + \lambda_{24} x_{24}^* \otimes y_{24} + \dots \\
T_3 &= \lambda_{31} x_{31}^* \otimes y_{31} + \lambda_{32} x_{32}^* \otimes y_{32} + \lambda_{33} x_{33}^* \otimes y_{33} + \lambda_{34} x_{34}^* \otimes y_{34} + \dots \\
T_4 &= \lambda_{41} x_{41}^* \otimes y_{41} + \lambda_{42} x_{42}^* \otimes y_{42} + \lambda_{43} x_{43}^* \otimes y_{43} + \lambda_{44} x_{44}^* \otimes y_{44} + \dots \\
&\vdots
\end{aligned}$$

Consider the sequences

$$(\gamma_n) = (\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{13}, \lambda_{22}, \lambda_{31}, \lambda_{14}, \lambda_{23}, \lambda_{32}, \lambda_{41}, \dots)$$

$$(z_n^*) = (x_{11}^*, x_{12}^*, x_{21}^*, x_{13}^*, x_{22}^*, x_{31}^*, x_{14}^*, x_{23}^*, x_{32}^*, x_{41}^*, \dots)$$

and $(w_n) = (y_{11}, y_{12}, y_{21}, y_{13}, y_{22}, y_{31}, y_{14}, y_{23}, y_{32}, y_{41}, \dots).$

Then the p -nuclear representation of T

$$T = \sum_{n=1}^{\infty} \gamma_n z_n^* \otimes w_n$$

satisfies that $(\gamma_n) \in \ell^p$, $(z_n^*) \in \ell^\infty(X^*)$ with $\|z_n^*\| = 1$ for all n , and $(w_n) \in \ell^{p^*} \langle Y \rangle$. Thus $\sum_{j=1}^{\infty} T_j \in \mathcal{N}_p(X, Y)$. Further,

$$\begin{aligned} \left\| T - \sum_{j=1}^n T_j \right\|_{\mathcal{N}_p} &= \left\| \sum_{j=n+1}^{\infty} T_j \right\|_{\mathcal{N}_p} \\ &\leq \|(\gamma_j)_{j=n+1}^\infty\|_{\alpha(p)} \| (w_j)_{j=n+1}^\infty \|_{\varepsilon(p^*)} \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned}$$

since $\sum_{j=1}^{\infty} \|T_j\|_{\mathcal{N}_p} < \infty$. $\square$

Theorem (Factorization Theorem) 2.3.8. *An operator $T \in L(X, Y)$ is p -nuclear if and only if there exists a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ A \downarrow & & \uparrow B \\ \ell^\infty & \xrightarrow{D} & \ell^p \end{array}$$

such that $D \in L(\ell^\infty, \ell^p)$ is a diagonal operator of the form $D(a_n) = (\lambda_n a_n)$ with $(\lambda_n) \in \ell^p$. Further, $A \in L(X, \ell^\infty)$ and $B \in L(\ell^p, Y)$.

In this case,

$$\|T\|_{\mathcal{N}_p} = \inf \|B\| \|D\| \|A\|$$

where the infimum is taken over all possible factorizations.

Proof. Let $T \in \mathcal{N}_p(X, Y)$ and $\epsilon > 0$. Consider a nuclear representation

$$T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n$$

such that $\|(\lambda_n)\|_{\alpha(p)} \leq \|T\|_{\mathcal{N}_p} + \epsilon$ and $\|(y_n)\|_{\epsilon(p^*)} = 1$. Define

$$\begin{aligned} A: X &\longrightarrow \ell^\infty \\ x &\longmapsto (\langle x_n^*, x \rangle). \end{aligned}$$

Then $\|Ax\| = \sup_n |\langle x_n^*, x \rangle| \leq \|x\|$. Hence $A \in L(X, \ell^\infty)$ and $\|A\| \leq 1$.

Next define

$$\begin{aligned} B: \ell^p &\longrightarrow Y \\ (b_n) &\longmapsto \sum_{n=1}^{\infty} b_n y_n. \end{aligned}$$

Since $\|B(b_n)\| = \sup_{y^* \in B_1(Y^*)} |\sum_{n=1}^{\infty} b_n \langle y^*, y_n \rangle| \leq \|(y_n)\|_{\epsilon(p^*)} \|(b_n)\|_{\alpha(p)}$, we have $B \in L(\ell^p, Y)$ and $\|B\| \leq 1$.

The diagonal operator

$$\begin{aligned} D: \ell^\infty &\longrightarrow \ell^p \\ (a_n) &\longmapsto (\lambda_n a_n) \end{aligned}$$

is continuous since $\|D(a_n)\| = (\sum_{n=1}^{\infty} |\lambda_n a_n|^p)^{\frac{1}{p}} \leq \|(\lambda_n)\|_{\alpha(p)} \| (a_n) \|_\infty$. So $D \in L(\ell^\infty, \ell^p)$ and $\|D\| \leq \|(\lambda_n)\|_{\alpha(p)}$. In addition, $\|D\| = \|(\lambda_n)\|_{\alpha(p)}$ since $\|D\| \geq \|D(1, 1, 1, \dots)\|$.

Now for every $x \in X$ we have

$$\begin{aligned} (BDA)(x) &= BD(\langle x_n^*, x \rangle) \\ &= B(\lambda_n \langle x_n^*, x \rangle) \\ &= \sum_{n=1}^{\infty} \lambda_n \langle x_n^*, x \rangle y_n. \end{aligned}$$

Consequently, there exists a factorization $T = BDA$ such that $\|A\| \|D\| \|B\| \leq \|(\lambda_n)\|_{\alpha(p)} \leq \|T\|_{\mathcal{N}_p} + \epsilon$.

For the converse, we need to show that every diagonal operator

$$\begin{aligned} D: \ell^\infty &\longrightarrow \ell^p \\ (a_n) &\longmapsto (\lambda_n a_n) \end{aligned}$$

where $(\lambda_n) \in \ell^p$, belongs to $\mathcal{N}_p(\ell^\infty, \ell^p)$. As D can be written in the form

$$D = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes \delta_n$$

where the first $\delta_n \in \ell^1$ and the other $\delta_n \in \ell^p$, we conclude that $D \in \mathcal{N}_p(\ell^\infty, \ell^p)$. Moreover, $\|D\|_{\mathcal{N}_p} \leq \|(\lambda_n)\|_{\alpha(p)} = \|D\|$. Since $\mathcal{N}_p$ is an operator ideal, $A \in L(X, \ell^\infty)$ and $B \in L(\ell^p, Y)$ we have $T = BDA \in \mathcal{N}_p(X, Y)$. Further, $\|T\|_{\mathcal{N}_p} \leq \|B\| \|D\|_{\mathcal{N}_p} \|A\| \leq \|B\| \|D\| \|A\|$. $\square$

Corollary 2.3.9. For $1 \leq p < \infty$, $\mathcal{N}_p \subseteq \Pi_p$.

Proof. Follows from Theorem 2.2.17 and Theorem 2.3.8. $\square$

REMARKS 2.3.10.

1. The series $\sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n$ converges in the norm topology of $L(X, Y)$. Therefore, every p -nuclear operator is compact. In other words $\mathcal{N}_p \subseteq \mathcal{K}$.
2. The natural inclusion map $J_p: C(I) \longrightarrow L^p(I, \mu)$ is in Π_p , however, J_p is not in $\mathcal{K}$. Therefore, $\Pi_p \neq \mathcal{N}_p$.

Theorem 2.3.11. If $p \leq q$, then $\mathcal{N}_p(X, Y) \subseteq \mathcal{N}_q(X, Y)$ and $\| \cdot \|_{\mathcal{N}_q} \leq \| \cdot \|_{\mathcal{N}_p}$.

Proof. Let $T \in \mathcal{N}_p(X, Y)$. Then, by Theorem 2.3.8, T admits a factorization, say

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ A \downarrow & & \uparrow B \\ \ell^\infty & \xrightarrow{D} & \ell^p \end{array}$$

where $D \in L(\ell^\infty, \ell^p)$ is a diagonal operator, $D(a_n) = (\lambda_n a_n)$ with $(\lambda_n) \in \ell^p$, $A \in L(X, \ell^\infty)$ and $B \in L(\ell^p, Y)$.

If $p \leq q$ then $1/p = 1/q + 1/r$ for some $r > 0$. Put $\gamma_n = \lambda_n^{1-p/r}$ for all n . Then $(\gamma_n) \in \ell^q$ since we have $\sum_{n=1}^{\infty} |\gamma_n|^q = \sum_{n=1}^{\infty} |\lambda_n|^p < \infty$. Define

$$\begin{aligned}\hat{D}: \ell^\infty &\longrightarrow \ell^q \\ (b_n) &\longmapsto (\gamma_n b_n)\end{aligned}$$

and

$$\begin{aligned}\tilde{D}: \ell^q &\longrightarrow \ell^p \\ (c_n) &\longmapsto (\lambda_n^{p/r} c_n).\end{aligned}$$

Then $\hat{D} \in L(\ell^\infty, \ell^q)$ is a diagonal operator and $\|\hat{D}\| = \|(\gamma_n)\|_{\alpha(q)}$. Also, by the Generalized Hölder's Inequality, we have

$$\left(\sum_{n=1}^{\infty} |\lambda_n^{p/r} c_n|^p \right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |\lambda_n|^p \right)^{\frac{1}{r}} \left(\sum_{n=1}^{\infty} |c_n|^q \right)^{\frac{1}{q}}.$$

Therefore $\tilde{D} \in L(\ell^q, \ell^p)$ and $\|\tilde{D}\| \leq (\sum_{n=1}^{\infty} |\lambda_n|^p)^{\frac{1}{r}}$. As $D = \tilde{D} \circ \hat{D}$, we have the following factorization for T :

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y \\ A \downarrow & & \uparrow B \\ \ell^\infty & \xrightarrow{\hat{D}} \ell^q & \xrightarrow{\tilde{D}} \ell^p \end{array}$$

Finally, letting $\hat{B} = B \circ \tilde{D}$ we get the factorization

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ A \downarrow & & \uparrow \hat{B} \\ \ell^\infty & \xrightarrow{\hat{D}} & \ell^q \end{array}$$

Hence $T = \hat{B}\hat{D}A \in \mathcal{N}_q(X, Y)$. Moreover,

$$\begin{aligned}
\|T\|_{\mathcal{N}_q} &= \|\hat{B}\hat{D}A\|_{\mathcal{N}_q} \\
&\leq \|B\| \|\tilde{D}\| \|\hat{D}\| \|A\| \\
&\leq \|B\| \left(\sum_{n=1}^{\infty} |\lambda_n|^p \right)^{\frac{1}{r}} \left(\sum_{n=1}^{\infty} |\lambda_n|^p \right)^{\frac{1}{q}} \|A\| \\
&= \|B\| \|(\lambda_n)\|_{\alpha(p)} \|A\| \\
&= \|B\| \|D\| \|A\|.
\end{aligned}$$

Taking infimum over all possible factorizations of T , gives $\|T\|_{\mathcal{N}_q} \leq \|T\|_{\mathcal{N}_p}$. $\square$

Theorem 2.3.12. $\mathcal{N}_1(X, Y) = X^* \otimes^{\pi} Y$. Further, $(\mathcal{N}_1(X, Y))^* = L(Y, X^{**})$.

Proof. Let $T \in X^* \otimes^{\pi} Y$. Then, by Theorem 1.3.3, given $\epsilon > 0$, there exist sequences $(\lambda_n) \in c_0$, $(x_n^*) \subset X^*$, $(y_n) \subset Y$ with $\|x_n^*\| = \|y_n\| = 1$ for all n , such that

$$T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n$$

and that

$$\|T\|_{\pi} \leq \sum_{n=1}^{\infty} \lambda_n \leq \|T\|_{\pi} + \epsilon.$$

Thus $(\lambda_n) \in \ell^1$, $(x_n^*) \in \ell^{\infty}(X^*)$ satisfies $\|x_n^*\| = 1$ for all n , and $(y_n) \in \ell^{\infty}\langle Y \rangle$. Hence, $T \in \mathcal{N}_1(X, Y)$ and $\|T\|_{\mathcal{N}_1} \leq \|(\lambda_n)\|_{\alpha(1)} \| (y_n) \|_{\epsilon(\infty)} \leq \|T\|_{\pi} + \epsilon$.

Conversely, let $T \in \mathcal{N}_1(X, Y)$. Then

$$T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n$$

where $(\lambda_n) \in \ell^1$, $(x_n^*) \in \ell^{\infty}(X^*)$ with $\|x_n^*\| = 1$ for all n , and $(y_n) \in \ell^{\infty}\langle Y \rangle$. Thus T can be written in the form

$$T = \sum_{n=1}^{\infty} \hat{x}_n^* \otimes y_n$$

where $(\hat{x}_n^*) \subset X^*$ and $(y_n) \subset Y$. Obviously, $T \in X^* \otimes^\pi Y$. Moreover,

$$\begin{aligned} \|T\|_\pi &\leq \sum_{n=1}^{\infty} \|\hat{x}_n^*\| \|y_n\| \\ &= \sum_{n=1}^{\infty} |\lambda_n| \|x_n^*\| \|y_n\| \\ &\leq \sum_{n=1}^{\infty} |\lambda_n| \|y_n\|. \end{aligned}$$

As $0 \neq y_n \in Y$, there exists $y_n^* \in Y^*$ such that $\|y_n^*\| = 1$ and $\langle y_n^*, y_n \rangle = \|y_n\|$. Since $(y_n) \in \ell^\infty \langle Y \rangle$, we have $\|(y_n)\|_{\varepsilon(\infty)} < \infty$. Consequently,

$$\begin{aligned} \|T\|_\pi &\leq \sum_{n=1}^{\infty} |\lambda_n| \langle y_n^*, y_n \rangle \\ &\leq \sum_{n=1}^{\infty} |\lambda_n| \sup_{\|y^*\| \leq 1} \sup_n |\langle y^*, y_n \rangle| \\ &= \|(\lambda_n)\|_{\alpha(1)} \|(y_n)\|_{\varepsilon(\infty)}. \end{aligned}$$

Taking the infimum over all nuclear representations of T in $\mathcal{N}_1(X, Y)$, we get $\|T\|_\pi \leq \|T\|_{\mathcal{N}_1}$. Since ϵ was arbitrary we have $\|T\|_\pi = \|T\|_{\mathcal{N}_1}$.

Finally, using Theorem 1.3.7, we get

$$(\mathcal{N}_1(X, Y))^* = (X^* \otimes^\pi Y)^* = (Y \otimes^\pi X^*)^* = L(Y, X^{**}). \quad \square$$

Theorem 2.3.13. *Let $1 \leq p, p^* < \infty$, $\frac{1}{p} + \frac{1}{p^*} = 1$. If X is reflexive, then*

$$(\mathcal{N}_p(X, Y), \|\cdot\|_{\mathcal{N}_p})^* = (\Pi_{p^*}(Y, X), \|\cdot\|_{\pi_{p^*}}).$$

Proof. Let $A \in \Pi_{p^*}(Y, X)$. Define

$$F_A: \mathcal{N}_p(X, Y) \longrightarrow \mathbb{R}$$

by

$$F_A\left(\sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n\right) = \sum_{n=1}^{\infty} \lambda_n \langle x_n^*, Ay_n \rangle = \text{trace}\left(\sum_{n=1}^{\infty} x_n^* \otimes A(\lambda_n y_n)\right).$$

Then F_A is a well-defined linear functional on $\mathcal{N}_p(X, Y)$. To see the boundedness of F_A , we have

$$\begin{aligned} \left| F_A \left(\sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n \right) \right| &= \left| \sum_{n=1}^{\infty} \lambda_n \langle x_n^*, Ay_n \rangle \right| \\ &\leq \|(\lambda_n)\|_{\alpha(p)} \|Ay_n\|_{\alpha(p^*)} \\ &\leq \|A\|_{\pi_{p^*}} \|(\lambda_n)\|_{\alpha(p)} \|y_n\|_{\varepsilon(p^*)}. \end{aligned}$$

Taking the infimum over all p -nuclear representations of $\sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n$ in $\mathcal{N}_p(X, Y)$, we get

$$\left| F_A \left(\sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n \right) \right| \leq \|A\|_{\pi_{p^*}} \|T\|_{\mathcal{N}_p}.$$

Hence $\|F_A\| \leq \|A\|_{\pi_{p^*}}$.

Conversely, let $F \in (\mathcal{N}_p(X, Y))^*$. Define

$$A: Y \longrightarrow X$$

by

$$\langle Ay, x^* \rangle = F(x^* \otimes y).$$

Note that $F = F_A$ since for an arbitrary $\sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n \in \mathcal{N}_p(X, Y)$ we have

$$F \left(\sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n \right) = \sum_{n=1}^{\infty} \lambda_n F(x_n^* \otimes y_n) = \sum_{n=1}^{\infty} \lambda_n \langle Ay_n, x_n^* \rangle = F_A \left(\sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n \right).$$

Further A is absolutely p^* -summing. Indeed, let $\{y_1, \dots, y_k\} \subseteq Y$. By the duality of ℓ^p and ℓ^{p^*} we have

$$\left(\sum_{n=1}^k \|Ay_n\|^{p^*} \right)^{\frac{1}{p^*}} = \sum_{n=1}^k \lambda_n \|Ay_n\|$$

for some $(\lambda_n) \in \ell^p$ with $\|(\lambda_n)\|_{\alpha(p)} = 1$. Since $Ay_n \in X$, there exists $x_n^* \in X^*$ such

that $\|x_n^*\| = 1$ and $\langle Ay_n, x^* \rangle = \|Ay_n\|$. Consequently,

$$\begin{aligned}
\left(\sum_{n=1}^k \|Ay_n\|^{p^*} \right)^{\frac{1}{p^*}} &= \sum_{n=1}^k \lambda_n \|Ay_n\| \\
&= \sum_{n=1}^k \lambda_n \langle Ay_n, x^* \rangle \\
&= \sum_{n=1}^k \lambda_n F_A(x_n^* \otimes y_n) \\
&= F_A \left(\sum_{n=1}^k \lambda_n x_n^* \otimes y_n \right) \\
&\leq \|F_A\| \left\| \sum_{n=1}^k \lambda_n x_n^* \otimes y_n \right\|_{\mathcal{N}_p} \\
&\leq \|F_A\| \left\| (y_n)_{n=1}^k \right\|_{\varepsilon(p^*)}.
\end{aligned}$$

Therefore, $A \in \Pi_{p^*}(Y, X)$ and $\|A\|_{\pi_{p^*}} \leq \|F_A\|$. Thus the mapping $A \mapsto F_A$ is an isometric isomorphism of $(\mathcal{N}_p(X, Y))^*$ onto $\Pi_{p^*}(Y, X)$. $\square$

§4. p -Integral Operators.

In our thesis all t

hat we need to know about p -integral operators is Definition 2.4.1. We shall only state some of the basic theorems on p -integral operators. For more details we refer to Pietsch and Persson [12].

Definition 2.4.1. Let $1 \leq p < \infty$. An operator $T \in L(X, Y)$ is p -integral if and only if it has a factorization of the form

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \xrightarrow{K_Y} & Y^{**} \\ A \downarrow & & & & \uparrow B \\ C(B_1(X^*)) & \xrightarrow{J_p} & L^p(B_1(X^*), \mu) & & \end{array}$$

where $\mu \in B_1(M(B_1(X^*)))$, $K_Y: Y \rightarrow Y^{**}$ is the canonical embedding of Y into Y^{**} , $J_p: C(B_1(X^*)) \rightarrow L^p(B_1(X^*), \mu)$ is the natural inclusion map and $A: X \rightarrow C(B_1(X^*))$ and $B: L^p(B_1(X^*), \mu) \rightarrow Y^{**}$ are bounded linear operators.

Notation 2.4.2. The class of all p -integral operators is denoted by $\mathcal{I}_p$.

REMARK 2.4.3. The set $\mathcal{I}_p(X, Y)$ of all p -integral operators from X into Y is a Banach space when provided with the norm

$$\|T\|_{\mathcal{I}_p} = \inf \|B\|$$

where the infimum is taken over all possible factorizations described by Definition 2.4.1. For the proof of this result and more we refer to [12, p.27].

Theorem 2.4.4. $\mathcal{I}_p$ is an operator ideal.

Proof. See [12, p.27].

Theorem 2.4.5. For $1 \leq p < \infty$, the inclusion map

$$J_p: C(I) \rightarrow L^p(I, \mu)$$

is p -integral.

Proof. See [12, p.29].

§5. The Injective Hull of an Operator Ideal.

In this section we study the injective hull of an operator ideal, which is an ideal constructed from a given one.

Definition (Pietsch) 2.5.1. A Banach space Y possesses the extension property if for every injection $J \in L(X_0, X)$ and every operator $T_0 \in L(X_0, Y)$ there exists an extension $T \in L(X, Y)$.

$$\begin{array}{ccc} X_0 & \xrightarrow{T_0} & Y \\ J \downarrow & \nearrow T & \\ X & & \end{array}$$

The metric extension property means that for every metric injection $J \in L(X_0, X)$ and every operator $T_0 \in L(X_0, Y)$ we can find $T \in L(X, Y)$ with $T_0 = TJ$ and $\|T\| = \|T_0\|$.

Definition 2.5.2. Let S be an arbitrary index set. Define

$$\ell^\infty(S) = \{ f \mid f: S \longrightarrow \mathbb{R}, f \text{ is bounded} \}$$

and for $f \in \ell^\infty(S)$,

$$\|f\|_\infty = \sup_{s \in S} |f(s)|.$$

REMARK 2.5.3. $(\ell^\infty(S), \|\cdot\|_\infty)$ is a Banach space.

Theorem 2.5.4. Let S be an index set. Then $\ell^\infty(S)$ has the metric extension property.

Proof. Let X_0 be a subspace of a Banach space $(X, \|\cdot\|)$. We claim that if

$$T_0: X_0 \subseteq X \longrightarrow \ell^\infty(S)$$

is a continuous linear operator, then T_0 extends to a continuous linear operator

$$T: X \longrightarrow \ell^\infty(S)$$

such that $\|T\|_X = \|T_0\|_{X_0}$.

Let $s \in S$. Then

$$\begin{aligned}\delta_s: \ell^\infty(S) &\longrightarrow \mathbb{R} \\ f &\longmapsto f(s)\end{aligned}$$

is a continuous linear functional on $\ell^\infty(S)$. Moreover, $\|\delta_s\| \leq 1$ since

$$|\delta_s(f)| = |f(s)| \leq \sup_{s \in S} |f(s)| = \|f\|_\infty.$$

Now

$$\delta_s \circ T_0: X_0 \subseteq X \longrightarrow \mathbb{R}$$

is a continuous linear functional on the subspace X_0 of the Banach space X . Thus, by the Hahn-Banach Theorem [9, p.221], there exists a continuous linear functional T_s on X which is an extension of $\delta_s \circ T_0$ to X and has the same norm. That is

$$\|T_s\|_X = \sup_{\|x\| \leq 1} |T_s(x)| = \sup_{\|x_0\| \leq 1} |(\delta_s \circ T_0)(x_0)| = \|\delta_s \circ T_0\|_{X_0}.$$

Finally, define

$$\begin{aligned}T: X &\longrightarrow \ell^\infty(S) \\ x &\longmapsto (T_s(x))_{s \in S}.\end{aligned}$$

Clearly, T is linear. Further, T is continuous since for all $x \in B_1(X)$ we have

$$\begin{aligned}\|T(x)\| &= \left\| (T_s(x))_{s \in S} \right\|_\infty \\ &= \sup_{s \in S} |T_s(x)| \\ &\leq \sup_{s \in S} \|T_s\|_X \\ &= \sup_{s \in S} \|\delta_s \circ T_0\|_{X_0} \\ &\leq \sup_{s \in S} \|\delta_s\| \|T_0\|_{X_0} \\ &\leq \|T_0\|_{X_0}.\end{aligned}$$

Taking the supremum over all $x \in B_1(X)$ we get $\|T\|_X \leq \|T_0\|_{X_0}$. Since T is an extension of T_0 to X we have $\|T\|_X \geq \|T_0\|_{X_0}$. Hence $\|T\|_X = \|T_0\|_{X_0}$. $\square$

REMARK 2.5.5. Let X be a Banach space. Then

$$\begin{aligned} J_X: X &\longrightarrow \ell^\infty(B_1(X^*)) \\ x &\longmapsto (\langle x^*, x \rangle)_{x^* \in B_1(X^*)} \end{aligned}$$

is a metric injection.

Definition 2.5.6. Let $\mathcal{O}$ be an operator ideal. An operator $T \in L(X, Y)$ belongs to the injective hull $\mathcal{O}^i$ if $J_Y T \in \mathcal{O}(X, \ell^\infty(B_1(Y^*)))$.

Lemma 2.5.7. $\mathcal{O}^i(X, Y)$ is a subspace of $L(X, Y)$.

Proof. Follows directly from the fact that $\mathcal{O}(X, Y)$ is a subspace of $L(X, Y)$. $\square$

Definition 2.5.8. Let $T \in \mathcal{O}^i(X, Y)$. Define

$$\|T\|_{\mathcal{O}^i} = \|J_Y T\|_{\mathcal{O}}.$$

Lemma 2.5.9. $\|\cdot\|_{\mathcal{O}^i}$ is a norm on $\mathcal{O}^i(X, Y)$. Moreover,

$$\|T\| \leq \|T\|_{\mathcal{O}^i} \quad \text{for all } T \in \mathcal{O}^i(X, Y).$$

Proof. The conditions needed for $\|\cdot\|_{\mathcal{O}^i}$ to be a norm are obviously satisfied. Further, let $T \in \mathcal{O}^i(X, Y)$, then, since $\|J_Y T\| = \|T\|$ and $J_Y T \in \mathcal{O}(X, \ell^\infty(B_1(Y^*)))$, we have $\|T\| = \|J_Y T\| \leq \|J_Y T\|_{\mathcal{O}} = \|T\|_{\mathcal{O}^i}$. $\square$

Theorem 2.5.10. $\mathcal{O}^i$ is an operator ideal.

Proof. By Lemma 2.5.7, $\mathcal{O}^i(X, Y)$ is a subspace of $L(X, Y)$. Let $x^* \in X^*$ and $y \in Y$. Then it is clear that $x^* \otimes y \in \mathcal{O}^i(X, Y)$. Since

$$\|x^*\| \|y\| = \|x^* \otimes y\| \leq \|x^* \otimes y\|_{\mathcal{O}^i} = \|J_Y(x^* \otimes y)\|_{\mathcal{O}} \leq \|x^* \otimes y\|_{\mathcal{O}} = \|x^*\| \|y\|$$

we have $\|x^* \otimes y\|_{\mathcal{O}^i} = \|x^*\| \|y\|$.

Finally, we show that $STR \in \mathcal{O}^i(W, Z)$ for all $R \in L(W, X)$, $T \in \mathcal{O}^i(X, Y)$ and $S \in L(Y, Z)$. As $T \in \mathcal{O}^i(X, Y)$, we have $J_Y T \in \mathcal{O}(X, \ell^\infty(B_1(Y^*)))$. Consider the following diagram :

$$\begin{array}{ccccc}
X & \xrightarrow{T} & Y & \xrightarrow{J_Y} & \ell^\infty(B_1(Y^*)) \\
R \uparrow & & \downarrow S & & \\
W & \xrightarrow{STR} & Z & \xrightarrow{J_Z} & \ell^\infty(B_1(Z^*))
\end{array}$$

We claim that $J_Z(STR) \in O(W, \ell^\infty(B_1(Z^*)))$. Indeed, since $\ell^\infty(B_1(Z^*))$ has the metric extension property, $J_Y \in L(Y, \ell^\infty(B_1(Y^*)))$ is a metric injection and $J_Z S \in L(Y, \ell^\infty(B_1(Z^*)))$, there exists $\hat{S} \in L(\ell^\infty(B_1(Y^*)), \ell^\infty(B_1(Z^*)))$ such that $\hat{S}J_Y = J_Z S$, as illustrated by the following diagram

$$\begin{array}{ccc}
Y & \xrightarrow{J_Z S} & \ell^\infty(B_1(Z^*)) \\
J_Y \downarrow & \nearrow \hat{S} & \\
\ell^\infty(B_1(Y^*)) & &
\end{array}$$

Now let us go back to the first diagram. Since $\hat{S} \in L(\ell^\infty(B_1(Y^*)), \ell^\infty(B_1(Z^*)))$, we obtain the commutative diagram

$$\begin{array}{ccccc}
X & \xrightarrow{T} & Y & \xrightarrow{J_Y} & \ell^\infty(B_1(Y^*)) \\
R \uparrow & & \downarrow S & & \downarrow \hat{S} \\
W & \xrightarrow{STR} & Z & \xrightarrow{J_Z} & \ell^\infty(B_1(Z^*))
\end{array}$$

Consequently, $J_Z(STR) = \hat{S}(J_Y T)R \in O(W, \ell^\infty(B_1(Z^*)))$ because O is an operator ideal and $J_Y T \in O(X, \ell^\infty(B_1(Y^*)))$. This proves that $STR \in \mathcal{O}^i(W, Z)$. $\square$

Theorem 2.5.11. $(\mathcal{O}^i(X, Y), \|\cdot\|_{\mathcal{O}^i})$ is a Banach space.

Proof. Let $T_1, T_2, \dots \in \mathcal{O}^i(X, Y)$ such that $\sum_{n=1}^{\infty} \|T_n\|_{\mathcal{O}^i} < \infty$. Our claim is that $\sum_{n=1}^{\infty} T_n \in \mathcal{O}^i(X, Y)$. By Definition 2.5.8, we have $\sum_{n=1}^{\infty} \|J_Y T_n\|_O < \infty$.

Hence $\sum_{n=1}^{\infty} J_Y T_n \in O(X, \ell^\infty(B_1(Y^*)))$ by completeness of $O(X, \ell^\infty(B_1(Y^*)))$. Since J_Y is continuous we have

$$\sum_{n=1}^{\infty} J_Y T_n = \lim_k \sum_{n=1}^k J_Y T_n = \lim_k J_Y \left(\sum_{n=1}^k T_n \right) = J_Y \left(\lim_k \sum_{n=1}^k T_n \right) = J_Y \left(\sum_{n=1}^{\infty} T_n \right).$$

So we get $J_Y \left(\sum_{n=1}^{\infty} T_n \right) \in O(X, \ell^\infty(B_1(Y^*)))$. Consequently $\sum_{n=1}^{\infty} T_n \in \mathcal{O}^i(X, Y)$. Further,

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} T_n - \sum_{n=1}^k T_n \right\|_{\mathcal{O}^i} &= \left\| \sum_{n=k+1}^{\infty} T_n \right\|_{\mathcal{O}^i} \\ &= \left\| \sum_{n=k+1}^{\infty} J_Y T_n \right\|_O \\ &\leq \sum_{n=k+1}^{\infty} \|J_Y T_n\|_O \\ &\longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \quad \square \end{aligned}$$

Theorem 2.5.12. *Let Y be a Banach space possessing the metric extension property. Then $\mathcal{O}^i(X, Y) = O(X, Y)$.*

Proof. We first show that $O(X, Y) \subseteq \mathcal{O}^i(X, Y)$. Let $T \in O(X, Y)$. Then $J_Y T \in O(X, \ell^\infty(B_1(Y^*)))$. Hence $T \in \mathcal{O}^i(X, Y)$.

For the converse we have $J_Y : Y \longrightarrow \ell^\infty(B_1(Y^*))$ is a metric injection, $I_Y : Y \longrightarrow Y \in L(Y, Y)$ and Y has the metric extension property. Consequently, there exists $B \in L(\ell^\infty(B_1(Y^*)), Y)$ such that $BJ_Y = I_Y$ as shown in the following diagram :

$$\begin{array}{ccc} Y & \xrightarrow{I_Y} & Y \\ J_Y \downarrow & \nearrow B & \\ \ell^\infty(B_1(Y^*)) & & \end{array}$$

Now let $T \in \mathcal{O}^i(X, Y)$. It follows that $J_Y T \in O(X, \ell^\infty(B_1(Y^*)))$. Since O is an operator ideal, we have $BJ_Y T \in O(X, Y)$. As $BJ_Y = I_Y$, we get $T \in O(X, Y)$. Hence $\mathcal{O}^i(X, Y) \subseteq O(X, Y)$. $\square$

Definition 2.5.13. An operator ideal $\mathcal{O}$ is called injective if $\mathcal{O}^i = \mathcal{O}$.

Lemma 2.5.14. For $1 \leq p < \infty$, Π_p is an injective ideal.

Proof. We claim that $\Pi_p^i = \Pi_p$. Let $T \in \Pi_p(X, Y)$. Then, since Π_p is an operator ideal we have $J_Y T \in \Pi_p(X, \ell^\infty(B_1(Y^*)))$. Hence $T \in \Pi_p^i(X, Y)$.

Conversely, let $T \in \Pi_p^i(X, Y)$. It follows that $J_Y T \in \Pi_p(X, \ell^\infty(B_1(Y^*)))$. Because J_Y is an isometry we have $T \in \Pi_p(X, Y)$. Consequently, $\Pi_p^i = \Pi_p$. $\square$

Theorem 2.5.15. For $1 \leq p < \infty$, $\mathcal{I}_p^i = \Pi_p$.

Proof. By Theorem 2.2.16, we have $T \in \Pi_p(X, \ell^\infty(B_1(Y^*)))$ if and only if it admits the factorization

$$\begin{array}{ccc} X & \xrightarrow{T} & \ell^\infty(B_1(Y^*)) \\ A \downarrow & & \uparrow B \\ C(B_1(X^*)) & \xrightarrow{J_p} & L^p(B_1(X^*), \mu) \end{array}$$

Since $J_p \in \mathcal{I}_p(C(B_1(X^*)), L^p(B_1(X^*), \mu))$ and $\mathcal{I}_p$ is an operator ideal, it follows that $T = BJ_pA \in \Pi_p(X, \ell^\infty(B_1(Y^*)))$.

On the other hand, by Definition 2.4.1, we have $T \in \mathcal{I}_p(X, \ell^\infty(B_1(Y^*)))$ if and only if it has the factorization

$$\begin{array}{ccccc} X & \xrightarrow{T} & \ell^\infty(B_1(Y^*)) & \xrightarrow{K_{\ell^\infty(B_1(Y^*))}} & (\ell^\infty(B_1(Y^*)))^{**} \\ A \downarrow & & & & \uparrow B \\ C(B_1(X^*)) & \xrightarrow{J_p} & & & L^p(B_1(X^*), \mu) \end{array}$$

As $J_p \in \Pi_p(C(B_1(X^*)), L^p(B_1(X^*), \mu))$ and Π_p is an operator ideal, we have $K_{\ell^\infty(B_1(Y^*))} T = BJ_pA \in \Pi_p(X, (\ell^\infty(B_1(Y^*)))^{**})$. Since $K_{\ell^\infty(B_1(Y^*))}$ is an isometric embedding, we get $T \in \Pi_p(X, \ell^\infty(B_1(Y^*)))$.

Hence, we conclude that

$$(\Pi_p(X, \ell^\infty(B_1(Y^*))), \| \cdot \|_{\pi_p}) = (\mathcal{I}_p(X, \ell^\infty(B_1(Y^*))), \| \cdot \|_{\mathcal{I}_p}).$$

Consequently, if $T \in \Pi_p(X, Y)$ we have $J_Y T \in \mathcal{I}_p(X, \ell^\infty(B_1(Y^*)))$ and hence $T \in \mathcal{I}_p^i(X, Y)$. Conversely, if $T \in \mathcal{I}_p^i(X, Y)$, then $J_Y T \in \mathcal{I}_p(X, \ell^\infty(B_1(Y^*)))$ and so $J_Y T \in \Pi_p(X, \ell^\infty(B_1(Y^*)))$. Because J_Y is an isometry, we have $T \in \Pi_p(X, Y)$. Hence $\mathcal{I}_p^i = \Pi_p$. $\square$

CHAPTER III

STABILITY

§1. Stability – The Definition.

In this chapter we give new and simple proofs for the stability of the p -summing, the p -nuclear and the p -integral operators with respect to $\|\cdot\|_\varepsilon$ -norm, as well as for the injective hull O^i of the ε -stable operator ideal O . The theorems of this chapter are proved using the factorization theorems studied in Chapter II together with the theorems on the p -nuclear tensor product on L^p -spaces studied in Chapter I.

Throughout, X_1, X_2, Y_1 , and Y_2 will denote real Banach spaces.

Definition 3.1.1. *An operator ideal O is said to be tensor stable with respect to the tensor norm $\|\cdot\|_\alpha$ (or α -tensor stable) if for all $A \in O(X_1, Y_1)$ and $B \in O(X_2, Y_2)$, $A \overset{\alpha}{\otimes} B \in O(X_1 \overset{\alpha}{\otimes} X_2, Y_1 \overset{\alpha}{\otimes} Y_2)$.*

Theorem 3.1.2. *L is ε - and π -stable.*

Proof. Follows from Theorem 1.5.4. $\square$

Theorem 3.1.3. *Let $O(X, Y)$ denote an ideal of operators. Then for any reasonable crossnorm $\|\cdot\|_\alpha \geq \|\cdot\|_\varepsilon$ and operators $T_1 : X_1 \rightarrow Y_1, T_2 : X_2 \rightarrow Y_2$, if $T_1 \overset{\alpha}{\otimes} T_2 \in O(X_1 \overset{\alpha}{\otimes} X_2, Y_1 \overset{\alpha}{\otimes} Y_2)$ and $T_2 \neq 0$ then $T_1 \in O(X_1, Y_1)$.*

Proof. Choose $x_2 \in X_2$ such that $\|T_2 x_2\| = 1$ and $y_2^* \in Y_2^*$, $\|y_2^*\| = 1$, such that $\langle T_2 x_2, y_2^* \rangle = 1$. Define the operators

$$\begin{aligned} P: X_1 &\longrightarrow X_1 \overset{\alpha}{\otimes} X_2 \\ x_1 &\longmapsto x_1 \otimes x_2 \end{aligned}$$

and

$$\begin{aligned} Q: Y_1 \overset{\alpha}{\otimes} Y_2 &\longrightarrow Y_1 \\ \sum_{i=1}^n x_i \otimes y_i &\longmapsto \sum_{i=1}^n \langle y_2^*, y_i \rangle x_i. \end{aligned}$$

Then P and Q are linear bounded operators since for all $x_1 \in X_1$

$$\|Px_1\| = \|x_1 \otimes x_2\|_\alpha = \|x_2\| \|x_1\|$$

and for all $\sum_{i=1}^n x_i \otimes y_i \in Y_1 \overset{\alpha}{\otimes} Y_2$

$$\begin{aligned} \left\| Q \left(\sum_{i=1}^n x_i \otimes y_i \right) \right\| &= \left\| \sum_{i=1}^n \langle y_2^*, y_i \rangle x_i \right\|_{Y_1} \\ &= \sup_{y_1^* \in B_1(Y_1^*)} \left| \sum_{i=1}^n \langle y_2^*, y_i \rangle \langle y_1^*, x_i \rangle \right| \\ &\leq \sup_{\substack{y_1^* \in B_1(Y_1^*) \\ y_2^* \in B_1(Y_2^*)}} \left| \sum_{i=1}^n \langle y_2^*, y_i \rangle \langle y_1^*, x_i \rangle \right| \\ &= \left\| \sum_{i=1}^n x_i \otimes y_i \right\|_\varepsilon \\ &\leq \left\| \sum_{i=1}^n x_i \otimes y_i \right\|_\alpha. \end{aligned}$$

Furthermore, for all $x_1 \in X_1$ we have

$$\begin{aligned} (Q(T_1 \overset{\alpha}{\otimes} T_2)P)(x_1) &= (Q(T_1 \overset{\alpha}{\otimes} T_2))(x_1 \otimes x_2) \\ &= Q(T_1 x_1 \otimes T_2 x_2) \\ &= T_1 x_1. \end{aligned}$$

Consequently, $Q(T_1 \overset{\alpha}{\otimes} T_2)P = T_1$. Since $T_1 \overset{\alpha}{\otimes} T_2 \in O(X_1 \overset{\alpha}{\otimes} X_2, Y_1 \overset{\alpha}{\otimes} Y_2)$ and O is an operator ideal, $T_1 \in O(X_1, Y_1)$. $\square$

Theorem 3.1.4. If $T_1: X_1 \longrightarrow Y_1$ and $T_2: X_2 \longrightarrow Y_2$ are compact operators, then so is $T_1 \overset{\pi}{\otimes} T_2: X_1 \overset{\pi}{\otimes} X_2 \longrightarrow Y_1 \overset{\pi}{\otimes} Y_2$.

Proof. Let B_1 and B_2 denote the closed unit balls $B_1(X_1)$ and $B_1(X_2)$ respectively. Our claim is that $(T_1 \overset{\pi}{\otimes} T_2)(B_1(X_1 \overset{\pi}{\otimes} X_2))$ is relatively compact.

Consider

$$T_1(B_1) \otimes T_2(B_2) = \{ T_1x \otimes T_2y \mid x \in B_1, y \in B_2 \}.$$

Since T_1 is compact, we have $\overline{T_1(B_1)}$ is compact. Similarly, $\overline{T_2(B_2)}$ is compact. It follows that $\overline{T_1(B_1)} \otimes \overline{T_2(B_2)}$ is also compact. Hence the closed convex hull $\overline{\text{co}}(\overline{T_1(B_1)} \otimes \overline{T_2(B_2)})$ is compact. Furthermore, we claim that

$$(T_1 \otimes T_2)(B_1(X_1 \overset{\pi}{\otimes} X_2)) \subseteq \overline{\text{co}}(\overline{T_1(B_1)} \otimes \overline{T_2(B_2)}).$$

Indeed, let $z \in B_1(X_1 \overset{\pi}{\otimes} X_2)$. Then, by Theorem 1.3.3, we have for all $k \in \mathbb{N}$ there exists $(\lambda_n^{(k)}) \in c_0$, $(x_n^{(k)}) \subset B_1$ and $(y_n^{(k)}) \subset B_2$ such that z has the representation

$$z = z^{(k)} = \sum_{n=1}^{\infty} \lambda_n^{(k)} x_n^{(k)} \otimes y_n^{(k)}$$

where

$$1 \leq \sum_{n=1}^{\infty} \lambda_n^{(k)} \leq 1 + \frac{1}{k}.$$

Consequently,

$$(T_1 \otimes T_2)(z) = \sum_{n=1}^{\infty} \lambda_n^{(k)} T_1 x_n^{(k)} \otimes T_2 y_n^{(k)}.$$

Let

$$\gamma_k = \sum_{n=1}^{\infty} \lambda_n^{(k)}.$$

Then $\gamma_k^{-1}(T_1 \otimes T_2)(z) = \gamma_k^{-1}(T_1 \otimes T_2)(z^{(k)}) \in \overline{\text{co}}(\overline{T_1(B_1)} \otimes \overline{T_2(B_2)})$ for all k . Therefore, $(T_1 \otimes T_2)(z) \in \overline{\text{co}}(\overline{T_1(B_1)} \otimes \overline{T_2(B_2)})$ since $\gamma_k \longrightarrow 1$. Thus we have proved that $(T_1 \otimes T_2)(B_1(X_1 \overset{\pi}{\otimes} X_2)) \subseteq \overline{\text{co}}(\overline{T_1(B_1)} \otimes \overline{T_2(B_2)})$. As $\overline{\text{co}}(\overline{T_1(B_1)} \otimes \overline{T_2(B_2)})$ is compact, we have $(T_1 \otimes T_2)(B_1(X_1 \overset{\pi}{\otimes} X_2))$ is relatively compact. $\square$

§2. Stability of p -Summing Operators.

J.R.Holub proved in 1970, [5], that the absolutely p -summing operators are ε -stable but not π -stable. In 1974, [6], Holub reproved the ε -stability of the absolutely p -summing operators using factorization techniques. This section gives details of the latter proof. It also studies that part of the former proof showing that the absolutely p -summing operators are not π -stable.

Lemma 3.2.1. *If $J_Y : Y \longrightarrow \ell^\infty(B_1(Y^*))$ is the canonical isometry, then*

$$J_{Y_1} \overset{\varepsilon}{\otimes} J_{Y_2} : Y_1 \overset{\varepsilon}{\otimes} Y_2 \longrightarrow \ell^\infty(B_1(Y_1^*)) \overset{\varepsilon}{\otimes} \ell^\infty(B_1(Y_2^*))$$

is an isometric embedding.

Proof. Let $\sum_{i=1}^n x_i \otimes y_i \in Y_1 \otimes Y_2$. Then, by Definitions 1.2.1 and 2.5.2, we have

$$\begin{aligned} \left\| (J_{Y_1} \overset{\varepsilon}{\otimes} J_{Y_2}) \left(\sum_{i=1}^n x_i \otimes y_i \right) \right\|_\varepsilon &= \left\| \sum_{i=1}^n J_{Y_1} x_i \otimes J_{Y_2} y_i \right\|_\varepsilon \\ &= \left\| \sum_{i=1}^n f_{x_i} \otimes f_{y_i} \right\|_\varepsilon \\ &= \sup_{\substack{\psi \in (\ell^\infty(B_1(Y_1^*)))^* \\ \|\psi\| \leq 1}} \left\| \sum_{i=1}^n \langle \psi, f_{x_i} \rangle f_{y_i} \right\|_{\ell^\infty(B_1(Y_2^*))} \\ &= \sup_{\substack{\psi \in (\ell^\infty(B_1(Y_1^*)))^* \\ \|\psi\| \leq 1}} \sup_{y_2^* \in B_1(Y_2^*)} \left| \sum_{i=1}^n \langle \psi, f_{x_i} \rangle \langle y_2^*, y_i \rangle \right| \\ &= \sup_{y_2^* \in B_1(Y_2^*)} \sup_{\substack{\psi \in (\ell^\infty(B_1(Y_1^*)))^* \\ \|\psi\| \leq 1}} \left| \left\langle \sum_{i=1}^n \langle y_2^*, y_i \rangle f_{x_i}, \psi \right\rangle \right| \\ &= \sup_{y_2^* \in B_1(Y_2^*)} \left\| \sum_{i=1}^n \langle y_2^*, y_i \rangle f_{x_i} \right\|_{\ell^\infty(B_1(Y_1^*))} \\ &= \sup_{y_2^* \in B_1(Y_2^*)} \sup_{y_1^* \in B_1(Y_1^*)} \left| \sum_{i=1}^n \langle y_2^*, y_i \rangle \langle y_1^*, x_i \rangle \right| \\ &= \left\| \sum_{i=1}^n x_i \otimes y_i \right\|_\varepsilon. \end{aligned}$$

This proves that $J_{Y_1} \overset{\varepsilon}{\otimes} J_{Y_2}$ is an isometric embedding. $\square$

Theorem 3.2.2. If $T_1: X_1 \longrightarrow Y_1$ and $T_2: X_2 \xrightarrow{\varepsilon} Y_2$ are absolutely p -summing, $1 \leq p < \infty$, then so is $T_1 \overset{\varepsilon}{\otimes} T_2: X_1 \overset{\varepsilon}{\otimes} X_2 \xrightarrow{\varepsilon} Y_1 \overset{\varepsilon}{\otimes} Y_2$.

Proof. Let K_1 and K_2 denote the closed unit balls in X_1^* and X_2^* respectively. Furthermore, let Γ_1 and Γ_2 denote the closed unit balls of Y_1^* and Y_2^* respectively.

Since $T_1 \in \Pi_p(X_1, Y_1)$, then, by Theorem 2.2.16 the following diagram commutes :

$$\begin{array}{ccccc} X_1 & \xrightarrow{T_1} & Y_1 & \xrightarrow{J_{Y_1}} & \ell^\infty(\Gamma_1) \\ A_1 \downarrow & & & & \uparrow B_1 \\ C(K_1) & \xrightarrow{J_{1p}} & & & L^p(K_1, \mu_1) \end{array}$$

where $\mu_1 \in B_1(M(K_1))$, $J_{Y_1}: Y_1 \longrightarrow \ell^\infty(\Gamma_1)$ is the natural isometric embedding, $J_{1p}: C(K_1) \longrightarrow L^p(K_1, \mu_1)$ is the natural inclusion map and $A_1: X_1 \longrightarrow C(K_1)$ and $B_1: L^p(K_1, \mu_1) \longrightarrow \ell^\infty(\Gamma_1)$ are bounded operators.

Similarly, $T_2 \in \Pi_p(X_2, Y_2)$ implies the following factorization :

$$\begin{array}{ccccc} X_2 & \xrightarrow{T_2} & Y_2 & \xrightarrow{J_{Y_2}} & \ell^\infty(\Gamma_2) \\ A_2 \downarrow & & & & \uparrow B_2 \\ C(K_2) & \xrightarrow{J_{2p}} & & & L^p(K_2, \mu_2) \end{array}$$

where $\mu_2 \in B_1(M(K_2))$, $J_{Y_2}: Y_2 \longrightarrow \ell^\infty(\Gamma_2)$ is the natural isometric embedding, $J_{2p}: C(K_2) \longrightarrow L^p(K_2, \mu_2)$ is the natural inclusion map and $A_2: X_2 \longrightarrow C(K_2)$ and $B_2: L^p(K_2, \mu_2) \longrightarrow \ell^\infty(\Gamma_2)$ are continuous operators.

Now, $T_1 \overset{\varepsilon}{\otimes} T_2$ can be easily seen to have the factorization

$$\begin{array}{ccccc}
X_1 \overset{\varepsilon}{\otimes} X_2 & \xrightarrow{T_1 \overset{\varepsilon}{\otimes} T_2} & Y_1 \overset{\varepsilon}{\otimes} Y_2 & \xrightarrow{J_{Y_1} \overset{\varepsilon}{\otimes} J_{Y_2}} & \ell^\infty(\Gamma_1) \overset{\varepsilon}{\otimes} \ell^\infty(\Gamma_2) \\
A_1 \overset{\varepsilon}{\otimes} A_2 \downarrow & & & & \uparrow B_1 \overset{\varepsilon}{\otimes} B_2 \\
C(K_1) \overset{\varepsilon}{\otimes} C(K_2) & \xrightarrow{J_{1p} \overset{\varepsilon}{\otimes} J_{2p}} & L^p(K_1, \mu_1) \overset{\varepsilon}{\otimes} L^p(K_2, \mu_2) & &
\end{array}$$

Next, we claim that $J_{1p} \overset{\varepsilon}{\otimes} J_{2p}$ can be factored as

$$\begin{array}{ccc}
C(K_1) \overset{\varepsilon}{\otimes} C(K_2) & \xrightarrow{J_{1p} \overset{\varepsilon}{\otimes} J_{2p}} & L^p(K_1, \mu_1) \overset{\varepsilon}{\otimes} L^p(K_2, \mu_2) \\
\downarrow \phi & & \uparrow I \\
& & L^p(K_1, \mu_1) \overset{\nu_p}{\otimes} L^p(K_2, \mu_2) \\
& & \uparrow \psi \\
C(K_1 \times K_2) & \xrightarrow{J_p} & L^p(K_1 \times K_2, \mu_1 \times \mu_2)
\end{array}$$

where

$$1. \quad \phi: C(K_1) \overset{\varepsilon}{\otimes} C(K_2) \longrightarrow C(K_1 \times K_2)$$

$$\sum_{i=1}^n f_i \otimes g_i \longmapsto h$$

$$h(x_1^*, x_2^*) = \sum_{i=1}^n f_i(x_1^*) g_i(x_2^*).$$

By Corollary 1.2.5, ϕ is an isometric isomorphism of $C(K_1) \overset{\varepsilon}{\otimes} C(K_2)$ onto $C(K_1 \times K_2)$.

$$2. \quad J_p: C(K_1 \times K_2) \longrightarrow L^p(K_1 \times K_2, \mu_1 \times \mu_2)$$

is the natural inclusion of C into L^p .

$$3. \quad \psi: L^p(K_1 \times K_2, \mu_1 \times \mu_2) \longrightarrow L^p(K_1, \mu_1) \overset{\nu_p}{\otimes} L^p(K_2, \mu_2)$$

$$h \longmapsto \sum_{i=1}^n f_i \otimes g_i$$

$$\sum_{i=1}^n f_i(x_1^*) g_i(x_2^*) = h(x_1^*, x_2^*).$$

By Theorem 1.4.5 ψ is an isometric isomorphism of $L^p(K_1 \times K_2, \mu_1 \times \mu_2)$ onto $L^p(K_1, \mu_1) \overset{\nu_p}{\otimes} L^p(K_2, \mu_2)$.

4. $I: L^p(K_1, \mu_1) \overset{\nu_p}{\otimes} L^p(K_2, \mu_2) \longrightarrow L^p(K_1, \mu_1) \overset{\varepsilon}{\otimes} L^p(K_2, \mu_2)$
is a continuous inclusion given in Corollary 1.4.4.

To verify our claim let $\sum_{i=1}^n f_i \otimes g_i \in C(K_1) \overset{\varepsilon}{\otimes} C(K_2)$. Then, we have

$$\begin{aligned} (J_{1p} \overset{\varepsilon}{\otimes} J_{2p}) \left(\sum_{i=1}^n f_i \otimes g_i \right) &= \sum_{i=1}^n J_{1p} f_i \otimes J_{2p} g_i \\ &= \sum_{i=1}^n f_i \otimes g_i. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (I \circ \psi \circ J_p \circ \phi) \left(\sum_{i=1}^n f_i \otimes g_i \right) &= (I \circ \psi \circ J_p)(h) \\ &= (I \circ \psi)(h) \\ &= I \left(\sum_{i=1}^n f_i \otimes g_i \right) \\ &= \sum_{i=1}^n f_i \otimes g_i. \end{aligned}$$

Therefore, $J_{1p} \overset{\varepsilon}{\otimes} J_{2p} = I \circ \psi \circ J_p \circ \phi$ as claimed. Since $J_p: C(K_1 \times K_2) \longrightarrow L^p(K_1 \times K_2, \mu_1 \times \mu_2)$ is absolutely p -summing, it follows that $J_{1p} \overset{\varepsilon}{\otimes} J_{2p}$ is absolutely p -summing.

As

$$(J_{Y_1} \overset{\varepsilon}{\otimes} J_{Y_2}) \circ (T_1 \overset{\varepsilon}{\otimes} T_2) = (B_1 \overset{\varepsilon}{\otimes} B_2) \circ (J_{1p} \overset{\varepsilon}{\otimes} J_{2p}) \circ (A_1 \overset{\varepsilon}{\otimes} A_2),$$

it follows that $(J_{Y_1} \overset{\varepsilon}{\otimes} J_{Y_2}) \circ (T_1 \overset{\varepsilon}{\otimes} T_2)$ is also absolutely p -summing. But, by Lemma 3.2.1, $J_{Y_1} \overset{\varepsilon}{\otimes} J_{Y_2}$ is an isometric embedding of $Y_1 \overset{\varepsilon}{\otimes} Y_2$ into $\ell^\infty(\Gamma_1) \overset{\varepsilon}{\otimes} \ell^\infty(\Gamma_2)$. Hence we conclude that $T_1 \overset{\varepsilon}{\otimes} T_2$ is an absolutely p -summing operator. $\square$

Lemma 3.2.3. *Let (e_n) denote the unit vector basis for ℓ^1 . Then ℓ^1 is isometrically isomorphic to $\overline{\text{Span}\{e_n \otimes e_n : n \in \mathbb{N}\}}$ in $\ell^1 \overset{\pi}{\otimes} \ell^1$.*

Proof. Let $X = \overline{\text{Span}\{e_n \otimes e_n : n \in \mathbb{N}\}}$ in $\ell^1 \overset{\pi}{\otimes} \ell^1$. Then every $A \in X$ can be written in the form

$$A = \sum_{n=1}^{\infty} a_n e_n \otimes e_n \quad \text{for some } (a_n) \subset \mathbb{R}.$$

Further, by Corollary 1.3.5, we have

$$\ell^1 \overset{\pi}{\otimes} \ell^1 = \ell^1(\mathbb{N} \times \mathbb{N}).$$

It follows that $\|A\|_{\ell^1 \overset{\pi}{\otimes} \ell^1} = \|A\|_{\ell^1(\mathbb{N} \times \mathbb{N})} = \sum_{n=1}^{\infty} |a_n| < \infty$.

Now, define

$$J: X \longrightarrow \ell^1$$

$$\sum_{n=1}^{\infty} a_n e_n \otimes e_n \longmapsto \sum_{n=1}^{\infty} a_n e_n.$$

Then clearly J is an isometric isomorphism of X onto ℓ^1 . $\square$

Lemma 3.2.4. *Let (e_n) denote the unit vector basis for ℓ^2 . Then ℓ^1 is isometrically isomorphic to $\overline{\text{Span}\{e_n \otimes e_n : n \in \mathbb{N}\}}$ in $\ell^2 \overset{\pi}{\otimes} \ell^2$.*

Proof. Let $Y = \overline{\text{Span}\{e_n \otimes e_n : n \in \mathbb{N}\}}$ in $\ell^2 \overset{\pi}{\otimes} \ell^2$. Then every $A \in Y$ has the form

$$A = \sum_{n=1}^{\infty} a_n e_n \otimes e_n \quad \text{for some } (a_n) \subset \mathbb{R}.$$

Let $\mathcal{C}_1(\ell^2)$ be the trace class operators [11, p.212]. Then, we have

$$\ell^2 \overset{\pi}{\otimes} \ell^2 = \mathcal{C}_1(\ell^2).$$

Now, define

$$J: Y \longrightarrow \ell^1$$

$$\sum_{n=1}^{\infty} a_n e_n \otimes e_n \longmapsto \sum_{n=1}^{\infty} a_n e_n.$$

Then, J is a linear onto map. Moreover,

$$\begin{aligned}
\left\| J\left(\sum_{n=1}^{\infty} a_n e_n \otimes e_n\right) \right\|_{\ell^1} &= \left\| \sum_{n=1}^{\infty} a_n e_n \right\|_{\ell^1} \\
&= \|(a_n)\|_{\alpha(1)} \\
&= \left\| \sum_{n=1}^{\infty} a_n e_n \otimes e_n \right\|_{\mathcal{C}_1(\ell^2)} \\
&= \left\| \sum_{n=1}^{\infty} a_n e_n \otimes e_n \right\|_{\ell^2 \overset{\pi}{\otimes} \ell^2}.
\end{aligned}$$

Therefore, J is an isometry. Consequently, J is an isometric isomorphism of Y onto ℓ^1 . $\square$

Proposition (Holub, 1970) 3.2.5. *Let $i: \ell^1 \longrightarrow \ell^2$ be the (absolutely summing) injection map. Then $i \overset{\pi}{\otimes} i: \ell^1 \overset{\pi}{\otimes} \ell^1 \longrightarrow \ell^2 \overset{\pi}{\otimes} \ell^2$ is not absolutely p -summing for any $1 \leq p < \infty$.*

Proof. Suppose that $i \overset{\pi}{\otimes} i$ is absolutely p -summing for some $1 \leq p < \infty$. Let X denote $\overline{\text{Span}\{e_n \otimes e_n : n \in \mathbb{N}\}}$ in $\ell^1 \overset{\pi}{\otimes} \ell^1$ and Y denote $\overline{\text{Span}\{e_n \otimes e_n : n \in \mathbb{N}\}}$ in $\ell^2 \overset{\pi}{\otimes} \ell^2$. Then since $(i \overset{\pi}{\otimes} i)(e_n \otimes e_n) = e_n \otimes e_n$, we have $(i \overset{\pi}{\otimes} i)(X) = Y$.

The restriction

$$i \overset{\pi}{\otimes} i \Big|_X : X \longrightarrow \ell^2 \overset{\pi}{\otimes} \ell^2$$

is an absolutely p -summing operator. As $X = \ell^1$ and $Y = \ell^1$ by Lemmas 3.2.3 and 3.2.4 respectively, $i \overset{\pi}{\otimes} i \Big|_X$ induces an isometric isomorphism

$$J: \ell^1 \longrightarrow \ell^1$$

which is absolutely p -summing. But, by Theorem 2.2.16, every absolutely p -summing operator is weakly compact. It follows that $B_1(\ell^1) = J(B_1(\ell^1))$ is weakly compact. However, ℓ^1 is not reflexive. This contradiction shows that $i \overset{\pi}{\otimes} i$ is not absolutely p -summing for any $1 \leq p < \infty$. $\square$

§3. Stability of p -Nuclear Operators.

In this section we show stability of p -nuclear operators with respect to $\|\cdot\|_\varepsilon$.

Theorem 3.3.1. *Let $1 \leq p < \infty$. If $T_1 \in \mathcal{N}_p(X_1, Y_1)$ and $T_2 \in \mathcal{N}_p(X_2, Y_2)$, then $T_1 \overset{\varepsilon}{\otimes} T_2 \in \mathcal{N}_p(X_1 \overset{\varepsilon}{\otimes} X_2, Y_1 \overset{\varepsilon}{\otimes} Y_2)$.*

Proof. Since $T_1 \in \mathcal{N}_p(X_1, Y_1)$, then, by Definition 2.3.1, T_1 has the p -nuclear representation

$$T_1 = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n$$

where $(\lambda_n) \in \ell^p$, $(x_n^*) \in \ell^\infty(X_1^*)$ with $\|x_n^*\| = 1$ for all n , and $(y_n) \in \ell^{p^*} \langle Y_1 \rangle$. Similarly, as $T_2: X_2 \rightarrow Y_2$ is p -nuclear, it has a representation of the form

$$T_2 = \sum_{m=1}^{\infty} \xi_m w_m^* \otimes z_m$$

where $(\xi_m) \in \ell^p$, $(w_m^*) \in \ell^\infty(X_2^*)$ with $\|w_m^*\| = 1$ for all m , and $(z_m) \in \ell^{p^*} \langle Y_2 \rangle$.

Further, by Theorem 2.3.8, T_1 has the factorization

$$\begin{array}{ccc} X_1 & \xrightarrow{T_1} & Y_1 \\ A_1 \downarrow & & \uparrow B_1 \\ \ell^\infty & \xrightarrow{D_1} & \ell^p \end{array}$$

where

$$A_1 \in L(X_1, \ell^\infty), A_1(x_1) = (\langle x_n^*, x_1 \rangle),$$

$$D_1 \in L(\ell^\infty, \ell^p), D_1(a_n) = (\lambda_n a_n)$$

$$\text{and } B_1 \in L(\ell^p, Y_1), B_1(b_n) = \sum_{n=1}^{\infty} b_n y_n.$$

Similarly T_2 has the factorization

$$\begin{array}{ccc} X_2 & \xrightarrow{T_2} & Y_2 \\ A_2 \downarrow & & \uparrow B_2 \\ \ell^\infty & \xrightarrow{D_2} & \ell^p \end{array}$$

where

$$A_2 \in L(X_2, \ell^\infty), A_2(x_2) = (\langle w_m^*, x_2 \rangle),$$

$$D_2 \in L(\ell^\infty, \ell^p), D_2(a_m) = (\xi_m a_m)$$

$$\text{and } B_2 \in L(\ell^p, Y_2), B_2(b_m) = \sum_{m=1}^{\infty} b_m z_m.$$

First we note that $T_1 \overset{\varepsilon}{\otimes} T_2$ admits the following factorization :

$$\begin{array}{ccc} X_1 \overset{\varepsilon}{\otimes} X_2 & \xrightarrow{T_1 \overset{\varepsilon}{\otimes} T_2} & Y_1 \overset{\varepsilon}{\otimes} Y_2 \\ A_1 \overset{\varepsilon}{\otimes} A_2 \downarrow & & \uparrow B_1 \overset{\varepsilon}{\otimes} B_2 \\ \ell^\infty \overset{\varepsilon}{\otimes} \ell^\infty & \xrightarrow{D_1 \overset{\varepsilon}{\otimes} D_2} & \ell^p \overset{\varepsilon}{\otimes} \ell^p \end{array}$$

Indeed, let $x_1 \in X_1$ and $x_2 \in X_2$. Then, we have

$$\begin{aligned} (T_1 \overset{\varepsilon}{\otimes} T_2)(x_1 \otimes x_2) &= T_1 x_1 \otimes T_2 x_2 \\ &= \left(\sum_{n=1}^{\infty} \lambda_n \langle x_n^*, x_1 \rangle y_n \right) \otimes \left(\sum_{m=1}^{\infty} \xi_m \langle w_m^*, x_2 \rangle z_m \right) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda_n \xi_m \langle x_n^*, x_1 \rangle \langle w_m^*, x_2 \rangle y_n \otimes z_m \end{aligned}$$

and

$$\begin{aligned} &((B_1 \overset{\varepsilon}{\otimes} B_2) \circ (D_1 \overset{\varepsilon}{\otimes} D_2) \circ (A_1 \overset{\varepsilon}{\otimes} A_2))(x_1 \otimes x_2) \\ &= ((B_1 \overset{\varepsilon}{\otimes} B_2) \circ (D_1 \overset{\varepsilon}{\otimes} D_2))(A_1 x_1 \otimes A_2 x_2) \\ &= ((B_1 \overset{\varepsilon}{\otimes} B_2) \circ (D_1 \overset{\varepsilon}{\otimes} D_2))(\langle x_n^*, x_1 \rangle \otimes \langle w_m^*, x_2 \rangle) \\ &= (B_1 \overset{\varepsilon}{\otimes} B_2)(D_1(\langle x_n^*, x_1 \rangle) \otimes D_2(\langle w_m^*, x_2 \rangle)) \\ &= (B_1 \overset{\varepsilon}{\otimes} B_2)(\lambda_n \langle x_n^*, x_1 \rangle \otimes \xi_m \langle w_m^*, x_2 \rangle) \\ &= B_1(\lambda_n \langle x_n^*, x_1 \rangle) \otimes B_2(\xi_m \langle w_m^*, x_2 \rangle) \\ &= \left(\sum_{n=1}^{\infty} \lambda_n \langle x_n^*, x_1 \rangle y_n \right) \otimes \left(\sum_{m=1}^{\infty} \xi_m \langle w_m^*, x_2 \rangle z_m \right) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda_n \xi_m \langle x_n^*, x_1 \rangle \langle w_m^*, x_2 \rangle y_n \otimes z_m. \end{aligned}$$

Consequently, we have

$$T_1 \overset{\varepsilon}{\otimes} T_2 = (B_1 \overset{\varepsilon}{\otimes} B_2) \circ (D_1 \overset{\varepsilon}{\otimes} D_2) \circ (A_1 \overset{\varepsilon}{\otimes} A_2).$$

Next, we claim that $D_1 \overset{\varepsilon}{\otimes} D_2$ can be factored as

$$\begin{array}{ccc} \ell^\infty \overset{\varepsilon}{\otimes} \ell^\infty & \xrightarrow{D_1 \overset{\varepsilon}{\otimes} D_2} & \ell^p \overset{\varepsilon}{\otimes} \ell^p \\ \downarrow I_1 & & \uparrow I_2 \\ & & \ell^p \overset{\nu_p}{\otimes} \ell^p \\ & & \uparrow J \\ \ell^\infty(\mathbb{N} \times \mathbb{N}) & \xrightarrow{D} & \ell^p(\mathbb{N} \times \mathbb{N}) \end{array}$$

where

$$1. \ I_1: \ell^\infty \overset{\varepsilon}{\otimes} \ell^\infty \longrightarrow \ell^\infty(\mathbb{N} \times \mathbb{N})$$

$$(a_j) \otimes (b_k) \longmapsto (a_j b_k).$$

I_1 is an inclusion map by theorem 1.4.6.

$$2. \ D: \ell^\infty(\mathbb{N} \times \mathbb{N}) \longrightarrow \ell^p(\mathbb{N} \times \mathbb{N})$$

$$(a_{nm}) \longmapsto (\lambda_n \xi_m a_{nm}).$$

D is a diagonal operator from $\ell^\infty(\mathbb{N} \times \mathbb{N})$ into $\ell^p(\mathbb{N} \times \mathbb{N})$. By Theorem 2.3.8, D is p -nuclear. Further, $\|D\| = \|(\lambda_n)\|_{\alpha(p)} \|(\xi_m)\|_{\alpha(p)}$.

To see the claim, consider

$$\phi: \ell^p \overset{\nu_p}{\otimes} \ell^p \longrightarrow \ell^p(\mathbb{N} \times \mathbb{N})$$

$$(a_j) \otimes (b_k) \longmapsto (a_j b_k).$$

By Theorem 1.4.5, ϕ is an isometric isomorphism of $\ell^p \overset{\nu_p}{\otimes} \ell^p$ onto $\ell^p(\mathbb{N} \times \mathbb{N})$. Let $J = \phi^{-1}$. Then $J \in L(\ell^p(\mathbb{N} \times \mathbb{N}), \ell^p \overset{\nu_p}{\otimes} \ell^p)$. Also, by Corollary 1.4.4, the inclusion map

$$\begin{aligned} I_2: \ell^p \overset{\nu_p}{\otimes} \ell^p &\longrightarrow \ell^p \overset{\varepsilon}{\otimes} \ell^p \\ (a_j) \otimes (b_k) &\longmapsto (a_j) \otimes (b_k) \end{aligned}$$

is continuous. Now, we have

$$\begin{aligned}(D_1 \overset{\varepsilon}{\otimes} D_2)((a_j) \otimes (b_k)) &= D_1(a_j) \otimes D_2(b_k) \\ &= (\lambda_j a_j) \otimes (\xi_k b_k).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(I_2 \circ J \circ D \circ I_1)((a_j) \otimes (b_k)) &= (I_2 \circ J \circ D)(a_j b_k) \\ &= (I_2 \circ J)(\lambda_j \xi_k a_j b_k) \\ &= I_2((\lambda_j a_j) \otimes (\xi_k b_k)) \\ &= (\lambda_j a_j) \otimes (\xi_k b_k).\end{aligned}$$

Hence, $D_1 \overset{\varepsilon}{\otimes} D_2 = I_2 \circ J \circ D \circ I_1$. Since D is p -nuclear, it follows that $D_1 \overset{\varepsilon}{\otimes} D_2$ is p -nuclear. As

$$T_1 \overset{\varepsilon}{\otimes} T_2 = (B_1 \overset{\varepsilon}{\otimes} B_2) \circ (D_1 \overset{\varepsilon}{\otimes} D_2) \circ (A_1 \overset{\varepsilon}{\otimes} A_2),$$

we conclude that $T_1 \overset{\varepsilon}{\otimes} T_2$ is also p -nuclear. This completes the proof. $\square$

§4. Stability of p -Integral Operators.

Here we study the ε -stability of p -integral operators. We start with:

Lemma 3.4.1. $Y_1^{**} \overset{\varepsilon}{\otimes} Y_2^{**} \subseteq (Y_1 \overset{\varepsilon}{\otimes} Y_2)^{**}$.

Proof. By [10, p.18], for any Banach spaces X and Y we have

$$X^* \overset{\varepsilon}{\otimes} Y^* = \overline{\mathcal{F}(X, Y^*)}$$

where the closure is taken in the norm topology of $L(X, Y^*)$. It follows, by Theorem 1.3.8, that

$$X^* \overset{\varepsilon}{\otimes} Y^* = \overline{\mathcal{F}(X, Y^*)} \subseteq L(X, Y^*) = (X \overset{\pi}{\otimes} Y)^*.$$

Further, we claim that

$$X^* \overset{\pi}{\otimes} Y^* \subseteq (X \overset{\varepsilon}{\otimes} Y)^*.$$

Let $\sum_{j=1}^m x_j^* \otimes y_j^* \in X^* \otimes Y^*$. Then for all $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$, we have

$$\begin{aligned}
\left| \left(\sum_{j=1}^m x_j^* \otimes y_j^* \right) (z) \right| &= \left| \sum_{j=1}^m \sum_{i=1}^n \langle x_j^*, x_i \rangle \langle y_j^*, y_i \rangle \right| \\
&\leq \sum_{j=1}^m \left| \sum_{i=1}^n \langle x_j^*, x_i \rangle \langle y_j^*, y_i \rangle \right| \\
&= \sum_{j=1}^m \|x_j^*\| \|y_j^*\| \left| \sum_{i=1}^n \left\langle \frac{x_j^*}{\|x_j^*\|}, x_i \right\rangle \left\langle \frac{y_j^*}{\|y_j^*\|}, y_i \right\rangle \right| \\
&\leq \left(\sum_{j=1}^m \|x_j^*\| \|y_j^*\| \right) \|z\|_\varepsilon.
\end{aligned}$$

Taking the infimum over all representations of $\sum_{j=1}^m x_j^* \otimes y_j^*$ in $X^* \otimes Y^*$, we obtain

$$\left| \left(\sum_{j=1}^m x_j^* \otimes y_j^* \right) (z) \right| \leq \left\| \sum_{j=1}^m x_j^* \otimes y_j^* \right\|_\pi \|z\|_\varepsilon.$$

Hence $\sum_{j=1}^m x_j^* \otimes y_j^* \in (X \overset{\varepsilon}{\otimes} Y)^*$ and we conclude that $X^* \overset{\pi}{\otimes} Y^* \subseteq (X \overset{\varepsilon}{\otimes} Y)^*$.

Thus, if we take $X = Y_1^*$ and $Y = Y_2^*$ in the above argument, we obtain

$$Y_1^{**} \overset{\varepsilon}{\otimes} Y_2^{**} \subseteq (Y_1^* \overset{\pi}{\otimes} Y_2^*)^* \subseteq (Y_1 \overset{\varepsilon}{\otimes} Y_2)^{**}. \quad \square$$

Theorem 3.4.2. *Let $1 \leq p < \infty$. If $T_1: X_1 \longrightarrow Y_1$ and $T_2: X_2 \longrightarrow Y_2$ are p -integral operators, then $T_1 \overset{\varepsilon}{\otimes} T_2: X_1 \overset{\varepsilon}{\otimes} X_2 \longrightarrow Y_1 \overset{\varepsilon}{\otimes} Y_2$ is also p -integral.*

Proof. Let K_1 and K_2 denote the closed unit balls $B_1(X_1^*)$ and $B_1(X_2^*)$, respectively. Since $T_1 \in \mathcal{I}_p(X_1, Y_1)$, then, by Definition 2.4.1, T_1 has the factorization

$$\begin{array}{ccccc}
X_1 & \xrightarrow{T_1} & Y_1 & \xrightarrow{K_{Y_1}} & Y_1^{**} \\
A_1 \downarrow & & & & \uparrow B_1 \\
C(K_1) & \xrightarrow{J_{1p}} & & & L^p(K_1, \mu_1)
\end{array}$$

where $\mu_1 \in B_1(M(K_1))$,
 $A_1 \in L(X_1, C(K_1))$, $A_1 x_1 = f_{x_1}$, $f_{x_1}(x_1^*) = \langle x_1^*, x_1 \rangle$,
 $J_{1p} \in L(C(K_1), L^p(K_1, \mu_1))$ is the inclusion map of C into L^p ,
 $B_1 \in L(L^p(K_1, \mu_1), Y_1^{**})$, $B_1(f_{x_1}) = f_{T_1 x_1}$ for $f_{x_1} \in \text{Range}(J_{1p} A_1)$,
and $K_{Y_1}: Y_1 \longrightarrow Y_1^{**}$, $K_{Y_1}(y_1) = f_{y_1}$, $f_{y_1}(y_1^*) = \langle y_1^*, y_1 \rangle$, K_{Y_1} is the canonical embedding of Y_1 into Y_1^{**} .

Similarly, as $T_2: X_2 \longrightarrow Y_2$ is p -integral, T_2 has the factorization

$$\begin{array}{ccccc} X_2 & \xrightarrow{T_2} & Y_2 & \xrightarrow{K_{Y_2}} & Y_2^{**} \\ A_2 \downarrow & & & & \uparrow B_2 \\ C(K_2) & \xrightarrow{J_{2p}} & L^p(K_2, \mu_2) & & \end{array}$$

where $\mu_2 \in B_1(M(K_2))$,
 $A_2 \in L(X_2, C(K_2))$, $A_2 x_2 = f_{x_2}$, $f_{x_2}(x_2^*) = \langle x_2^*, x_2 \rangle$,
 $J_{2p} \in L(C(K_2), L^p(K_2, \mu_2))$ is the inclusion map of C into L^p ,
 $B_2 \in L(L^p(K_2, \mu_2), Y_2^{**})$, $B_2(f_{x_2}) = f_{T_2 x_2}$ for $f_{x_2} \in \text{Range}(J_{2p} A_2)$,
and $K_{Y_2}: Y_2 \longrightarrow Y_2^{**}$, $K_{Y_2}(y_2) = f_{y_2}$, $f_{y_2}(y_2^*) = \langle y_2^*, y_2 \rangle$, K_{Y_2} is the canonical embedding of Y_2 into Y_2^{**} .

Now, our first claim is that $T_1 \overset{\varepsilon}{\otimes} T_2$ admits the factorization

$$\begin{array}{ccccc} X_1 \overset{\varepsilon}{\otimes} X_2 & \xrightarrow{T_1 \overset{\varepsilon}{\otimes} T_2} & Y_1 \overset{\varepsilon}{\otimes} Y_2 & \xrightarrow{K_{Y_1} \overset{\varepsilon}{\otimes} K_{Y_2}} & Y_1^{**} \overset{\varepsilon}{\otimes} Y_2^{**} \\ A_1 \overset{\varepsilon}{\otimes} A_2 \downarrow & & & & \uparrow B_1 \overset{\varepsilon}{\otimes} B_2 \\ C(K_1) \overset{\varepsilon}{\otimes} C(K_2) & \xrightarrow{J_{1p} \overset{\varepsilon}{\otimes} J_{2p}} & L^p(K_1, \mu_1) \overset{\varepsilon}{\otimes} L^p(K_2, \mu_2) & & \end{array}$$

Indeed, let $x_1 \in X_1$ and $x_2 \in X_2$. Then we have

$$\begin{aligned} ((K_{Y_1} \overset{\varepsilon}{\otimes} K_{Y_2}) \circ (T_1 \overset{\varepsilon}{\otimes} T_2))(x_1 \otimes x_2) &= (K_{Y_1} \overset{\varepsilon}{\otimes} K_{Y_2})(T_1 x_1 \otimes T_2 x_2) \\ &= K_{Y_1}(T_1 x_1) \otimes K_{Y_2}(T_2 x_2) \\ &= f_{T_1 x_1} \otimes f_{T_2 x_2}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& ((B_1 \overset{\varepsilon}{\otimes} B_2) \circ (J_{1p} \overset{\varepsilon}{\otimes} J_{2p}) \circ (A_1 \overset{\varepsilon}{\otimes} A_2))(x_1 \otimes x_2) \\
&= ((B_1 \overset{\varepsilon}{\otimes} B_2) \circ (J_{1p} \overset{\varepsilon}{\otimes} J_{2p}))(A_1 x_1 \otimes A_2 x_2) \\
&= ((B_1 \overset{\varepsilon}{\otimes} B_2) \circ (J_{1p} \overset{\varepsilon}{\otimes} J_{2p}))(f_{x_1} \otimes f_{x_2}) \\
&= (B_1 \overset{\varepsilon}{\otimes} B_2)(J_{1p} f_{x_1} \otimes J_{2p} f_{x_2}) \\
&= (B_1 \overset{\varepsilon}{\otimes} B_2)(f_{x_1} \otimes f_{x_2}) \\
&= B_1 f_{x_1} \otimes B_2 f_{x_2} \\
&= f_{T_1 x_1} \otimes f_{T_2 x_2}.
\end{aligned}$$

Thus, we conclude that $(K_{Y_1} \overset{\varepsilon}{\otimes} K_{Y_2}) \circ (T_1 \overset{\varepsilon}{\otimes} T_2) = (B_1 \overset{\varepsilon}{\otimes} B_2) \circ (J_{1p} \overset{\varepsilon}{\otimes} J_{2p}) \circ (A_1 \overset{\varepsilon}{\otimes} A_2)$ as claimed.

Further, we claim that $J_{1p} \overset{\varepsilon}{\otimes} J_{2p}$ can be factored as

$$\begin{array}{ccc}
C(K_1) \overset{\varepsilon}{\otimes} C(K_2) & \xrightarrow{J_{1p} \overset{\varepsilon}{\otimes} J_{2p}} & L^p(K_1, \mu_1) \overset{\varepsilon}{\otimes} L^p(K_2, \mu_2) \\
\downarrow \phi & & \uparrow I \\
& & L^p(K_1, \mu_1) \overset{\nu_p}{\otimes} L^p(K_2, \mu_2) \\
& & \uparrow \psi \\
C(K_1 \times K_2) & \xrightarrow{J_p} & L^p(K_1 \times K_2, \mu_1 \times \mu_2)
\end{array}$$

where

$$1. \quad \phi: C(K_1) \overset{\varepsilon}{\otimes} C(K_2) \longrightarrow C(K_1 \times K_2)$$

$$\sum_{i=1}^n f_i \otimes g_i \longmapsto h$$

$$h(x_1^*, x_2^*) = \sum_{i=1}^n f_i(x_1^*) g_i(x_2^*).$$

ϕ , by Corollary 1.2.5, is an isometric isomorphism of $C(K_1) \overset{\varepsilon}{\otimes} C(K_2)$ onto $C(K_1 \times K_2)$.

$$2. \quad J_p: C(K_1 \times K_2) \longrightarrow L^p(K_1 \times K_2, \mu_1 \times \mu_2)$$

is the usual inclusion map of C into L^p .

$$\begin{aligned}
3. \quad \psi: L^p(K_1 \times K_2, \mu_1 \times \mu_2) &\longrightarrow L^p(K_1, \mu_1) \overset{\nu_p}{\otimes} L^p(K_2, \mu_2) \\
h &\longmapsto \sum_{i=1}^n f_i \otimes g_i \\
\sum_{i=1}^n f_i(x_1^*) g_i(x_2^*) &= h(x_1^*, x_2^*).
\end{aligned}$$

By Theorem 1.4.5, ψ is an isometric isomorphism of $L^p(K_1 \times K_2, \mu_1 \times \mu_2)$ onto $L^p(K_1, \mu_1) \overset{\nu_p}{\otimes} L^p(K_2, \mu_2)$.

$$\begin{aligned}
4. \quad I: L^p(K_1, \mu_1) \overset{\nu_p}{\otimes} L^p(K_2, \mu_2) &\longrightarrow L^p(K_1, \mu_1) \overset{\varepsilon}{\otimes} L^p(K_2, \mu_2) \\
I &\text{ is the inclusion map and by Corollary 1.4.4 it is continuous.}
\end{aligned}$$

Thus our claim is that $J_{1p} \overset{\varepsilon}{\otimes} J_{2p} = I \circ \psi \circ J_p \circ \phi$. To see this, let $\sum_{i=1}^n f_i \otimes g_i \in C(K_1) \overset{\varepsilon}{\otimes} C(K_2)$. Then, we have

$$\begin{aligned}
(J_{1p} \overset{\varepsilon}{\otimes} J_{2p}) \left(\sum_{i=1}^n f_i \otimes g_i \right) &= \sum_{i=1}^n J_{1p} f_i \otimes J_{2p} g_i \\
&= \sum_{i=1}^n f_i \otimes g_i.
\end{aligned}$$

On the other hand

$$\begin{aligned}
(I \circ \psi \circ J_p \circ \phi) \left(\sum_{i=1}^n f_i \otimes g_i \right) &= (I \circ \psi \circ J_p)(h) \\
&= (I \circ \psi)(h) \\
&= I \left(\sum_{i=1}^n f_i \otimes g_i \right) \\
&= \sum_{i=1}^n f_i \otimes g_i.
\end{aligned}$$

Consequently, $J_{1p} \overset{\varepsilon}{\otimes} J_{2p} = I \circ \psi \circ J_p \circ \phi$ as claimed.

Now, by Lemma 3.4.1, we know that $Y_1^{**} \overset{\varepsilon}{\otimes} Y_2^{**} \subseteq (Y_1 \overset{\varepsilon}{\otimes} Y_2)^{**}$. Let

$$J: Y_1^{**} \overset{\varepsilon}{\otimes} Y_2^{**} \longrightarrow (Y_1 \overset{\varepsilon}{\otimes} Y_2)^{**}$$

denote the inclusion map of $Y_1^{**} \overset{\varepsilon}{\otimes} Y_2^{**}$ into $(Y_1 \overset{\varepsilon}{\otimes} Y_2)^{**}$. Further, let $A = \phi \circ (A_1 \overset{\varepsilon}{\otimes} A_2)$ and $B = J \circ (B_1 \overset{\varepsilon}{\otimes} B_2) \circ I \circ \psi$. Our final step is establishing the following commutative diagram

$$\begin{array}{ccccc}
X_1 \overset{\varepsilon}{\otimes} X_2 & \xrightarrow{T_1 \overset{\varepsilon}{\otimes} T_2} & Y_1 \overset{\varepsilon}{\otimes} Y_2 & \xrightarrow{K_{Y_1 \overset{\varepsilon}{\otimes} Y_2}} & (Y_1 \overset{\varepsilon}{\otimes} Y_2)^{**} \\
A \downarrow & & & & \uparrow B \\
C(K_1 \times K_2) & \xrightarrow{J_p} & L^p(K_1 \times K_2, \mu_1 \times \mu_2) & &
\end{array}$$

Let $x_1 \in X_1$ and $x_2 \in X_2$. Then, we have

$$(K_{Y_1 \overset{\varepsilon}{\otimes} Y_2} \circ (T_1 \overset{\varepsilon}{\otimes} T_2))(x_1 \otimes x_2) = K_{Y_1 \overset{\varepsilon}{\otimes} Y_2}(T_1 x_1 \otimes T_2 x_2).$$

On the other hand

$$\begin{aligned}
(B \circ J_p \circ A)(x_1 \otimes x_2) &= (B \circ J_p)(\phi(A_1 x_1 \otimes A_2 x_2)) \\
&= (B \circ J_p)(\phi(f_{x_1} \otimes f_{x_2})) \\
&= (B \circ J_p)(h) \\
&= B(h) \\
&= (J \circ (B_1 \overset{\varepsilon}{\otimes} B_2) \circ I)(f_{x_1} \otimes f_{x_2}) \\
&= J(B_1 f_{x_1} \otimes B_2 f_{x_2}) \\
&= J(f_{T_1 x_1} \otimes f_{T_2 x_2}) \\
&= J(K_{Y_1}(T_1 x_1) \otimes K_{Y_2}(T_2 x_2)) \\
&= (J \circ (K_{Y_1} \overset{\varepsilon}{\otimes} K_{Y_2}))(T_1 x_1 \otimes T_2 x_2).
\end{aligned}$$

As $K_{Y_1 \overset{\varepsilon}{\otimes} Y_2} = J \circ (K_{Y_1} \overset{\varepsilon}{\otimes} K_{Y_2})$, it follows that

$$(K_{Y_1 \overset{\varepsilon}{\otimes} Y_2}) \circ (T_1 \overset{\varepsilon}{\otimes} T_2) = B \circ J_p \circ A.$$

Hence, by Definition 2.4.1, we conclude that $T_1 \overset{\varepsilon}{\otimes} T_2$ is indeed p -integral. $\square$

§5. Injective Hull Stability.

In this section we prove that the injective hull $\mathcal{O}^i$ of an operator ideal O is ε -stable if O is.

Theorem 3.5.1. *Let O be an ε -stable operator ideal. If $T_1 \in \mathcal{O}^i(X_1, Y_1)$ and $T_2 \in \mathcal{O}^i(X_2, Y_2)$, then $T_1 \overset{\varepsilon}{\otimes} T_2 \in \mathcal{O}^i(X_1 \overset{\varepsilon}{\otimes} X_2, Y_1 \overset{\varepsilon}{\otimes} Y_2)$.*

Proof. We claim that $J_{Y_1 \overset{\varepsilon}{\otimes} Y_2} \circ (T_1 \overset{\varepsilon}{\otimes} T_2) \in O(X_1 \overset{\varepsilon}{\otimes} X_2, \ell^\infty(B_1((Y_1 \overset{\varepsilon}{\otimes} Y_2)^*)))$. As $T_1 \in \mathcal{O}^i(X_1, Y_1)$, it follows that $J_{Y_1} T_1 \in O(X_1, \ell^\infty(B_1(Y_1^*)))$. Similarly, $J_{Y_2} T_2 \in O(X_2, \ell^\infty(B_1(Y_2^*)))$. Stability of O with respect to $\| \cdot \|_\varepsilon$ gives

$$J_{Y_1} T_1 \overset{\varepsilon}{\otimes} J_{Y_2} T_2 \in O(X_1 \overset{\varepsilon}{\otimes} X_2, \ell^\infty(B_1(Y_1^*)) \overset{\varepsilon}{\otimes} \ell^\infty(B_1(Y_2^*))).$$

Now consider the following diagram :

$$\begin{array}{ccc} X_1 \overset{\varepsilon}{\otimes} X_2 & \xrightarrow{J_{Y_1} T_1 \overset{\varepsilon}{\otimes} J_{Y_2} T_2} & \ell^\infty(B_1(Y_1^*)) \overset{\varepsilon}{\otimes} \ell^\infty(B_1(Y_2^*)) \\ T_1 \overset{\varepsilon}{\otimes} T_2 \downarrow & \nearrow J_{Y_1} \overset{\varepsilon}{\otimes} J_{Y_2} & \\ Y_1 \overset{\varepsilon}{\otimes} Y_2 & & \\ J_{Y_1 \overset{\varepsilon}{\otimes} Y_2} \downarrow & & \\ \ell^\infty(B_1((Y_1 \overset{\varepsilon}{\otimes} Y_2)^*)) & & \end{array}$$

The space $\ell^\infty(B_1((Y_1 \overset{\varepsilon}{\otimes} Y_2)^*))$ has the metric extension property. By Lemma 3.2.1 $J_{Y_1} \overset{\varepsilon}{\otimes} J_{Y_2} \in L(Y_1 \overset{\varepsilon}{\otimes} Y_2, \ell^\infty(B_1(Y_1^*)) \overset{\varepsilon}{\otimes} \ell^\infty(B_1(Y_2^*)))$ is a metric injection. Further, since $J_{Y_1 \overset{\varepsilon}{\otimes} Y_2} \in L(Y_1 \overset{\varepsilon}{\otimes} Y_2, \ell^\infty(B_1((Y_1 \overset{\varepsilon}{\otimes} Y_2)^*)))$ then, by Definition 2.5.1, there exists an extension $S \in L(\ell^\infty(B_1(Y_1^*)) \overset{\varepsilon}{\otimes} \ell^\infty(B_1(Y_2^*)), \ell^\infty(B_1((Y_1 \overset{\varepsilon}{\otimes} Y_2)^*)))$ such that $S \circ (J_{Y_1} \overset{\varepsilon}{\otimes} J_{Y_2}) = J_{Y_1 \overset{\varepsilon}{\otimes} Y_2}$ as illustrated by the following diagram :

$$\begin{array}{ccc} Y_1 \overset{\varepsilon}{\otimes} Y_2 & \xrightarrow{J_{Y_1 \overset{\varepsilon}{\otimes} Y_2}} & \ell^\infty(B_1((Y_1 \overset{\varepsilon}{\otimes} Y_2)^*)) \\ J_{Y_1} \overset{\varepsilon}{\otimes} J_{Y_2} \downarrow & \nearrow S & \\ \ell^\infty(B_1(Y_1^*)) \overset{\varepsilon}{\otimes} \ell^\infty(B_1(Y_2^*)) & & \end{array}$$

Hence, we have the following factorization for $J_{Y_1 \overset{\varepsilon}{\otimes} Y_2} \circ (T_1 \overset{\varepsilon}{\otimes} T_2)$:

$$\begin{array}{ccc}
 X_1 \overset{\varepsilon}{\otimes} X_2 & \xrightarrow{J_{Y_1} T_1 \overset{\varepsilon}{\otimes} J_{Y_2} T_2} & \ell^\infty(B_1(Y_1^*)) \overset{\varepsilon}{\otimes} \ell^\infty(B_1(Y_2^*)) \\
 T_1 \overset{\varepsilon}{\otimes} T_2 \downarrow & \nearrow J_{Y_1} \overset{\varepsilon}{\otimes} J_{Y_2} & \\
 Y_1 \overset{\varepsilon}{\otimes} Y_2 & & \\
 J_{Y_1 \overset{\varepsilon}{\otimes} Y_2} \downarrow & \nwarrow S & \\
 \ell^\infty(B_1((Y_1 \overset{\varepsilon}{\otimes} Y_2)^*)) & &
 \end{array}$$

Consequently,

$$J_{Y_1 \overset{\varepsilon}{\otimes} Y_2} \circ (T_1 \overset{\varepsilon}{\otimes} T_2) = S \circ (J_{Y_1} T_1 \overset{\varepsilon}{\otimes} J_{Y_2} T_2).$$

As $J_{Y_1} T_1 \overset{\varepsilon}{\otimes} J_{Y_2} T_2 \in O(X_1 \overset{\varepsilon}{\otimes} X_2, \ell^\infty(B_1(Y_1^*)) \overset{\varepsilon}{\otimes} \ell^\infty(B_1(Y_2^*)))$ and O is an operator ideal, we have

$$J_{Y_1 \overset{\varepsilon}{\otimes} Y_2} \circ (T_1 \overset{\varepsilon}{\otimes} T_2) \in O(X_1 \overset{\varepsilon}{\otimes} X_2, \ell^\infty(B_1((Y_1 \overset{\varepsilon}{\otimes} Y_2)^*))).$$

Hence, by Definition 2.5.6, $T_1 \overset{\varepsilon}{\otimes} T_2 \in \mathcal{O}^i(X_1 \overset{\varepsilon}{\otimes} X_2, Y_1 \overset{\varepsilon}{\otimes} Y_2)$. $\square$

Corollary 3.5.2. For $1 \leq p < \infty$, $\Pi_p = \mathcal{I}_p^i$ is ε -stable.

Proof. Follows from Theorems 3.5.1, 3.4.2 and 2.5.15. $\square$

NOTATION

The symbols listed below are followed by a brief statement of their meaning and by the number of the page on which they are defined.

Symbol	Stands for ...	Page #
X^*	dual space of a normed space X	1
$L(X, Y)$	space of bounded linear operators from X into Y	1
$\otimes$	tensor product	1
$\langle \cdot, \cdot \rangle$	evaluation	1
$\text{Span } M$	span of the set M	1
$\mathbb{R}$	real line	2
$\ \cdot \ $	norm	2
$\ \cdot \ _d$	operator-norm on the dual of $(X \otimes Y, \ \cdot \)$	2
X^{**}	second dual space of a normed space X	4
$B_1(X)$	closed unit ball of the space X	4
$\ \cdot \ _{dd}$	operator-norm on the second dual of $(X \otimes Y, \ \cdot \)$	5
$\ \cdot \ _\varepsilon$	least of the reasonable crossnorms	6
$\ \cdot \ _\alpha$	generic reasonable crossnorm	6
$\overset{\varepsilon}{\otimes}$	completion of tensor product with $\ \cdot \ _\varepsilon$	7
$C(I, X)$	continuous functions from I into X	8
$C(I)$	space of continuous functions on I	8
$\text{Supp}(h)$	Support of the function h	9
$\ \cdot \ _\pi$	greatest of the reasonable crossnorms	11
$\overset{\pi}{\otimes}$	completion of tensor product with $\ \cdot \ _\pi$	13
c_0	space of sequences tending to zero	14

$L^p(I, X)$	space of L^p -maps from I into X	16
$L^p(I)$	Lebesgue space on I	16
$L^\infty(I, X)$	space of essentially bounded functions from I into X	16
$L^\infty(I)$	space of essentially bounded functions on I	16
χ_E	characteristic function of the set E	16
$S(I, X)$	space of simple maps from I into X	16
$S(I)$	space of simple functions on I	17
$\text{trace}(T)$	trace of the operator T	18
p^*	dual exponent, $\frac{1}{p} + \frac{1}{p^*} = 1$	22
$\ \cdot \ _{\nu_p}$	p -nuclear crossnorm	22
$\overset{\nu_p}{\otimes}$	completion of tensor product with $\ \cdot \ _{\nu_p}$	26
$\mathcal{F}(X, Y)$	finite rank operators from X into Y	26
$A \otimes B$	tensor product of operators	27
$\overset{\alpha}{\otimes}$	completion of tensor product with $\ \cdot \ _\alpha$	28
A^*	adjoint of an operator A	29
$A \overset{\alpha}{\otimes} B$	continuous extension of $A \otimes B$	28
L	Ideal of all bounded linear operators	31
$\text{rank}(T)$	rank of an operator T	32
$\dim X$	dimension of X	32
$\mathcal{K}$	ideal of compact operators	33
$\mathbb{N}$	set of natural numbers	33
$\ell^p(X)$	space of absolutely p -summable sequences in X	35
$\ell^p\langle X \rangle$	space of weakly p -summable sequences in X	35
$\ \cdot \ _{\alpha(p)}$	norm on $\ell^p(X)$	35
$\ \cdot \ _{\varepsilon(p)}$	norm on $\ell^p\langle X \rangle$	35
Π_p	ideal of absolutely p -summing operators	36
$\ \cdot \ _{\pi_p}$	p -summing norm on $\Pi_p(X, Y)$	36
J_p	canonical map from $C(K)$ into $L^p(K, \mu)$	41
$M(X)$	space of regular Borel measures on X	42
$L^p(K, \mu)$	p -integrable functions on (K, μ)	42
$\ell^\infty(S)$	bounded nets on S	42
J_Y	canonical injection of Y into $\ell^\infty(B_1(Y^*))$	42

$\overline{M}$	closure of the set M in the norm topology	43
$L^\infty(K, \mu)$	space of essentially bounded μ -measurable functions on K	44
$\mathcal{N}_p$	ideal of p -nuclear operators	46
$\ \cdot \ _{\mathcal{N}_p}$	p -nuclear norm on $\mathcal{N}_p(X, Y)$	47
δ_n	the sequence $(0, \dots, 0, 0, \underset{\substack{\downarrow \\ \text{nth coordinate}}}{1}, 0, 0, \dots)$	54
$\mathcal{I}_p$	ideal of p -integral operators	59
$\ \cdot \ _{\mathcal{I}_p}$	p -integral norm on $\mathcal{I}_p(X, Y)$	59
K_Y	canonical embedding of Y into Y^{**}	59
$\mathcal{O}^i$	Injective hull of an operator ideal $\mathcal{O}$	62
$\ \cdot \ _{\mathcal{O}^i}$	norm on the injective hull $\mathcal{O}^i(X, Y)$	62
$\ \cdot \ _{\mathcal{O}}$	norm on the component $\mathcal{O}(X, Y)$ of the operator ideal $\mathcal{O}$	62
$\overline{\text{co}}(A)$	closed convex hull of the set A	66
$\mathcal{C}_1(H)$	trace class on H	72

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