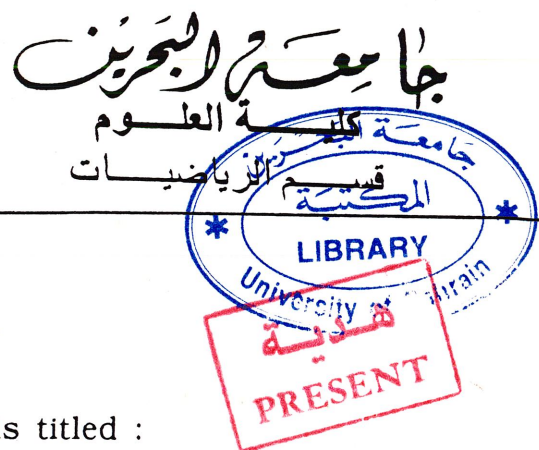


*Asymptotic Solutions for
Second-Order
Differential Equations*

By

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INTRODUCTION

We shall examine the asymptotic form of the solutions to the second-order differential equation

$$(py')' + ry' - qy = 0 \quad (1)$$

as $x \rightarrow \infty$, where x is the independent variable and the prime denotes $\frac{d}{dx}$. The functions p , r and q are defined on the interval $[a, \infty)$ and are not necessarily real-valued, while p is nowhere zero and $a \geq 0$ in this interval.

We shall consider the situation where r is small compared to p and q as $x \rightarrow \infty$ in the sense

$$r = o\{(pq)^{1/2}\} \quad (x \rightarrow \infty), \quad (2)$$

where q is nowhere zero in $[a, \infty)$.

In this situation we identify two cases, the first case is

$$\frac{(pq)'}{(pq)} = o\left\{\left(\frac{q}{p}\right)^{1/2}\right\} \quad (x \rightarrow \infty) \quad (3)$$

which gives us solutions y_1 and y_2 such that

$$y_1 \sim (pq)^{-1/4} \exp\left(\int_a^x \left\{\left(\frac{(pq)'}{4pq}\right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2\right\}^{1/2} dt\right) \quad (4a)$$

$$py_1' \sim (pq)^{1/4} \exp\left(\int_a^x \left\{\left(\frac{(pq)'}{4pq}\right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2\right\}^{1/2} dt\right), \quad (4b)$$

and

$$y_2 \sim (pq)^{-1/4} \exp \left(- \int_a^x \left\{ \left(\frac{(pq)'}{4pq} \right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2} \right)^2 \right\}^{1/2} dt \right) \quad (4c)$$

$$py_2' \sim (pq)^{1/4} \exp \left(- \int_a^x \left\{ \left(\frac{(pq)'}{4pq} \right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2} \right)^2 \right\}^{1/2} dt \right) \quad (4d)$$

where

$$\lambda_1 - \lambda_2 = 2 \left(\frac{q}{p} \right)^{1/2} \left(1 + O \left(\frac{r}{(pq)^{1/2}} \right) \right) \quad (x \rightarrow \infty).$$

The second case is

$$\frac{(pq)'}{(pq)} \sim (\text{const.}) \left(\frac{q}{p} \right)^{1/2} \quad (x \rightarrow \infty) \quad (5)$$

which gives us solutions y_1 and y_2 satisfying

$$y_1 \sim (pq)^c \exp \left(\frac{1}{4} \int_a^x \left\{ \left(\frac{(pq)'}{pq} \right) \Phi \right\} dt \right) \quad (6a)$$

$$py_1' \sim ck(pq)^{1/2+c} \exp \left(\frac{1}{4} \int_a^x \left\{ \left(\frac{(pq)'}{pq} \right) \Phi \right\} dt \right), \quad (6b)$$

and

$$y_2 \sim (pq)^{-1/2-c} \exp \left(- \frac{1}{4} \int_a^x \left\{ \left(\frac{(pq)'}{pq} \right) \Phi \right\} dt \right) \quad (6c)$$

$$py_2' \sim -(ck)^{-1} (pq)^{-c} \exp \left(- \frac{1}{4} \int_a^x \left\{ \left(\frac{(pq)'}{pq} \right) \Phi \right\} dt \right), \quad (6d)$$

where

$$c = -\frac{1}{4} + \frac{1}{4}(1 + 16k^{-2})^{1/2} \quad (7)$$

and

$$\Phi = -16k^{-2}(1 + 16k^{-2})^{-1/2} \phi + O\left(\phi \frac{r}{(pq)^{1/2}}\right) + O(\phi^2), \quad (8)$$

where $\phi = o(1)$ as $x \rightarrow \infty$.

Here (5) is the Euler case for (1) which is given in Section 3.4.

Walker [7] considered (1) with $p=1$ under some conditions on r and q . Similar second-order equations to (1) with $r = 0$ have been considered previously by Eastham [6, Chapter 2] and Cassell [4]. Al-Hammadi [3] considered (1) where p and q are small compared to r in the sense that

$$pq = o(r^2) \quad (x \rightarrow \infty). \quad (9)$$

The methods used in this thesis take the advantage of recent developments in asymptotic theory and are much simpler than those in [4] and [7]. In particular we use the recent asymptotic theorem of Eastham [6, Theorem 1.6.1] which, although a consequence of the basic Levinson theorem [6, Theorem 1.3.1], is in a form suitable for many applications. Third-order equations with $r=0$ were considered by [1] and [2].

The thesis consists of four chapters. In the first one I give notations and basic theory.

Chapter two deals with some basic theorems and results. In this chapter, the Levinson and Eastham Theorems [6] are stated without proofs.

The general features of our method are given in Section 3.1 and the main theorems for (1) in Sections 3.3 and 3.4.

Finally, some examples and remarks are given in Chapter 4.

CHAPTER ONE

Notation and Basic Theory :

There are several basic facts concerning linear algebra and linear differential systems which will be used occasionally, and it is convenient to have these results available here for reference. Some notations that will be used throughout are also introduced.

Notation: For an $m \times n$ matrix $A=(a_{ij})$, we define the modulus $|A|$ by

$$|A| = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|. \quad (1.1)$$

For a square matrix $A=(a_{ij})$, we denote by dgA or, more fully, $dg(a_{11}, \dots, a_{nn})$, the diagonal matrix formed by the diagonal entries a_{ii} in A . We also denote A^t the transpose matrix defined by $A^t = (a_{ji})$.

For matrices whose entries are differentiable functions of x , we define

$$A'(x) = \left(a'_{ij}(x) \right), \quad (1.2)$$

and similarly for the integral of $A(x)$ over an interval (a, b) .

For a function f we use $f \in L(a, \infty)$ to indicate that

$$\int_a^\infty |f(x)| dx < \infty. \quad (1.3)$$

For any pair of real continuous functions $f(x)$ and $g(x)$, $x \in [a, \infty)$, the behaviour of these functions, as $x \rightarrow \infty$, can be compared by using the classical Landau [8] order symbols O and o , which are defined as follows:

(i) $f = O(g)$ ($x \rightarrow \infty$) if there exists positive constant k such that $|f(x)| \leq k |g(x)|$ for $x \in [a, \infty)$, where $g(x) \neq 0 \forall x \in [a, \infty)$.

(ii) $f = o(g)$, where $g(x) \neq 0 \forall x \in [a, \infty)$, if $\lim_{x \rightarrow \infty} \left\{ \frac{f(x)}{g(x)} \right\}$ exists and equals zero.

We shall use the result, from linear algebra, that an $n \times n$ matrix A with n distinct eigenvalues λ_k ($1 \leq k \leq n$) can be expressed in the diagonal form

$$T^{-1}AT = \Lambda, \quad (1.4)$$

where $\Lambda = \text{dg}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and T has the corresponding eigenvectors v_k as its columns. The form of T^{-1} can be obtained directly from the form of T .

CHAPTER TWO

Basic Theorems and Results

2.1 Introduction:

The basic problem posed by a differential system [6]

$$\dot{Y}(x) = A(x) Y(x) \quad (2.1.1)$$

is that the solution vector $Y(x)$ cannot normally be written as explicit expressions involving the entries of the given square matrix $A(x)$. It is of course this difficulty which creates the challenge and interest in developing a wide range of techniques for investigating the properties of solutions, where (2.1.1) alone is taken as the starting point. The general aim of the asymptotic theory of (2.1.1) is to identify those types of systems whose solutions can nevertheless be approximated by such explicit expressions as $x \rightarrow \infty$. Therefore, we need (2.1.1) to be defined on the interval $[a, \infty)$, where $a \geq 0$.

An obvious starting point for the asymptotic theory of (2.1.1) is to examine the case where (2.1.1) does for large x approximate to a simple system

$$\dot{Y}(x) = A_0(x) Y(x) \quad (2.1.2)$$

which can be solved explicitly.

We now introduce two simple systems [6]

(i) *The constant coefficient system*

Here $A_0(x)$ is a constant matrix C , and (2.1.2) is

$$\dot{Y}(x) = C Y(x). \quad (2.1.3)$$

(ii) *The diagonal system*

Here $A_0(x)$ is a diagonal matrix $\Lambda(x)$ and (2.1.2) is

$$\dot{Y}(x) = \Lambda(x) Y(x). \quad (2.1.4)$$

Moving on now to systems (2.1.1) which approximate to (2.1.3) or (2.1.4) when x is large, we consider [6]

$$Y'(x) = \{ C + R(x) \} Y(x) \quad (2.1.5)$$

or

$$Y'(x) = \{ \Lambda(x) + R(x) \} Y(x), \quad (2.1.6)$$

where C is a constant matrix, $\Lambda(x)$ is a diagonal matrix and $R(x)$ is an $n \times n$ matrix which is small in some sense as $x \rightarrow \infty$. These systems are called asymptotically constant and asymptotically diagonal, respectively. The condition on $R(x)$ is

$$\int_a^\infty |R(x)| \, dx < \infty. \quad (2.1.7)$$

Now we give the statements of two basic theorems on asymptotically diagonal systems (2.1.6), the first theorem is due to Levinson which was proved in 1948 and the second theorem is due to Eastham.

2.2 The Levinson Theorem:

In this section we give the statement of the Levinson theorem [6, Theorem 1.3.1] on an asymptotically diagonal system (2.1.6).

Theorem (2.2.1) :

Let $\Lambda(x)$ be an $n \times n$ diagonal matrix,

$$\Lambda(x) = \text{dg}(\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x)), \quad (2.2.1)$$

which satisfies the following two-part condition L.

L. For each pair of integers i and j in $[1, n]$ ($i \neq j$) and for all x and t such that $a \leq t \leq x < \infty$, either

$$(a) \quad \int_t^x \text{re} \{ \lambda_i(s) - \lambda_j(s) \} ds \leq k_1, \quad (2.2.2)$$

or

$$(b) \quad \int_t^x \text{re} \{ \lambda_i(s) - \lambda_j(s) \} ds \geq k_2, \quad (2.2.3)$$

where k_1 and k_2 are constants.

Let (2.1.7) hold. Then, as $x \rightarrow \infty$, the system (2.1.6) has solutions $Y_k(x)$ ($1 \leq k \leq n$) with the asymptotic form

$$Y_k(x) = \{ e_k + o(1) \} \exp \left(\int_a^x \lambda_k(t) dt \right), \quad (2.2.4)$$

where e_k is the coordinate vector whose k th component is unity and other components zero.

Condition L is called the dichotomy condition [6, Section 1.3] which can be written in a stronger form as

Dichotomy condition :

For each pair (i,j) , let

$$\operatorname{re}\{\mu_i(x) - \mu_j(x)\} = f(x) + g(x), \quad (2.2.5)$$

where $g(x) \in L(a, \infty)$ and either $f(x) \geq 0$ in $[a, \infty)$ or $f(x) \leq 0$ in $[a, \infty)$.

i.e.

$f(x)$ have one sign in $[a, \infty)$.

2.3 The Eastham Theorem:

The theorem in this section [6, Theorem 1.6.1] gives the asymptotic form of the solutions of (2.1.6) subject to a pointwise condition on $R(x)$ as well as an integral condition, with the latter relating to $R'(x)$ rather than $R(x)$. The condition on $R(x)$ is not simply $R(x) \rightarrow 0$ as $x \rightarrow \infty$ but rather that $R(x)$ should be small compared to $\Lambda(x)$ in the sense specified in (2.3.1).

Theorem (2.3.1) :

Let $\Lambda(x)$ be an $n \times n$ diagonal matrix,

$$\Lambda(x) = \text{dg}(\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x)),$$

in which $\lambda_i(x) - \lambda_j(x)$ is nowhere zero in some interval $[a, \infty)$ for each pair of unequal integers i and j in $[1, n]$. Let the $n \times n$ matrix $R(x)$ satisfy:

$$(i) \quad \{\lambda_i(x) - \lambda_j(x)\}^{-1} R(x) \rightarrow 0 \text{ as } x \rightarrow \infty; \quad (2.3.1)$$

$$(ii) \quad \left(\{\lambda_i(x) - \lambda_j(x)\}^{-1} R(x) \right)' \in L(a, \infty) \quad (2.3.2)$$

for each unequal i and j in $[1, n]$. Also, let the eigenvalues $\mu_k(x)$ ($1 \leq k \leq n$) of $\Lambda(x) + R(x)$ satisfy the dichotomy condition (2.2.5). Then, as $x \rightarrow \infty$, (2.1.6) has solutions $Y_k(x)$ with the asymptotic form

$$Y_k(x) = \{e_k + o(1)\} \exp \left(\int_a^x \mu_k(t) dt \right), \quad (2.3.3)$$

where e_k is the coordinate vector whose k th component is unity and other components zero.

2.4 Basic Results:

The aim of this section is to consider systems which can nevertheless be transformed into the asymptotically diagonal case covered in Sections 2 and 3. The asymptotic form of the solutions for these classes of systems is then obtained by applying the theorems in Sections 2 and 3 and transforming back. An important feature of the transformations to be introduced is that they involve only the solution vector of the system and not the independent variable x .

Eastham [6, Chapter 2] considered the familiar second-order equation

$$(p(x)y'(x))' - q(x)y(x) = 0 \quad (2.4.1)$$

and obtained the form of solutions under various conditions on $p(x)$ and $q(x)$ as in the following theorems, where the condition (2.4.8) in the second theorem is due to Euler. We state here the results obtained and proofs can be found in [6, Chapter 2].

Theorem (2.4.1) :

Let $p(x)$ and $q(x)$ be nowhere zero and $C^{(2)}[a, \infty)$. Let

$$\frac{(pq)'}{(pq)} = o\left\{\left(\frac{q}{p}\right)^{1/2}\right\} \quad (x \rightarrow \infty) \quad (2.4.2)$$

and let

$$\left\{ p^{-1/2} q^{-3/2} (pq)' \right\}' \in L(a, \infty). \quad (2.4.3)$$

Let

$$\operatorname{re} \left(\frac{q}{p} + u^2 \right)^{1/2} \text{ have one sign in } [a, \infty), \quad (2.4.4)$$

where

$$u = \frac{(pq)'}{4pq}. \quad (2.4.5)$$

Then (2.4.1) has solutions y_1 and y_2 such that

$$y_1 \sim (pq)^{-1/4} \exp \left(\int_a^x \left(\frac{q}{p} + u^2 \right)^{1/2} dt \right), \quad (2.4.6)$$

$$p y_1' \sim (pq)^{1/4} \exp \left(\int_a^x \left(\frac{q}{p} + u^2 \right)^{1/2} dt \right), \quad (2.4.7)$$

with similar formulae for y_2 containing $-(\dots)^{1/2}$ in the exponential term.

Theorem (2.4.2) :

Let $p(x)$ and $q(x)$ be nowhere zero and $C^{(2)}[a, \infty)$. Let

$$\frac{(pq)'}{pq} = k \left(\frac{q}{p} \right)^{1/2(1+\phi)}, \quad (2.4.8)$$

where k is a non-zero constant with $k^2 \neq -16$,

$$\phi = o(1) \quad (x \rightarrow \infty), \quad (2.4.9)$$

and

$$\phi' \in L(a, \infty). \quad (2.4.10)$$

Let

$$\operatorname{re} \left(\frac{q}{p} + u^2 \right)^{1/2} \text{ have one sign in } [a, \infty), \quad (2.4.11)$$

where u is as in (2.4.5). Then (2.4.1) has solutions y_1 and y_2 such that

$$y_1 \sim (pq)^c \exp\{I(x)\}, \quad (2.4.12)$$

$$p y_1' \sim c k (pq)^{1/2+c} \exp\{I(x)\},$$

$$y_2 \sim (pq)^{-1/2-c} \exp\{-I(x)\}, \quad (2.4.13)$$

$$p y_2' \sim -(ck)^{-1} (pq)^{-c} \exp\{-I(x)\},$$

where

$$c = -\frac{1}{4} + \frac{1}{4} (1 + 16k^{-2})^{1/2}, \quad (2.4.14)$$

and

$$I(x) = \frac{1}{4} \int_a^x \frac{(pq)'}{pq} \Phi dt \quad (2.4.15)$$

with

$$\Phi = -16k^{-2} (1 + 16k^{-2})^{-1/2} \phi + O(\phi^2). \quad (2.4.16)$$

The simplest example considered to theorems (2.4.1) and (2.4.2) is by taking the coefficients

$$p(x) = Ax^\alpha, \quad q(x) = Bx^\beta, \quad (2.4.17)$$

where $\alpha, \beta, A(\neq 0)$ and $B(\neq 0)$ are real constants with $AB > 0$. Then (2.4.2) holds if

$$\alpha - \beta < 2. \quad (2.4.18)$$

Also (2.4.8) holds if

$$\alpha - \beta = 2. \quad (2.4.19)$$

Hence (2.4.8) represents a borderline case in which asymptotic formulae of algebraic type appear in the case where (2.4.19) holds as we shall see by Example 2 in Chapter 4.

CHAPTER THREE

3.1 Introduction:

In this chapter we consider a class of systems which are not covered by the results in Section 4 of the previous chapter. The situation to be considered is the second-order differential equation

$$(p(x)y'(x))' + r(x)y'(x) - q(x)y(x) = 0 \quad (3.1.1)$$

where $p(x)$, $r(x)$ and $q(x)$ are defined on the interval $[a, \infty)$ and are not necessarily real-valued, while $p(x)$ is nowhere zero and $a \geq 0$ in the interval.

The equation (3.1.1) can be written in a standard form [6] as a first-order system

$$Y' = AY, \quad (3.1.2a)$$

by first writing equation (3.1.1) as

$$(py')' = qy - ry' \quad (3.1.2b)$$

Now define

$$y_1 = y, \quad y_2 = py'.$$

Then

$$y_1' = y' \quad \text{and} \quad y_2' = (py')' = qy - ry'.$$

Hence (3.1.2b) can be written as (3.1.2a) by taking

$$Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \quad (3.1.3)$$

and

$$A = \begin{pmatrix} 0 & p^{-1} \\ q & -rp^{-1} \end{pmatrix}. \quad (3.1.4)$$

We express A in its diagonal form

$$T^{-1}AT = \Lambda \quad (3.1.5)$$

and we therefore require the eigenvalues λ_j and eigenvectors v_j ($j=1,2$) of A. The characteristic equation of A is given by

$$p\lambda^2 + r\lambda - q = 0. \quad (3.1.6)$$

The solution of equation (3.1.6) is given by

$$\lambda_j = \{ -r \pm (r^2 + 4pq)^{1/2} \} / 2p \quad (j=1,2).$$

Now the equation

$$Av = \lambda v$$

can be written as

$$\begin{pmatrix} 0 & p^{-1} \\ q & -rp^{-1} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \lambda \begin{pmatrix} u \\ w \end{pmatrix}.$$

which gives us

$$p^{-1}w = \lambda u$$

and

$$qu = (rp^{-1} + \lambda)w.$$

Define $u = 1$, then $w = p\lambda$ and $q = (\lambda + rp^{-1})p\lambda$.

Therefore an eigenvector v_j of A corresponding to λ_j ($j=1,2$) is

$$v_j = \begin{pmatrix} 1 & p\lambda_j \end{pmatrix}^t, \quad (3.1.7)$$

where the superscript t denotes the transpose. We assume at this stage that the λ_j are distinct and we define the matrix T in (3.1.5) by

$$T = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \quad (3.1.8a)$$

Then by, (3.1.7) and (3.1.8a),

$$T = \begin{pmatrix} 1 & 1 \\ p\lambda_1 & p\lambda_2 \end{pmatrix} \quad (3.1.8b)$$

The inverse of T is

$$T^{-1} = \{p(\lambda_2 - \lambda_1)\}^{-1} \begin{pmatrix} p\lambda_2 & -1 \\ -p\lambda_1 & 1 \end{pmatrix} \quad (3.1.9)$$

Differentiating (3.1.8b) we obtain

$$T' = \begin{pmatrix} 0 & 0 \\ (p\lambda_1)' & (p\lambda_2)' \end{pmatrix} \quad (3.1.10)$$

By (3.1.5), the transformation

$$Y = TZ \quad (3.1.11)$$

takes (3.1.2a) into

$$Z' = (\Lambda + R)Z, \quad (3.1.12)$$

where $\Lambda = \text{dg}(\lambda_1, \lambda_2)$. (3.1.13)

From (3.1.7) - (3.1.10), we obtain

$$\begin{aligned} R &= -T^{-1}T' \\ &= -\{p(\lambda_2 - \lambda_1)\}^{-1} \begin{pmatrix} -(p\lambda_1)' & -(p\lambda_2)' \\ (p\lambda_1)' & (p\lambda_2)' \end{pmatrix}. \end{aligned} \quad (3.1.14)$$

Now we need to work out (3.1.13) and (3.1.14) in some detail in terms of p , q and r in order to determine the form of (3.1.12) and then to apply the general asymptotic theorem of Eastham [6, Section 1.6]. We then transform back to Y by means of (3.1.11) and (3.1.8b).

3.2 The Matrices Λ and R :

In our analysis, the basic conditions required to express the fact that r is small compared to p and q as $x \rightarrow \infty$, are as follows:

(1) p and q are nowhere zero in some interval $[a, \infty)$ and

$$r = o\{(pq)^{1/2}\} \quad (x \rightarrow \infty) \quad (3.2.1)$$

and we write

$$\delta = \frac{r}{(pq)^{1/2}} = o(1). \quad (3.2.2)$$

Also, we require that p , q and r to be $C^{(2)} [a, \infty)$ such that the functions

$$(2) \quad \frac{r'}{(pq)^{1/2}}, \frac{q'}{q} \delta, \frac{p'}{p} \delta \quad \text{are all } L(a, \infty). \quad (3.2.3)$$

Now subject to (3.2.1), the characteristic equation (3.1.6) can be solved as in [1] to obtain

$$\lambda_j = \{ -r \pm (r^2 + 4pq)^{1/2} \} / 2p \quad (j=1,2). \quad (3.2.4)$$

Using

$$\begin{aligned} r^2 + 4pq &= 4pq \left(1 + \frac{r^2}{4pq} \right) \\ &= 4pq \left(1 + \frac{\delta^2}{4} \right), \end{aligned}$$

where δ is as in (3.2.2), we obtain

$$(r^2 + 4pq)^{1/2} = \left\{ 4pq \left(1 + \frac{\delta^2}{4} \right) \right\}^{1/2}. \quad (3.2.5)$$

But

$$\left(1 + \frac{\delta^2}{4}\right)^{1/2} = 1 + \frac{\delta^{2/4}}{1 + \left(1 + \delta^{2/4}\right)^{1/2}},$$

therefore (3.2.5) becomes

$$(r^2 + 4pq)^{1/2} = 2(pq)^{1/2} \left\{ 1 + \frac{\delta^{2/4}}{1 + \left(1 + \delta^{2/4}\right)^{1/2}} \right\}$$

and hence

$$\lambda_j = \left\{ -r \pm 2(pq)^{1/2} \left\{ 1 + \frac{\delta^{2/4}}{1 + \left(1 + \delta^{2/4}\right)^{1/2}} \right\} \right\} / 2p$$

i.e.

$$\begin{aligned} \lambda_j &= -\frac{r}{2p} \pm \left(\frac{q}{p}\right)^{1/2} \left\{ 1 + \frac{\delta^{2/4}}{1 + \left(1 + \delta^{2/4}\right)^{1/2}} \right\} \\ &= \pm \left(\frac{q}{p}\right)^{1/2} \left\{ \left\{ 1 + \frac{\delta^{2/4}}{1 + \left(1 + \delta^{2/4}\right)^{1/2}} \right\} \mp \frac{r}{2(pq)^{1/2}} \right\} \\ &= \pm \left(\frac{q}{p}\right)^{1/2} \left\{ \left\{ 1 + \frac{\delta^{2/4}}{1 + \left(1 + \delta^{2/4}\right)^{1/2}} \right\} \mp \frac{\delta}{2} \right\}. \\ &= \pm \left(\frac{q}{p}\right)^{1/2} \left\{ \left(1 + \frac{\delta^2}{4}\right)^{1/2} \mp \frac{\delta}{2} \right\}. \end{aligned}$$

Hence

$$\lambda_j = \pm \left(\frac{q}{p}\right)^{1/2} \left\{ 1 + \delta_j \right\} \quad (j=1,2), \quad (3.2.6)$$

where

$$\delta_j = o(\delta) \quad (j=1,2). \quad (3.2.7)$$

From (3.2.6) we see that the eigenvalues $\lambda_1 \sim (\text{const.}) \lambda_2$, i.e. the eigenvalues having same order for large x .

By (3.2.6), we have

$$p\lambda_j = \pm (pq)^{1/2}(1 + \delta_j) \quad (j=1,2). \quad (3.2.8)$$

Differentiating (3.2.8) to get

$$(p\lambda_j)' = \pm \frac{(pq)'}{2(pq)^{1/2}} \pm (pq)^{1/2} \left\{ \delta_j' + O\left(\delta_j \frac{(pq)'}{pq}\right) \right\} \quad (j=1,2) \quad (3.2.9)$$

$$\text{Let } B = \delta_j' + O\left(\delta_j \frac{(pq)'}{pq}\right). \quad (j=1,2) \quad (3.2.10)$$

On substituting (3.2.6) into (3.1.6) we obtain

$$p \frac{q}{p} (1 + \delta_j)^2 + r \left(\frac{q}{p}\right)^{1/2} (1 + \delta_j) - q = 0$$

which can be simplified to get

$$(1 + \delta_j)^2 + \delta (1 + \delta_j) - 1 = 0. \quad (3.2.11)$$

Differentiating (3.2.11) we get

$$2(1 + \delta_j)\delta_j' + \delta'(1 + \delta_j) + \delta\delta_j' = 0 \quad (3.2.12)$$

i.e.

$$2\delta_j' \left\{ 1 + \delta_j + \frac{\delta}{2} \right\} = -\delta'(1 + \delta_j). \quad (3.2.13)$$

Solving (3.2.13) for δ_j' we get

$$\delta_j' = -\frac{1}{2} \delta'(1 + \delta_j) \left\{ 1 + \delta_j + \frac{\delta}{2} \right\}^{-1} \quad (j=1,2). \quad (3.2.14)$$

Hence

$$\begin{aligned}\delta_j' &= O(\delta') \\ &= O\left(\frac{r'}{(pq)^{1/2}}\right) + O\left(\frac{q'}{q}\delta\right) + O\left(\frac{p'}{p}\delta\right) \quad (j=1,2).\end{aligned}\quad (3.2.15)$$

Therefore, by (3.2.3) and (3.2.15),

$$\delta_j' \text{ is in } L(a, \infty). \quad (3.2.16)$$

So (3.2.9) becomes

$$(p\lambda_j)' = \pm \frac{(pq)'}{2(pq)^{1/2}} \pm (pq)^{1/2} B. \quad (3.2.17)$$

Hence (3.1.12) can be written as

$$Z' = (\Lambda + Q + S)Z, \quad (3.2.18)$$

where

$$Q = -\frac{(pq)'}{4pq} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad (3.2.19)$$

and

$$S = O\left(\frac{r'}{(pq)^{1/2}}\right) + O\left(\frac{q'}{q}\delta\right) + O\left(\frac{p'}{p}\delta\right) \quad (j=1,2). \quad (3.2.20)$$

By (3.2.16), S is in $L(a, \infty)$.

Now we shall consider the two cases (3) and (5) for the system (3.2.18) in some detail.

3.3 Case I:

Let

$$\frac{(pq)'}{(pq)} = o \left\{ \left(\frac{q}{p} \right)^{1/2} \right\} \quad (x \rightarrow \infty). \quad (3.3.1)$$

Also let

$$\left\{ p^{-1/2} q^{-3/2} (pq)' \right\}' \in L(a, \infty). \quad (3.3.2)$$

Now we can apply the asymptotic theorem of Eastham [6, Theorem 1.6.1] to the system (3.2.18) provided that Λ and Q satisfy the conditions in [6, Theorem 1.6.1]. By (3.2.19) and (3.2.6), the conditions (i) and (ii) in Theorem (2.3.1) in Chapter 2 are satisfied by (3.3.1) and (3.3.2). We now state our main theorem for (3.1.1).

Theorem (3.3.1) :

Let p and q be nowhere zero and $C^{(2)}[a, \infty)$. Let (3.2.1), (3.2.3), (3.3.1) and (3.3.2) hold. Also let

$$\text{re} \left\{ \left(\frac{(pq)'}{4pq} \right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2} \right)^2 \right\}^{1/2} \text{ have one sign in } [a, \infty). \quad (3.3.3)$$

Then (3.1.1) has solutions y_1 and y_2 such that

$$y_1 \sim (pq)^{-1/4} \exp \left(\int_a^x \left\{ \left(\frac{(pq)'}{4pq} \right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2} \right)^2 \right\}^{1/2} dt \right) \quad (3.3.4a)$$

$$py_1' \sim (pq)^{1/4} \exp \left(\int_a^x \left\{ \left(\frac{(pq)'}{4pq} \right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2} \right)^2 \right\}^{1/2} dt \right), \quad (3.3.4b)$$

while

$$y_2 \sim (pq)^{-1/4} \exp \left(- \int_a^x \left\{ \left(\frac{(pq)'}{4pq} \right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2} \right)^2 \right\}^{1/2} dt \right) \quad (3.3.5a)$$

$$py_2' \sim (pq)^{1/4} \exp \left(- \int_a^x \left\{ \left(\frac{(pq)'}{4pq} \right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2} \right)^2 \right\}^{1/2} dt \right). \quad (3.3.5b)$$

Proof: Before applying the Eastham theorem(2.3.1), we show that Λ and Q satisfy the conditions (2.3.1) and (2.3.2). By (3.2.18), we first require that

$$\frac{(pq)'}{(pq)} = o \left\{ \left(\frac{q}{p} \right)^{1/2} \right\} \quad (x \rightarrow \infty),$$

this being (2.3.1) for our system and it is implied by (3.3.1). We also require that

$$\left((\lambda_2 - \lambda_1)^{-1} \frac{(pq)'}{(pq)} \right)' \in L(a, \infty)$$

which is (2.3.2) for our system and it is implied by (3.3.2). Moreover, we show that the eigenvalues of $\Lambda + Q$ satisfy the dichotomy condition (2.2.5). Now let μ_j ($j=1,2$) be the eigenvalues of $\Lambda + Q$. Then μ_j can be found from the characteristic equation

$$\det(\Lambda + Q - \mu I) = 0 \quad (3.3.6)$$

where Λ and Q are as in (3.1.13) and (3.2.19), respectively, and I is the 2×2 identity matrix. Therefore, (3.3.6) becomes

$$(\lambda_1 - u - \mu)(\lambda_2 - u - \mu) - u^2 = 0,$$

where u is as in (2.4.5), which can be simplified to get

$$\mu^2 + b\mu + c = 0, \quad (3.3.7)$$

where

$$b = 2u - \lambda_1 - \lambda_2, \text{ and } c = \lambda_1 \lambda_2 - u(\lambda_1 + \lambda_2).$$

Solving the second-degree equation (3.3.7) and using the facts that

$$\lambda_1 + \lambda_2 = O\left(\left(\frac{q}{p}\right)^{1/2} \delta\right), \quad \lambda_1 - \lambda_2 = 2\left(\frac{q}{p}\right)^{1/2} (1 + O(\delta))$$

and

$$(4u^2 + \lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2)^{1/2} = 2 \left\{ u^2 + \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 \right\}^{1/2},$$

we obtain

$$\mu_j = -u \pm \left\{ u^2 + \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 \right\}^{1/2} + O\left(\left(\frac{q}{p}\right)^{1/2} \delta\right) \quad (j=1,2). \quad (3.3.8)$$

Then

$$\operatorname{re} \{ \mu_1 - \mu_2 \} = \operatorname{re} \left\{ \left(\frac{pq}{4pq}\right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 \right\}^{1/2}.$$

So, by (3.3.3), the dichotomy condition holds. Since (3.2.18) satisfies all the conditions for the asymptotic Theorem (2.3.1), it follows that, as $x \rightarrow \infty$, (3.2.18) has two linearly independent solutions:

$$Z_j(x) = \{e_j + o(1)\} \exp\left(\int_a^x \mu_{j(t)} dt\right) \quad (3.3.9)$$

where e_j ($j=1,2$) is the coordinate vector with j^{th} component unity and other components zero. Finally, on transforming back via (3.1.8b), (3.1.11), (3.1.3) and (3.3.9), we get

$$\begin{pmatrix} y_j \\ py'_j \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ p\lambda_1 & p\lambda_2 \end{pmatrix} \{e_j + o(1)\} \exp \left(\int_a^x \mu_j(t) dt \right).$$

For $j = 1$,

$$\begin{pmatrix} y_1 \\ py'_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ p\lambda_1 & p\lambda_2 \end{pmatrix} \begin{pmatrix} 1 + o(1) \\ o(1) \end{pmatrix} \exp \left(\int_a^x \mu_1(t) dt \right)$$

i.e.

$$\begin{pmatrix} y_1 \\ py'_1 \end{pmatrix} = \begin{pmatrix} 1 + o(1) \\ p\lambda_1 (1 + o(1)) + o(p\lambda_2) \end{pmatrix} \exp \left(\int_a^x \mu_1(t) dt \right),$$

which gives

$$y_1 = (1 + o(1)) \exp \left(\int_a^x \mu_1(t) dt \right), \quad (3.3.10a)$$

and

$$py'_1 = (pq)^{1/2} (1 + o(1)) \exp \left(\int_a^x \mu_1(t) dt \right). \quad (3.3.10b)$$

Integrating (3.3.8) and taking the exponential we obtain

$$\exp \left(\int_a^x \mu_1(t) dt \right) = (pq)^{-1/4} \exp \left(\int_a^x \left\{ \left(\frac{(pq)'}{4pq} \right)^2 + \left(\frac{\lambda_1 - \lambda_2}{2} \right)^2 \right\}^{1/2} dt \right). \quad (3.3.11)$$

By substituting (3.3.11) into (3.3.10a) and (3.3.10b), we obtain (3.3.4a) and (3.3.4b) after adjusting y_1 by a constant multiple. Similarly for $j = 2$.

Corollary (3.3.2) :

Let $p(x)$ and $q(x)$ be nowhere zero and $C^{(2)}[a, \infty)$. Let (3.3.1) and (3.3.2) hold and, in addition, let

$$rp^{-1} \in L(a, \infty), \quad (3.3.12)$$

$$p^{-3/2} q^{-5/2} (pq)'^2 \in L(a, \infty). \quad (3.3.13)$$

Let

$$\operatorname{re} \left\{ \left(\frac{q}{p} \right)^{1/2} \right\} \text{ have one sign in } [a, \infty). \quad (3.3.14)$$

Then (3.1.1) has solutions y_1 and y_2 such that

$$y_1 \sim (pq)^{-1/4} \exp \left(\int_a^x \left\{ \left(\frac{q}{p} \right) \right\}^{1/2} dt \right), \quad (3.3.15)$$

$$py_1' \sim (pq)^{1/4} \exp \left(\int_a^x \left\{ \left(\frac{q}{p} \right) \right\}^{1/2} dt \right), \quad (3.3.16)$$

with similar formulae for y_2 containing $-\left(\frac{q}{p}\right)^{1/2}$ in the exponential term.

Proof: By (3.3.1) and (2.4.5), we have $u = o\left\{\left(\frac{q}{p}\right)^{1/2}\right\}$ and therefore, in (3.3.8), we can write

$$\begin{aligned} \left\{ u^2 + \left(\frac{\lambda_1 - \lambda_2}{2} \right)^2 \right\}^{1/2} &= \frac{\lambda_1 - \lambda_2}{2} + O\left(u^2 (\lambda_1 - \lambda_2)^{-1} \right) \\ &= \left(\frac{q}{p} \right)^{1/2} (1 + O(\delta)) + O\left(u^2 \left(\frac{p}{q} \right)^{1/2} \right) \\ &= \left(\frac{q}{p} \right)^{1/2} + O\left(\left(\frac{q}{p} \right)^{1/2} \delta \right) + O\left(u^2 \left(\frac{p}{q} \right)^{1/2} \right). \end{aligned}$$

The O-terms here are $L(a, \infty)$ by (3.3.12) and (3.3.13). Then it follows from (3.3.14) that μ_1 and μ_2 satisfy the dichotomy condition (2.2.5). Also, (3.3.15) and (3.3.16) follow from (3.3.4a) and (3.3.4b) when y_1 is adjusted by a constant multiple.

4. Case II (Euler Type):

Now to deal with the condition (5) for the system (3.2.18), we write (5) as

$$\frac{(pq)'}{pq} = k \left(\frac{q}{p} \right)^{1/2} (1 + \phi) \quad (3.4.1)$$

where k is a non-zero constant with $k^2 \neq -16$,

$$\phi = o(1) \quad (x \rightarrow \infty) \quad (3.4.2)$$

and

$$\phi' \in L(a, \infty). \quad (3.4.3)$$

Now it is easy to see that the first condition of Eastham theorem [6, Section 1.6] is not true subject to (3.4.1). So it is necessary to carry out a second diagonalization. To do this, we first use (3.4.1), (3.2.19) and (3.2.6) to write the system (3.2.18) in the form

$$Z' = \left(\frac{q}{p} \right)^{1/2} (C + R_1 + S)Z, \quad (3.4.4)$$

where C is a constant matrix, S as in (3.2.20) and $R_1 \rightarrow 0$ as $x \rightarrow \infty$.

To obtain the system (3.4.4), we write

$$\Lambda + Q = \begin{pmatrix} \left(\frac{q}{p} \right)^{1/2} (1 + \delta_1) - \frac{(pq)'}{4pq} & \frac{(pq)'}{4pq} \\ \frac{(pq)'}{4pq} & - \left(\frac{q}{p} \right)^{1/2} (1 + \delta_2) - \frac{(pq)'}{4pq} \end{pmatrix}. \quad (3.4.5)$$

Our aim now is to write the matrix on the right-hand side of (3.4.5) as

$$\left(\frac{q}{p} \right)^{1/2} (C + R_1),$$

and this can be done as follows

$$\Lambda + Q = \left(\frac{q}{p}\right)^{1/2} \begin{pmatrix} 1 + \delta_1 - \frac{(pq)'}{4pq} \left(\frac{p}{q}\right)^{1/2} & \frac{(pq)'}{4pq} \left(\frac{p}{q}\right)^{1/2} \\ \frac{(pq)'}{4pq} \left(\frac{p}{q}\right)^{1/2} & -1 - \delta_2 - \frac{(pq)'}{4pq} \left(\frac{p}{q}\right)^{1/2} \end{pmatrix}$$

using (3.4.1), we get

$$= \left(\frac{q}{p}\right)^{1/2} \begin{pmatrix} 1 + \delta_1 - \frac{1}{4} k(1 + \phi) & \frac{1}{4} k(1 + \phi) \\ \frac{1}{4} k(1 + \phi) & -1 - \delta_2 - \frac{1}{4} k(1 + \phi) \end{pmatrix}$$

$$= \left(\frac{q}{p}\right)^{1/2} \begin{pmatrix} 1 + \delta_1 - \frac{1}{4} k - \frac{1}{4} k\phi & \frac{1}{4} k + \frac{1}{4} k\phi \\ \frac{1}{4} k + \frac{1}{4} k\phi & -1 - \delta_2 - \frac{1}{4} k - \frac{1}{4} k\phi \end{pmatrix}$$

$$= \left(\frac{q}{p}\right)^{1/2} (C + R_1),$$

where C is a constant matrix given by

$$C = \begin{pmatrix} 1 - \frac{1}{4} k & \frac{1}{4} k \\ \frac{1}{4} k & -1 - \frac{1}{4} k \end{pmatrix} \quad (3.4.6)$$

and $R_1 = o(1)$ as $x \rightarrow \infty$, where

$$R_1 = \begin{pmatrix} \delta_1 - \frac{1}{4} k\phi & \frac{1}{4} k\phi \\ \frac{1}{4} k\phi & -\delta_2 - \frac{1}{4} k\phi \end{pmatrix}. \quad (3.4.7)$$

As in [6] , since C is not a diagonal matrix, we have to express C in its diagonal form and then apply Eastham theorem. Thus, by (3.4.6), the eigenvalues of C are given by

$$\alpha_j = -\frac{1}{4}k \pm \frac{1}{4}k(1 + 16k^{-2})^{1/2}. \quad (j=1,2) \quad (3.4.8)$$

Hence we can use the transformation

$$Z = T_1 W, \quad (3.4.9)$$

in (3.4.4), where

$$T_1 = \begin{pmatrix} \frac{1}{4}k & -\frac{1}{4}k \\ -1 + \frac{1}{4}k(1 + 16k^{-2})^{1/2} & 1 + \frac{1}{4}k(1 + 16k^{-2})^{1/2} \end{pmatrix}, \quad (3.4.10)$$

T_1 is non-singular when $k \neq 0$ and $k^2 \neq -16$ and it diagonalizes the matrix C . Thus, (3.4.4) becomes, after differentiating (3.4.9),

$$W' = \left(\frac{q}{p}\right)^{1/2} (\Lambda_1 + M + T_1^{-1} S T_1) W, \quad (3.4.11)$$

where

$$\Lambda_1 = T_1^{-1} C T_1 = \text{dg}(\alpha_1, \alpha_2), \quad (3.4.12)$$

and

$$M = T_1^{-1} R_1 T_1, \quad (3.4.13)$$

while

$$T_1^{-1} S T_1 \in L(a, \infty). \quad (3.4.14)$$

Now to apply the asymptotic theorem of Eastham [6, Section 1.6], we show that Λ_1 and M satisfy the conditions (i) and (ii) in theorem (2.3.1). First we note that by (3.4.8), α_j ($j=1,2$) are distinct. Second we note that, by (3.4.12) and (3.4.13),

$$(\alpha_2 - \alpha_1)^{-1} M = \left\{ -\frac{1}{2} k(1 + 16k^{-2})^{1/2} \right\}^{-1} T_1^{-1} R_1 T_1 = o(1). \quad (x \rightarrow \infty) \quad (3.4.15)$$

Third, we need to show that

$$\{(\alpha_2 - \alpha_1)^{-1} M\}' \in L(a, \infty). \quad (3.4.16)$$

Now

$$\{(\alpha_2 - \alpha_1)^{-1} M\}' = \left\{ -\frac{1}{2} k(1 + 16k^{-2})^{1/2} \right\}^{-1} T_1^{-1} R_1' T_1. \quad (3.4.17)$$

Hence (3.4.16) holds if we show that

$$R_1' \in L(a, \infty). \quad (3.4.18)$$

Thus, by (3.2.16) and (3.4.3), (3.4.18) holds and consequently (3.4.16). Hence the conditions (i) and (ii) of theorem (2.3.1) are satisfied. Now we state the theorem for the Euler case.

Theorem (3.4.1) :

Let p and q be nowhere zero and $C^{(2)} [a, \infty)$. Let (3.2.1), (3.2.3), (3.3.3), (3.4.1) and (3.4.3) hold.

Then (3.1.1) has solutions y_1 and y_2 such that

$$y_1 \sim (pq)^c \exp \left(\frac{1}{4} \int_a^x \left\{ \left(\frac{(pq)'}{pq} \right) \Phi \right\} dt \right) \quad (3.4.20)$$

$$py_1' \sim ck(pq)^{1/2+c} \exp \left(\frac{1}{4} \int_a^x \left\{ \left(\frac{(pq)'}{pq} \right) \Phi \right\} dt \right) \quad (3.4.21)$$

and

$$y_2 \sim (pq)^{-1/2-c} \exp \left(-\frac{1}{4} \int_a^x \left\{ \left(\frac{(pq)'}{pq} \right) \Phi \right\} dt \right) \quad (3.4.22)$$

$$py_2' \sim -(ck)^{-1} (pq)^{-c} \exp \left(-\frac{1}{4} \int_a^x \left\{ \left(\frac{(pq)'}{pq} \right) \Phi \right\} dt \right), \quad (3.4.23)$$

where

$$c = -\frac{1}{4} + \frac{1}{4} (1 + 16k^{-2})^{1/2}, \quad (3.4.24)$$

and

$$\Phi = -16k^{-2}(1 + 16k^{-2})^{-1/2} \phi + O(\phi\delta) + O(\phi^2) \quad (3.4.25)$$

Proof: By (3.4.15) and (3.4.16), the conditions (i) and (ii) of the Eastham theorem (2.3.1) are satisfied. Now we show that the eigenvalues of $\Lambda_1 + M$ satisfy the dichotomy condition (2.2.5). The eigenvalues of $\Lambda_1 + M$ are just the eigenvalues of

$$\left(\frac{q}{p}\right)^{1/2} (C + R_1) = \Lambda + Q$$

Thus, by (3.3.8) and (3.4.19), the dichotomy condition holds. Since (3.4.11) satisfies all the conditions of the asymptotic theorem of Eastham (2.3.1), it follows that, as $x \rightarrow \infty$, (3.4.11) has two linearly independent solutions:

$$W_j(x) = \{e_j + o(1)\} \exp\left(\int_a^x \mu_j(t) dt\right), \quad (3.4.26)$$

where e_j ($j=1,2$) is as in the proof of Theorem (3.3.1). Now by transforming back to Y by means of (3.4.9), (3.4.10), (3.1.11), (3.1.8b), (3.1.3) and (3.4.26), we obtain (3.4.20), (3.4.21), (3.4.22) and (3.4.23).

Using (3.4.1), the expression (3.3.8) for μ_1 and μ_2 can also be written as

$$\frac{(pq)'}{pq} \left(-\frac{1}{4} \pm \frac{1}{4} \{1 + 16k^{-2}(1 + \phi)^{-2}(1 + O(\delta))^2\}^{1/2} \right). \quad (3.4.27)$$

To obtain (3.4.25), we note that the exact expression for Φ given by (3.4.26) and (3.4.27) is

$$\Phi = \{1 + 16k^{-2}(1 + \phi)^{-2}(1 + O(\delta))^2\}^{1/2} - \{1 + 16k^{-2}\}^{1/2}$$

we then expand in powers of ϕ up to $\phi\delta$ and ϕ^2 . Because

$$(1 + \phi)^{-2} = 1 - 2\phi + 3\phi^2 + \dots,$$

it follows that

$$16k^{-2}(1 + \phi)^{-2} = 16k^{-2} - 16k^{-2} \cdot 2\phi + O(\phi^2).$$

Then

$$\begin{aligned} 16k^{-2}(1+\phi)^{-2}(1+O(\delta))^2 &= 16k^{-2}(1+\phi)^{-2}(1+O(\delta)) \\ &= 16k^{-2} - 16k^{-2} \cdot 2\phi + O(\phi\delta) + O(\phi^2). \end{aligned}$$

Therefore

$$\begin{aligned} \left\{ 1 + 16k^{-2}(1+\phi)^{-2}(1+O(\delta))^2 \right\}^{1/2} &= \left\{ 1 + 16k^{-2} \right\}^{1/2} - 16k^{-2}(1+16k^{-2})^{-1/2} \phi \\ &\quad + O(\phi\delta) + O(\phi^2). \end{aligned}$$

Hence the result follows.

CHAPTER FOUR

Examples and Remarks

Example (1). We consider theorem (3.3.1) applied to the coefficients

$$p(x) = Ax^\alpha, \quad q(x) = Bx^\beta, \quad r(x) = Cx^\gamma, \quad (4.1)$$

where $\alpha, \beta, \gamma, A(\neq 0), B(\neq 0)$ and C are real constants with $AB > 0$. To show that the conditions of theorem (3.3.1) are all satisfied, we first note the following

$$\begin{aligned} \text{(i)} \quad \delta &= \frac{r}{(pq)^{1/2}} \\ &= \frac{C}{\sqrt{AB}} x^{\gamma - (\alpha + \beta)/2}. \end{aligned}$$

Thus, $\delta \rightarrow 0$ as $x \rightarrow \infty$ if $\gamma - \frac{1}{2}(\alpha + \beta) < 0$, that is, if $\alpha + \beta > 2\gamma$.

Therefore (3.2.1) is satisfied if

$$\alpha + \beta > 2\gamma. \quad (4.2)$$

$$\text{(ii)} \quad \frac{r'}{(pq)^{1/2}} = \frac{C\gamma}{\sqrt{AB}} x^{\gamma - 1 - (\alpha + \beta)/2}.$$

Thus, $\frac{r'}{(pq)^{1/2}} \in L(a, \infty)$ if

$$\alpha + \beta > 2\gamma. \quad (4.3a)$$

$$\text{(iii)} \quad \frac{q'}{q} = \beta x^{-1}$$

Therefore, $\frac{q'}{q} \delta = \frac{C\beta}{\sqrt{AB}} x^{\gamma - 1 - (\alpha + \beta)/2}.$

Hence $\frac{q'}{q} \delta \in L(a, \infty)$ if

$$\alpha + \beta > 2\gamma. \quad (4.3b)$$

$$(iv) \frac{p'}{p} = \alpha x^{-1}$$

$$\text{Therefore, } \frac{p'}{p} \delta = \frac{C\alpha}{\sqrt{AB}} x^{\gamma-1-(\alpha+\beta)/2}.$$

$$\text{Hence } \frac{p'}{p} \delta \in L(a, \infty) \text{ if}$$

$$\alpha + \beta > 2\gamma. \quad (4.3c)$$

Therefore, (3.2.3) is satisfied if

$$\alpha + \beta > 2\gamma. \quad (4.4)$$

Now equation (3.3.1) gives us

$$\frac{(pq)'}{pq} = (\text{const.}) x^{-1}$$

and

$$\left(\frac{p}{q}\right)^{1/2} = \left(\frac{A}{B}\right)^{1/2} x^{(\alpha-\beta)/2}.$$

$$\text{Hence (3.3.1) is satisfied if } \frac{1}{2}(\alpha - \beta) - 1 < 0.$$

$$\text{i.e. } \alpha - \beta < 2. \quad (4.5)$$

The condition (3.3.2) gives us

$$p^{-1/2} q^{-3/2} (pq)' = (\text{const.}) x^{(\alpha-\beta)/2-1}.$$

Hence (3.3.2) is satisfied if

$$\alpha - \beta < 2. \quad (4.6)$$

Therefore the conditions of theorem (3.3.1) are all satisfied under the two conditions $\alpha + \beta > 2\gamma$ and $\alpha - \beta < 2$.

Example (2). We consider theorem (3.4.1) applied to the coefficients in (4.1), where $\alpha + \beta \neq 0$. Then (3.2.1) and (3.2.3) are satisfied if (4.2) and (4.5) hold, respectively.

Now equation (3.4.1) gives us

$$\frac{(pq)'}{pq} \left(\frac{p}{q}\right)^{1/2} = \left(\frac{A}{B}\right)^{1/2} (\alpha + \beta) x^{(\alpha - \beta)/2 - 1}.$$

Then (3.4.1) holds if $\frac{1}{2}(\alpha - \beta) - 1 = 0$.

$$\text{i.e. } \alpha - \beta = 2. \quad (4.7)$$

The constant k in (3.4.1) is given by

$$k = (\alpha + \beta) \left(\frac{A}{B}\right)^{1/2}.$$

Here $\phi = 0$, so $\Phi = 0$.

Therefore, the asymptotic formulae give

$$y_j \sim (x^{\alpha + \beta})^{-\frac{1}{4} \pm \frac{1}{4} \left[1 + \frac{4B}{A}(1 + \beta)^{-2}\right]^{1/2}} \quad (j=1,2) \quad (4.8)$$

$$py_1' \sim ck (x^{\alpha + \beta})^{-\frac{1}{4} + \frac{1}{4} \left[1 + \frac{4B}{A}(1 + \beta)^{-2}\right]^{1/2}} \quad (4.9)$$

and

$$py_2' \sim -(ck)^{-1} (x^{\alpha + \beta})^{-\frac{1}{4} - \frac{1}{4} \left[1 + \frac{4B}{A}(1 + \beta)^{-2}\right]^{1/2}}. \quad (4.10)$$

Example (3). Consider theorem (3.3.1) applied to the coefficients

$$p(x) = Ax^\alpha \exp(x^m), q(x) = Bx^\beta \exp(x^n), r(x) = Cx^\gamma \exp\left(\frac{1}{2}x^m\right),$$

where $\alpha, \beta, \gamma, m, n, A(\neq 0), B(\neq 0)$ and C are real constants with $AB > 0$ and $n > m \geq 0$. To show that the conditions of theorem (3.3.1) are all satisfied, we first note the following:

$$(i) \quad \delta = \frac{r}{(pq)^{1/2}}$$

$$= (\text{const.}) x^{\gamma - (\alpha + \beta)/2} \exp\left(-\frac{1}{2}x^n\right).$$

Thus, $\delta \rightarrow 0$ as $x \rightarrow \infty$.

$$(ii) \quad \frac{r'}{(pq)^{1/2}} = (\text{const.}) x^{\gamma-1 - (\alpha + \beta)/2} \exp\left(-\frac{1}{2}x^n\right)$$

$$+ (\text{const.}) x^{\gamma+m-1 - (\alpha + \beta)/2} \exp\left(-\frac{1}{2}x^n\right).$$

Thus, $\frac{r'}{(pq)^{1/2}} \in L(a, \infty)$.

$$(iii) \quad \frac{q'}{q} = \beta x^{-1} + nx^{n-1}$$

Therefore

$$\frac{q'}{q} \delta = \left[(\text{const.}) x^{\gamma-1 - (\alpha + \beta)/2} + (\text{const.}) x^{\gamma+n-1 - (\alpha + \beta)/2} \right] \exp\left(-\frac{1}{2}x^n\right).$$

Hence, $\frac{q'}{q} \delta \in L(a, \infty)$.

$$(iv) \quad \frac{p'}{p} = \alpha x^{-1} + mx^{m-1}$$

Therefore

$$\frac{p'}{p} \delta = \left[(\text{const.}) x^{\gamma-1 - (\alpha + \beta)/2} + (\text{const.}) x^{\gamma+m-1 - (\alpha + \beta)/2} \right] \exp\left(-\frac{1}{2}x^n\right).$$

Hence, $\frac{p'}{p} \delta \in L(a, \infty)$.

Therefore, (3.2.1) and (3.2.3) are all satisfied.

Now equation (3.3.1) gives us

$$\frac{(pq)'}{pq} = (\alpha + \beta) x^{-1} + mx^{m-1} + nx^{n-1} \quad (4.11)$$

and

$$\left(\frac{p}{q}\right)^{1/2} = \left(\frac{A}{B}\right)^{1/2} x^{(\alpha - \beta)/2} \exp\left(\frac{1}{2} (x^m - x^n)\right). \quad (4.12)$$

Therefore, we get

$$\frac{(pq)'}{pq} \left(\frac{p}{q}\right)^{1/2} = \left(\frac{A}{B}\right)^{1/2} x^{(\alpha - \beta)/2 - 1} \{ (\alpha + \beta) + mx^m + nx^n \} \exp\left(\frac{1}{2} (x^m - x^n)\right).$$

Thus (3.3.1) is satisfied.

The condition (3.3.2) gives us

$$p^{-1/2} q^{-3/2} (pq)' = \{ (\text{const.}) + (\text{const.})x^m + (\text{const.})x^n \} x^{(\alpha - \beta)/2 - 1} \exp\left(\frac{1}{2} (x^m - x^n)\right). \quad (4.13)$$

Hence (3.3.2) is satisfied.

Example (4). Consider theorem (3.4.1) applied to the coefficients

$$p(x) = Ax^\alpha \exp(x^m), q(x) = Bx^\beta \exp(x^m), r(x) = Cx^\gamma \exp(x^n),$$

where $\alpha, \beta, \gamma, m, n, A(\neq 0), B(\neq 0)$ and C are real constants with $AB > 0$, $m > n \geq 0$ and $\alpha + \beta \neq 0$. To show that the conditions of theorem (3.4.1) are all satisfied, we first note the following:

$$\begin{aligned} \text{(i)} \quad \delta &= \frac{r}{(pq)^{1/2}} \\ &= (\text{const.}) x^{\gamma - (\alpha + \beta)/2} \exp(x^n - x^m). \end{aligned} \quad (4.14)$$

Thus, $\delta \rightarrow 0$ as $x \rightarrow \infty$.

$$\begin{aligned} \text{(ii)} \quad \frac{r'}{(pq)^{1/2}} &= (\text{const.}) x^{\gamma - 1 - (\alpha + \beta)/2} \exp(x^n - x^m) \\ &\quad + (\text{const.}) x^{\gamma + n - 1 - (\alpha + \beta)/2} \exp(x^n - x^m). \end{aligned}$$

$$\text{Thus, } \frac{r'}{(pq)^{1/2}} \in L(a, \infty). \quad (4.15)$$

$$\text{(iii)} \quad \frac{q'}{q} = \beta x^{-1} + mx^{m-1}$$

Therefore

$$\begin{aligned} \frac{q'}{q} \delta &= (\text{const.}) x^{\gamma - 1 - (\alpha + \beta)/2} \exp(x^n - x^m) \\ &\quad + (\text{const.}) x^{\gamma + m - 1 - (\alpha + \beta)/2} \exp(x^n - x^m). \end{aligned}$$

$$\text{Hence, } \frac{q'}{q} \delta \in L(a, \infty).$$

$$(iv) \frac{p'}{p} = \alpha x^{-1} + mx^{m-1}$$

Therefore

$$\begin{aligned} \frac{p'}{p} \delta &= (\text{const.}) x^{\gamma-1-(\alpha+\beta)/2} \exp(x^n - x^m) \\ &\quad + (\text{const.}) x^{\gamma+m-1-(\alpha+\beta)/2} \exp(x^n - x^m). \end{aligned}$$

$$\text{Hence } \frac{p'}{p} \delta \in L(a, \infty) .$$

Hence (3.2.1) and (3.2.3) are satisfied .

Now equation (3.4.1) gives us

$$\frac{(pq)'}{pq} = (\alpha + \beta) x^{-1} + 2mx^{m-1} \quad (4.16)$$

and

$$\left(\frac{p}{q}\right)^{1/2} = \left(\frac{A}{B}\right)^{1/2} x^{(\alpha-\beta)/2} .$$

Therefore, we get

$$\frac{(pq)'}{pq} \left(\frac{p}{q}\right)^{1/2} = 2 \left(\frac{A}{B}\right)^{1/2} m x^{m-1+(\alpha-\beta)/2} \left\{ 1 + \frac{(\alpha+\beta)}{2m} x^{-m} \right\} .$$

Hence (3.4.1) is satisfied if $m-1 = \frac{1}{2}(\beta-\alpha)$

$$\text{i.e.} \quad \beta - \alpha = 2(m-1). \quad (4.17)$$

In full, (3.4.1) is

$$(\alpha + \beta) x^{-1} + 2mx^{m-1} = k \left(\frac{B}{A}\right)^{1/2} x^{(\beta-\alpha)/2} (1 + \phi), \quad (4.18)$$

after simplifying, we get

$$2mx^{m-1} \left\{ 1 + \frac{(\alpha + \beta)}{2m} x^{-m} \right\} = k \left(\frac{B}{A} \right)^{1/2} x^{(\beta - \alpha)/2} (1 + \phi) \quad (4.19)$$

which gives

$$k = 2m \left(\frac{A}{B} \right)^{1/2}, \quad (4.20)$$

and

$$\phi(x) = \frac{(\alpha + \beta)}{2m} x^{-m}. \quad (4.21)$$

It is clear that ϕ satisfies (3.4.2) and (3.4.3). Then (3.4.25) gives, using (4.14) and (4.21),

$$\begin{aligned} \Phi(x) &= -16k^{-2}(1 + 16k^{-2})^{-1/2} \phi + O(\phi\delta) + O(\phi^2) \\ &= -16k^{-2}(1 + 16k^{-2})^{-1/2} \frac{(\alpha + \beta)}{2m} x^{-m} \\ &\quad + O(x^{\gamma - m - (\alpha + \beta)/2}) \exp(x^n - x^m) \\ &\quad + O(x^{-2m}). \end{aligned} \quad (4.22)$$

Then, using (4.16),

$$\begin{aligned} \frac{(pq)'}{pq} \Phi(x) &= -16k^{-2}(1 + 16k^{-2})^{-1/2} \frac{(\alpha + \beta)^2}{2m} x^{-m-1} \\ &\quad - 16k^{-2}(1 + 16k^{-2})^{-1/2} (\alpha + \beta) x^{-1} \\ &\quad + O\{(x^{\gamma - m - 1 - (\alpha + \beta)/2} + x^{\gamma - 1 - (\alpha + \beta)/2}) \exp(x^n - x^m)\} \\ &\quad + O(x^{-m-1}). \end{aligned} \quad (4.23)$$

The O-terms in the right-hand of (4.23) are $L(a, \infty)$. Thus

$$\begin{aligned}
 I(x) &= \frac{1}{4} \int_a^x \left(\frac{(pq)'}{pq} \right) \Phi \, dt \\
 &= -4k^{-2} (1 + 16k^{-2})^{-1/2} (\alpha + \beta) \int_a^x t^{-1} \, dt + o(1) \\
 &= -4k^{-2} (1 + 16k^{-2})^{-1/2} (\alpha + \beta) \log x + \text{const.} + o(1).
 \end{aligned} \tag{4.24}$$

By taking the exponential function for (4.24) and a djusting constant multiples, (3.4.20), (3.4.21), (3.4.22) and (3.4.23) give

$$\begin{aligned}
 y_1(x) &\sim (x)^{-(\alpha + \beta)(1 + 16k^{-2})^{-1/2} c} \exp(2cx^m) \\
 y_2(x) &\sim (x)^{-(\alpha + \beta)(1 + 16k^{-2})^{-1/2} (c+1/2)} \exp(-(2c + 1)x^m).
 \end{aligned}$$

Remark 1. We extend the results obtained by Eastham [6] for the equation (3.1.1) with $r = 0$. AL-Hammadi [3] considered (3.1.1) under the case where

$$pq = o(r^2), \quad (x \rightarrow \infty) \tag{4.25}$$

If $p(x) = \text{const. } x^\alpha$, $q(x) = \text{const. } x^\beta$, $r(x) = \text{const. } x^\gamma$, then (4.25) is given by $\alpha + \beta < 2\gamma$ which is the opposite case of (4.4).

Remark 2. The methods in Sections 3.1 to 3.3 can also be applied to the equation

$$y'' + ry' + qy = 0 \tag{4.26}$$

with $p = 1$. Here (4.26) is the equation considered by Walker [7]. We obtain asymptotic results for (4.26) which are extension to those of Walker [7].

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