

TITLE OF THESIS



ASYMPTOTIC THEORY FOR THIRD-ORDER DIFFERENTIAL EQUATIONS

AUTHOR:

ABDUL AZIZ SULTAN ALI HASAN AL-HAMADI

DEGREE:

PH.D. IN MATHEMATICS  
LONDON UNIVERSITY  
KING'S COLLEGE



TO

My mother, father, wife and

my daughter Odhoob

### Abstract

In this Ph.D. thesis we investigate the form of three linearly independent solutions of two types of the third-order linear differential equations as  $x \rightarrow \infty$ .

Asymptotic theory will be developed for these two differential equations with large coefficients that are like powers of the independent variable. The first type is the differential equation

$$\{s^{\frac{1}{2}}(s^{\frac{1}{2}} y')'\}' + qy' + ry = 0, \quad (a \leq x < \infty) \quad (E_1)$$

and we shall establish the asymptotic formula

$$y_k(x) \sim s^{-\frac{1}{6}}(x) r^{-\frac{1}{3}}(x) \exp \left\{ \int_a^x \left[ \omega_k \left( \frac{r}{s} \right)^{\frac{1}{3}}(t) - \frac{1}{3} \bar{\omega}_k \frac{q}{r^{\frac{1}{3}} s^{\frac{2}{3}}}(t) \right] dt \right\}$$

as  $x \rightarrow \infty$ , for three solutions of the differential equation  $(E_1)$  subject to certain conditions on  $s(x)$ ,  $q(x)$  and  $r(x)$  as stated in section 4, Theorem 4.1.

The second type is the differential equation

$$\{s^{\frac{1}{2}}(s^{\frac{1}{2}} y')'\}' - (py')' - ry = 0, \quad (a \leq x < \infty) \quad (E_2)$$

and for this differential equation we shall consider two cases, case (1) is given in section 9, Theorem 9.1, and under this Theorem we obtain three linearly independent solutions  $y_k(x)$  of the differential equation  $(E_2)$  such that

$$y_k(x) \sim \{p(x)r(x)\}^{-\frac{1}{4}} \exp \left\{ \int_a^x \left[ \pm i(r/p)^{\frac{1}{2}}(t) - \frac{1}{2} \frac{sr}{p^2}(t) \right] dt \right\}$$

as  $x \rightarrow \infty$  ( $k = 1, 2$ )

and

$$y_3(x) = o\left\{\left(\frac{p(x)}{s(x)r(x)}\right)^{\frac{1}{2}} \exp\left[\int_a^x \frac{p}{s}(t)dt + \int_a^x \frac{sr}{p^2}(t)dt\right]\right\} \text{ as } x \rightarrow \infty.$$

Also case (2) of the differential equation ( $E_2$ ) is given in Section 13, Theorem 13.1, and under this theorem we obtain three linearly independent solutions  $y_k(x)$  of the differential equation ( $E_2$ ) such that

$$y_1(x) = o\{(p(x)r(x))^{-\frac{1}{2}}\}, \quad \text{as } x \rightarrow \infty$$

$$y_2(x) \sim (p(x)r(x))^{-\frac{1}{2}} \exp\left(2\sigma \int_a^x \frac{p}{s}(1+\phi)dt\right), \text{ as } x \rightarrow \infty$$

$$y_3(x) = o\left\{(p(x)r(x))^{-\frac{1}{2}} \exp\left[\omega \int_a^x \frac{p}{s}\psi dt + (1+\omega) \int_a^x \frac{p}{s}dt\right]\right\}$$

as  $x \rightarrow \infty$ .

A C K N O W L E D G E M E N T S

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## Section 1: INTRODUCTION

(A) Previous work

A recently asymptotic theory for systems of linear differential equations has been developed by many mathematicians, here before we give our problem for studying the asymptotic theory for the third-order differential equations we mentioned some of those mathematicians which are given in our bibliography.

There were certain developments in the 1950's and 1960's [29, 20, 24], but a major advance was made by Walker [26, 28] and Devinatz [21] in 1971-1972. They considered the fourth-order equation

$$(ry'')'' - (py')' + qy = \lambda y \quad (a \leq x < \infty) \quad (1.1)$$

and were able to deal with situations where  $p$  is large compared to  $q$  and  $r$  as  $x \rightarrow \infty$ . The method involved transformations of (1.1) into suitable first-order systems, and this theory is remarkable not only for the advance which it marked at the time but also for the complexity of the transformations and computations that are involved. These difficulties in the fourth-order case have, with the exception of [31], apparently deterred progress to higher-order equations and other differential systems. Harris and Lutz [9,10,11] in 1974 considered the system

$$Y' = (\Lambda_0 + V)Y \quad (1.2)$$

where  $\Lambda_0$  is a constant diagonal matrix with distinct eigenvalues and  $V(x) \rightarrow 0$  as  $x \rightarrow \infty$ . They transformed (1.2) into

$$W' = (\Lambda + \tilde{V})W \quad (1.3)$$

and they used two basic Lemmas as given in [9, Lemma 1 and Lemma 2] for studying this system and expressing it in the Levinson form (see Part C). Of course in [9, Theorem D] they obtained a result of Devinatz [20, P. 354]. Eastham [12] in 1982 considered (1.1) (with  $\lambda = 0$ ) and he identified a relation between  $p$  and  $q$  that represents a critical situation in general asymptotic theory of (1.1) (with  $\lambda = 0$ ), this critical case is given by

$$\frac{(pq)'}{pq} \sim (\text{const.}) (q/p)^{\frac{1}{2}} \quad (1.4)$$

as  $x \rightarrow \infty$ , and he showed that (1.4) identified a borderline between situations where all solutions have asymptotically a certain exponential character in terms of the coefficients and where only two solutions have this character.

Eastham [13] in 1982 investigated the asymptotic form of  $n$  linearly independent solutions of the  $n$ -th order differential equation

$$(r_{n-1} \dots (r_2 (r_1 y')') \dots)' - qy = 0, \quad (a \leq x < \infty) \quad (1.5)$$

where  $q, r_1, \dots, r_{n-1}$  are all nowhere zero in  $[a, \infty)$ , and he considered the following two cases of (1.5).

Case A. As  $x \rightarrow \infty$ , let

$$(Qr_k)' / Qr_k = o(Q)$$

for  $1 \leq k \leq n-1$ , where

$$Q = \{q / (r_1 \dots r_{n-1})\}^{1/n}.$$

Case B. Let  $(Qr_k)'$  be nowhere zero in  $[a, \infty)$  and, as  $x \rightarrow \infty$  let

$$Q = o\{(Qr_k)'/Qr_k\}$$

for  $1 \leq k \leq n-1$ .

Eastham gave case A in [13, Theorem 1] and case B in [13, Theorem 2]. Under Theorem 1 he showed that (1.5) has solution  $y_k$  ( $1 \leq k \leq n$ ) with asymptotic forms, as  $x \rightarrow \infty$ ,

$$y_k \sim Q^{-(n-1)/2} \left( r_1^{n-1} r_2^{n-2} \dots r_{n-2}^2 r_{n-1} \right)^{-1/n} \exp \left( \omega_k \int_a^x Q(t) dt \right) \quad (1.6)$$

where  $\omega_k = \exp \{2(k-1)\pi/n\}$  ( $1 \leq k \leq n$ ), as he showed that the asymptotic formula (1.6) agrees with Hinton's result [24, Theorem 1]. Eastham [14] in 1985 considered the linear system

$$Y'(x) = \{\Lambda(x) + R(x)\} Y(x),$$

this system is given below in the asymptotic Theorem 1.2 (Eastham [19, 14 p. 133]), and under this theorem he obtained the result of M.V. Fedorjuk [25, Theorem 1]. Finally we refer to Eastham [19] for the methods for repeating transformation in the asymptotic theory.

### (B) Subject of the Thesis

In this Ph.D. thesis we investigate the form of three linearly independent solutions of two types of third-order differential equations as  $x \rightarrow \infty$ .

Asymptotic theory will be developed here for these two differential equations with large coefficients that are like powers of the independent variable. The first type is the differential equation

$$\{s^{\frac{1}{2}}(s^{\frac{1}{2}}y')'\}' + qy' + ry = 0, \quad (a \leq x < \infty) \quad (1.7)$$

and we shall investigate the asymptotic form of three linearly independent solutions of (1.7) as  $x \rightarrow \infty$ , where  $x$  is the independent variable and the prime  $\frac{d}{dx}$ . The functions  $s, q$  and  $r$  are defined on an interval  $[a, \infty)$ , where  $s$  and  $r$  are nowhere zero there. This differential equation will be given in Part (a) of this Ph.D. thesis from Section 2 to Section 7, and our aim is to discuss the following case:-

As  $x \rightarrow \infty$ , let

$$\left. \begin{aligned} \frac{q}{r^{1/3}s^{2/3}} &= o \left\{ \left( \frac{r}{s} \right)^{1/3} \right\}, \\ \frac{r'}{r} &= o \left\{ \left( \frac{r}{s} \right)^{1/3} \right\}, \\ \frac{s'}{s} &= o \left\{ \left( \frac{r}{s} \right)^{1/3} \right\}. \end{aligned} \right\} \quad (1.8)$$

This case will be given in Section 4, Theorem 4.1 and the proof of Theorem 4.1 starts from Section 4 to Section 7 and two examples are given at the end of Section 7. The interesting point of studying the above case for the differential equation (1.7) is to generalize the asymptotic form which was obtained by G.W. Pefeifer [8, Theorem 4] in 1971, Pefeifer considered (1.7) with  $s = 1$  and he proved that (1.7) has solutions  $y_k(x)$  ( $1 \leq k \leq 3$ ) with asymptotic form, as  $x \rightarrow \infty$ ,

$$y_k \sim r^{-1/3} \exp \left( \int_a^x \left\{ \omega_k r^{1/3} - \frac{1}{3} \bar{\omega}_k \frac{q}{r^{1/3}} \right\} dt \right), \quad (1.9)$$

where  $\omega_k$  ( $1 \leq k \leq 3$ ) are the 3-rd roots of  $(-1)$ , subject to certain conditions as given in [8, Theorem 4] also these conditions are given in Section 2, conditions (a) and (b). Pefeifer's [8] takes  $r$  and  $q$  are real-valued functions, but here in our

investigation of (1.7) we shall consider general  $s$ , and we shall not restrict ourselves by considering  $s, q$  and  $r$  to be real-valued functions. The method in outline for the above case of the differential equation (1.7) is given in Section 2, also discussion between our method and Pefeifer's [8] is given at the end of Section 7. The theorem which was used to establish the asymptotic form of three solutions of (1.7) under this case is Theorem 1.2 [Eastham, 14 p. 133] which is stated below, and we shall see that Theorem 1.2 applicable in our method when we consider the above case.

The second type of the third-order differential equation is

$$\{s^{\frac{1}{2}}(s^{\frac{1}{2}} y')'\}' - (py')' - ry = 0, \quad (a \leq x < \infty) \quad (1.10)$$

and we shall investigate the form of three linearly independent solutions of (1.10) as  $x \rightarrow \infty$ , where  $x$  is the independent variable and the prime  $\frac{d}{dx}$ . The functions  $s, p$  and  $r$  are defined on an interval  $[a, \infty)$  and they are nowhere zero there. This differential equation will be considered in Part (b) of this Ph.D. thesis from Section 8 to Section 16, and our aim is to discuss the following two cases for (1.10):-

#### Case (1)

As  $x \rightarrow \infty$ , Let

$$\left. \begin{aligned} \left(\frac{r}{p}\right)^{\frac{1}{2}} &= o\left(\frac{p}{s}\right), \\ \frac{p'}{p} &= o\left\{\left(\frac{r}{p}\right)^{\frac{1}{2}}\right\}, \\ \frac{r'}{r} &= o\left\{\left(\frac{r}{p}\right)^{\frac{1}{2}}\right\}, \\ \frac{s'}{s} &= o\left\{\left(\frac{r}{p}\right)^{\frac{1}{2}}\right\}. \end{aligned} \right\} \quad (1.11)$$

Case (2) (The Critical Case)

As  $x \rightarrow \infty$ , let the critical case

$$\left. \begin{aligned} \frac{(pr)'}{pr} &\sim (\text{const.}) \frac{p}{s} \\ \frac{(ps^{-\frac{1}{2}})'}{ps^{-\frac{1}{2}}} &\sim (\text{const.}) \frac{p}{s} \end{aligned} \right\} \quad (1.12)$$

hold.

Case (1) will be given in Section 9, Theorem 9.1 and the proof of Theorem 9.1 starts from Section 9 to Section 12 and for this case we shall obtain three linearly independent solutions of (1.10) as given in Theorem 9.1. For this case we shall also apply Theorem 1.2 [Eastham 14 p. 133] to obtain the form of the solutions of (1.10), two examples for this case are given at the end of Section 12 and a discussion about a different case than case (1) and case (2) for (1.10) is given at the end of Section 12.

Case (2) will be given in Section 13, Theorem 13.1 and also for this case we give a method in outline in the same section and the proof of Theorem 13.1 starts from Section 13 to Section 15 and as we shall see this critical case represents a borderline between situations where two solutions of (1.10) are small compared to algebraic functions and where two solutions are small compared to an exponential functions while the third solution has algebraic behaviour in both situations. Again for this case we shall make use of Theorem 1.2 [Eastham 14, p. 133] to obtain the form of the solutions of (1.10) under this critical case. Two examples for this case are given in Section 16, also at the end of Section 16 there is a discussion about a critical case of the differential equation (1.7) and a discussion about a second critical case of (1.10).

(C) General Asymptotic Theorems

Now before we give Part (a) of our Ph.D. thesis we start by giving the definition of the norm  $|A|$  of a matrix  $A = (a_{ij})$  and by giving Levinson Theorem [23, 1 pp. 92-95, 14 pp. 131-132] and Eastham Theorem [19, 14 p. 133].

Definition 1.1

We define the norm  $|A|$  of a matrix  $A = (a_{ij})$  in our work by

$$|A| = \sum_{i,j} |a_{ij}|.$$

Levinson's Theorem 1.1 [23, 1 pp. 92-95, 14 pp. 131-132].

Let  $\Lambda(x)$  be an  $n \times n$  diagonal matrix

$$\Lambda(x) = \text{dg} (\lambda_1(x), \dots, \lambda_n(x))$$

which satisfies the Dichotomy condition that, for each pair of integers  $i$  and  $j$  in  $[1, n]$ , either

$$\text{I} \int_a^x \text{Re} \{ \lambda_i(s) - \lambda_j(s) \} ds \rightarrow -\infty \quad (x \rightarrow \infty)$$

and

$$\int_t^x \text{Re} \{ \lambda_i(s) - \lambda_j(s) \} ds \leq k_1 \quad (a \leq t \leq x < \infty)$$

or

$$\text{II} \int_t^x \text{Re} \{ \lambda_i(s) - \lambda_j(s) \} ds \geq k_2 \quad (a \leq t \leq x < \infty)$$

where  $k_1$  and  $k_2$  are constants. Also, let the  $n \times n$  matrix  $R(x)$  satisfy

$$\int_a^\infty |R(x)| dx < \infty. \quad (1.13)$$

Then, as  $x \rightarrow \infty$ , the system

$$Y'(x) = \{\Lambda(x) + R(x)\} Y(x) \quad (1.14)$$

has solutions  $Y_k(x)$  ( $1 \leq k \leq n$ ) with the asymptotic form

$$Y_k(x) = \{e_k + o(1)\} \exp\left[\int_a^x \lambda_k(t) dt\right]. \quad (1.15)$$

Here  $e_k$  denotes the coordinate vector with  $k$ -th component unity and other components zero.

We shall refer to system (1.14) in which (1.13) holds as having the Levinson form. A special case of the Dichotomy condition which is useful in applications occurs when

$$\operatorname{Re} \{\lambda_i(x) - \lambda_j(x)\} = f(x) + g(x) \quad (1.16)$$

where  $f(x)$  does not change sign in  $[a, \infty)$  that is, either  $f(x) \geq 0$  through  $[a, \infty)$  or  $f(x) \leq 0$  through  $[a, \infty)$  and  $g(x)$  is  $L(a, \infty)$ .

The asymptotic Theorem 1.2 (Eastham [19, 14 p. 133])

Let  $\Lambda(x)$  be an  $n \times n$  diagonal matrix

$$\Lambda(x) = \operatorname{dg} (\lambda_1(x), \dots, \lambda_n(x))$$

in which  $\lambda_i(x) - \lambda_j(x)$  is nowhere zero in some interval  $[a, \infty)$  for each pair of integers  $i$  and  $j$  in  $[1, n]$  with  $i \neq j$ .

Let  $R(x)$  be an  $n \times n$  matrix such that

$$(i) \quad R(x) = o \{\lambda_i(x) - \lambda_j(x)\} \quad (x \rightarrow \infty) \quad (1.17)$$

$$(ii) \quad \{(\lambda_i(x) - \lambda_j(x))^{-1} R(x)\}' \text{ is } L(a, \infty) \quad (1.18)$$

for each  $i$  and  $j$  in  $[1, n]$  with  $i \neq j$ . Also let the eigenvalues  $\mu_k(x)$  ( $1 \leq k \leq n$ ) of  $\Lambda(x) + R(x)$  satisfy the Dichotomy condition I-II in Levinson's Theorem 1.1, then as  $x \rightarrow \infty$ , the system

$$Y'(x) = \{\Lambda(x) + R(x)\}Y(x)$$

has solutions  $Y_k(x)$  with the asymptotic form

$$Y_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t) dt\right). \quad (1.19)$$

We emphasise that, in the statement of Theorem 1.2, it is the  $\mu_k(x)$  rather than  $\lambda_k(x)$  which appear in (1.19) and which are obliged to satisfy the Dichotomy condition in Levinson's Theorem. However, in practice, it may not be easy to calculate the  $\mu_k(x)$  and therefore it is useful to know that, if the additional condition

$$|\lambda_i(x) - \lambda_j(x)|^{-1} |R(x)|^2 \text{ is } L(a, \infty) \quad (1.20)$$

holds for all  $i \neq j$ ,  $\mu_k(x)$  can be replaced by the simple expression  $\lambda_k(x) + r_{kk}(x)$  in the statement of theorem and in the asymptotic formula, where

$$\mu_k = \lambda_k + r_{kk} + O\left(\max_{\ell \neq k} |R|^2 / |\lambda_\ell - \lambda_k|\right) \quad (1.21)$$

and the O-term here is  $L(a, \infty)$  if (1.20) holds.

Finally we mention that Theorem 1.2 also applies to the more general system

$$Y'(x) = \{\Lambda(x) + R(x) + S(x)\}Y(x) \quad (1.22)$$

where  $S(x)$  is  $L(a, \infty)$ . Because the transformation  $Y = (I+Q)Z$  was used in [14], where  $Q$  as defined there when  $S(x) = 0$ . The transformation system then contains the extra term  $(I+Q)^{-1}S(I+Q)$ , which being  $L(a, \infty)$ , contributes to the Levinson form and does not influence the asymptotic formula (1.19).

## Section 2. The Method in Outline

In this section we present the main features of our method and then, in subsequent sections, we investigate the asymptotic form of three linearly independent solutions of 3-rd order differential equation

$$\{s^{\frac{1}{2}}(s^{\frac{1}{2}} y')'\}' + qy' + ry = 0 \quad (a \leq x < \infty) \quad (2.1)$$

as  $x \rightarrow \infty$ , where  $x$  is the independent variable and the prime denotes  $\frac{d}{dx}$ . The coefficients  $s, q$  and  $r$  are defined on an interval  $[a, \infty)$  such that  $s$  and  $r$  are nowhere zero there.

We begin by writing (2.1) as a first-order system

$$Y' = AY \quad (2.2)$$

where  $A$  is  $3 \times 3$  matrix.

The relationship between (2.2) and (2.1) is that the first component of  $Y$  is  $y$ . The aim is to transform (2.2) into the standard form

$$Z' = (\Lambda + R_1 + R_2)Z \quad (2.3)$$

where  $\Lambda$  is a diagonal matrix, say

$$\Lambda(x) = \text{dg}(\lambda_1(x), \lambda_2(x), \lambda_3(x)) \quad (2.4)$$

such that  $R_2$  is  $L(a, \infty)$  for some  $a \geq 0$ , and the matrix  $\Lambda + R_1$  satisfies the hypotheses of Theorem 1.2, then Theorem 1.2, is applicable and it follows that (2.3) has solutions  $Z_k$  ( $1 \leq k \leq 3$ ) with asymptotic forms

$$Z_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t) dt\right) \quad (k = 1, 2, 3) \quad (2.5)$$

as  $x \rightarrow \infty$ , where  $e_k$  is the coordinate vector with  $k$ -th component unity and other components zero, and  $\mu_k(x)$  ( $1 \leq k \leq 3$ ) are the eigenvalues of the matrix  $\Lambda + R_1$ .

The asymptotic forms of the solutions of (2.2), and hence of (2.1), are then obtained by transforming back.

The first step in the transformation of (2.3) is to diagonalize  $A$ ; that is, we define a suitable matrix  $T$  such that

$$T^{-1}AT = \Lambda = \text{dg}(\lambda_1, \lambda_2, \lambda_3) \quad (2.6)$$

say, the  $\lambda_k(x)$  ( $1 \leq k \leq 3$ ) being the eigenvalues of  $A$ .

The substitution

$$Y = TZ \quad (2.7)$$

then takes (2.2) into

$$Z' = (\Lambda - T^{-1}T')Z \quad (2.8)$$

where  $\Lambda(x)$  is given by (2.6), and the next step is to make a detailed analysis of the matrix  $T^{-1}T'$ , then as we shall see (2.8) is in the required form (2.3) by writing

$$-T^{-1}T' = R_1 + R_2 \quad (2.9)$$

such that  $R_2$  is  $L(a, \infty)$  and the matrix  $\Lambda + R_1$  satisfies Theorem 1.2.

We note the system (2.3) is in the same form as (1.22) because  $R_2$  is  $L(a, \infty)$  and this is the reason we only required that  $\Lambda + R_1$  in the above method to satisfy Theorem 1.2. In the case where  $s = 1$ , (2.1) is reduced to the differential equation

$$y'''(x) + q(x)y'(x) + r(x)y(x) = 0 \quad (a \leq x < \infty) \quad (2.10)$$

This differential equation has been considered by G.W. Pfeifer [8] in 1971, and he proved that (2.10) has solutions  $y_k$  ( $1 \leq k \leq 3$ ) with asymptotic forms, as  $x \rightarrow \infty$ ,

$$y_k(x) \sim r^{-1/3}(x) \exp \left[ \int_a^x \left\{ \omega_k r^{1/3}(t) - \frac{1}{3} \bar{\omega}_k \frac{q}{r^{1/3}}(t) \right\} dt \right], \quad (2.11)$$

where

$$\omega_1 = -1, \quad \omega_2 = \frac{1}{2} + i \frac{\sqrt{3}}{2}, \quad \omega_3 = \frac{1}{2} - i \frac{\sqrt{3}}{2},$$

subject to the following conditions:-

- (a)  $r$  and  $q$  are real-valued functions with  $r$  is  $C^2[a, \infty)$  and nowhere zero in  $[a, \infty)$ , and  $q$  is  $C^1[a, \infty)$ , such that
- (b)  $\frac{r''}{r^{4/3}}$ ,  $\frac{q'}{r^{2/3}}$  and  $\frac{q^2}{r}$  are  $L(a, \infty)$ .

If the coefficients  $q(x)$  and  $r(x)$  are given by

$$q(x) = Bx^\beta, \quad r(x) = Cx^\gamma, \quad (2.12)$$

where  $\beta, \gamma, B$  and  $C (\neq 0)$  are real constants, then condition (b) is satisfied when

$$\frac{1}{3} \gamma + 1 > 0, \quad \gamma - 2\beta - 1 > 0. \quad (2.13)$$

Here we consider the problem of developing an asymptotic theory for (2.1), under certain conditions on  $s, q$  and  $r$  as stated later on in Theorem 4.1.

## Section 3: A Transformation of the Differential Equation

Now we start our work in detail, and hence (2.1) can be written as a first-order system (2.2) as follows.

Take

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & s^{-\frac{1}{2}} & 0 \\ 0 & 0 & s^{-\frac{1}{2}} \\ -r & -qs^{-\frac{1}{2}} & 0 \end{pmatrix} \quad (3.1)$$

then by (2.2)

$$\begin{aligned} Y_1' &= s^{-\frac{1}{2}} Y_2 \\ Y_2' &= s^{-\frac{1}{2}} Y_3 \\ Y_3' &= -rY_1 - qs^{-\frac{1}{2}} Y_2 \end{aligned}$$

Define

$$Y_1 = Y$$

hence

$$Y_2 = s^{\frac{1}{2}} Y'$$

and

$$Y_3 = s^{\frac{1}{2}} (s^{\frac{1}{2}} Y')'$$

i.e.

$$\{s^{\frac{1}{2}}(s^{\frac{1}{2}} y')'\}' = -ry - qy'$$

$$\{s^{\frac{1}{2}}(s^{\frac{1}{2}} y')'\}' + qy' + ry = 0$$

Hence (2.2) reduced to (2.1) if we take A as in (3.1) and

$$Y(x) = \begin{pmatrix} y \\ s^{\frac{1}{2}} y' \\ s^{\frac{1}{2}}(s^{\frac{1}{2}} y')' \end{pmatrix} \quad (3.2)$$

Now the first step in the transformation to (2.3) is to diagonalize A; hence we require the eigenvalues  $\lambda_j$  ( $1 \leq j \leq 3$ ) and the eigenvectors  $v_j$  ( $1 \leq j \leq 3$ ).

The characteristic equation of A is given by

$$\det(\lambda_j I - A) = \begin{vmatrix} \lambda_j & -s^{-\frac{1}{2}} & 0 \\ 0 & \lambda_j & -s^{-\frac{1}{2}} \\ r & qs^{-\frac{1}{2}} & \lambda_j \end{vmatrix} = 0$$

i.e.

$$s\lambda_j^3 + q\lambda_j + r = 0 \quad (1 \leq j \leq 3) \quad (3.3)$$

Now the eigenvector  $v_j$  of the matrix A corresponding to  $\lambda_j$  is given by

$$Av_j = \lambda_j v_j. \quad (3.4)$$

Taking

$$v_j = \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

then (3.4) gives

$$\begin{pmatrix} 0 & s^{-\frac{1}{2}} & 0 \\ 0 & 0 & s^{-\frac{1}{2}} \\ -r & -qs^{-\frac{1}{2}} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \lambda_j \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

Hence

$$s^{-\frac{1}{2}} v = \lambda_j u \quad (3.5)$$

$$s^{-\frac{1}{2}} w = \lambda_j v \quad (3.6)$$

$$-r u - qs^{-\frac{1}{2}} v = \lambda_j w \quad (3.7)$$

Define

$$u = 1 \quad (3.8)$$

then by (3.8), (3.5) gives

$$v = s^{\frac{1}{2}} \lambda_j \quad (3.9)$$

Also by (3.9), (3.6) gives

$$w = s \lambda_j^2 \quad (3.10)$$

We note that (3.7) reduced to (3.3) if we substitute for  $u = 1$ ,  $v = s^{\frac{1}{2}} \lambda_j$  and  $w = s \lambda_j^2$ . Hence by (3.8), (3.9) and (3.10) we can define the eigenvector  $v_j$  of the matrix  $A$  corresponding to  $\lambda_j$  which satisfy the relationship (3.4) by

$$v_j = \begin{pmatrix} 1 \\ s^{\frac{1}{2}} \lambda_j \\ s \lambda_j^2 \end{pmatrix} \quad (1 \leq j \leq 3) \quad (3.11)$$

Hence we define the suitable matrix  $T$  which satisfy (2.6) by

$$T = (v_1 \quad v_2 \quad v_3) \quad (3.12)$$

then by (3.11), (3.12) becomes

$$T = \begin{pmatrix} 1 & 1 & 1 \\ s^{\frac{1}{2}}\lambda_1 & s^{\frac{1}{2}}\lambda_2 & s^{\frac{1}{2}}\lambda_3 \\ s\lambda_1^2 & s\lambda_2^2 & s\lambda_3^2 \end{pmatrix} \quad (3.13)$$

Hence the transformation (2.7) will take (2.2) into the system (2.8).

Now we need to work out the matrix  $T^{-1}T'$ . Writing

$$T^{-1}T' = (t_{jk}) \quad (1 \leq j, k \leq 3). \quad (3.14)$$

First we consider  $T^{-1}$ , now

$$T^{-1} = \frac{\text{adj}T}{\det T}. \quad (3.15)$$

Hence by (3.3)

$$T^{-1} = \frac{s^{\frac{1}{2}}}{\det T} \begin{bmatrix} s(\lambda_2\lambda_3^2 - \lambda_2^2\lambda_3) & s^{\frac{1}{2}}(\lambda_2^2 - \lambda_3^2) & \lambda_3 - \lambda_2 \\ s(\lambda_1^2\lambda_3 - \lambda_1\lambda_3^2) & s^{\frac{1}{2}}(\lambda_3^2 - \lambda_1^2) & \lambda_1 - \lambda_3 \\ s(\lambda_1\lambda_2^2 - \lambda_1^2\lambda_2) & s^{\frac{1}{2}}(\lambda_1^2 - \lambda_2^2) & \lambda_2 - \lambda_1 \end{bmatrix} \quad (3.16)$$

Also by (3.13)

$$\det T = s^{\frac{3}{2}}(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2). \quad (3.17)$$

Second we consider  $T'$ , then by (3.13)

$$T' = \begin{bmatrix} 0 & 0 & 0 \\ (s^{\frac{1}{2}}\lambda_1)' & (s^{\frac{1}{2}}\lambda_2)' & (s^{\frac{1}{2}}\lambda_3)' \\ (s\lambda_1^2)' & (s\lambda_2^2)' & (s\lambda_3^2)' \end{bmatrix} \quad (3.18)$$

Hence by (3.16), (3.18) and (3.14)

$$t_{1k} = \frac{s^{\frac{1}{2}}}{\det T} \left[ s^{\frac{1}{2}}(\lambda_2^2 - \lambda_3^2)(s^{\frac{1}{2}}\lambda_k)' + (\lambda_3 - \lambda_2)(s\lambda_k^2)' \right] \quad (k = 1, 2, 3)$$

i.e.

$$t_{1k} = \frac{s^{\frac{1}{2}}}{\det T} \left[ s^{\frac{1}{2}}(\lambda_2^2 - \lambda_3^2)(s^{\frac{1}{2}}\lambda_k)' + (\lambda_3 - \lambda_2) \{ (s^{\frac{1}{2}}\lambda_k)^2 \}' \right]$$

$$t_{1k} = \frac{s(s^{\frac{1}{2}}\lambda_k)'}{\det T} \left[ (\lambda_2 - \lambda_3)(\lambda_2 + \lambda_3) + 2\lambda_k(\lambda_3 - \lambda_2) \right],$$

hence by (3.17)

$$t_{1k} = \frac{(s^{\frac{1}{2}}\lambda_k)'}{s^{\frac{1}{2}}(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} \left[ 2\lambda_k - (\lambda_2 + \lambda_3) \right] \quad (k = 1, 2, 3) \quad (3.19)$$

Similarly by (3.16), (3.18), (3.17) and (3.14), we obtain

$$t_{2k} = \frac{(s^{\frac{1}{2}}\lambda_k)'}{s^{\frac{1}{2}}(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} \left[ \lambda_3 + \lambda_1 - 2\lambda_k \right] \quad (k = 1, 2, 3) \quad (3.20)$$

and

$$t_{3k} = \frac{(s^{\frac{1}{2}}\lambda_k)'}{s^{\frac{1}{2}}(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)} \left[ 2\lambda_k - (\lambda_1 + \lambda_2) \right] \quad (k = 1, 2, 3) \quad (3.21)$$

#### 4. The Asymptotic Theorem

Now we shall establish the asymptotic formula

$$y_k \sim s^{-1/6}(x) r^{-1/3}(x) \exp \left\{ \int_a^x \left[ \omega_k \left( \frac{r}{s} \right)^{1/3}(t) - \frac{1}{3} \bar{\omega}_k \frac{q}{r^{1/3} s^{2/3}}(t) \right] dt \right\} \quad (4.1)$$

as  $x \rightarrow \infty$ , where

$$\omega_1 = -1, \quad \omega_2 = \frac{1}{2} + i \frac{\sqrt{3}}{2}, \quad \omega_3 = \frac{1}{2} - i \frac{\sqrt{3}}{2}$$

for three solutions of the differential equation (2.1) subject to certain conditions on  $s(x)$ ,  $q(x)$  and  $r(x)$  as stated below in Theorem 4.1. This is the case for the differential equation (2.1) as mentioned in the introduction section. Of course if  $s=1$  in (4.1), then for the particular equation (2.10), the asymptotic formula (4.1) agrees with Pfeifer result (2.11). Discussion between Pfeifer [8] result and our result for the differential equation (2.1) with  $s=1$  is given at the end of §7.

##### Theorem 4.1

Let the coefficients  $s$  and  $r$  in (2.1) be  $C^2[a, \infty)$  and nowhere zero in  $[a, \infty)$ , also let  $q$  in (2.1) be  $C^1[a, \infty)$ . Let

$$\frac{q}{r^{1/3} s^{2/3}} = o \left\{ \left( \frac{r}{s} \right)^{1/3} \right\} \quad \text{as } x \rightarrow \infty \quad (4.2)$$

$$\frac{r'}{r} = o \left\{ \left( \frac{r}{s} \right)^{1/3} \right\} \quad \text{as } x \rightarrow \infty \quad (4.3)$$

$$\frac{s'}{s} = o \left\{ \left( \frac{r}{s} \right)^{1/3} \right\} \quad \text{as } x \rightarrow \infty \quad (4.4)$$

Let

$$\frac{q'}{r^{2/3} s^{1/3}}, \quad \frac{r''}{r} \left( \frac{s}{r} \right)^{1/3}, \quad \frac{r'^2}{r^2} \left( \frac{s}{r} \right)^{1/3}, \quad \frac{s''}{s} \left( \frac{s}{r} \right)^{1/3}, \quad \frac{s'^2}{s^2} \left( \frac{s}{r} \right)^{1/3}$$

and  $\frac{q^2}{sr}$  be  $L(a, \infty)$ .

(4.5)

Finally let one of the three following cases (I), (II) and (III) hold.

Case (I)

$q$ ,  $s$  and  $r$  are real-valued functions.

Case (II)

$q$ ,  $s$  and  $r$  are complex-valued functions and

$$\operatorname{im} \frac{q}{r^{1/3} s^{2/3}} = o \left\{ \operatorname{im} \left( \frac{r}{s} \right)^{1/3} \right\}, \quad \left\{ \operatorname{im} \left( \frac{r}{s} \right)^{1/3} \neq 0 \right\}$$

as  $x \rightarrow \infty$ ,

$$\operatorname{im} \left( \frac{r}{s} \right)^{1/3} = o \left\{ \operatorname{re} \left( \frac{r}{s} \right)^{1/3} \right\}, \quad \left\{ \operatorname{re} \left( \frac{r}{s} \right)^{1/3} \neq 0 \right\}$$

as  $x \rightarrow \infty$ .

Case (III)

$q, s$  and  $r$  are complex-valued functions and

$$\operatorname{re} \left( \frac{r}{s} \right)^{1/3} = o \left\{ \operatorname{im} \left( \frac{r}{s} \right)^{1/3} \right\}, \quad \left\{ \operatorname{im} \left( \frac{r}{s} \right)^{1/3} \neq 0 \right\}$$

as  $x \rightarrow \infty$ .

Then the differential equation (2.1) has solutions  $y_k(x)$  ( $1 \leq k \leq 3$ ) with the asymptotic form (4.1).

We note as a general comment in the above theorem that conditions (4.2), (4.3) and (4.4) are equivalent to say that the functions  $\frac{q}{r^{1/3} s^{2/3}}$ ,  $\frac{r'}{r}$  and  $\frac{s'}{s}$  are all small compared to the function  $\left(\frac{r}{s}\right)^{1/3}$  as  $x \rightarrow \infty$ . Of course condition (4.2) implies that the  $\text{re } \frac{q}{r^{1/3} s^{2/3}}$  and the  $\text{im } \frac{q}{r^{1/3} s^{2/3}}$  are both small compared to  $\left(\frac{r}{s}\right)^{1/3}$  as  $x \rightarrow \infty$ , similarly for conditions (4.3) and (4.4). In the proof of Theorem 4.1, we require this remark.

#### Remark 4.1

Since by (4.5)  $\frac{r'^2}{r^2} \left(\frac{s}{r}\right)^{1/3}$ ,  $\frac{s'^2}{s^2} \left(\frac{r}{s}\right)^{1/3}$  and  $\frac{q^2}{sr}$  are  $L(a, \infty)$ , hence  $\frac{r'}{r} \delta$  and  $\frac{s'}{s} \delta$  are both  $L(a, \infty)$  where  $\delta$  as given below in (4.12).

To show this we use schwarz inequality

$$\int_a^x |f(t)g(t)| dt \leq \left\{ \left( \int_a^x |f(t)|^2 dt \right) \cdot \left( \int_a^x |g(t)|^2 dt \right) \right\}^{1/2} \quad (4.6)$$

by taking  $f = \frac{q}{(sr)^{1/2}}$  and  $g = \frac{r'}{r} \left(\frac{s}{r}\right)^{1/6}$ , then  $\frac{r'}{r} \delta$  is  $L(a, \infty)$  by (4.6).

Similarly  $\frac{s'}{s} \delta$  is  $L(a, \infty)$ .

#### Proof of Theorem 4.1

The first step in the proof is to solve (3.3) asymptotically as  $x \rightarrow \infty$ , and to make a detail analysis of  $T^{-1}T'$ .

Since  $s$  and  $r$  are nowhere zero in  $[a, \infty)$  and (4.2) holds, hence (3.3) has distinct roots,  $\lambda_j(x)$  ( $1 \leq j \leq 3$ ) such that

$$\lambda_j = \omega_j \left(\frac{r}{s}\right)^{1/3} (1 + \delta_j) \quad (1 \leq j \leq 3) \quad (4.7)$$

where

$$\omega_j^3 = -1 \quad (1 \leq j \leq 3) \quad (4.8)$$

and so

$$\omega_1 = -1, \omega_2 = \frac{1}{2} + i \frac{\sqrt{3}}{2}, \omega_3 = \frac{1}{2} - i \frac{\sqrt{3}}{2} \quad (4.9)$$

where

$$\delta_j(x) = o(1). \quad (1 \leq j \leq 3) \text{ as } x \rightarrow \infty \quad (4.10)$$

Now we need to show that (4.10) holds, and then to work out

$\delta_j(x)$  ( $1 \leq j \leq 3$ ). Now by (4.7) and (4.8), (3.3) gives

$$-r(1 + \delta_j)^3 + \omega_j \frac{qr^{1/3}}{s^{1/3}} (1 + \delta_j) + r = 0 \quad (1 \leq j \leq 3)$$

i.e.

$$-(1 + \delta_j)^3 + \omega_j \frac{q}{s^{1/3} r^{2/3}} (1 + \delta_j) + 1 = 0. \quad (1 \leq j \leq 3) \quad (4.11)$$

Let

$$\delta = \frac{q}{s^{1/3} r^{2/3}}, \quad (4.12)$$

then by (4.2)

$$\delta = o(1). \quad \text{as } x \rightarrow \infty \quad (4.13)$$

Now equation (4.11) is of the form  $f(x, \delta_j) = 0$ , where

$f(\infty, 0) = 0$  and  $f_{\delta_j}(\infty, 0) \neq 0$ ; hence the standard implicit function

theorem [7, p.64] guarantees the existence of a unique solution

$\delta_j = \delta_j(x)$  to this equation such that  $\delta_j(\infty) = 0$ , thus (4.10) holds.

Now to work out  $\delta_j$ , we consider (4.11), thus (4.11) gives

$$-(1+3\delta_j + O(\delta_j^2)) + \omega_j \frac{q}{s^{1/3} r^{2/3}} (1+\delta_j) + 1 = 0$$

i.e.

$$-3\delta_j + O(\delta_j^2) + \omega_j \frac{q}{s^{1/3} r^{2/3}} (1+\delta_j) = 0 \quad (1 \leq j \leq 3) \quad (4.14)$$

Now we solve (4.14) in term of the dominant term  $\delta_j$ , then (4.14) gives

$$\delta_j = \frac{\omega_j}{3} \frac{q}{r^{1/3} s^{2/3}} \left( 1 - \frac{\omega_j}{3} \frac{q}{r^{2/3} s^{1/3}} \right)^{-1} \quad (1 \leq j \leq 3) \quad (4.15)$$

Hence by (4.12), (4.15) gives

$$\delta_j = \frac{\omega_j}{3} \delta \left( 1 - \frac{\omega_j}{3} \delta \right)^{-1} \quad (1 \leq j \leq 3) \quad (4.16)$$

Hence by previous work, the diagonal matrix  $\Lambda(x)$  in (2.6) is given by (4.7), (4.9) and (4.16).

We mention here that since (4.7) and (4.10) holds, hence the case which is considered for  $\lambda_j (1 \leq j \leq 3)$  is given by

$$\lambda_j \sim \omega_j \left( \frac{r}{s} \right)^{1/3} \quad (1 \leq j \leq 3) \text{ as } x \rightarrow \infty \quad (4.17)$$

i.e.

$\lambda_j(x)$  ( $1 \leq j \leq 3$ ) having same size as  $x \rightarrow \infty$ .

Now before working in  $T^{-1} T'$  in some detail, We need to find  $(s^{1/2} \lambda_k)'$ ; now by (4.7)

$$s^{1/2} \lambda_k = \omega_k r^{1/3} s^{1/6} (1 + \delta_k). \quad (1 \leq k \leq 3) \quad (4.18)$$

Differentiating (4.18), then

$$(s^{\frac{1}{2}} \lambda_k)' = \omega_k \{r^{\frac{1}{3}} s^{\frac{1}{6}}\}' (1 + \delta_k) + \omega_k r^{\frac{1}{3}} s^{\frac{1}{6}} \delta_k'.$$

Since

$$\{r^{\frac{1}{3}} s^{\frac{1}{6}}\}' = \frac{1}{3} r^{\frac{1}{3}} s^{\frac{1}{6}} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right].$$

Hence

$$(s^{\frac{1}{2}} \lambda_k)' = \frac{\omega_k}{3} r^{\frac{1}{3}} s^{\frac{1}{6}} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] (1 + \delta_k) + \omega_k r^{\frac{1}{3}} s^{\frac{1}{6}} \delta_k'. \quad (1 \leq k \leq 3) \quad (4.19)$$

Now we have to work out  $\delta_k'(x)$  ( $1 \leq k \leq 3$ ) in some detail in terms of  $s, r$  and  $q$ , hence by (4.7) and (4.8), (3.3) becomes after multiplying by  $r^{-1}$

$$-(1 + \delta_k)^3 + \omega_k \frac{q}{r^{\frac{2}{3}} s^{\frac{1}{3}}} (1 + \delta_k) + 1 = 0. \quad (1 \leq k \leq 3)$$

Differentiating this equation, we obtain

$$\begin{aligned} -3(1 + \delta_k)^2 \delta_k' + \omega_k (1 + \delta_k) \left\{ \frac{q'}{s^{\frac{1}{3}} r^{\frac{2}{3}}} - \frac{\omega_k}{3} (1 + \delta_k) \left\{ \frac{s' q}{s^{\frac{4}{3}} r^{\frac{2}{3}}} + 2 \frac{r' q}{s^{\frac{1}{3}} r^{\frac{5}{3}}} \right\} \right. \\ \left. + \omega_k \frac{q}{r^{\frac{2}{3}} s^{\frac{1}{3}}} \delta_k' \right\} = 0 \end{aligned} \quad (1 \leq k \leq 3)$$

Hence

$$\begin{aligned} -3 \delta_k' \left\{ 1 - \frac{\omega_k}{3} \frac{q}{r^{\frac{2}{3}} s^{\frac{1}{3}}} + 2 \delta_k + O(\delta_k^2) \right\} + \omega_k (1 + \delta_k) \frac{q'}{s^{\frac{1}{3}} r^{\frac{2}{3}}} \\ - \frac{\omega_k}{3} (1 + \delta_k) \left\{ \frac{s'}{s} + 2 \frac{r'}{r} \right\} \frac{q}{r^{\frac{2}{3}} s^{\frac{1}{3}}} = 0 \end{aligned}$$

i.e.

$$\delta'_k = \frac{\omega_k}{3} \frac{q'}{s^{1/3} r^{2/3}} (1 + \delta_k) \left\{ 1 - \frac{\omega_k}{3} \frac{q}{r^{2/3} s^{1/3}} + 2\delta_k + O(\delta_k^2) \right\}^{-1} \\ - \frac{\omega_k}{9} \frac{q}{r^{2/3} s^{1/3}} \left\{ \frac{s'}{s} + 2\frac{r'}{r} \right\} (1 + \delta_k) \left\{ 1 - \frac{\omega_k}{3} \frac{q}{r^{2/3} s^{1/3}} + 2\delta_k + O(\delta_k^2) \right\}^{-1}$$

Thus after making use of (4.12), we obtain

$$\delta'_k = O\left(\frac{q'}{s^{1/3} r^{2/3}}\right) + O\left(\frac{s'}{s} \delta\right) + O\left(\frac{r'}{r} \delta\right). \quad (1 \leq k \leq 3)$$

(4.20)

We note that the first 0-term on the right hand side of (4.20) is  $L(a, \infty)$  by (4.5), also the last 0-terms are  $L(a, \infty)$  subject to remark 4.1. Thus

$$\delta'_k(x) \text{ is } L(a, \infty). \quad (1 \leq k \leq 3) \quad (4.21)$$

We have now to work out (3.19), (3.20) and (3.21) in some detail in terms of  $s, r$  and  $q$  in order to determine the form of (2.8) and then make progress towards (2.1).

5. The Matrix  $T^{-1}T'$ 

Considering first  $k = 1$  in (3.19), we have

$$t_{11} = \frac{(s^{\frac{1}{2}}\lambda_1)'}{s^{\frac{1}{2}}(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} [2\lambda_1 - \lambda_2 - \lambda_3]. \quad (5.1)$$

Hence by (4.7)

$$\begin{aligned} \lambda_3 - \lambda_1 &= \omega_3 \left(\frac{r}{s}\right)^{1/3} (1 + \delta_3) - \omega_1 \left(\frac{r}{s}\right)^{1/3} (1 + \delta_1) \\ &= \left(\frac{r}{s}\right)^{1/3} (\omega_3 - \omega_1) \left\{ 1 + \frac{1}{\omega_3 - \omega_1} (\omega_3 \delta_3 - \omega_1 \delta_1) \right\}. \end{aligned}$$

Using (4.16), then

$$\lambda_3 - \lambda_1 = (\omega_3 - \omega_1) \left(\frac{r}{s}\right)^{1/3} \left\{ 1 + \frac{\delta}{3} (\omega_3 + \omega_1) + O(\delta^3) \right\}. \quad (5.2)$$

Also by (4.7)

$$\lambda_2 - \lambda_1 = (\omega_2 - \omega_1) \left(\frac{r}{s}\right)^{1/3} + \left(\frac{r}{s}\right)^{1/3} (\omega_2 \delta_2 - \omega_1 \delta_1)$$

i.e.

$$\lambda_2 - \lambda_1 = (\omega_2 - \omega_1) \left(\frac{r}{s}\right)^{1/3} \left\{ 1 + \frac{1}{\omega_2 - \omega_1} (\omega_2 \delta_2 - \omega_1 \delta_1) \right\}.$$

Hence by (4.16)

$$\lambda_2 - \lambda_1 = (\omega_2 - \omega_1) \left(\frac{r}{s}\right)^{1/3} \left\{ 1 + \frac{\delta}{3} (\omega_2 + \omega_1) + O(\delta^3) \right\}. \quad (5.3)$$

We note that by (5.2) and (5.3)

$$\lambda_3(x) - \lambda_1(x) \sim (\omega_3 - \omega_1) \left(\frac{r}{s}\right)^{1/3} \quad \text{as } x \rightarrow \infty$$

$$\lambda_2(x) - \lambda_1(x) \sim (\omega_2 - \omega_1) \left(\frac{r}{s}\right)^{1/3} \quad \text{as } x \rightarrow \infty$$

Now by (5.2) and (5.3)

$$(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) = \left(\frac{r}{s}\right)^{2/3} (\omega_3 - \omega_1)(\omega_2 - \omega_1) \left\{ 1 + \frac{\delta}{3} (\omega_3 + \omega_2 + 2\omega_1) \right\} + O(\delta^2).$$

Hence

$$s^{1/2}(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) = 3 r^{2/3} s^{-1/6} \left\{ 1 - \frac{\delta}{3} + O(\delta^2) \right\}. \quad (5.4)$$

Now by (4.19)

$$(s^{1/2} \lambda_1)' = \frac{\omega_1}{3} r^{1/3} s^{1/6} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] (1 + \delta_1) + \omega_1 r^{1/3} s^{1/6} \delta_1',$$

where  $\delta_1$  is given by (4.16) and  $\delta_1'$  is given by (4.20), hence by (4.16)

$$(s^{1/2} \lambda_1)' = \frac{\omega_1}{3} r^{1/3} s^{1/6} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{\omega_1}{3} \delta + O(\delta^2) \right\} + \omega_1 r^{1/3} s^{1/6} \delta_1'. \quad (5.5)$$

Now by (4.7)

$$2\lambda_1 - \lambda_2 - \lambda_3 = (2\omega_1 - 1) \left(\frac{r}{s}\right)^{1/3} + \left(\frac{r}{s}\right)^{1/3} (2\omega_1 \delta_1 - \omega_2 \delta_2 - \omega_3 \delta_3).$$

Hence by (4.16) and (4.8)

$$2\lambda_1 - \lambda_2 - \lambda_3 = (2\omega_1 - 1) \left(\frac{r}{s}\right)^{1/3} + \left(\frac{r}{s}\right)^{1/3} \left\{ \frac{\delta}{3} (2\omega_1^2 - \omega_2^2 - \omega_3^2) + O(\delta^3) \right\},$$

and so by (4.9) after doing simple calculation we obtain

$$2\lambda_1 - \lambda_2 - \lambda_3 = -3 \left(\frac{r}{s}\right)^{1/3} \left\{ 1 - \frac{\delta}{3} + O(\delta^3) \right\}. \quad (5.6)$$

Hence by (5.5) and (5.6)

$$(s^{\frac{1}{2}}\lambda_1)'(2\lambda_1 - \lambda_2 - \lambda_3) = r^{2/3} s^{-1/6} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{\omega_1}{3} \delta + O(\delta^2) \right\} \left\{ 1 - \frac{1}{3} \delta + O(\delta^3) \right\} \\ - 3\omega_1 r^{2/3} s^{-1/6} \delta_1' \left\{ 1 - \frac{1}{3} \delta + O(\delta^3) \right\},$$

i.e.

$$(s^{\frac{1}{2}}\lambda_1)'(2\lambda_1 - \lambda_2 - \lambda_3) = r^{2/3} s^{-1/6} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 - \frac{2}{3} \delta + O(\delta^2) \right\} \\ + 3 r^{2/3} s^{-1/6} \delta_1' \left\{ 1 - \frac{1}{3} \delta + O(\delta^3) \right\}. \quad (5.7)$$

Hence by (4.20), (5.4) and (5.7), (5.1) becomes

$$t_{11} = \frac{1}{3} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] + O(\delta \frac{s'}{s}) + O(\delta \frac{r'}{r}) + O\left(\frac{q'}{r^{2/3} s^{1/3}}\right). \quad (5.8)$$

Next we consider  $k = 2$  in (3.19), we have

$$t_{12} = \frac{(s^{\frac{1}{2}}\lambda_2)'(\lambda_2 - \lambda_3)}{s^{\frac{1}{2}}(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}. \quad (5.9)$$

Now by (4.19)

$$(s^{\frac{1}{2}}\lambda_2)' = \frac{\omega_2}{3} r^{1/3} s^{1/6} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] (1 + \delta_2) + \omega_2 r^{1/3} s^{1/6} \delta_2',$$

where  $\delta_2$  is given by (4.16) and  $\delta_2'$  is given by (4.20), hence by (4.16).

$$(s^{\frac{1}{2}}\lambda_2)' = \frac{\omega_2}{3} r^{1/3} s^{1/6} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{\omega_2}{3} \delta + O(\delta^2) \right\} + \omega_2 r^{1/3} s^{1/6} \delta_2' \quad (5.10)$$

Now by (4.7)

$$\begin{aligned}\lambda_2 - \lambda_3 &= (\omega_2 - \omega_3) \left( \frac{r}{s} \right)^{1/3} + \left( \frac{r}{s} \right)^{1/3} (\omega_2 \delta_2 - \omega_3 \delta_3) \\ &= \left( \frac{r}{s} \right)^{1/3} (\omega_2 - \omega_3) \left\{ 1 + \frac{1}{\omega_2 - \omega_3} (\omega_2 \delta_2 - \omega_3 \delta_3) \right\}.\end{aligned}$$

Hence by (4.16) and (4.8)

$$\lambda_2 - \lambda_3 = \left( \frac{r}{s} \right)^{1/3} (\omega_2 - \omega_3) \left\{ 1 + \frac{\delta}{3} (\omega_2 + \omega_3) + O(\delta^3) \right\}$$

and so

$$\lambda_2 - \lambda_3 = \left( \frac{r}{s} \right)^{1/3} (\omega_2 - \omega_3) \left\{ 1 + \frac{\delta}{3} + O(\delta^3) \right\}. \quad (5.11)$$

i.e.

$$\lambda_2(x) - \lambda_3(x) \sim (\omega_2 - \omega_3) \left( \frac{r}{s} \right)^{1/3} \quad \text{as } x \rightarrow \infty.$$

Hence by (5.10) and (5.11)

$$\begin{aligned}(s^{1/2} \lambda_2)' (\lambda_2 - \lambda_3) &= \frac{1}{3} r^{2/3} s^{-1/6} (\omega_2^2 - \omega_2 \omega_3) \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{\delta}{3} (1 + \omega_2) + O(\delta^2) \right\} \\ &\quad + r^{2/3} s^{-1/6} (\omega_2^2 - \omega_2 \omega_3) \delta_2' \left\{ 1 + \frac{\delta}{3} + O(\delta^3) \right\}\end{aligned}$$

and so by (4.9), we obtain

$$\begin{aligned}(s^{1/2} \lambda_2)' (\lambda_2 - \lambda_3) &= -\frac{1}{3} (\omega_3 + 1) r^{2/3} s^{-1/6} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{\delta}{3} (1 + \omega_2) + O(\delta^2) \right\} \\ &\quad - (\omega_3 + 1) r^{2/3} s^{-1/6} \delta_2' \left\{ 1 + \frac{1}{3} \delta + O(\delta^3) \right\}.\end{aligned}$$

(5.12)

Hence by (4.20), (5.4) and (5.12), (5.9) becomes

$$t_{12} = -\frac{1}{9}(\omega_3+1) \left[ \frac{r'}{r} + \frac{1}{2} + \frac{s'}{s} \right] + O(\delta \frac{s'}{s}) + O(\delta \frac{r'}{r}) + O\left(\frac{q'}{r^{2/3} s^{1/3}}\right). \quad (5.13)$$

Now putting  $k = 3$  in (3.19), hence

$$t_{13} = \frac{(s^{\frac{1}{2}\lambda_3})' (\lambda_3 - \lambda_2)}{s^{\frac{1}{2}(\lambda_2 - \lambda_1)} (\lambda_3 - \lambda_1)}. \quad (5.14)$$

Now by (4.19),

$$(s^{\frac{1}{2}\lambda_3})' = \frac{\omega_3}{3} r^{1/3} s^{1/6} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] (1 + \delta_3) + \omega_3 r^{1/3} s^{1/6} \delta_3',$$

where  $\delta_3$  is given by (4.16) and  $\delta_3'$  is given by (4.20), hence by (4.16)

$$(s^{\frac{1}{2}\lambda_3})' = \frac{\omega_3}{3} r^{1/3} s^{1/6} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{\omega_3}{3} \delta + O(\delta^2) \right\} + \omega_3 r^{1/3} s^{1/6} \delta_3'. \quad (5.15)$$

Now by (5.11)

$$\lambda_3 - \lambda_2 = (\omega_3 - \omega_2) \left( \frac{r}{s} \right)^{1/3} \left\{ 1 + \frac{1}{3} \delta + O(\delta^3) \right\}. \quad (5.16)$$

Hence by (5.15), (5.16) and (4.9)

$$\begin{aligned} (s^{\frac{1}{2}\lambda_3})' (\lambda_3 - \lambda_2) &= -\frac{1}{3} \omega_3 (\omega_2 + 1) r^{2/3} s^{-1/6} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{1}{3} \delta (1 + \omega_3) + O(\delta^2) \right\} \\ &\quad - (\omega_2 + 1) r^{2/3} s^{-1/6} \delta_3' \left\{ 1 + \frac{1}{3} \delta + O(\delta^3) \right\}. \end{aligned} \quad (5.17)$$

Hence by (4.20), (5.4) and (5.17), (5.14) becomes

$$t_{13} = -\frac{1}{9}(\omega_2+1) \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] + O(\delta \frac{s'}{s}) + O(\delta \frac{r'}{r}) + O\left(\frac{g'}{r^{2/3} s^{1/3}}\right). \quad (5.18)$$

Now considering (3.20), then for  $k = 1$ , (3.20) gives

$$t_{21} = \frac{(s^{\frac{1}{2}}\lambda_1)'(\lambda_3-\lambda_1)}{s^{\frac{1}{2}}(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)}. \quad (5.19)$$

Now by (5.3) and (5.16)

$$(\lambda_2-\lambda_1)(\lambda_3-\lambda_2) = \left(\frac{r}{s}\right)^{2/3} (\omega_2-\omega_1)(\omega_3-\omega_2) \left\{ 1 + \frac{\delta}{3} (1+\omega_2+\omega_1) + O(\delta^2) \right\}.$$

Hence

$$s^{\frac{1}{2}}(\lambda_2-\lambda_1)(\lambda_3-\lambda_2) = (1+2\omega_3-\omega_2) r^{2/3} s^{-1/6} \left\{ 1 + \frac{\omega_2}{3} \delta + O(\delta^2) \right\}. \quad (5.20)$$

Now by (5.2) and (5.5)

$$\begin{aligned} (s^{\frac{1}{2}}\lambda_1)'(\lambda_3-\lambda_1) &= \frac{1}{3}\omega_1(\omega_3-\omega_1) r^{2/3} s^{-1/6} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{\delta}{3} (\omega_3+\omega_1) + O(\delta^3) \right\} \left\{ 1 + \right. \\ &\quad \left. \frac{\omega_1}{3} \delta + O(\delta^2) \right\} + \\ &\quad + \omega_1(\omega_3-\omega_1) r^{2/3} s^{-1/6} \delta_1' \left\{ 1 + \frac{\delta}{3} (\omega_3+\omega_1) + O(\delta^3) \right\} \end{aligned}$$

i.e.

$$\begin{aligned} (s^{\frac{1}{2}}\lambda_1)'(\lambda_3-\lambda_1) &= -\frac{1}{3}(\omega_3+1) r^{2/3} s^{-1/6} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{\delta}{3} (\omega_3-2) + O(\delta^2) \right\} \\ &\quad - (\omega_3+1) r^{2/3} s^{-1/6} \delta_1' \left\{ 1 + \frac{\delta}{3} (\omega_3-1) + O(\delta^3) \right\} \end{aligned}$$

Hence by (4.20), (5.20) and (5.21), (5.19) becomes

$$t_{21} = -\frac{1}{9} \frac{(\omega_3+1)}{\omega_3} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] + O(\delta \frac{s'}{s}) + O(\delta \frac{r'}{r}) + O\left(\frac{q'}{r^{2/3} s^{1/3}}\right).$$

i.e.

$$t_{21} = -\frac{1}{18} (i\sqrt{3}+3) \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] + O(\delta \frac{s'}{s}) + O(\delta \frac{r'}{r}) + O\left(\frac{q'}{r^{2/3} s^{1/3}}\right).$$

(5.22)

Next putting  $k=2$  in (3.20), then

$$t_{22} = \frac{(s^{\frac{1}{2}\lambda_2})' (\lambda_3 + \lambda_1 - 2\lambda_2)}{s^{\frac{1}{2}(\lambda_2 - \lambda_1)} (\lambda_3 - \lambda_2)} . \quad (5.23)$$

Now by (4.7)

$$\lambda_3 + \lambda_1 - 2\lambda_2 = (\omega_3 + \omega_1 - 2\omega_2) \left(\frac{r}{s}\right)^{1/3} + \left(\frac{r}{s}\right)^{1/3} (\omega_3 \delta_3 + \omega_1 \delta_1 - 2\omega_2 \delta_2).$$

Hence by (4.16) and (4.8)

$$\lambda_3 + \lambda_1 - 2\lambda_2 = (\omega_3 + \omega_1 - 2\omega_2) \left(\frac{r}{s}\right)^{1/3} + \left(\frac{r}{s}\right)^{1/3} \left\{ \frac{\delta}{3} (\omega_3^2 + \omega_1^2 - 2\omega_2^2) + O(\delta^3) \right\}$$

so by (4.9)

$$\lambda_3 + \lambda_1 - 2\lambda_2 = (\omega_3 + \omega_1 - 2\omega_2) \left(\frac{r}{s}\right)^{1/3} \left\{ 1 + \frac{\delta}{3} \left( \frac{1 + 2\omega_3 - \omega_2}{\omega_3 + \omega_1 - 2\omega_2} \right) + O(\delta^3) \right\}. \quad (5.24)$$

i.e.

$$\lambda_3 + \lambda_1 - 2\lambda_2 \sim (\omega_3 + \omega_1 - 2\omega_2) \left(\frac{r}{s}\right)^{1/3} \quad \text{as } x \rightarrow \infty .$$

Hence by (5.10) and (5.24)

$$\begin{aligned}
 (s^{\frac{1}{2}}\lambda_2)'(\lambda_3+\lambda_1-2\lambda_2) &= \frac{\omega_2}{3}(\omega_3+\omega_1-2\omega_2) r^{\frac{2}{3}} s^{-\frac{1}{6}} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{2}{3} \delta \left( \frac{2\omega_3-\omega_2+1}{\omega_3+\omega_1-2\omega_2} \right) + O(\delta^2) \right\} \\
 &\quad + \omega_3(\omega_3+\omega_1-2\omega_2) r^{\frac{2}{3}} s^{-\frac{1}{6}} \delta_2' \left\{ 1 + \frac{\delta}{3} \left( \frac{2\omega_3-\omega_2+1}{\omega_3+\omega_1-2\omega_2} \right) + O(\delta^3) \right\}.
 \end{aligned}
 \tag{5.25}$$

Hence by (4.20), (5.20) and (5.25), (5.23) becomes

$$t_{22} = \frac{1}{3} \omega_2 \frac{(\omega_3+\omega_1+2\omega_2)}{(1+2\omega_3-\omega_2)} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] + O(\delta \frac{s'}{s}) + O(\delta \frac{r'}{r}) + O\left( \frac{q'}{r^{\frac{2}{3}} s^{\frac{1}{3}}} \right).$$

Now by (4.9)

$$\omega_2(\omega_3+\omega_1-2\omega_2) = 1+2\omega_3-\omega_2.$$

Hence

$$t_{22} = \frac{1}{3} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] + O(\delta \frac{s'}{s}) + O(\delta \frac{r'}{r}) + O\left( \frac{q'}{r^{\frac{2}{3}} s^{\frac{1}{3}}} \right). \tag{5.26}$$

Now taking  $k=3$  in (3.20), hence

$$t_{23} = \frac{(s^{\frac{1}{2}}\lambda_3)'(\lambda_1-\lambda_3)}{s^{\frac{1}{2}}(\lambda_2-\lambda_3)(\lambda_3-\lambda_2)}. \tag{5.27}$$

Now by (5.15) and (5.2)

$$\begin{aligned}
 (s^{\frac{1}{2}}\lambda_3)'(\lambda_1-\lambda_3) &= \frac{\omega_3}{3}(\omega_1-\omega_3) r^{\frac{2}{3}} s^{-\frac{1}{6}} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{\delta}{3} (2\omega_3+\omega_1) + O(\delta^2) \right\} \\
 &\quad + \omega_3(\omega_1-\omega_3) r^{\frac{2}{3}} s^{-\frac{1}{6}} \delta_3' \left\{ 1 + \frac{1}{3} \delta (\omega_3+\omega_1) + O(\delta^3) \right\}
 \end{aligned}$$

i.e.

$$\begin{aligned}
 (s^{\frac{1}{2}}\lambda_3)'(\lambda_1-\lambda_3) &= \frac{1}{3}(\omega_2-\omega_3)r^{\frac{2}{3}}s^{-\frac{1}{6}}\left[\frac{r'}{r} + \frac{1}{2}\frac{s'}{s}\right]\left\{1 + \frac{\delta}{3}(2\omega_3-1) + O(\delta^2)\right\} \\
 &+ (\omega_2-\omega_3)r^{\frac{2}{3}}s^{-\frac{1}{6}}\delta_3'\left\{1 + \frac{1}{3}\delta(\omega_3-1) + O(\delta^3)\right\}.
 \end{aligned}
 \tag{5.28}$$

Hence by (4.20), (5.20) and (5.28), (5.27) becomes

$$t_{23} = \frac{1}{3} \frac{(\omega_2-\omega_3)}{(1+2\omega_3-\omega_2)} \left[\frac{r'}{r} + \frac{1}{2}\frac{s'}{s}\right] + O(\delta\frac{r'}{r}) + O(\delta\frac{s'}{s}) + O\left(\frac{q'}{r^{\frac{2}{3}}s^{\frac{1}{3}}}\right)$$

i.e.

$$t_{23} = \frac{1}{18}(i\sqrt{3}-3)\left[\frac{r'}{r} + \frac{1}{2}\frac{s'}{s}\right] + O(\delta\frac{r'}{r}) + O(\delta\frac{s'}{s}) + O\left(\frac{q'}{r^{\frac{2}{3}}s^{\frac{1}{3}}}\right). \tag{5.29}$$

Now we consider (3.21), hence putting  $k=1$  in (3.21), then

$$t_{31} = \frac{(s^{\frac{1}{2}}\lambda_1)'(\lambda_1-\lambda_2)}{s^{\frac{1}{2}}(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}. \tag{5.30}$$

Now by (5.2) and (5.11)

$$(\lambda_3-\lambda_1)(\lambda_3-\lambda_2) = (\omega_3-\omega_1)(\omega_3-\omega_2)\left(\frac{r}{s}\right)^{\frac{2}{3}}\left\{1 + \frac{\omega_3}{3}\delta + O(\delta^2)\right\}.$$

Hence

$$s^{\frac{1}{2}}(\lambda_3-\lambda_1)(\lambda_3-\lambda_2) = (\omega_3-\omega_1)(\omega_3-\omega_2)\frac{r^{\frac{2}{3}}}{s^{\frac{1}{6}}}\left\{1 + \frac{\omega_3}{3}\delta + O(\delta^2)\right\}. \tag{5.31}$$

By (5.3) and (5.5)

$$\begin{aligned} (s^{\frac{1}{2}\lambda_1})'(\lambda_1 - \lambda_2) &= -\frac{1}{3}\omega_1(\omega_2 - \omega_1) r^{\frac{2}{3}} s^{-\frac{1}{6}} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{\omega_1}{3}\delta + \frac{1}{3}\delta(\omega_2 + \omega_1) + O(\delta^2) \right\} \\ &\quad - \omega_1(\omega_2 - \omega_1) r^{\frac{2}{3}} s^{-\frac{1}{6}} \delta_1' \left\{ 1 + \frac{1}{3}\delta(\omega_2 + \omega_1) + O(\delta^2) \right\} \end{aligned}$$

i.e.

$$\begin{aligned} (s^{\frac{1}{2}\lambda_1})'(\lambda_1 - \lambda_2) &= \frac{1}{3}(\omega_2 + 1) r^{\frac{2}{3}} s^{-\frac{1}{6}} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{\delta}{3}(\omega_2 - 2) + O(\delta^2) \right\} \\ &\quad + (\omega_2 + 1) r^{\frac{2}{3}} s^{-\frac{1}{6}} \delta_1' \left\{ 1 + \frac{1}{3}\delta(\omega_2 - 1) + O(\delta^3) \right\}. \quad (5.32) \end{aligned}$$

Hence by (4.20), (5.31) and (5.32), (5.30) becomes

$$t_{31} = \frac{1}{3} \frac{(\omega_2 + 1)}{(\omega_3 - \omega_1)(\omega_3 - \omega_2)} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] + O(\delta \frac{r'}{r}) + O(\delta \frac{s'}{s}) + O\left(\frac{q'}{r^{\frac{2}{3}} s^{\frac{1}{3}}}\right),$$

i.e.

$$t_{31} = \frac{1}{18} (i\sqrt{3} - 3) \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] + O(\delta \frac{r'}{r}) + O(\delta \frac{s'}{s}) + O\left(\frac{q'}{r^{\frac{2}{3}} s^{\frac{1}{3}}}\right). \quad (5.33)$$

Next taking  $k=2$  in (3.21), then

$$t_{32} = \frac{(s^{\frac{1}{2}\lambda_2})'(\lambda_2 - \lambda_1)}{s^{\frac{1}{2}}(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}. \quad (5.34)$$

Now by (5.3) and (5.10)

$$\begin{aligned} (s^{\frac{1}{2}\lambda_2})'(\lambda_2 - \lambda_1) &= \frac{1}{3}\omega_2(\omega_2 - \omega_1) r^{\frac{2}{3}} s^{-\frac{1}{6}} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{\omega_2}{3}\delta \right. \\ &\quad \left. + \frac{\delta}{3}(\omega_2 + \omega_1) + O(\delta^2) \right\} \\ &\quad + \omega_2(\omega_2 - \omega_1) \delta_2' r^{\frac{2}{3}} s^{-\frac{1}{6}} \left\{ 1 + \frac{\delta}{3}(\omega_2 + \omega_1) + O(\delta^2) \right\} \end{aligned}$$

i.e.

$$\begin{aligned}
 (s^{\frac{1}{2}}\lambda_2)'(\lambda_2-\lambda_1) &= \frac{1}{3}(\omega_2-\omega_3) r^{\frac{2}{3}} s^{-\frac{1}{6}} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] \left\{ 1 + \frac{\delta}{3} (2\omega_2-1) + O(\delta^3) \right\} \\
 &+ (\omega_2-\omega_3) \delta_2' r^{\frac{2}{3}} s^{-\frac{1}{6}} \left\{ 1 + \frac{\delta}{3} (\omega_2-1) + O(\delta^3) \right\}.
 \end{aligned}
 \tag{5.35}$$

Hence by (4.20), (5.31) and (5.35), (5.34) becomes

$$t_{32} = \frac{1}{3} \frac{(\omega_2-\omega_3)}{(\omega_3-\omega_1)(\omega_3-\omega_2)} \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] + O(\delta \frac{s'}{s}) + O(\delta \frac{r'}{r}) + O\left(\frac{q'}{r^{\frac{2}{3}} s^{\frac{1}{3}}}\right)$$

i.e.

$$t_{32} = -\frac{1}{18}(3+i\sqrt{3}) \left[ \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right] + O(\delta \frac{s'}{s}) + O(\delta \frac{r'}{r}) + O\left(\frac{q'}{r^{\frac{2}{3}} s^{\frac{1}{3}}}\right). \tag{5.36}$$

Finally taking  $k=3$  in (3.21), then

$$t_{33} = \frac{(s^{\frac{1}{2}}\lambda_3)'(2\lambda_3-\lambda_1-\lambda_2)}{s^{\frac{1}{2}}(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}. \tag{5.37}$$

Now by (4.7)

$$2\lambda_3-\lambda_1-\lambda_2 = (2\omega_3-\omega_1-\omega_2) \left( \frac{r}{s} \right)^{\frac{1}{3}} + \left( \frac{r}{s} \right)^{\frac{1}{3}} (2\omega_3\delta_3-\omega_1\delta_1-\omega_2\delta_2)$$

Hence by (4.16) and (4.8)

$$2\lambda_3-\lambda_1-\lambda_2 = (2\omega_3-\omega_1-\omega_2) \left( \frac{r}{s} \right)^{\frac{1}{3}} + \left( \frac{r}{s} \right)^{\frac{1}{3}} \left\{ \frac{\delta}{3} (2\omega_3^2-\omega_1^2-\omega_2^2) + O(\delta^3) \right\}$$

so by (4.9)

$$2\lambda_3+\lambda_1-\lambda_2 = \left( \frac{r}{s} \right)^{\frac{1}{3}} (2\omega_3-\omega_2+1) \left\{ 1 + \frac{\delta}{3} \left( \frac{\omega_3-2\omega_2-1}{2\omega_3-\omega_2+1} \right) + O(\delta^3) \right\}. \tag{5.38}$$

Now by (5.15) and (5.38)

$$\begin{aligned}
 (s^{\frac{1}{2}}\lambda_3)'(2\lambda_3-\lambda_1-\lambda_2) &= \frac{1}{3}\omega_3(2\omega_3-\omega_2+1)r^{\frac{2}{3}}s^{-\frac{1}{6}}\left[\frac{r'}{r} + \frac{1}{2}\frac{s'}{s}\right] \times \\
 &\times \left\{ 1 + \frac{2\delta}{3}\left(\frac{\omega_3-2\omega_2-1}{2\omega_3-\omega_2+1}\right) + O(\delta^2) \right\} + \omega_3(2\omega_3-\omega_2+1)r^{\frac{2}{3}}s^{-\frac{1}{6}}\delta_3' \times \\
 &\times \left\{ 1 + \frac{\delta}{3}\left(\frac{\omega_3-2\omega_2+1}{2\omega_3-\omega_2+1}\right) + O(\delta^3) \right\}
 \end{aligned}$$

i.e.

$$\begin{aligned}
 (s^{\frac{1}{2}}\lambda_3)'(2\lambda_3-\lambda_1-\lambda_2) &= \frac{1}{3}(\omega_3-2\omega_2-1)r^{\frac{2}{3}}s^{-\frac{1}{6}}\left[\frac{r'}{r} + \frac{1}{2}\frac{s'}{s}\right]\left\{1 + \frac{2\delta}{3}\left(\frac{\omega_3-2\omega_2-1}{2\omega_3-\omega_2+1}\right)\right. \\
 &\quad \left.+ O(\delta^2)\right\} + (\omega_3-2\omega_2-1)r^{\frac{2}{3}}s^{-\frac{1}{6}}\delta_3'\left\{1 + \frac{\delta}{3}\left(\frac{\omega_3-2\omega_2+1}{2\omega_3-\omega_2+1}\right)\right. \\
 &\quad \left.+ O(\delta^3)\right\}. \quad (5.39)
 \end{aligned}$$

Hence by (4.20), (5.31) and (5.39), (5.37) becomes

$$t_{33} = \frac{1}{3} \frac{(\omega_3-2\omega_2-1)}{(\omega_3-\omega_1)(\omega_3-\omega_2)} \left[\frac{r'}{r} + \frac{1}{2}\frac{s'}{s}\right] + O(\delta\frac{r'}{r}) + O(\delta\frac{s'}{s}) + O\left(\frac{q'}{r^{\frac{2}{3}}s^{\frac{1}{3}}}\right).$$

Now by (4.9)

$$(\omega_3-\omega_1)(\omega_3-\omega_2) = \omega_3-2\omega_2-1.$$

Hence

$$t_{33} = \frac{1}{3}\left[\frac{r'}{r} + \frac{1}{2}\frac{s'}{s}\right] + O(\delta\frac{r'}{r}) + O(\delta\frac{s'}{s}) + O\left(\frac{q'}{r^{\frac{2}{3}}s^{\frac{1}{3}}}\right). \quad (5.40)$$

Now by previous work in this section we complete our discussion of the matrix  $T^{-1}T'$ .

6. The system  $Z' = (\Lambda + R_1 + R_2)Z$

Now putting together (5.8)-(5.40), we can write (2.8) as

$$Z' = (\Lambda + R_1 + R_2)Z \quad (6.1)$$

where

$$R_2(x) = (\tilde{r}_{ij}^{(2)}), \quad (1 \leq i, j \leq 3)$$

$$\tilde{r}_{ij} = O(\delta \frac{s'}{s}) + O(\delta \frac{r'}{r}) + O\left(\frac{q'}{r^{2/3} s^{1/3}}\right) \quad (1 \leq i, j \leq 3) \quad (6.2)$$

and

$$R_1(x) = (r_{ij}(x)), \quad (1 \leq i, j \leq 3)$$

such that

$$\begin{aligned} r_{11} &= -\frac{1}{3} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \\ r_{12} &= \frac{1}{9} (\omega_3 + 1) \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \\ r_{13} &= \frac{1}{9} (\omega_2 + 1) \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \\ r_{21} &= \frac{1}{18} (i\sqrt{3} + 3) \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \\ r_{22} &= -\frac{1}{3} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \\ r_{23} &= \frac{1}{18} (3 - i\sqrt{3}) \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \\ r_{31} &= \frac{1}{18} (3 - i\sqrt{3}) \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \\ r_{32} &= \frac{1}{18} (3 + i\sqrt{3}) \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \\ r_{33} &= -\frac{1}{3} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \end{aligned} \quad (6.3)$$

Now the last O-term on the right hand side of (6.2) is  $L(a, \infty)$  by (4.5). Also the first two O-terms on the right of (6.2) are  $L(a, \infty)$  by remark 4.1 and so  $\tilde{r}_{ij}$  is  $L(a, \infty)$ . Hence  $R_2(x)$  is  $L(a, \infty)$  and so (6.1) has the required form (2.3) if  $\Lambda + R_1$  satisfies the hypothesis of Theorem 1.2.

Now we shall show that the system (6.1) satisfies Theorem 1.2, in particular the matrix  $\Lambda + R_1$  because  $R_2$  is  $L(a, \infty)$ . To do this, first we need to show that the eigenvalues  $\lambda_j(x)$  ( $1 \leq j \leq 3$ ) of  $\Lambda(x)$  are distinct, this is obvious by (4.7) and (4.9).

Second we need to show that condition (i) of Theorem 1.2, holds, in particular we need to show that

$$R_1(x) = o\{\lambda_i(x) - \lambda_j(x)\} \text{ as } x \rightarrow \infty \quad (6.4)$$

for  $(i \neq j, 1 \leq i, j \leq 3)$ .

Now by (4.7), for  $(i \neq j, 1 \leq i, j \leq 3)$

$$\lambda_i(x) - \lambda_j(x) = \sigma_{ij} \left( \frac{r}{s} \right)^{1/3} \left\{ 1 + \sigma_{ij}^{-1} \omega_i \delta_i - \sigma_{ij}^{-1} \omega_j \delta_j \right\} \sim \sigma_{ij} \left( \frac{r}{s} \right)^{1/3} \quad (6.5)$$

where  $\sigma_{ij} = \omega_i - \omega_j$ . ( $i \neq j, 1 \leq i, j \leq 3$ )

Hence by (4.3), (4.4), (6.3) and (6.5) implies that (6.4) holds for  $(i \neq j, 1 \leq i, j \leq 3)$ , which means that  $\lambda_i - \lambda_j$  ( $i \neq j, 1 \leq i, j \leq 3$ ) dominate the matrix  $R_1$ .

Next we need to show that (ii) holds i.e.

$$\{(\lambda_i - \lambda_j)^{-1} R_1\}' \text{ is } L(a, \infty) \quad (i \neq j, 1 \leq i, j \leq 3).$$

(6.6)

Hence subject to (6.3), it suffices to show that

$$\left\{ (\lambda_i - \lambda_j)^{-1} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \right\}' \text{ is } L(a, \infty) \quad (i \neq j, 1 \leq i, j \leq 3). \quad (6.7)$$

Now by (6.5)

$$\begin{aligned} \left\{ (\lambda_i - \lambda_j)^{-1} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \right\}' &= \left\{ \sigma_{ij}^{-1} \left( \frac{s}{r} \right)^{1/3} (1 + \sigma_{ij}^{-1} \omega_i \delta_i - \sigma_{ij}^{-1} \omega_j \delta_j)^{-1} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \right\}' \\ &= \sigma_{ij}^{-1} \left( \frac{s}{r} \right)^{1/3} \left( \frac{1}{6} \frac{s' r'}{s r} - \frac{1}{3} \frac{s'^2}{s^2} + \frac{1}{2} \frac{s''}{s} + \frac{r''}{r} \right. \\ &\quad \left. - \frac{4}{3} \frac{r'^2}{r^2} \right) \left( 1 + \sigma_{ij}^{-1} \omega_i \delta_i - \sigma_{ij}^{-1} \omega_j \delta_j \right)^{-1} \\ &\quad - \sigma_{ij}^{-2} \left( \frac{s}{r} \right)^{1/3} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \left( 1 + \sigma_{ij}^{-1} \omega_i \delta_i - \sigma_{ij}^{-1} \omega_j \delta_j \right)^{-1} \times \\ &\quad \times (\omega_i \delta_i' - \omega_j \delta_j') \\ &\quad (i \neq j, 1 \leq i, j \leq 3). \end{aligned}$$

i.e.

$$\begin{aligned} \left\{ (\lambda_i - \lambda_j)^{-1} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \right\}' &= O \left( \left( \frac{s}{r} \right)^{1/3} \frac{r''}{r} \right) + O \left( \left( \frac{s}{r} \right)^{1/3} \frac{s''}{s} \right) \\ &\quad + O \left( \left( \frac{s}{r} \right)^{1/3} \frac{r'^2}{r^2} \right) + O \left( \left( \frac{s}{r} \right)^{1/3} \frac{s'^2}{s^2} \right) \\ &\quad + O \left( \left( \frac{s}{r} \right)^{1/3} \frac{r'}{r} \delta_i' \right) + O \left( \left( \frac{s}{r} \right)^{1/3} \frac{s'}{s} \delta_i' \right) + O \left( \left( \frac{s}{r} \right)^{1/3} \frac{r'}{r} \delta_j' \right) \\ &\quad + O \left( \left( \frac{s}{r} \right)^{1/3} \frac{s'}{s} \delta_j' \right) \quad (i \neq j, 1 \leq i, j \leq 3). \quad (6.8) \end{aligned}$$

Now let

$$\gamma_1 = \frac{r'}{r} \left( \frac{s}{r} \right)^{1/3}, \quad (6.9)$$

$$\gamma_2 = \frac{s'}{s} \left( \frac{s}{r} \right)^{1/3}. \quad (6.10)$$

then by (4.3) and (4.4),  $\gamma_i = o(1)$  as  $x \rightarrow \infty$  and  $\gamma_2 = o(1)$  as  $x \rightarrow \infty$ . Hence by (6.9) and (6.10), (6.8) becomes

$$\begin{aligned} \left\{ (\lambda_i - \lambda_j)^{-1} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \right\}' &= O \left( \left( \frac{s}{r} \right)^{1/3} \frac{r''}{r} \right) + O \left( \left( \frac{s}{r} \right)^{1/3} \frac{s''}{s} \right) \\ &+ O \left( \left( \frac{s}{r} \right)^{1/3} \frac{s'^2}{s^2} \right) + O \left( \left( \frac{s}{r} \right)^{1/3} \frac{r'^2}{r^2} \right) \\ &+ O(\gamma_1 \delta_i') + O(\gamma_2 \delta_i') + O(\gamma_1 \delta_j') \\ &+ O(\gamma_2 \delta_j') \quad (i \neq j, 1 \leq i, j \leq 3). \end{aligned}$$

(6.11)

Now the first four terms on the right hand side of (6.11) are  $L(a, \infty)$  by (4.5), also by (4.21)  $\delta_i'$  and  $\delta_j'$  are  $L(a, \infty)$ . Hence the last four terms on (6.11) are  $L(a, \infty)$ . Thus (6.7) holds which makes (6.6) also holds for  $(i \neq j, 1 \leq i, j \leq 3)$ .

Finally we need to show that the eigenvalues of  $\Lambda + R_1$  satisfy the Dichotomy condition I-II in Levinson's Theorem 1.1. Now with  $R_1$  as given in (6.3), it's so difficult to calculate the eigenvalues of  $\Lambda + R_1$ , hence we can use the simplification (1.21), and so if  $\mu_k(x)$  ( $1 \leq k \leq 3$ ) are the eigenvalues of  $\Lambda + R_1$ , then subject to (1.21),

$$\mu_k = \lambda_k + r_{kk} + O \left( \max_{\ell \neq k} |R_1|^2 / |\lambda_\ell - \lambda_k| \right) \quad (1 \leq k \leq 3),$$

(6.12)

Now the Dichotomy condition I-II of Levinson's Theorem 1.1, holds if we show that (1.16) holds. i.e.

$$\operatorname{Re} \{ \mu_j(x) - \mu_k(x) \} = f(x) + g(x), \quad (6.13)$$

where  $f(x)$  does not change sign in  $[a, \infty)$  and  $g(x)$  is  $L(a, \infty)$ .

Hence we need to show that the O-term on the right hand side of (6.12) is  $L(a, \infty)$ , and so we need the additional condition (1.20) holds i.e.

$$|\lambda_i - \lambda_j|^{-1} |R_1|^2 \text{ is } L(a, \infty) \quad (\text{for all } i \neq j). \quad (6.14)$$

Thus (6.14) holds if we show that

$$(\lambda_i - \lambda_j)^{-1} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right)^2 \text{ is } L(a, \infty) \quad (\text{for all } i \neq j). \quad (6.15)$$

Now by (6.5),

$$(\lambda_i - \lambda_j)^{-1} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right)^2 = \sigma_{ij}^{-1} \left( \frac{s}{r} \right)^{1/3} \left( 1 + \sigma_{ij}^{-1} \omega_i \delta_i - \sigma_{ij}^{-1} \omega_j \delta_j \right)^{-1} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right)^2$$

i.e.

$$(\lambda_i - \lambda_j)^{-1} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right)^2 = O \left( \left( \frac{s}{r} \right)^{1/3} \frac{r'^2}{r^2} \right) + O \left( \left( \frac{s}{r} \right)^{1/3} \frac{s'^2}{s^2} \right) \quad (i \neq j, 1 \leq i, j \leq 3).$$

(6.16)

Now the O-terms on the right hand side of (6.16) are  $L(a, \infty)$  by (4.5). Hence (6.15) holds which makes the additional condition (6.14) also holds, and so the O-term on the right hand side of (6.12) is  $L(a, \infty)$ .

Now first we need to show (6.13) holds under case I of Theorem 4.1, i.e. when  $s, r$  and  $q$  are real-valued functions. Now (6.12) can be written as

$$\mu_k = \lambda_k + r_{kk} + O(f_k) \quad (1 \leq k \leq 3), \quad (6.17)$$

where

$$f_k = \max_{\ell \neq k} |R_1|^2 / |\lambda_\ell - \lambda_k| \quad (1 \leq k \leq 3) \quad (6.18)$$

such that

$$f_k \text{ is } L(a, \infty) \quad (1 \leq k \leq 3) \quad (6.19)$$

subject to (6.14).

Now by (6.17)

$$\begin{aligned} \mu_j - \mu_k &= (\lambda_j - \lambda_k) + (r_{jj} - r_{kk}) + O(f_j) + O(f_k) \\ &\quad (j \neq k, 1 \leq j, k \leq 3). \end{aligned}$$

since by (6.3)

$$r_{kk} = -\frac{1}{3} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) \quad (1 \leq k \leq 3).$$

Hence

$$r_{jj} - r_{kk} = 0 \quad (j \neq k, 1 \leq j, k \leq 3),$$

and so

$$\begin{aligned} \mu_j - \mu_k &= \lambda_j - \lambda_k + O(f_j) + O(f_k) \quad (j \neq k, 1 \leq j, k \leq 3). \\ &\quad (6.20) \end{aligned}$$

since

$$\lambda_j = \omega_j \left( \frac{r}{s} \right)^{1/3} (1 + \delta_j),$$

where

$$\delta_j = \frac{\omega_j}{3} \delta (1 - \frac{\omega_j}{3} \delta)^{-1},$$

$$\omega_j^3 = -1.$$

Hence

$$\lambda_j - \lambda_k = \left(\frac{r}{s}\right)^{1/3} (\omega_j - \omega_k) \left\{ 1 + \frac{1}{3} (\omega_j + \omega_k) \delta + O(\delta^3) \right\} \quad (j \neq k, 1 \leq j, k \leq 3),$$

since

$$\delta = \frac{q}{r^{2/3} s^{1/3}}.$$

Hence

$$\lambda_j - \lambda_k = \left(\frac{r}{s}\right)^{1/3} (\omega_j - \omega_k) + \frac{1}{3} \frac{q}{r^{1/3} s^{2/3}} (\omega_j^2 - \omega_k^2) + O\left(\delta \frac{q^2}{sr}\right)$$

$$(j \neq k, 1 \leq j, k \leq 3). \quad (6.21)$$

since by (4.5)  $\frac{q^2}{sr}$  is  $L(a, \infty)$ , hence the 0-term on the right of (6.21) is  $L(a, \infty)$ . Now by (6.21), (6.20) gives

$$\mu_j - \mu_k = \left(\frac{r}{s}\right)^{1/3} (\omega_j - \omega_k) + \frac{1}{3} \frac{q}{r^{1/3} s^{2/3}} (\omega_j^2 - \omega_k^2) + O\left(\delta \frac{q^2}{sr}\right) + O(f_j)$$

$$+ O(f_k) \quad (j \neq k, 1 \leq j, k \leq 3).$$

$$(6.22)$$

since  $r, s$  and  $q$  are real-valued functions, hence (6.22) gives

$$\operatorname{re}(\mu_j - \mu_k) = \left(\frac{r}{s}\right)^{1/3} \operatorname{re}(\omega_j - \omega_k) + \frac{1}{3} \frac{q}{r^{1/3} s^{2/3}} \operatorname{re}(\omega_j^2 - \omega_k^2) + O\left(\delta \frac{q^2}{sr}\right) + O(f_j) + O(f_k)$$

$$(j \neq k, 1 \leq j, k \leq 3). \quad (6.23)$$

Now putting  $j=1$  and  $k=2$  in (6.23), then

$$\operatorname{re}(\mu_1 - \mu_2) = \left(\frac{r}{s}\right)^{1/3} \operatorname{re}(\omega_1 - \omega_2) + \frac{1}{3} \frac{q}{r^{1/3} s^{2/3}} \operatorname{re}(\omega_1^2 - \omega_2^2) + O\left(\delta \frac{q^2}{sr}\right) + O(f_1) + O(f_2)$$

Now by (4.9)

$$\operatorname{re}(\omega_1 - \omega_2) = -\frac{3}{2},$$

$$\operatorname{re}(\omega_1^2 - \omega_2^2) = \frac{3}{2},$$

hence

$$\operatorname{re}(\mu_1 - \mu_2) = -\frac{3}{2} \left( \frac{r}{s} \right)^{1/3} + \frac{1}{2} \frac{q}{r^{1/3} s^{2/3}} + O\left( \delta \frac{q^2}{sr} \right) + O(f_1) + O(f_2).$$

Hence by (4.2)

$$\operatorname{re}(\mu_1 - \mu_2) = -\frac{3}{2} \left( \frac{r}{s} \right)^{1/3} (1 + o(1)) + O\left( \delta \frac{q^2}{sr} \right) + O(f_1) + O(f_2).$$

Now putting  $j=1$  and  $k=3$  in (6.23), then

$$\operatorname{re}(\mu_1 - \mu_3) = \left( \frac{r}{s} \right)^{1/3} \operatorname{re}(\omega_1 - \omega_3) + \frac{1}{3} \frac{q}{r^{1/3} s^{2/3}} (\omega_1^2 - \omega_3^2) + O\left( \delta \frac{q^2}{sr} \right) + O(f_1) + O(f_3).$$

Again by (4.9)

$$\operatorname{re}(\omega_1 - \omega_3) = -\frac{3}{2}, \quad \operatorname{re}(\omega_1^2 - \omega_3^2) = \frac{3}{2}.$$

Hence

$$\operatorname{re}(\mu_1 - \mu_3) = -\frac{3}{2} \left( \frac{r}{s} \right)^{1/3} + \frac{1}{2} \frac{q}{r^{1/3} s^{2/3}} + O\left( \delta \frac{q^2}{rs} \right) + O(f_1) + O(f_3).$$

Hence by (4.2)

$$\operatorname{re}(\mu_1 - \mu_3) = -\frac{3}{2} \left( \frac{r}{s} \right)^{1/3} (1 + o(1)) + O\left( \delta \frac{q^2}{rs} \right) + O(f_1) + O(f_3).$$

Now putting  $j=2$  and  $k=3$  in (6.23), then

$$\operatorname{re}(\mu_2 - \mu_3) = \left( \frac{r}{s} \right)^{1/3} \operatorname{re}(\omega_2 - \omega_3) + \frac{1}{3} \frac{q}{r^{1/3} s^{2/3}} \operatorname{re}(\omega_2^2 - \omega_3^2) + O\left( \delta \frac{q^2}{sr} \right) + O(f_2) + O(f_3)$$

Now by 4.9)

$$\operatorname{re}(\omega_2 - \omega_3) = 0$$

$$\operatorname{re}(\omega_2^2 - \omega_3^2) = 0$$

Hence

$$\operatorname{re}(\mu_2 - \mu_3) = O\left(\delta \frac{q^2}{sr}\right) + O(f_2) + O(f_3)$$

Hence by previous work (6.13) holds which implies that the Dichotomy condition of Levinson's Theorem 1.1 holds under case I if Theorem 4.1.

Next we need to show (6.13) also holds under case II of Theorem 4.1 since  $r, s$  and  $q$  are complex-valued functions, hence (6.22) gives

$$\begin{aligned} \operatorname{re}(\mu_j - \mu_k) &= \operatorname{re}\left(\frac{r}{s}\right)^{1/3} \cdot \operatorname{re}(\omega_j - \omega_k) - \operatorname{im}\left(\frac{r}{s}\right)^{1/3} \cdot \operatorname{im}(\omega_j - \omega_k) \\ &\quad + \frac{1}{3} \operatorname{re} \frac{q}{r^{1/3} s^{2/3}} \cdot \operatorname{re}(\omega_j^2 - \omega_k^2) - \frac{1}{3} \operatorname{im} \frac{q}{r^{1/3} s^{2/3}} \cdot \operatorname{im}(\omega_j^2 - \omega_k^2) \\ &\quad + O\left(\delta \frac{q^2}{sr}\right) + O(f_j) + O(f_k). \quad (j \neq k, 1 \leq j, k \leq 3) \quad (6.24) \end{aligned}$$

Now by (4.9)

$$\begin{aligned} \operatorname{re}(\omega_1 - \omega_2) &= -\frac{3}{2}, & \operatorname{im}(\omega_1 - \omega_2) &= -\frac{\sqrt{3}}{2} \\ \operatorname{re}(\omega_1^2 - \omega_2^2) &= \frac{3}{2}, & \operatorname{im}(\omega_1^2 - \omega_2^2) &= -\frac{\sqrt{3}}{2}. \end{aligned}$$

Hence (6.24), becomes for  $j=1$  and  $k=2$

$$\begin{aligned} \operatorname{re}(\mu_1 - \mu_2) &= -\frac{3}{2} \operatorname{re}\left(\frac{r}{s}\right)^{1/3} + \frac{\sqrt{3}}{2} \operatorname{im}\left(\frac{r}{s}\right)^{1/3} + \frac{1}{2} \operatorname{re} \frac{q}{r^{1/3} s^{2/3}} \\ &\quad + \frac{\sqrt{3}}{6} \operatorname{im} \frac{q}{r^{1/3} s^{2/3}} + O\left(\delta \frac{q^2}{sr}\right) + O(f_1) + O(f_2). \end{aligned}$$

since by case II and (4.2) we have

$$\operatorname{re} \frac{q}{r^{1/3} s^{2/3}} = o\left\{\operatorname{re}\left(\frac{r}{s}\right)^{1/3}\right\}, \quad \left\{\operatorname{re}\left(\frac{r}{s}\right)^{1/3} \neq 0\right\}, \quad \text{as } x \rightarrow \infty$$

$$\operatorname{im} \frac{q}{r^{1/3} s^{2/3}} = o \left\{ \operatorname{re} \left( \frac{r}{s} \right)^{1/3} \right\}, \quad \text{as } x \rightarrow \infty$$

$$\operatorname{im} \left( \frac{r}{s} \right)^{1/3} = o \left\{ \operatorname{re} \left( \frac{r}{s} \right)^{1/3} \right\}. \quad \text{as } x \rightarrow \infty$$

Hence

$$\operatorname{re}(\mu_1 - \mu_2) = -\frac{3}{2} \operatorname{re} \left( \frac{r}{s} \right)^{1/3} (1 + o(1)) + o \left( \delta \frac{q^2}{sr} \right) + o(f_1) + o(f_2). \quad (6.25)$$

Now by (4.9)

$$\begin{aligned} \operatorname{re}(\omega_1 - \omega_3) &= -\frac{3}{2}, & \operatorname{im}(\omega_1 - \omega_3) &= \frac{\sqrt{3}}{2} \\ \operatorname{re}(\omega_1^2 - \omega_3^2) &= \frac{3}{2}, & \operatorname{im}(\omega_1^2 - \omega_3^2) &= \frac{\sqrt{3}}{2}. \end{aligned}$$

Hence (6.24), becomes for  $j=1$  and  $k=3$

$$\begin{aligned} \operatorname{re}(\mu_1 - \mu_3) &= -\frac{3}{2} \operatorname{re} \left( \frac{r}{s} \right)^{1/3} - \frac{\sqrt{3}}{2} \operatorname{im} \left( \frac{r}{s} \right)^{1/3} + \frac{1}{2} \operatorname{re} \frac{q}{r^{1/3} s^{2/3}} - \frac{\sqrt{3}}{6} \operatorname{im} \frac{q}{r^{1/3} s^{2/3}} \\ &\quad + o \left( \delta \frac{q^2}{sr} \right) + o(f_1) + o(f_3). \end{aligned}$$

Thus by the above relations

$$\operatorname{re}(\mu_1 - \mu_3) = -\frac{3}{2} \operatorname{re} \left( \frac{r}{s} \right)^{1/3} (1 + o(1)) + o \left( \delta \frac{q^2}{sr} \right) + o(f_1) + o(f_3). \quad (6.26)$$

Now considering  $j=2$  and  $k=3$  in (6.24), then

$$\begin{aligned} \operatorname{re}(\mu_2 - \mu_3) &= \operatorname{re} \left( \frac{r}{s} \right)^{1/3} \cdot \operatorname{re}(\omega_2 - \omega_3) - \operatorname{im} \left( \frac{r}{s} \right)^{1/3} \cdot \operatorname{im}(\omega_2 - \omega_3) \\ &\quad + \frac{1}{3} \operatorname{re} \frac{q}{r^{1/3} s^{2/3}} \cdot \operatorname{re}(\omega_2^2 - \omega_3^2) - \frac{1}{3} \operatorname{im} \frac{q}{r^{1/3} s^{2/3}} \cdot \operatorname{im}(\omega_2^2 - \omega_3^2) \\ &\quad + o \left( \delta \frac{q^2}{sr} \right) + o(f_2) + o(f_3) \end{aligned}$$

Now by (4.9)

$$\begin{aligned} \operatorname{re}(\omega_2 - \omega_3) &= 0, & \operatorname{im}(\omega_2 - \omega_3) &= \sqrt{3} \\ \operatorname{re}(\omega_2^2 - \omega_3^2) &= 0, & \operatorname{im}(\omega_2^2 - \omega_3^2) &= \sqrt{3}. \end{aligned}$$

Hence

$$\operatorname{re}(\mu_2 - \mu_3) = -\sqrt{3} \operatorname{im}\left(\frac{r}{s}\right)^{1/3} - \frac{\sqrt{3}}{3} \operatorname{im} \frac{q}{r^{1/3} s^{2/3}} + O\left(\delta \frac{q^2}{sr}\right) + O(f_2) + O(f_3).$$

Thus by case II.

$$\operatorname{re}(\mu_2 - \mu_3) = -\sqrt{3} \operatorname{im}\left(\frac{r}{s}\right)^{1/3} (1 + o(1)) + O\left(\delta \frac{q^2}{sr}\right) + O(f_2) + O(f_3). \quad (6.27)$$

Hence by (6.25), (6.26) and (6.27), (6.13) holds which implies that the Dichotomy condition of Levinson's Theorem 1.1 holds under case II of Theorem 4.1.

Finally we need to show (6.13) holds under case III of Theorem 4.1. Similar work for this case can be done, but here we obtain

$$\operatorname{re}(\mu_1 - \mu_2) = \frac{\sqrt{3}}{2} \operatorname{im}\left(\frac{r}{s}\right)^{1/3} (1 + o(1)) + O\left(\delta \frac{q^2}{sr}\right) + O(f_1) + O(f_2), \quad (6.28)$$

$$\operatorname{re}(\mu_1 - \mu_3) = -\frac{\sqrt{3}}{2} \operatorname{im}\left(\frac{r}{s}\right)^{1/3} (1 + o(1)) + O\left(\delta \frac{q^2}{sr}\right) + O(f_1) + O(f_3), \quad (6.29)$$

$$\operatorname{re}(\mu_2 - \mu_3) = -\sqrt{3} \operatorname{im}\left(\frac{r}{s}\right)^{1/3} (1 + o(1)) + O\left(\delta \frac{q^2}{sr}\right) + O(f_2) + O(f_3). \quad (6.30)$$

Hence by (6.28), (6.29) and (6.30), (6.13) holds and consequently the Dichotomy condition of Levinson's Theorem 1.1 holds under case III of Theorem 4.1.

## 7. The Asymptotic Form of Solutions

Now that (6.1) has been shown to have the form (2.3), hence Theorem 1.2, is applicable and it follows that as  $x \rightarrow \infty$ , (6.1) has three linearly independent solutions  $Z_k(x)$  such that

$$Z_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t) dt\right) \quad (1 \leq k \leq 3) \quad (7.1)$$

where  $e_k$  is the coordinate vector with  $k$ th component unity and other components zero. Now we need to obtain the three linearly independent solutions of our original system (2.2). Hence by (2.7), (2.2) has corresponding solutions,

$$Y_k(x) = T(x) Z_k(x). \quad (1 \leq k \leq 3) \quad (7.2)$$

Hence by (3.13) and (7.1), (7.2) becomes

$$Y_k(x) = \begin{pmatrix} 1 & 1 & 1 \\ s^{\frac{1}{2}} \lambda_1 & s^{\frac{1}{2}} \lambda_2 & s^{\frac{1}{2}} \lambda_3 \\ s \lambda_1^2 & s \lambda_2^2 & s \lambda_3^2 \end{pmatrix} \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t) dt\right) \quad \text{as } x \rightarrow \infty \quad (1 \leq k \leq 3) \quad (7.3)$$

and, taking the first component on each side of (7.3), we obtain three solutions  $y_k(x)$  of (2.1) with the asymptotic forms

$$y_k(x) \sim \exp\left(\int_a^x \mu_k(t) dt\right) \quad \text{as } x \rightarrow \infty \quad (1 \leq k \leq 3) \quad (7.4)$$

Now by (4.7), (4.16) and (4.12)

$$\lambda_k = \omega_k \left( \frac{r}{s} \right)^{1/3} + \frac{\omega_k^2}{3} \cdot \frac{q}{r^{1/3} s^{2/3}} + O\left(\frac{q^2}{sr}\right)$$

$$\lambda_k = \omega_k \left( \frac{r}{s} \right)^{1/3} - \frac{1}{3} \bar{\omega}_k \frac{q}{r^{1/3} s^{2/3}} + O\left(\frac{q^2}{sr}\right) \quad (1 \leq k \leq 3) \quad (7.5)$$

where the 0-term on the right hand side of (7.5) is  $L(a, \infty)$  by (4.5).

Also by (6.3)

$$r_{kk} = -\frac{1}{3} \left( \frac{r'}{r} + \frac{1}{2} \frac{s'}{s} \right) = \frac{\left( s^{-1/6} r^{-1/3} \right)'}{s^{-1/6} r^{-1/3}}, \quad (1 \leq k \leq 3) \quad (7.6)$$

hence by (7.5) and (7.6), (6.19) becomes

$$\mu_k = \omega_k \left( \frac{r}{s} \right)^{1/3} - \frac{1}{3} \bar{\omega}_k \frac{q}{r^{1/3} s^{2/3}} + \frac{\left( s^{-1/6} r^{-1/3} \right)'}{s^{-1/6} r^{-1/3}} + O\left(\frac{q^2}{sr}\right) + O(f_k) \quad (1 \leq k \leq 3). \quad (7.7)$$

Hence by (7.7), (7.4) becomes after adjusting a constant multiple in  $y_k(x)$  ( $1 \leq k \leq 3$ ), we obtain the asymptotic forms

$$y_k(x) \sim s^{-1/6}(x) r^{-1/3}(x) \exp \left\{ \int_a^x \left[ \omega_k \left( \frac{r}{s} \right)^{1/3} (t) - \frac{1}{3} \bar{\omega}_k \frac{q}{r^{1/3} s^{2/3}} (t) \right] dt \right\} \\ \text{as } x \rightarrow \infty \quad (1 \leq k \leq 3) \quad (7.8)$$

of the differential equation (2.1). The proof of Theorem 4.1, is now complete.

We shall now consider two examples of the differential equation (2.1).

## Example (1)

In the familiar case we consider

$$s(x) = x^\alpha, \quad q(x) = Bx^\beta, \quad r(x) = Cx^\gamma$$

where  $\alpha, \beta, \gamma, B$  and  $C (\neq 0)$  are real constants, and  $(a \leq x < \infty)$  excluding  $a=0$ . Now

$$\frac{q}{r^{2/3} s^{1/3}} = BC^{-2/3} x^{\beta - \frac{2}{3}\gamma - \frac{1}{3}\alpha}.$$

Hence (4.2) holds if

$$2\gamma + \alpha - 3\beta > 0. \quad (7.9)$$

Now

$$\begin{aligned} r' &= C \gamma x^{\gamma-1}, \\ r'' &= C \gamma (\gamma-1) x^{\gamma-2}, \\ r'^2 &= C^2 \gamma^2 x^{2\gamma-2}. \end{aligned}$$

Also

$$\begin{aligned} s' &= \alpha x^{\alpha-1}, \\ s'' &= \alpha (\alpha-1) x^{\alpha-2}, \\ s'^2 &= \alpha^2 x^{2(\alpha-1)} \end{aligned}$$

Hence

$$\frac{r'}{r} = \gamma x^{-1}, \quad \frac{s'}{s} = \alpha x^{-1}, \quad \left( \frac{s}{r} \right)^{1/3} = C^{-1/3} x^{1/3} (\alpha - \gamma)$$

Now

$$\frac{r'}{r} \left( \frac{s}{r} \right)^{1/3} = \gamma C^{-1/3} x^{\frac{1}{3}(\alpha-\gamma)-1}$$

Hence (4.3) holds if

$$\frac{1}{3} (\gamma-\alpha) + 1 > 0 \quad (7.10)$$

Similarly (4.4) holds if (7.10) holds.

Now

$$\frac{q'}{r^{2/3} s^{1/3}} = B \rho C^{-2/3} x^{-\frac{1}{3}(\alpha+2\gamma)+\beta-1}$$

Hence  $\frac{q'}{r^{2/3} s^{1/3}}$  is  $L(a, \infty)$  if (7.9) holds.

Also

$$\frac{r''}{r} \left( \frac{s}{r} \right)^{1/3} = \gamma(\gamma-1) C^{-1/3} x^{\frac{1}{3}(\alpha-\gamma)-2}$$

Hence  $\frac{r''}{r} \left( \frac{s}{r} \right)^{1/3}$  is  $L(a, \infty)$  if (7.10) holds.

Similarly

$$\frac{r'^2}{r^2} \left( \frac{s}{r} \right)^{1/3}, \quad \frac{s''}{s} \left( \frac{s}{r} \right)^{1/3} \text{ and } \frac{s'^2}{s^2} \left( \frac{s}{r} \right)^{1/3}$$

are all  $L(a, \infty)$  under condition (7.10).

Finally

$$\frac{q^2}{sr} = B^2 C^{-1} x^{2\beta-\alpha-\gamma},$$

is  $L(a, \infty)$  if

$$\gamma + \alpha - 2\beta - 1 > 0 \quad (7.11)$$

Hence putting together (7.9), (7.10) and (7.11), then conditions (4.2), (4.3), (4.4) and (4.5) are all satisfied under a set of two conditions in  $\alpha, \beta$  and  $\gamma$  which is given by

$$\left. \begin{aligned} \frac{1}{3} (\gamma - \alpha) + 1 &> 0 \\ \gamma + \alpha - 2\beta - 1 &> 0 \end{aligned} \right\} \quad (7.12)$$

Also we mention that case I of Theorem 4.1, holds for this example.

Thus (4.1) gives after adjusting a constant multiple in  $y_k$

$$\begin{aligned} y_k(x) \sim x^{-1/3 (\gamma + \frac{1}{2}\alpha)} \exp \left\{ \omega_k C^{1/3} \int_a^x t^{1/3 (\gamma - \alpha)} dt \right. \\ \left. - \frac{1}{3} \bar{\omega}_k C^{-1/3} B \int_a^x t^{\beta - \frac{1}{3}(\gamma + 2\alpha)} dt \right\} \\ (1 \leq k \leq 3), \text{ as } x \rightarrow \infty \quad (7.13) \end{aligned}$$

Example (2).

Here we give different class of coefficients covered by our analysis, we consider

$$s(x) = x^\alpha \exp x^b, \quad q(x) = Bx^\beta \exp x^b,$$

$$r(x) = Cx^\gamma \exp x^d$$

where  $\alpha, \beta, \gamma, b (\geq 0), d (> b), B$  and  $C (\neq 0)$  are real constants and our interval  $(a \leq x < \infty)$  excluding  $a=0$ .

Now (4.2) is given by

$$\frac{q}{r^{2/3} s^{1/3}} = B C^{-2/3} x^{\beta - \frac{2}{3}\gamma - \frac{1}{3}\alpha} \exp \left( \frac{2}{3}x^b - \frac{2}{3}x^d \right) = o(1)$$

as  $x \rightarrow \infty$ , because  $d > b$ .

Now

$$s' = \{\alpha + b x^b\} x^{\alpha-1} \exp x^b,$$

$$s'' = \{\alpha(\alpha-1) + b(2\alpha + b-1)x^b + b^2 x^{2b}\} x^{\alpha-2} \exp x^b.$$

Also

$$r' = \{\gamma + d x^d\} C x^{-1} \exp x^d,$$

$$r'' = \{\gamma(\gamma-1) + d(2\gamma + d-1) x^d + d^2 x^{2d}\} C x^{\gamma-2} \exp x^d.$$

Now

$$\frac{s'}{s} = x^{-1} \{\alpha + b x^b\}, \quad \frac{r'}{r} = x^{-1} \{\gamma + d x^d\}, \quad \left(\frac{s}{r}\right)^{1/3} = C^{-1/3} x^{1/3 (\alpha-\gamma)} x$$

$$x \exp\left(\frac{1}{3}x^b - \frac{1}{3}x^d\right).$$

Hence

$$\frac{r'}{r} \left(\frac{s}{r}\right)^{1/3} = C^{-1/3} x^{1/3 (\alpha-\gamma)-1} \exp\left(\frac{1}{3}x^b - \frac{1}{3}x^d\right) \{\gamma + d x^d\}.$$

Thus  $\frac{r'}{r} \left(\frac{s}{r}\right)^{1/3}$  tends to zero as  $x \rightarrow \infty$  i.e. (4.3) holds.

Also

$$\frac{s'}{s} \left(\frac{s}{r}\right)^{1/3} = C^{-1/3} x^{1/3 (\alpha-\gamma)-1} \exp\left(\frac{1}{3}x^b - \frac{1}{3}x^d\right) \{\alpha + b x^b\} = o(1)$$

as  $x \rightarrow \infty$ . Thus (4.4) holds.

Also

$$\frac{q'}{r^{2/3} s^{1/3}} = C^{-2/3} B\{\beta + b x^b\} x^{-1/3 (2\gamma+\alpha)+\beta-1} \exp\left(\frac{2}{3}x^b - \frac{2}{3}x^d\right).$$

Thus  $\frac{q'}{r^{2/3} s^{1/3}}$  is  $L(a, \infty)$ .

Also

$$\frac{s''}{s} \left( \frac{s}{r} \right)^{1/3} = C^{-1/3} \left\{ \alpha(\alpha-1) + b(2\alpha + b-1)x^b + b^2 x^{2b} \right\} x^{1/3 (\alpha-\gamma)-2} \exp\left(\frac{1}{3}x^b - \frac{1}{3}x^d\right)$$

Thus  $\frac{s''}{s} \left( \frac{s}{r} \right)^{1/3}$  is  $L(a, \infty)$ .

Also

$$\frac{r''}{r} \left( \frac{s}{r} \right)^{1/3} = C^{-1/3} x^{1/3 (\alpha-\gamma)-2} \left\{ \gamma(\gamma-1) + d(2\gamma+d-1)x^d + d^2 x^{2d} \right\} \times \\ \times \exp\left(\frac{1}{3}x^b - \frac{1}{3}x^d\right).$$

Thus  $\frac{r''}{r} \left( \frac{s}{r} \right)^{1/3}$  is  $L(a, \infty)$ . Similarly  $\frac{r'^2}{r^2} \left( \frac{s}{r} \right)^{1/3}$  and  $\frac{s'^2}{s^2} \left( \frac{s}{r} \right)^{1/3}$  are all  $L(a, \infty)$ . Also

$$\frac{q^2}{sr} = B^2 C^{-1} x^{2\beta-\alpha-\gamma} \exp(x^b - x^d)$$

which is  $L(a, \infty)$ . We note case I holds for this example. Since by previous work the conditions of Theorem 4.1, are all satisfied, hence (4.1) holds with  $s, r$  and  $q$  as given in example (2).

Discussion:

Now if  $s=1$  in (4.1), then for the particular equation (2.10), the asymptotic formula (4.1) agrees with Pfeifer<sup>[8]</sup> result which is given by (2.11), of course Pfeifer's result obtained under conditions (a) and (b) as given in §2, it's obvious that condition (a) holds by our hypothesis in Theorem 4.1, and as mentioned in condition (a) we have  $r$  and  $q$  as real-valued functions which is related with case I of Theorem 4.1, so

condition (4.5) will be reduced to Pfeifer condition (b) in §2, of course here we have  $\frac{r'^2}{r^{2/3}}$  is  $L(a, \infty)$  also, but we can omit this condition under case I by using the same theorem which is used by Pfeifer for this condition this theorem is given in [8, Theorem 3] and so subject to this theorem if  $r$  nowhere zero in  $[a, \infty)$  and  $\frac{r''}{r^{4/3}}$  is  $L(a, \infty)$ , then  $\frac{r'^2}{r^{2/3}}$  is  $L(a, \infty)$ , hence (4.5) does reduce to condition (b) in this case, and so the only difference between our conditions and Pfeifer [8] conditions is that we have extra conditions (4.2) and (4.3) with  $s=1$ , i.e.  $\frac{q}{r^{2/3}} \rightarrow 0$  as  $x \rightarrow \infty$  and  $\frac{r'}{r^{4/3}} \rightarrow 0$  as  $x \rightarrow \infty$ .

Now conditions (4.2) - (4.5) do however reduce to (2.13) when the coefficients are (2.12).

Also we mention that in Pfeifer work for the differential equation (2.10), he defines the matrix  $T$  to be

$$T = \text{dg} \left( r^{1/3}, 1, r^{-1/3} \right) \quad (7.14)$$

but in ~~our~~ case we define  $T$  as given in (3.13). Also in our method we avoid changing the independent variable and this allow us to consider a complex-valued functions as given in case II and III of Theorem 4.1, where Pfeifer [8] did change the independent variable, and this is the reason he considered only real-valued functions.

## 8. The second type of the third-order differential equation

In this section until §16 we present part (b) of our thesis by considering a different class of 3-rd order differential equation than (2.1). We are concerned with problem of developing an asymptotic theory for the differential equation

$$\{s^{\frac{1}{2}}(s^{\frac{1}{2}}y')'\}' - (py')' - ry = 0 \quad (a \leq x < \infty) \quad (8.1)$$

as  $x \rightarrow \infty$ , where  $x$  is the independent variable and the prime denotes  $\frac{d}{dx}$ . The coefficients  $s, p$  and  $r$  are defined on an interval  $[a, \infty)$  and are all nowhere zero there. Under part (b) we have case (1) and case (2) of the differential equation (8.1) which are mentioned in §1. Case (1) will be given in Theorem 9.1, §9 and case (2) will be given in Theorem 13.1, §13. The proof of each case is given in subsequent sections. Discussion of other two cases of the differential equation (8.1) are given at the end of §12 and at the end of §16. Now our aim is to determine the three linearly independent solutions of (8.1) as  $x \rightarrow \infty$ , under case (1) and case (2). First we carry out the same method which was developed for the differential equation (2.1) in §2, and the steps of the method from (2.2) until (2.9) will be the same here for (8.1). Now (8.1) can be written as a first-order system (2.2) as follows.

We take

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & s^{-\frac{1}{2}} & 0 \\ 0 & ps^{-1} & s^{-\frac{1}{2}} \\ r & 0 & 0 \end{pmatrix}. \quad (8.2)$$

Then (2.2) gives

$$y'_1 = s^{-\frac{1}{2}} y_2$$

$$y'_2 = ps^{-1} y_2 + s^{-\frac{1}{2}} y_3$$

$$y'_3 = ry_1.$$

Define

$$y_1 = y$$

hence

$$y_2 = s^{\frac{1}{2}} y'$$

and

$$y_3 = s^{\frac{1}{2}} (s^{\frac{1}{2}} y')' - py'$$

i.e.

$$\{s^{\frac{1}{2}} (s^{\frac{1}{2}} y')'\}' - (py')' - ry = 0.$$

Hence (2.2) reduced to (8.1) if we take A as in (8.2) and

$$Y(x) = \begin{pmatrix} y \\ s^{\frac{1}{2}} y' \\ s^{\frac{1}{2}} (s^{\frac{1}{2}} y')' - py' \end{pmatrix}. \quad (8.3)$$

Now the first step in the transformation to (2.3) is to diagonalize the matrix A; hence we require the eigenvalues  $\lambda_j$  ( $1 \leq j \leq 3$ ) and the eigenvectors  $\mathbf{v}_j$  ( $1 \leq j \leq 3$ ).

The characteristic equation of A is given by

$$\det(\lambda_j I_3 - A) = \begin{vmatrix} \lambda_j & -s^{-\frac{1}{2}} & 0 \\ 0 & \lambda_j - ps^{-1} & -s^{-\frac{1}{2}} \\ -r & 0 & \lambda_j \end{vmatrix} = 0$$

i.e.

$$s\lambda_j^3 - p\lambda_j^2 - r = 0 \quad (1 \leq j \leq 3). \quad (8.4)$$

Now the eigenvector  $\mathbf{v}_j$  of the matrix A corresponding  $\lambda_j$  is given by

$$A\mathbf{v}_j = \lambda_j \mathbf{v}_j. \quad (8.5)$$

Taking

$$\mathbf{v}_j = \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

then (8.5) gives

$$\begin{pmatrix} 0 & s^{-\frac{1}{2}} & 0 \\ 0 & ps^{-1} & s^{-\frac{1}{2}} \\ r & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \lambda_j \begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad (8.6)$$

Define

$$u = 1, \quad (8.7)$$

then (8.6) gives

$$v = s^{\frac{1}{2}} \lambda_j, \quad (8.8)$$

$$w = r \lambda_j^{-1}. \quad (8.9)$$

Hence by (8.7), (8.8) and (8.9) we can define the eigenvector  $v_j$  of the matrix A corresponding  $\lambda_j$  which satisfy the relationship (8.5) by

$$v_j = \begin{pmatrix} 1 \\ s^{\frac{1}{2}} \lambda_j \\ r \lambda_j^{-1} \end{pmatrix} \quad (1 \leq j \leq 3). \quad (8.10)$$

Hence we define the suitable matrix T which satisfy (2.6) by

$$T = (m_1^{-1} v_1 \quad m_2^{-1} v_2 \quad m_3^{-1} v_3) \quad (8.11)$$

where  $m_j$  ( $1 \leq j \leq 3$ ) are scalar factors to be specified in a moment.

Hence the transformation (2.7) will take (2.2) into the system (2.8).

Now we note that the matrix A which is given by (8.2) satisfy the relation

$$(EA)^t = EA, \quad (8.12)$$

where the superscript  $t$  denotes the transpose and the  $3 \times 3$  matrix  $E$  is defined by the elements  $e_{jk}$  are all zero except when  $j+k=4$  ( $1 \leq j \leq 3$ ) and these elements are unity.

i.e.

$$E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (8.13)$$

Now subject to [16 p. 260] we have

$$(Ev_k)^t v_j = 0 \quad (k \neq j) \quad (8.14)$$

where the superscript  $t$  denotes the transpose.

Now we define the scalar  $m_j$  ( $1 \leq j \leq 3$ ) as the products

$$m_j = (Ev_j)^t v_j \quad (1 \leq j \leq 3). \quad (8.15)$$

Let

$$r_j = (Ev_j)^t \quad (1 \leq j \leq 3). \quad (8.16)$$

Then we have by (8.11), (8.14) and (8.15)

$$T^{-1} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}. \quad (8.17)$$

To express  $m_j$  ( $1 \leq j \leq 3$ ) in terms of our original coefficients we consider (8.15). Hence by (8.10) and (8.13)

$$E\mathbf{v}_j = \begin{pmatrix} r \lambda_j^{-1} \\ \lambda_j s^{\frac{1}{2}} \\ 1 \end{pmatrix},$$

hence

$$(E\mathbf{v}_j)^t = (r\lambda_j^{-1} \quad \lambda_j s^{\frac{1}{2}} \quad 1). \quad (8.18)$$

Hence by (8.18) and (8.10), (8.15) gives

$$m_j = (r\lambda_j^{-1} \quad \lambda_j s^{\frac{1}{2}} \quad 1) \begin{pmatrix} 1 \\ \lambda_j s^{\frac{1}{2}} \\ r\lambda_j^{-1} \end{pmatrix}$$

i.e.

$$m_j = 2r\lambda_j^{-1} + s\lambda_j^2 \quad (1 \leq j \leq 3). \quad (8.19)$$

Now by (8.4)

$$2r\lambda_j^{-1} = 2s\lambda_j^2 - 2p\lambda_j,$$

hence (8.19) gives

$$m_j = 3s\lambda_j^2 - 2p\lambda_j \quad (1 \leq j \leq 3). \quad (8.20)$$

Now by (8.16) and (8.18), the row vector  $r_j$  is given by

$$r_j = (r\lambda_j^{-1} \quad \lambda_j s^{\frac{1}{2}} \quad 1). \quad (8.21)$$

Now we need to work out the matrix  $T^{-1}T'$ . Writing

$$T^{-1}T' = (t_{jk}) \quad (1 \leq j, k \leq 3). \quad (8.22)$$

Hence by (8.17) and (8.11)

$$t_{jk} = r_j (m_k^{-1} v_k)', \quad (1 \leq j, k \leq 3). \quad (8.23)$$

First we shall consider the diagonal elements in (8.23) i.e. when  $j=k$ , then (8.23) gives

$$t_{jj} = r_j (m_j^{-1} v_j)', \quad (1 \leq j \leq 3). \quad (8.24)$$

Now by (8.15) and (8.16)

$$m_j = r_j v_j \quad (1 \leq j \leq 3). \quad (8.25)$$

Now consider

$$(Em_j^{-1} v_j)^t (m_j^{-1} v_j) = m_j^{-2} r_j v_j \quad (1 \leq j \leq 3). \quad (8.26)$$

Hence by (8.25), (8.26) gives

$$(Em_j^{-1} v_j)^t (m_j^{-1} v_j) = m_j^{-1} \quad (1 \leq j \leq 3). \quad (8.27)$$

Differentiating (8.27), then

$$(Em_j^{-1} v_j)^t (m_j^{-1} v_j)' + [(Em_j^{-1} v_j)^t]' (m_j^{-1} v_j) = - \frac{m_j'}{m_j^2}. \quad (1 \leq j \leq 3). \quad (8.28)$$

Now by (8.18)

$$(Em_j^{-1} v_j)^t = (m_j^{-1} r \lambda_j^{-1} m_j^{-1} \lambda_j s^{\frac{1}{2}} m_j^{-1}). \quad (1 \leq j \leq 3) \quad (8.29)$$

Also by (8.10)

$$m_j^{-1} v_j = \begin{pmatrix} m_j^{-1} \\ m_j^{-1} \lambda_j s^{\frac{1}{2}} \\ m_j^{-1} r \lambda_j^{-1} \end{pmatrix} \quad (1 \leq j \leq 3). \quad (8.30)$$

Hence by (8.29) and (8.30)

$$\begin{aligned} [(Em_j^{-1} v_j)^t]' (m_j^{-1} v_j) &= (m_j^{-1} r \lambda_j^{-1})' m_j^{-1} \\ &\quad + (m_j^{-1} \lambda_j s^{\frac{1}{2}})' (m_j^{-1} \lambda_j s^{\frac{1}{2}}) \\ &\quad + (m_j^{-1})' (m_j^{-1} r \lambda_j) \\ &= (m_j^{-1} r \lambda_j \quad m_j^{-1} \lambda_j s^{\frac{1}{2}} \quad m_j^{-1}) \begin{pmatrix} (m_j^{-1})' \\ (m_j^{-1} \lambda_j s^{\frac{1}{2}})' \\ (m_j^{-1} r \lambda_j^{-1})' \end{pmatrix} \end{aligned}$$

i.e.

$$[(Em_j^{-1} v_j)^t]' (m_j^{-1} v_j) = (Em_j^{-1} v_j)^t (m_j^{-1} v_j)', \quad (1 \leq j \leq 3). \quad (8.31)$$

Hence by (8.31), (8.28) gives

$$2 (Em_j^{-1} v_j)^t (m_j^{-1} v_j)' = - \frac{m_j^{-1}}{m_j^2}$$

i.e.

$$r_j (m_j^{-1} v_j)' = - \frac{1}{2} \frac{m_j'}{m_j} \quad (1 \leq j \leq 3). \quad (8.32)$$

Hence by (8.32), (8.24) becomes

$$t_{jj} = -\frac{1}{2} \frac{m'_j}{m_j} \quad (1 \leq j \leq 3) \quad (8.33)$$

Now for the non-diagonal elements we use (8.23), hence (8.23) gives for  $k \neq j$

$$\begin{aligned} t_{jk} &= r_{jm_k}^{-1} v'_k - r_{jm_k}^{-2} m'_k v_k \\ m_k^2 t_{jk} &= r_{jm_k} v'_k - r_{jm_k} m'_k v_k \quad (j \neq k, 1 \leq j, k \leq 3). \end{aligned} \quad (8.34)$$

Now by (8.21) and (8.10)

$$r_j v'_k = (r \lambda_j^{-1} \quad \lambda_j s^{\frac{1}{2}} \quad 1) \begin{pmatrix} 0 \\ \lambda'_k s^{\frac{1}{2}} + \frac{\lambda_k s'}{2s^{\frac{1}{2}}} \\ (r \lambda_k^{-1})' \end{pmatrix}$$

i.e.

$$r_j v'_k = s \lambda_j \lambda'_k + \frac{1}{2} \lambda_k \lambda_j s' + (r \lambda_k^{-1})'. \quad (8.35)$$

Now by (8.4)

$$r = s \lambda_k^3 - p \lambda_k^2$$

Hence

$$r \lambda_k^{-1} = s \lambda_k^2 - p \lambda_k.$$

Differentiating, then

$$(r\lambda_k^{-1})' = s' \lambda_k^2 + 2s \lambda_k \lambda'_k - p' \lambda_k - p \lambda'_k. \quad (8.36)$$

Now by (8.20)

$$m_k = 3s \lambda_k^2 - 2p \lambda_k.$$

Hence

$$m'_k = 3s' \lambda_k^2 + 6s \lambda_k \lambda'_k - 2p' \lambda_k - 2p \lambda'_k$$

i.e.

$$\frac{m'_k}{2} = \frac{3s'}{2} \lambda_k^2 + 3s \lambda_k \lambda'_k - p' \lambda_k - p \lambda'_k,$$

then

$$-(p' \lambda_k + p \lambda'_k) = -\frac{3}{2} s' \lambda_k^2 - 3s \lambda_k \lambda'_k + \frac{m'_k}{2}. \quad (8.37)$$

Hence by (8.37), (8.36) gives

$$(r\lambda_k^{-1})' = \frac{m'_k}{2} + s' \lambda_k^2 + 2s \lambda_k \lambda'_k - \frac{3}{2} s' \lambda_k^2 - 3s \lambda_k \lambda'_k$$

i.e.

$$(r\lambda_k^{-1})' = \frac{m'_k}{2} - \frac{1}{2} s' \lambda_k^2 - s \lambda_k \lambda'_k. \quad (8.38)$$

Hence by (8.38), (8.35) gives

$$r_j v'_k = s \lambda_j \lambda'_k + \frac{1}{2} \lambda_k \lambda'_j s' + \frac{m'_k}{2} - \frac{1}{2} s' \lambda_k^2 - s \lambda_k \lambda'_k$$

$$r_j v'_k = \frac{m'_k}{2} + (\lambda_j - \lambda_k) (s \lambda'_k + \frac{1}{2} \lambda_k s').$$

Hence

$$r_j m_k v'_k = \frac{m'_k}{2} m_k + m_k (\lambda_j - \lambda_k) (s \lambda'_k + \frac{1}{2} \lambda_k s'). \quad (8.39)$$

Now by (8.21) and (8.10)

$$m'_k r_j v_k = m'_k (r \lambda_j^{-1} + s \lambda_j \lambda_k + r \lambda_k^{-1}). \quad (8.40)$$

Hence by (8.39) and (8.40), (8.34) gives

$$t_{jk} = \frac{m'_k}{2m_k} + \frac{(\lambda_j - \lambda_k)}{m_k} (s \lambda'_k + \frac{1}{2} \lambda_k s') - \frac{m'_k}{m_k^2} (r \lambda_j^{-1} + s \lambda_j \lambda_k + r \lambda_k^{-1}). \quad (8.41)$$

( $j \neq k, 1 \leq j, k \leq 3$ )

## 9. Asymptotic Theory Under Case (1)

In this section until §12 we discuss case (1) of the differential equation (8.1), this case will be given in the statement of Theorem 9.1, which will be stated below. The proof of Theorem 9.1 will be given through subsequent sections, two examples on Theorem 9.1 are given in §12, during these two examples we discuss two different class of the coefficients  $s, p$  and  $r$  and we shall show that Theorem 9.1 applicable in each example. Discussion of different case also given at the end of §12.

## Theorem 9.1

Let the coefficients  $s, p$  and  $r$  in (8.1) be  $C^2[a, \infty)$  and nowhere zero in  $[a, \infty)$ . Let

$$\left(\frac{r}{p}\right)^{\frac{1}{2}} = o\left(\frac{p}{s}\right) \quad \text{as } x \rightarrow \infty, \quad (9.1)$$

$$\frac{p'}{p} = o\left\{\left(\frac{r}{p}\right)^{\frac{1}{2}}\right\} \quad \text{as } x \rightarrow \infty, \quad (9.2)$$

$$\frac{r'}{r} = o\left\{\left(\frac{r}{p}\right)^{\frac{1}{2}}\right\} \quad \text{as } x \rightarrow \infty, \quad (9.3)$$

$$\frac{s'}{s} = o\left\{\left(\frac{r}{p}\right)^{\frac{1}{2}}\right\} \quad \text{ad } x \rightarrow \infty, \quad (9.4)$$

Let

$$\frac{p'^2}{p^2} \left(\frac{p}{r}\right)^{\frac{1}{2}}, \quad \frac{r'^2}{r^2} \left(\frac{p}{r}\right)^{\frac{1}{2}}, \quad \frac{s'^2}{s^2} \left(\frac{p}{r}\right)^{\frac{1}{2}}, \quad \frac{p''}{p} \left(\frac{p}{r}\right)^{\frac{1}{2}},$$

$$\frac{r''}{r} \left(\frac{p}{r}\right)^{\frac{1}{2}}, \quad \frac{s''}{s} \left(\frac{p}{r}\right)^{\frac{1}{2}} \quad \text{and} \quad \frac{s^2 r^{3/2}}{p^{7/2}} \quad \text{be } L(a, \infty). \quad (9.5)$$

Finally let one of the four following cases A,B,C and D hold.

(A)  $r, p$  and  $s$  are real-valued functions with  $\frac{r}{p} > 0$ .

(B)  $r, p$  and  $s$  are real-valued functions with  $\frac{r}{p} < 0$ .

(C)  $s = i S$ , where  $S$  real-valued function and  $p, r$  are real-valued functions with  $\frac{r}{p} > 0$  and

$$\left\{ |p|^{\frac{3}{4}} |S|^{\frac{1}{2}} |r|^{-\frac{1}{4}} \right\},$$

does not change sign in  $[a, \infty)$ .

(D)  $s = i S$ , where  $S$  real-valued function and  $p, r$  are real-valued functions with  $\frac{r}{p} < 0$ .

Then (8.1) has three linearly independent solutions  $y_k(x)$  ( $1 \leq k \leq 3$ ) such that

$$y_k(x) \sim \{p(x)r(x)\}^{-1/4} \exp\left\{\int_a^x \left[\pm i(r/p)^{1/2}(t) - \frac{1}{2} \frac{sr}{p^2}(t)\right] dt\right\}$$

as  $x \rightarrow \infty$  ( $k=1,2$ ) (9.6)

and

$$y_3(x) = o\left\{\left[\frac{p(x)}{s(x)r(x)}\right]^{1/2} \exp\left[\int_a^x \frac{p}{s}(t) dt + \int_a^x \frac{sr}{p^2}(t) dt\right]\right\} \text{ as } x \rightarrow \infty.$$

(9.7)

We note in the above theorem we require no more differentiability on  $s, p$  and  $r$  than that they are  $C^2 [a, \infty)$ , also we note from conditions (9.1), (9.2), (9.3) and (9.4) that the function  $\frac{p}{s}$  dominate the functions  $\left(\frac{r}{p}\right)^{1/2}$ ,  $\frac{p'}{p}$ ,  $\frac{r'}{r}$  and  $\frac{s'}{s}$  as  $x \rightarrow \infty$ .

Also our theorem covered two cases where  $s$  is pure imaginary as given in cases (C) and (D).

In proof of Theorem 9.1, we require two remarks.

Remark 9.1

Since by (9.5)  $\frac{p'^2(p)}{p^2(r)}^{\frac{1}{2}}$ ,  $\frac{r'^2(p)}{r^2(r)}^{\frac{1}{2}}$  and  $\frac{s'^2(p)}{s^2(r)}^{\frac{1}{2}}$  are  $L(a, \infty)$ , hence  $\frac{p'r'}{p r} \left(\frac{p}{r}\right)^{\frac{1}{2}}$ ,  $\frac{p's'}{p s} \left(\frac{p}{r}\right)^{\frac{1}{2}}$  and  $\frac{s'r'}{s r} \left(\frac{p}{r}\right)^{\frac{1}{2}}$  are  $L(a, \infty)$ .

To show this we use Schwarz inequality (4.6), then  $\frac{p'r'}{p r} \left(\frac{p}{r}\right)^{\frac{1}{2}}$  is  $L(a, \infty)$ , by taking  $f = \frac{p'}{p} \left(\frac{p}{r}\right)^{\frac{1}{4}}$  and  $g = \frac{r'}{r} \left(\frac{p}{r}\right)^{\frac{1}{4}}$ . Similarly

$\frac{p's'}{p s} \left(\frac{p}{r}\right)^{\frac{1}{2}}$  and  $\frac{s'r'}{s r} \left(\frac{p}{r}\right)^{\frac{1}{2}}$  are both  $L(a, \infty)$ .

Remark 9.2

Since by (9.5)  $\frac{p'^2(p)}{p^2(r)}^{\frac{1}{2}}$ ,  $\frac{r'^2(p)}{r^2(r)}^{\frac{1}{2}}$ ,  $\frac{s'^2(p)}{s^2(r)}^{\frac{1}{2}}$  and  $\frac{s^2 r^{3/2}}{p^{7/2}}$  are  $L(a, \infty)$ , hence  $\frac{p'}{p} \delta$ ,  $\frac{r'}{r} \delta$  and  $\frac{s'}{s} \delta$  are  $L(a, \infty)$ , where  $\delta$  as given below in (9.14).

To show this again we make use of Schwarz inequality (4.6), then  $\frac{p'}{p} \delta$  is  $L(a, \infty)$ , by taking  $f = \frac{p'}{p} \left(\frac{p}{r}\right)^{\frac{1}{4}}$  and  $g = \frac{s r^{3/4}}{p^{7/4}}$ . Similarly  $\frac{r'}{r} \delta$  and  $\frac{s'}{s} \delta$  are both  $L(a, \infty)$ .

Proof of Theorem 9.1.

The first step in the proof is to solve (8.4) asymptotically as  $x \rightarrow \infty$  and to make a detailed analysis of  $T^{-1}T'$ .

Since  $s, p$  and  $r$  are nowhere zero in  $[a, \infty)$  and (9.1) holds, hence (8.4) has distinct roots  $\lambda_j(x)$  ( $1 \leq j \leq 3$ ) such that

$$\lambda_1 = i (r/p)^{\frac{1}{2}} (1 + \delta_1), \quad (9.8)$$

$$\lambda_2 = -i(r/p)^{\frac{1}{2}} (1 + \delta_2), \quad (9.9)$$

and

$$\lambda_3 = \frac{p}{s}(1 + \delta_3) \quad (9.10)$$

where

$$\delta_j(x) = o(1). \quad (1 \leq j \leq 3) \quad \text{as } x \rightarrow \infty \quad (9.11)$$

Hence subject to (9.1)

$$\lambda_j = o(\lambda_3) \quad \text{as } x \rightarrow \infty \quad (j=1,2), \quad (9.12)$$

i.e.  $\lambda_3$  dominate  $\lambda_j$  ( $j=1,2$ ) as  $x \rightarrow \infty$ . Now we need to show that (9.1) holds, and then to work out  $\delta_j(x)$  ( $1 \leq j \leq 3$ ). Starting with  $\delta_1$  we use (9.8) and (8.4), we substitute from (9.8) into (8.4) for  $\lambda_1$ , then

$$-i \frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}} (1 + \delta_1)^3 + (1 + \delta_1)^2 - 1 = 0. \quad (9.13)$$

Let

$$\delta = \frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}}, \quad (9.14)$$

then by (9.1)

$$\delta = o(1). \quad \text{as } x \rightarrow \infty \quad (9.15)$$

Now equation (9.13) is of the form  $f(x, \delta_1) = 0$ , where  $f(\infty, 0) = 0$  and  $f_{\delta_1}(\infty, 0) \neq 0$ ; ; hence the standard implicit function theorem [7 p.64] guarantees the existence of a unique solution  $\delta_1 = \delta_1(x)$  to this equation such that  $\delta_1(\infty) = 0$ , thus (9.11) holds for  $j=1$ .

Next we consider  $\delta_2$ , then by (9.9) and (8.4)

$$i \frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}} (1 + \delta_2)^3 + (1 + \delta_2)^2 - 1 = 0. \quad (9.16)$$

Again equation (9.16) is of the form  $f(x, \delta_2) = 0$ , where  $f(\infty, 0) = 0$  and  $f_{\delta_1}(\infty, 0) \neq 0$ ; hence the standard implicit function theorem [7 p.64] guarantees the existence of a unique solution  $\delta_2 = \delta_2(x)$  to this equation such that  $\delta_2(\infty) = 0$ , thus (9.11) holds  $j=2$ . Finally we consider  $\delta_3$ , hence by (9.10) and (8.4).

$$(1 + \delta_3)^3 - (1 + \delta_3)^2 - \frac{s^2 r}{p^3} = 0. \quad (9.17)$$

Again equation (9.17) is of the form  $f(x, \delta_3) = 0$ , where  $f(\infty, 0) = 0$  and  $f_{\delta_3}(\infty, 0) \neq 0$ ; hence the standard implicit function theorem [7 p.64] guarantees the existence of a unique solution  $\delta_3 = \delta_3(x)$  to this equation such that  $\delta_3(\infty) = 0$ , thus (9.11) holds for  $j=3$ . Hence by previous work (9.11) holds for  $(1 \leq j \leq 3)$ .

Now to work out  $\delta_j$  ( $1 \leq j \leq 3$ ), we start with  $\delta_1$ , then by (9.13)

$$-i \frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}} \left\{ 1 + 3\delta_1 + O(\delta_1^2) \right\} + 2\delta_1 + O(\delta_1^2) = 0 \quad (9.18).$$

Now we solve (9.18) in term of the dominant term  $\delta_1$ , then (9.18) gives

$$\delta_1 = \frac{i}{2} \frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}} \left( 1 - \frac{3}{2} i \frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}} \right)^{-1}. \quad (9.19)$$

Hence by (9.14), (9.19) gives

$$\delta_1 = \frac{i}{2} \delta \left( 1 - \frac{3}{2} i \delta \right)^{-1}. \quad (9.20)$$

Similarly by (9.16) and (9.14)

$$\delta_2 = -\frac{i}{2} \delta \left( 1 + \frac{3}{2} i \delta \right)^{-1}. \quad (9.21)$$

Finally we consider  $\delta_3$ , then by (9.17)

$$\delta_3 + O(\delta_3^2) - \frac{s^2 r}{p^3} = 0 \quad (9.22)$$

and solving (9.22) in term of the dominant term  $\delta_3$ , then (9.22) gives

$$\delta_3 = \frac{s^2 r}{p^3} . \quad (9.23)$$

Hence by (9.14), (9.23) gives

$$\delta_3 = \delta^2 . \quad (9.24)$$

Now by previous work, the diagonal matrix  $\Lambda(x)$  in (2.6) is given by (9.8), (9.9), (9.10), (9.20), (9.21) and (9.24). We note by (9.8), (9.9), (9.10) and (9.11)

$$\lambda_1 \sim i(r/p)^{\frac{1}{2}}, \quad \text{as } x \rightarrow \infty \quad (9.25)$$

$$\lambda_2 \sim -i(r/p)^{\frac{1}{2}}, \quad \text{as } x \rightarrow \infty \quad (9.26)$$

$$\lambda_3 \sim \frac{p}{s} . \quad \text{as } x \rightarrow \infty \quad (9.27)$$

We mention here that since (9.25), (9.26), (9.27) and (9.1) holds, hence the case which is considered for  $\lambda_j$  ( $1 \leq j \leq 3$ ) is given by

$$\lambda_j = o(\lambda_3) \quad \text{as } x \rightarrow \infty \quad (j=1,2)$$

where  $\lambda_j$  ( $j=1,2$ ) having same size as  $x \rightarrow \infty$  . i.e.

$$\lambda_j \sim \pm i\left(\frac{r}{p}\right)^{\frac{1}{2}} \quad \text{as } x \rightarrow \infty \quad (j=1,2). \quad (9.28)$$

Now before working  $T^{-1}T'$  in some detail, we need to find  $\lambda'_k$  ( $k=1,2,3$ ).

Starting with  $\lambda'_1$ , and so differentiating (9.8) we obtain

$$\lambda'_1 = \frac{1}{2} i (r/p)^{\frac{1}{2}} \left[ \frac{r'}{r} - \frac{p'}{p} \right] (1 + \delta_1) + i (r/p)^{\frac{1}{2}} \delta'_1. \quad (9.29)$$

Similarly differentiating (9.9), then

$$\lambda'_2 = -\frac{1}{2} i (r/p)^{\frac{1}{2}} \left[ \frac{r'}{r} - \frac{p'}{p} \right] (1 + \delta_2) - i (r/p)^{\frac{1}{2}} \delta'_2. \quad (9.30)$$

Also differentiating (9.10), then

$$\lambda'_3 = \frac{p}{s} \left[ \frac{p'}{p} - \frac{s'}{s} \right] (1 + \delta_3) + \frac{p}{s} \delta'_3. \quad (9.31)$$

Now we need to work out  $\delta'_k(x)$  ( $1 \leq k \leq 3$ ) in some detail in terms of  $s, r$  and  $p$ , hence starting with  $\delta'_1$ , and so differentiating (9.13), we obtain

$$\begin{aligned} -i \left[ \frac{s'}{s} + \frac{1}{2} \frac{r'}{r} - \frac{3}{2} \frac{p'}{p} \right] \frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}} (1 + \delta_1)^3 - 3i \frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}} (1 + \delta_1)^2 \delta'_1 \\ + 2(1 + \delta_1) \delta'_1 = 0 \\ \delta'_1 = \frac{i}{2} \left[ \frac{s'}{s} + \frac{1}{2} \frac{r'}{r} - \frac{3}{2} \frac{p'}{p} \right] \frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}} (1 + \delta_1)^2 \left\{ 1 - \frac{3}{2} i \frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}} (1 + \delta_1) \right\}^{-1} \end{aligned}$$

i.e.

$$\delta'_1 = O(\delta \frac{p'}{p}) + O(\delta \frac{r'}{r}) + O(\delta \frac{s'}{s}), \quad (9.32)$$

and we make use of (9.14).

Similar work can be done for  $\delta'_2$ , then

$$\delta'_2 = O(\delta \frac{p'}{p}) + O(\delta \frac{r'}{r}) + O(\delta \frac{s'}{s}) \quad (9.33)$$

Finally we need to work out  $\delta'_3$ , hence differentiating (9.17), we obtain

$$3(1 + \delta_3)^2 \delta'_3 - 2(1 + \delta_3) \delta'_3 + \frac{s^2 r}{p^3} \left[ 3 \frac{p'}{p} - \frac{r'}{r} - 2 \frac{s'}{s} \right] = 0$$

$$\delta'_2 = - \frac{s^2 r}{p^3} \left[ 3 \frac{p'}{p} - \frac{r'}{r} - 2 \frac{s'}{s} \right] \left\{ 1 + 4\delta_3 + 3\delta_3^2 \right\}^{-1}$$

i.e.

$$\delta'_3 = O(\delta^2 \frac{p'}{p}) + O(\delta^2 \frac{r'}{r}) + O(\delta^2 \frac{s'}{s}) \quad (9.34)$$

and we make use of (9.14)

We note that by remark 9.2

$$\delta'_k(x) \text{ is } L(a, \infty) \quad (1 \leq k \leq 3) \quad (9.35)$$

We have now to work out (8.33) and (9.38) in some detail in terms of  $s, r$  and  $p$  in order to determine the form of (2.8) and then make progress towards (8.1).

10. The Matrix  $T^{-1}T'$ 

Considering first  $j=1$  in (8.33), then

$$t_{11} = -\frac{1}{2} \frac{m'_1}{m_1} \quad (10.1)$$

Now by (8.20)

$$m_1 = 3 s \lambda_1^2 - 2 \bar{p} \lambda_1 \quad (10.2)$$

Hence by (9.8), (10.2) gives

$$m_1 = -3 \frac{s r^{\frac{1}{2}}}{p^{\frac{3}{2}}} (\bar{p} r)^{\frac{1}{2}} (1 + \delta_1)^2 - 2i (\bar{p} r)^{\frac{1}{2}} (1 + \delta_1)$$

since

$$\delta = \frac{s r^{\frac{1}{2}}}{p^{\frac{3}{2}}}$$

hence

$$m_1 = -2i (\bar{p} r)^{\frac{1}{2}} \left\{ 1 + \delta_1 + \frac{3}{2i} \delta (1 + \delta_1)^2 \right\}$$

Hence by (9.20)

$$m_1 = -2i (\bar{p} r)^{\frac{1}{2}} \left\{ 1 + \frac{i}{2} \delta (1 - \frac{3}{2} i \delta)^{-1} + \frac{3}{2i} \delta \left[ 1 + \frac{i}{2} \delta (1 - \frac{3}{2} i \delta)^{-1} \right]^2 \right\}$$

i.e.,

$$m_1 = -2i (\bar{p} r)^{\frac{1}{2}} \{ 1 - i \delta + O(\delta^2) \} \quad (10.3)$$

and so

$$m_1 \sim -2i (\bar{p} r)^{\frac{1}{2}} \quad \text{as } x \rightarrow \infty \quad (10.4)$$

Now by (10.2)

$$m'_1 = \lambda'_1 \{ 6s \lambda_1 - 2p \} + \lambda_1 \{ 3s' \lambda'_1 - 2p' \} \quad (10.5)$$

Now by (9.8), (9.14) and (9.20)

$$6s\lambda_1 - 2p = -2p \{1 - 3i\delta + O(\delta^2)\}. \quad (10.6)$$

Also by (9.8) and (9.20)

$$\lambda_1 (3s'\lambda_1 - 2p') = -3 \frac{sr}{p} \left\{ 1 + i\delta + O(\delta^2) \right\} - 2i \frac{p'}{p} (pr)^{\frac{1}{2}} \left\{ 1 + \frac{1}{2}\delta + O(\delta^2) \right\} \quad (10.7)$$

Now by (9.29) and (9.20)

$$\lambda'_1 = \frac{1}{2}i (r/p)^{\frac{1}{2}} \left[ \frac{r'}{r} - \frac{p'}{p} \right] \left\{ 1 + \frac{1}{2}\delta + O(\delta^2) \right\} + i(r/p)^{\frac{1}{2}} \delta'_1 \quad (10.8)$$

where  $\delta'_1$  is given by (9.32). Hence by (10.6) and (10.8)

$$\begin{aligned} \lambda'_1 (6s\lambda_1 - 2p) &= -i(pr)^{\frac{1}{2}} \left[ \frac{r'}{r} - \frac{p'}{p} \right] \left\{ 1 - \frac{5}{2}i\delta + O(\delta^2) \right\} \\ &\quad - 2i(pr)^{\frac{1}{2}} \delta'_1 \left\{ 1 - 3i\delta + O(\delta^2) \right\}. \end{aligned} \quad (10.9)$$

Hence by (10.7) and (10.9), (10.5) becomes

$$\begin{aligned} m'_1 &= -3 \frac{sr}{p} \left\{ 1 + i\delta + O(\delta^2) \right\} - 2i \frac{p'}{p} (pr)^{\frac{1}{2}} \left\{ 1 + \frac{1}{2}\delta + O(\delta^2) \right\} \\ &\quad - i(pr)^{\frac{1}{2}} \left[ \frac{r'}{r} - \frac{p'}{p} \right] \left\{ 1 - \frac{5}{2}i\delta + O(\delta^2) \right\} \\ &\quad - 2i(pr)^{\frac{1}{2}} \delta'_1 \left\{ 1 - 3i\delta + O(\delta^2) \right\}. \end{aligned}$$

i.e.

$$\begin{aligned} m'_1 &= -3 \frac{sr}{p} \left\{ 1 + i\delta + O(\delta^2) \right\} - i \frac{p'}{p} (pr)^{\frac{1}{2}} \left\{ 1 + \frac{7}{2}i\delta + O(\delta^2) \right\} \\ &\quad - i(pr)^{\frac{1}{2}} \frac{r'}{r} \left\{ 1 - \frac{5}{2}i\delta + O(\delta^2) \right\} - 2i(pr)^{\frac{1}{2}} \delta'_1 \left\{ 1 - 3i\delta + O(\delta^2) \right\}. \end{aligned} \quad (10.10)$$

Hence by (10.3), (10.10) and (9.14)

$$\begin{aligned} \frac{m'_1}{m_1} &= \frac{3}{2i} \frac{s'}{s} \delta \left\{ 1 + 2i\delta + O(\delta^2) \right\} + \frac{1}{2} \frac{p'}{p} \left\{ 1 + \frac{9}{2}i\delta + O(\delta^2) \right\} \\ &+ \frac{1}{2} \frac{r'}{r} \left\{ 1 - \frac{3}{2}i\delta + O(\delta^2) \right\} + \delta'_1 \left\{ 1 - 2i\delta + O(\delta^2) \right\}. \end{aligned} \quad (10.11)$$

Hence by (10.11) and (9.32), (10.1) gives

$$t_{11} = -\frac{1}{4} \left[ \frac{r'}{r} + \frac{p'}{p} \right] + O(\delta \frac{p'}{p}) + O(\delta \frac{r'}{r}) + O(\delta \frac{s'}{s}). \quad (10.12)$$

Next we consider  $j=2$  in (8.33), then

$$t_{22} = -\frac{1}{2} \frac{m'_2}{m_2}. \quad (10.13)$$

Now by (8.20)

$$m_2 = 3s\lambda_2^2 - 2p\lambda_2. \quad (10.14)$$

Hence by (9.9) and (9.14), (10.14) gives

$$\begin{aligned} m_2 &= -3\delta (pr)^{\frac{1}{2}} (1 + \delta_2)^2 + 2i(pr)^{\frac{1}{2}} (1 + \delta_2) \\ m_2 &= 2i (pr)^{\frac{1}{2}} \left\{ 1 + \delta_2 - \frac{3}{2i}\delta (1 + \delta_2)^2 \right\}. \end{aligned}$$

Hence by (9.21)

$$\begin{aligned} m_2 &= 2i(pr)^{\frac{1}{2}} \left\{ 1 - \frac{i}{2}\delta (1 + \frac{3}{2}i\delta)^{-1} - \frac{3}{2i}\delta \left[ 1 - \frac{i}{2}\delta (1 + \frac{3}{2}i\delta)^{-1} \right]^2 \right\} \\ m_2 &= 2i(pr)^{\frac{1}{2}} \left\{ 1 + i\delta + O(\delta^2) \right\}. \end{aligned} \quad (10.15)$$

i.e.

$$m_2 \sim 2i (pr)^{\frac{1}{2}} \quad \text{as } x \rightarrow \infty \quad (10.16)$$

Now by (10.14)

$$m'_2 = \lambda'_2 \{6s\lambda_2 - 2p\} + \lambda_2 \{3s'\lambda_2 - 2p'\}. \quad (10.17)$$

Now by (9.9), (9.14) and (9.21)

$$6s\lambda_2 - 2p = -2p \left\{ 1 + 3i\delta + O(\delta^2) \right\}. \quad (10.18)$$

Also by (9.9) and (9.21)

$$\begin{aligned} \lambda_2 (3s'\lambda_2 - 2p') &= -3 \frac{s'r}{p} \left\{ 1 - i\delta + O(\delta^2) \right\} \\ &\quad + 2i \frac{p'}{p} (pr)^{\frac{1}{2}} \left\{ 1 - \frac{i}{2}\delta + O(\delta^2) \right\}. \end{aligned} \quad (10.19)$$

Now by (9.30) and (9.21)

$$\lambda_2' = -\frac{1}{2}i (r/p)^{\frac{1}{2}} \left[ \frac{r'}{r} - \frac{p'}{p} \right] \left\{ 1 - \frac{i}{2}\delta + O(\delta^2) \right\} - i(r/p)^{\frac{1}{2}} \delta_2' \quad (10.20)$$

where  $\delta_2'$  is given by (9.33).

Hence by (10.18) and (10.20)

$$\begin{aligned} \lambda_2' \{6s\lambda_2 - 2p\} &= i(rp)^{\frac{1}{2}} \left[ \frac{r'}{r} - \frac{p'}{p} \right] \left\{ 1 + \frac{5}{2}i\delta + O(\delta^2) \right\} \\ &\quad + 2i (rp)^{\frac{1}{2}} \delta_2' \left\{ 1 + 3i\delta + O(\delta^2) \right\}. \end{aligned} \quad (10.21)$$

Hence by (10.21) and (10.19), (10.17) gives

$$\begin{aligned} m_2' &= i(rp)^{\frac{1}{2}} \left[ \frac{r'}{r} - \frac{p'}{p} \right] \left\{ 1 + \frac{5}{2}i\delta + O(\delta^2) \right\} \\ &\quad + 2i (rp)^{\frac{1}{2}} \delta_2' \left\{ 1 + 3i\delta + O(\delta^2) \right\} - 3 \frac{s'r}{p} \left\{ 1 - i\delta + O(\delta^2) \right\} \\ &\quad + 2i \frac{p'}{p} (rp)^{\frac{1}{2}} \left\{ 1 - \frac{i}{2}\delta + O(\delta^2) \right\}. \end{aligned}$$

i.e.

$$\begin{aligned} m_2' &= i (rp)^{\frac{1}{2}} \frac{r'}{r} \left\{ 1 + \frac{5}{2}i\delta + O(\delta^2) \right\} + 2i (rp)^{\frac{1}{2}} \delta_2' \left\{ 1 + 3i\delta + O(\delta^2) \right\} \\ &\quad - 3 \frac{s'r}{p} \left\{ 1 - i\delta + O(\delta^2) \right\} + i \frac{p'}{p} (pr)^{\frac{1}{2}} \left\{ 1 + \frac{7}{2}i\delta + O(\delta^2) \right\}. \end{aligned}$$

(10.22)

Hence by (10.15), (10.22) and (9.14)

$$\begin{aligned} \frac{m'_2}{m_2} = & \frac{1}{2} \frac{r'}{r} \left\{ 1 + \frac{3}{2} i \delta + O(\delta^2) \right\} + \delta'_2 \left\{ 1 + 2i\delta + O(\delta^2) \right\} \\ & - \frac{3}{2i} \frac{s'}{s} \delta \left\{ 1 - 2i\delta + O(\delta^2) \right\} + \frac{1}{2} \frac{p'}{p} \left\{ 1 - \frac{9}{2} i \delta + O(\delta^2) \right\}. \end{aligned} \quad (10.23)$$

Hence by (10.23) and (9.33), (10.13) gives

$$t_{22} = -\frac{1}{4} \left[ \frac{r'}{r} + \frac{p'}{p} \right] + O(\delta \frac{p'}{p}) + O(\delta \frac{r'}{r}) + O(\delta \frac{s'}{s}). \quad (10.24)$$

Finally we consider  $j=3$  in (8.33), then

$$t_{33} = -\frac{1}{2} \frac{m'_3}{m_3}. \quad (10.25)$$

Now by (8.20)

$$m_3 = 3 s \lambda_3^2 - 2 p \lambda_3. \quad (10.26)$$

Hence by (9.10), (10.26) gives

$$m_3 = 3 \frac{p^2}{s} (1 + \delta_3)^2 - 2 \frac{p^2}{s} (1 + \delta_3)$$

and so by (9.24)

$$m_3 = \frac{p^2}{s} \left\{ 1 + 4\delta^2 + O(\delta^4) \right\} \quad (10.27)$$

i.e.

$$m_3 \sim \frac{p^2}{s}. \quad \text{as } x \rightarrow \infty \quad (10.28)$$

Now by (10.26)

$$m'_3 = \lambda'_3 \left\{ 6s\lambda_3 - 2p \right\} + \lambda_3 \left\{ 3s'\lambda_3 - 2p' \right\}. \quad (10.29)$$

Now by (9.10) and (9.24)

$$6s\lambda_3 - 2p = 4p \left\{ 1 + \frac{3}{2} \delta^2 \right\}. \quad (10.30)$$

Also by (9.10) and (9.24)

$$\lambda_3 (3s' \lambda_3 - 2p') = 3 \frac{s' p^2}{s^2} \left\{ 1 + 2\delta^2 + O(\delta^4) \right\} - 2 \frac{p' p}{s} \left\{ 1 + \delta^2 \right\}. \quad (10.31)$$

Now by (9.31) and (9.24)

$$\lambda'_3 = \frac{p}{s} \left[ \frac{p'}{p} - \frac{s'}{s} \right] (1 + \delta^2) + \frac{p}{s} \delta'_3 \quad (10.32)$$

where  $\delta'_3$  is given by (9.34). Hence by (10.30) and (10.32)

$$\lambda'_3 \left\{ 6s \lambda_3 - 2p \right\} = 4 \frac{p^2}{s} \left[ \frac{p'}{p} - \frac{s'}{s} \right] \left\{ 1 + \frac{5}{2} \delta^2 + O(\delta^4) \right\} + 4 \frac{p^2}{s} \delta'_3 \left\{ 1 + \frac{3}{2} \delta^2 \right\}. \quad (10.33)$$

Hence by (10.33) and (10.31)

$$\begin{aligned} m'_3 &= 4 \frac{p^2}{s} \left[ \frac{p'}{p} - \frac{s'}{s} \right] \left\{ 1 + \frac{5}{2} \delta^2 + O(\delta^4) \right\} + 4 \frac{p^2}{s} \delta'_3 \left\{ 1 + \frac{3}{2} \delta^2 \right\} \\ &\quad + 3 \frac{s' p^2}{s^2} \left\{ 1 + 2\delta^2 + O(\delta^4) \right\} - 2 \frac{p' p}{s} \left\{ 1 + \delta^2 \right\} \end{aligned}$$

i.e.

$$\begin{aligned} m'_3 &= 2 \frac{p p'}{s} \left\{ 1 + 4\delta^2 + O(\delta^4) \right\} - \frac{p^2 s'}{s^2} \left\{ 1 + 4\delta^2 + O(\delta^4) \right\} \\ &\quad + 4 \frac{p^2}{s} \delta'_3 \left\{ 1 + \frac{3}{2} \delta^2 \right\}. \end{aligned} \quad (10.34)$$

Hence by (10.27) and (10.34)

$$\frac{m'_3}{m_3} = 2 \frac{p'}{p} \left\{ 1 + O(\delta^4) \right\} - \frac{s'}{s} \left\{ 1 + O(\delta^4) \right\} + 4 \delta'_3 \left\{ 1 - \frac{5}{2} \delta^2 + O(\delta^4) \right\}. \quad (10.35)$$

Hence by (10.35) and (9.34), (10.25) gives

$$t_{33} = \frac{1}{2} \frac{s'}{s} - \frac{p'}{p} + O(\delta^2 \frac{p'}{p}) + O(\delta^2 \frac{r'}{r}) + O(\delta^2 \frac{s'}{s}). \quad (10.36)$$

Now for nondiagonal elements we consider (8.41), hence putting  $j=1$  and  $k=2$  in (8.41), then

$$t_{12} = \frac{m'_2}{2m} + \frac{(\lambda_1 - \lambda_2)}{m_2} (s\lambda'_2 + \frac{1}{2}\lambda_2 s') - \frac{m'_2}{m_2} (r\lambda_1^{-1} + s\lambda_1\lambda_2 + r\lambda_2^{-1}). \quad (10.37)$$

Now by (9.8) and (9.9)

$$\lambda_1 - \lambda_2 = 2i (r/p)^{\frac{1}{2}} \left\{ 1 + \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2 \right\}. \quad (10.38)$$

Hence by (9.20) and (9.21), (10.38) gives

$$\lambda_1 - \lambda_2 = 2i (r/p)^{\frac{1}{2}} \left\{ 1 - \frac{3}{4}\delta^2 (1 + \frac{9}{4}\delta^2)^{-1} \right\} = 2i (r/p)^{\frac{1}{2}} \left\{ 1 + O(\delta^2) \right\}. \quad (10.39)$$

Now by (9.9) and (9.30)

$$\begin{aligned} s\lambda'_2 + \frac{1}{2}\lambda_2 s' &= -\frac{1}{2}i \frac{sr^{\frac{1}{2}}}{p^{\frac{1}{2}}} \left[ \frac{r'}{r} - \frac{p'}{p} \right] (1 + \delta_2) - i \frac{sr^{\frac{1}{2}}}{p^{\frac{1}{2}}} \delta'_2 \\ &\quad - \frac{1}{2}i \frac{s'}{s} \cdot \frac{sr^{\frac{1}{2}}}{p^{\frac{1}{2}}} (1 + \delta_2) \end{aligned}$$

i.e.

$$s\lambda'_2 + \frac{1}{2}\lambda_2 s' = -\frac{1}{2}i \frac{sr^{\frac{1}{2}}}{p^{\frac{1}{2}}} \left[ \frac{r'}{r} - \frac{p'}{p} + \frac{s'}{s} \right] (1 + \delta_2) - i \frac{sr^{\frac{1}{2}}}{p^{\frac{1}{2}}} \delta'_2.$$

Hence by (9.21)

$$s\lambda'_2 + \frac{1}{2}\lambda_2 s' = -\frac{1}{2}i \frac{sr^{\frac{1}{2}}}{p^{\frac{1}{2}}} \left[ \frac{r'}{r} - \frac{p'}{p} + \frac{s'}{s} \right] \left\{ 1 - \frac{i}{2}\delta + O(\delta^2) \right\} - i \frac{sr^{\frac{1}{2}}}{p^{\frac{1}{2}}} \delta'_2.$$

(10.40)

Now by (10.15) and (10.39)

$$\frac{\lambda_1 - \lambda_2}{m_2} = \frac{1}{p} \left\{ 1 - i\delta + O(\delta^2) \right\}. \quad (10.41)$$

Hence by (9.14), (10.40) and (10.41)

$$\begin{aligned} \frac{(\lambda_1 - \lambda_2)}{m_2} (s\lambda'_2 + \frac{1}{2}\lambda_2 s') &= -\frac{1}{2}i\delta \left[ \frac{r'}{r} - \frac{p'}{p} + \frac{s'}{s} \right] \left\{ 1 - \frac{3}{2}i\delta + O(\delta^2) \right\} \\ &\quad - i\delta \delta'_2 \left\{ 1 - i\delta + O(\delta^2) \right\}. \end{aligned} \quad (10.42)$$

Now by (10.14)

$$m_2^2 = -4 \text{ pr} \left\{ 1 + 2i\delta + O(\delta^2) \right\}. \quad (10.43)$$

Hence by (10.22) and (10.43)

$$\begin{aligned} \frac{m'_2}{m_2^2} &= -\frac{i}{4} \cdot \frac{1}{(\text{pr})^{\frac{1}{2}}} \cdot \frac{r'}{r} \left\{ 1 + \frac{i}{2}\delta + O(\delta^2) \right\} - \frac{i}{2} \cdot \frac{1}{(\text{pr})^{\frac{1}{2}}} \cdot \delta'_2 \left\{ 1 + i\delta + O(\delta^2) \right\} \\ &\quad - \frac{i}{4} \cdot \frac{1}{(\text{pr})^{\frac{1}{2}}} \cdot \frac{p'}{p} \left\{ 1 + \frac{11}{2i}\delta + O(\delta^2) \right\} + \frac{3}{4} \frac{s'}{p^2} \left\{ 1 - 3i\delta + O(\delta^2) \right\} \end{aligned}$$

(10.44)

Now by (9.8)

$$r\lambda_1^{-1} = -i (rp)^{\frac{1}{2}} (1 + \delta_1)^{-1}. \quad (10.45)$$

Also by (9.8) and (9.9)

$$s\lambda_1\lambda_2 = \frac{sr}{p} (1 + \delta_1)(1 + \delta_2), \quad (10.46)$$

$$r\lambda_2^{-1} = i (rp)^{\frac{1}{2}} (1 + \delta_2)^{-1}. \quad (10.47)$$

Hence by (10.45), (10.46), (10.45) and (9.14)

$$\begin{aligned} r\lambda_1^{-1} + s\lambda_1\lambda_2 + r\lambda_2^{-1} &= -i (rp)^{\frac{1}{2}} \left\{ (1 + \delta_1)^{-1} - (1 + \delta_2)^{-1} \right. \\ &\quad \left. + i\delta (1 + \delta_1)(1 + \delta_2) \right\}. \end{aligned}$$

Hence by (9.20) and (9.21)

$$\begin{aligned} r\lambda_1^{-1} + s\lambda_1\lambda_2 + r\lambda_2^{-1} &= -i (rp)^{\frac{1}{2}} \left\{ \left[ 1 + \frac{i}{2}\delta (1 - \frac{3}{2}i\delta)^{-1} \right]^{-1} \right. \\ &\quad \left. - \left[ 1 - \frac{i}{2}\delta (1 + \frac{3}{2}i\delta)^{-1} \right]^{-1} + i\delta \left[ 1 + \frac{i}{2}\delta (1 - \frac{3}{2}i\delta)^{-1} \right] \right. \\ &\quad \left. \times \left[ 1 - \frac{i}{2}\delta (1 + \frac{3}{2}i\delta)^{-1} \right] \right\} \end{aligned}$$

i.e.

$$r\lambda_1^{-1} + s\lambda_1\lambda_2 + r\lambda_2^{-1} = O((rp)^{\frac{1}{2}}\delta^3). \quad (10.48)$$

Hence by (9.14), (10.44), (10.48) and (9.33)

$$\frac{m_2'}{m_2} (r\lambda_1^{-1} + s\lambda_1\lambda_2 + r\lambda_2^{-1}) = O\left(\frac{r'}{r}\delta^3\right) + O\left(\frac{s'}{s}\delta^4\right) + O\left(\frac{p'}{p}\delta^3\right). \quad (10.49)$$

Hence by (10.23), (10.42), (10.49) and (9.33), (10.37) gives

$$t_{12} = \frac{1}{4} \left[ \frac{r'}{r} + \frac{p'}{p} \right] + o(\delta \frac{r'}{r}) + o(\delta \frac{p'}{p}) + o(\delta \frac{s'}{s}). \quad (10.50)$$

Next we consider  $j=2$  and  $k=1$  in (8.41) then

$$t_{21} = \frac{m'_1}{2m_1} + \frac{(\lambda_2 - \lambda_1)}{m_1} (s\lambda'_1 + \frac{1}{2} \lambda_1 s') - \frac{m'_1}{m_1^2} (r\lambda_2^{-1} + s\lambda_1\lambda_2 + r\lambda_1^{-1}). \quad (10.51)$$

Now by (10.39)

$$\lambda_2 - \lambda_1 = -2i (pr)^{\frac{1}{2}} \left\{ 1 + o(\delta^2) \right\}. \quad (10.52)$$

Also by (9.8) and (9.29)

$$s\lambda'_1 + \frac{1}{2} \lambda_1 s' = \frac{i}{2} \frac{sr^{\frac{1}{2}}}{p^{\frac{1}{2}}} \left[ \frac{s'}{s} + \frac{r'}{r} - \frac{p'}{p} \right] (1 + \delta_1) + i \frac{sr^{\frac{1}{2}}}{p^{\frac{1}{2}}} \delta'_1.$$

Hence by (9.20)

$$s\lambda'_1 + \frac{1}{2} \lambda_1 s' = \frac{i}{2} \frac{sr^{\frac{1}{2}}}{p^{\frac{1}{2}}} \left[ \frac{s'}{s} + \frac{r'}{r} - \frac{p'}{p} \right] \left\{ 1 + \frac{1}{2} \delta + o(\delta^2) \right\} + i \frac{sr^{\frac{1}{2}}}{p^{\frac{1}{2}}} \delta'_1. \quad (10.53)$$

Also by (10.3) and (10.52)

$$\frac{(\lambda_2 - \lambda_1)}{m_1} = \frac{1}{p} \left\{ 1 + i\delta + o(\delta^2) \right\}. \quad (10.54)$$

Hence by (9.14), (10.53) and (10.54)

$$\begin{aligned} \frac{(\lambda_2 - \lambda_1)}{m_1} (s\lambda'_1 + \frac{1}{2} \lambda_1 s') &= \frac{1}{2} i\delta \left[ \frac{s'}{s} + \frac{r'}{r} - \frac{p'}{p} \right] \left\{ 1 + \frac{3}{2} i\delta + o(\delta^2) \right\} \\ &\quad + i\delta\delta'_1 \left\{ 1 + i\delta + o(\delta^2) \right\}, \end{aligned} \quad (10.55)$$

where  $\delta'_1$  is given by (9.32).

Now by (10.3)

$$m_1^2 = -4pr \left\{ 1 - 2i\delta + O(\delta^2) \right\}. \quad (10.56)$$

Hence by (10.10) and (10.56)

$$\begin{aligned} \frac{m'_1}{m_1^2} &= \frac{3}{4} \frac{s'}{p^2} \left\{ 1 + 3i\delta + O(\delta^2) \right\} + \frac{i}{4} \frac{p'}{p} \cdot \frac{1}{(pr)^{\frac{1}{2}}} \left\{ 1 + \frac{11}{2i}\delta + O(\delta^2) \right\} \\ &+ \frac{i}{4} \cdot \frac{r'}{r} \cdot \frac{1}{(pr)^{\frac{1}{2}}} \left\{ 1 - \frac{i}{2}\delta + O(\delta^2) \right\} + \frac{i}{2}\delta'_1 \cdot \frac{1}{(pr)^{\frac{1}{2}}} \left\{ 1 - i\delta + O(\delta^2) \right\} \end{aligned} \quad (10.57)$$

Hence by (9.14), (10.48), (10.57) and (9.32)

$$\frac{m'_1}{m_1^2} (r\lambda_2^{-1} + s\lambda_1\lambda_2 + r\lambda_1^{-1}) = O\left(\frac{s'}{s}\delta^4\right) + O\left(\frac{p'}{p}\delta^3\right) + O\left(\frac{r'}{r}\delta^3\right). \quad (10.58)$$

Hence by (10.11), (10.55), (10.58) and (9.32), (10.52) gives

$$t_{21} = \frac{1}{4} \left[ \frac{r'}{r} + \frac{p'}{p} \right] + O\left(\delta \frac{s'}{s}\right) + O\left(\delta \frac{p'}{p}\right) + O\left(\delta \frac{r'}{r}\right). \quad (10.59)$$

Now we consider  $j=1$  and  $k=3$  in (8.41) then

$$t_{13} = \frac{m'_3}{2m_3} + \frac{(\lambda_1 - \lambda_3)}{m_3} (s\lambda'_3 + \frac{1}{2}\lambda_3 s') - \frac{m'_3}{m_3^2} (r\lambda_1^{-1} + s\lambda_1\lambda_3 + r\lambda_3^{-1}). \quad (10.60)$$

Now by (9.8), (9.10) and (9.14)

$$\lambda_1 - \lambda_3 = i \frac{p}{s} \delta (1 + \delta_1) - \frac{p}{s} (1 + \delta_3)$$

i.e.

$$\lambda_1 - \lambda_3 = - \frac{p}{s} \left\{ 1 - i\delta - i\delta\delta_1 + \delta_3 \right\}. \quad (10.61)$$

Hence by (9.20) and (9.24), (10.61) gives

$$\lambda_1 - \lambda_3 = -\frac{p}{s} \left\{ 1 - i\delta + O(\delta^2) \right\}, \quad (10.62)$$

Now by (9.10) and (9.31)

$$s\lambda'_3 + \frac{1}{2}\lambda_3 s' = p \left[ \frac{p'}{p} - \frac{s'}{s} \right] (1 + \delta_3) + p\delta'_3 + \frac{1}{2} \frac{s'}{s} p(1 + \delta_3)$$

i.e.

$$s\lambda'_3 + \frac{1}{2}\lambda_3 s' = p \left[ \frac{p'}{p} - \frac{1}{2} \frac{s'}{s} \right] (1 + \delta_3) + p\delta'_3.$$

Hence by (9.24)

$$s\lambda'_3 + \frac{1}{2}\lambda_3 s' = p \left[ \frac{p'}{p} - \frac{1}{2} \frac{s'}{s} \right] (1 + \delta^2) + p\delta'_3 \quad (10.63)$$

where  $\delta'_3$  is given by (9.34).

Also by (10.27) and (10.62)

$$\frac{(\lambda_1 - \lambda_3)}{m_3} = -\frac{1}{p} \left\{ 1 + i\delta + O(\delta^2) \right\}. \quad (10.64)$$

Hence by (10.63) and (10.64)

$$\begin{aligned} \frac{(\lambda_1 - \lambda_3)}{m_3} (s\lambda'_3 + \frac{1}{2}\lambda_3 s') &= \left[ \frac{1}{2} \frac{s'}{s} - \frac{p'}{p} \right] \left\{ 1 + i\delta + O(\delta^2) \right\} \\ &\quad - \delta'_3 \left\{ 1 + i\delta + O(\delta^2) \right\}. \end{aligned} \quad (10.65)$$

Now by (10.27)

$$\frac{2}{m_3} = \frac{p^4}{s^2} \left\{ 1 + 8\delta^2 + O(\delta^4) \right\}. \quad (10.66)$$

Hence by (10.34) and (10.66)

$$\begin{aligned} \frac{m'_3}{m_3} &= 2 \frac{s p'_3}{p_3} \left\{ 1 - 4\delta^2 + O(\delta^4) \right\} - \frac{s'_2}{p_2} \left\{ 1 - 4\delta^2 + O(\delta^4) \right\} \\ &+ 4\delta'_3 \frac{s}{p_2} \left\{ 1 - \frac{13}{2}\delta^2 + O(\delta^4) \right\}. \end{aligned} \quad (10.67)$$

Now by (9.8) and (9.10)

$$s\lambda_1\lambda_3 = i(rp)^{\frac{1}{2}} (1 + \delta_1) (1 + \delta_3). \quad (10.68)$$

Also by (9.10) and (9.14)

$$r\lambda_3^{-1} = (rp)^{\frac{1}{2}}\delta (1 + \delta_3)^{-1} \quad (10.69)$$

$$\begin{aligned} r\lambda_1^{-1} + s\lambda_1\lambda_3 + r\lambda_3^{-1} &= i(rp)^{\frac{1}{2}} \left\{ -(1 + \delta_1)^{-1} + (1 + \delta_1)(1 + \delta_3) \right. \\ &\quad \left. - i\delta (1 + \delta_3)^{-1} \right\}, \end{aligned}$$

and so by (9.20) and (9.24) as in (10.48) we obtain

$$r\lambda_1^{-1} + s\lambda_1\lambda_3 + r\lambda_3^{-1} = O((rp)^{\frac{1}{2}}\delta^2). \quad (10.70)$$

Hence by (9.14), (10.67), (10.70) and (9.34)

$$\frac{m'_3}{m_3} (r\lambda_1^{-1} + s\lambda_1\lambda_3 + r\lambda_3^{-1}) = O\left(\frac{p'}{p}\delta^3\right) + O\left(\frac{s'}{s}\delta^3\right) + O\left(\frac{r'}{r}\delta^5\right). \quad (10.71)$$

Hence by (10.35), (10.65), (10.71) and (9.34), (10.60) gives

$$t_{13} = O\left(\frac{p'}{p}\delta\right) + O\left(\delta\frac{s'}{s}\right) + O\left(\frac{r'}{r}\delta^2\right). \quad (10.72)$$

Next we consider  $j=2$  and  $k=3$  in (8.41), then

$$t_{23} = \frac{m'_3}{2m_3} + \frac{(\lambda_2 - \lambda_3)}{m_3} (s\lambda'_3 + \frac{1}{2}\lambda_3 s') - \frac{m'_3}{m_3} (r\lambda_2^{-1} + s\lambda_2\lambda_3 + r\lambda_3^{-1}). \quad (10.73)$$

Now by (9.9), (9.10) and (9.14)

$$\lambda_2 - \lambda_3 = -\frac{p}{s} \left\{ 1 + i\delta + \delta\delta_2 + \delta_3 \right\}. \quad (10.74)$$

Hence by (9.21) and (9.24), (10.74) becomes

$$\lambda_2 - \lambda_3 = -\frac{p}{s} \left\{ 1 + i\delta + O(\delta^2) \right\}. \quad (10.75)$$

Now by (10.27) and (10.75)

$$\frac{(\lambda_2 - \lambda_3)}{m_3} = -\frac{1}{p} \left\{ 1 + i\delta + O(\delta^2) \right\}. \quad (10.76)$$

Hence by (10.63) and (10.76)

$$\begin{aligned} \frac{(\lambda_2 - \lambda_3)}{m_3} (s\lambda'_3 + \frac{1}{2} \lambda_3 s') &= \left[ \frac{1}{2} \frac{s'}{s} - \frac{p'}{p} \right] \left\{ 1 + i\delta + O(\delta^2) \right\} \\ &- \delta'_3 \left\{ 1 + i\delta + O(\delta^2) \right\}. \end{aligned} \quad (10.77)$$

Also by (9.9) and (9.10)

$$s\lambda_2\lambda_3 = -i (rp)^{\frac{1}{2}} (1 + \delta_2) (1 + \delta_3) \quad (10.78)$$

Hence by (10.47), (10.69) and (10.78)

$$\begin{aligned} r\lambda_2^{-1} + s\lambda_2\lambda_3 + r\lambda_3^{-1} &= i (rp)^{\frac{1}{2}} \left\{ (1 + \delta_2)^{-1} - i\delta (1 + \delta_3)^{-1} \right. \\ &\quad \left. - (1 + \delta_2)(1 + \delta_3) \right\}, \end{aligned}$$

and so by (9.21) and (9.24) as in (10.70) we obtain

$$r\lambda_2^{-1} + s\lambda_2\lambda_3 + r\lambda_3^{-1} = O((rp)^{\frac{1}{2}}\delta^2). \quad (10.79)$$

Hence by (9.14), (10.67), (10.79) and (9.34)

$$\frac{m'_3}{m_3^2} (r\lambda_2^{-1} + s\lambda_2\lambda_3 + r\lambda_3^{-1}) = O(\frac{p'}{p}\delta^3) + O(\frac{s'}{s}\delta^3) + O(\frac{r'}{r}\delta^5). \quad (10.80)$$

Hence by (10.35), (10.77), (10.80) and (9.34), (10.73) gives

$$t_{23} = O(\frac{s'}{s}\delta) + O(\frac{p'}{p}\delta) + O(\frac{r'}{r}\delta^2). \quad (10.81)$$

Now we consider  $j=3$  and  $k=1$  in (8.41), then

$$t_{31} = \frac{m'_1}{2m_1} + \frac{(\lambda_3 - \lambda_1)}{m_1} (s\lambda'_1 + \frac{1}{2}\lambda_1 s') - \frac{m'_1}{m_1^2} (r\lambda_3^{-1} + s\lambda_3\lambda_1 + r\lambda_1^{-1}). \quad (10.82)$$

Now by (10.62)

$$\lambda_3 - \lambda_1 = \frac{p}{s} \left\{ 1 - i\delta + O(\delta^2) \right\}. \quad (10.83)$$

Hence by (10.3) and (10.83)

$$\frac{(\lambda_3 - \lambda_1)}{m_1} = -\frac{1}{2i} \cdot \frac{p^{\frac{1}{2}}}{sr^{\frac{1}{2}}} \left\{ 1 + O(\delta^2) \right\}. \quad (10.84)$$

so by (10.53) and (10.84)

$$\begin{aligned} \frac{(\lambda_3 - \lambda_1)}{m_1} (s\lambda'_1 + \frac{1}{2}\lambda_1 s') &= -\frac{1}{4} \left[ \frac{s'}{s} + \frac{r'}{r} - \frac{p'}{p} \right] \left\{ 1 + \frac{i}{2}\delta + O(\delta^2) \right\} \\ &\quad - \frac{1}{2} \delta'_1 \left\{ 1 + O(\delta^2) \right\}. \end{aligned} \quad (10.85)$$

Also by (9.14), (10.57), (10.70) and (9.32)

$$\frac{m'_1}{m_1^2} (r\lambda_3^{-1} + s\lambda_3\lambda_1 + r\lambda_1^{-1}) = O(\frac{s'}{s}\delta^3) + O(\frac{p'}{p}\delta^2) + O(\frac{r'}{r}\delta^2). \quad (10.86)$$

Hence by (10.11), (10.85), (10.86) and (9.32), (10.82) gives

$$t_{31} = \frac{1}{2} \frac{p'}{p} - \frac{1}{4} \frac{s'}{s} + O(\delta \frac{s'}{s}) + O(\delta \frac{r'}{r}) + O(\delta \frac{p'}{p}) \quad (10.87)$$

Finally we consider  $j=3$  and  $k=2$  in (8.41), then

$$t_{32} = \frac{m'_2}{2m_2} + \frac{(\lambda_3 - \lambda_2)}{m_2} (s\lambda'_2 + \frac{1}{2} \lambda_3 s') - \frac{m'_2}{m_2^2} (r\lambda_3^{-1} + s\lambda_3\lambda_2 + r\lambda_2^{-1}). \quad (10.88)$$

Now by (10.75)

$$\lambda_3 - \lambda_2 = \frac{p}{s} \left\{ 1 + i\delta + O(\delta^2) \right\}. \quad (10.89)$$

Hence by (10.15) and (10.89)

$$\frac{(\lambda_3 - \lambda_2)}{m_2} = \frac{1}{2i} \frac{p^{\frac{1}{2}}}{sr^{\frac{1}{2}}} \left\{ 1 + O(\delta^2) \right\} \quad (10.90)$$

So by (10.40) and (10.90)

$$\begin{aligned} \frac{(\lambda_3 - \lambda_2)}{m_2} (s\lambda'_2 + \frac{1}{2} \lambda_2 s') &= -\frac{1}{4} \left[ \frac{r'}{r} - \frac{p'}{p} + \frac{s'}{s} \right] \left\{ 1 - \frac{i}{2} \delta + O(\delta^2) \right\} \\ &\quad - \frac{1}{2} \delta'_2 \left\{ 1 + O(\delta^2) \right\}. \end{aligned} \quad (10.91)$$

Also by (9.14), (10.44), (10.79) and (9.33)

$$\frac{m'_2}{m_2^2} (r\lambda_3^{-1} + s\lambda_3\lambda_2 + r\lambda_2^{-1}) = O(\frac{r'}{r} \delta^2) + O(\frac{s'}{s} \delta^3) + O(\frac{p'}{p} \delta^2). \quad (10.92)$$

Hence, by (10.23), (10.91), (10.92) and (9.33), (10.88) gives

$$t_{32} = \frac{1}{2} \frac{p'}{p} - \frac{1}{4} \frac{s'}{s} + O(\frac{r'}{r} \delta) + O(\frac{s'}{s} \delta) + O(\frac{p'}{p} \delta) \quad (10.93)$$

Now by previous work in this section we complete our discussion of the matrix  $T^{-1}T'$ .

# 11. The System $Z' = (\Lambda + R_1 + R_2)Z$

Now putting together (10.12)-(10.93), we can write (2.8) as

$$Z' = (\Lambda + R_1 + R_2)Z \quad (11.1)$$

where

$$R_1 = \begin{pmatrix} \eta & -\eta & 0 \\ -\eta & \eta & 0 \\ -\frac{1}{2}\theta & -\frac{1}{2}\theta & \theta \end{pmatrix} \quad (11.2)$$

such that

$$\eta = \frac{1}{4} \left( \frac{p'}{p} + \frac{r'}{r} \right) = \frac{1}{4} \frac{(pr)'}{pr} \quad (11.3)$$

$$\theta = \frac{p'}{p} - \frac{1}{2} \frac{s'}{s} = \frac{(ps^{-\frac{1}{2}})'}{ps^{-\frac{1}{2}}} \quad (11.4)$$

and the matrix  $R_2(x)$  is  $L(a, \infty)$  subject to remark 9.2, and so (11.1) has the required form (2.3) if  $\Lambda + R_1$  satisfies the hypothesis of Theorem 1.2.

Now we shall show that the matrix  $\Lambda + R_1$  satisfying Theorem 1.2. To do this, first we need to show that the eigenvalues  $\lambda_j(x)$  ( $1 \leq j \leq 3$ ) of  $\Lambda(x)$  are distinct, this is obvious by (9.8), (9.9) and (9.10).

Second we need to show that condition (i) of Theorem 1.2, holds, in particular we need to show that

$$R_1(x) = o\left\{ \lambda_i(x) - \lambda_j(x) \right\} \quad \text{as } x \rightarrow \infty \quad (11.5)$$

for  $(i \neq j, 1 \leq i, j \leq 3)$ .

Now by (10.38), (10.61), (10.74), (9.11) and (9.15) we have

$$\lambda_1 - \lambda_2 = 2i \left(\frac{r}{p}\right)^{\frac{1}{2}} \{1 + o(1)\} \quad \text{as } x \rightarrow \infty \quad (11.6)$$

$$\lambda_1 - \lambda_3 = -\frac{p}{s} \{1 + o(1)\} \quad \text{as } x \rightarrow \infty \quad (11.7)$$

$$\lambda_2 - \lambda_3 = -\frac{p}{s} \{1 + o(1)\} \quad \text{as } x \rightarrow \infty \quad (11.8)$$

Hence by (9.1), (9.2), (9.3), (9.4), (11.2), (11.3), (11.4), (11.6), (11.7) and (11.8) implies that (11.5) holds for  $(i \neq j, 1 \leq j, j \leq 3)$  and so  $\lambda_i - \lambda_j$  ( $i \neq j, 1 \leq i, j \leq 3$ ) dominate the matrix  $R_1$ .

Next we need to show that (ii) holds. i.e.

$$\{(\lambda_i - \lambda_j)^{-1} R_1\}' \quad \text{is } L(a, \infty) \quad (i \neq j, 1 \leq i, j \leq 3). \quad (11.9)$$

Hence subject to (11.2), it suffices to show that

$$\{(\lambda_i - \lambda_j)^{-1} \eta\}' \quad \text{is } L(a, \infty) \quad (i \neq j, 1 \leq j, j \leq 3) \quad (11.10)$$

and

$$\{(\lambda_i - \lambda_j)^{-1} \theta\}' \quad \text{is } L(a, \infty) \quad (i \neq j, 1 \leq i, j \leq 3). \quad (11.11)$$

Now by (10.38)

$$\begin{aligned} \{(\lambda_1 - \lambda_2)^{-1} \eta\}' &= -\frac{i}{2} \left\{ \left(\frac{p}{r}\right)^{\frac{1}{2}} \eta (1 + \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2)^{-1} \right\}' \\ &= -\frac{i}{2} \left\{ \left(\frac{p}{r}\right)^{\frac{1}{2}} \eta \right\}' (1 + \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2)^{-1} \\ &\quad + \frac{i}{4} \left(\frac{p}{r}\right)^{\frac{1}{2}} (1 + \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2)^{-2} (\delta_1' + \delta_2') \end{aligned}$$

i.e.

$$\{(\lambda_1 - \lambda_2)^{-1} \eta\}' = O\left\{\left(\left(\frac{p}{r}\right)^{\frac{1}{2}} \eta\right)'\right\} + O\left(\left(\frac{p}{r}\right)^{\frac{1}{2}} \eta \delta_1'\right) + O\left(\left(\frac{p}{r}\right)^{\frac{1}{2}} \eta \delta_2'\right). \quad (11.12)$$

We note by (11.3), (9.2) and (9.3),  $\left(\frac{p}{r}\right)^{\frac{1}{2}} \eta = o(1)$  as  $x \rightarrow \infty$ .

Now

$$\left(\left(\frac{p}{r}\right)^{\frac{1}{2}} \eta\right)' = \left(\frac{p}{r}\right)^{\frac{1}{2}} \eta' + \eta \left(\left(\frac{p}{r}\right)^{\frac{1}{2}}\right)'. \quad (11.13)$$

Also by (11.3) we have that

$$\left(\frac{p}{r}\right)^{\frac{1}{2}} \eta' = \frac{1}{4} \left(\frac{p}{r}\right)^{\frac{1}{2}} \left(\frac{p''}{p} - \frac{p'^2}{p^2} + \frac{r''}{r} - \frac{r'^2}{r^2}\right), \quad (11.14)$$

$$\eta \left(\left(\frac{p}{r}\right)^{\frac{1}{2}}\right)' = \frac{1}{8} \left(\frac{p}{r}\right)^{\frac{1}{2}} \left(\frac{p'^2}{p^2} - \frac{r'^2}{r^2}\right). \quad (11.15)$$

Hence by (11.14), (11.15) and (11.13)

$$\left(\left(\frac{p}{r}\right)^{\frac{1}{2}} \eta\right)' = O\left(\frac{p''}{p} \left(\frac{p}{r}\right)^{\frac{1}{2}}\right) + O\left(\frac{r''}{r} \left(\frac{p}{r}\right)^{\frac{1}{2}}\right) + O\left(\frac{p'^2}{p^2} \left(\frac{p}{r}\right)^{\frac{1}{2}}\right) + O\left(\frac{r'^2}{r^2} \left(\frac{p}{r}\right)^{\frac{1}{2}}\right).$$

Thus, by (9.5),

$$\left(\left(\frac{p}{r}\right)^{\frac{1}{2}} \eta\right)' \quad \text{is } L(a, \infty). \quad (11.16)$$

Now by (11.16) and (9.35), all the terms on the right hand side of (11.12) are  $L(a, \infty)$ . Thus

$$\{(\lambda_1 - \lambda_2)^{-1} \eta\}' \quad \text{is } L(a, \infty). \quad (11.17)$$

Next by (10.38)

$$\{(\lambda_1 - \lambda_2)^{-1}\theta\}' = -\frac{i}{2} \left\{ \left( \frac{p}{r} \right)^{\frac{1}{2}} \theta (1 + \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2)^{-1} \right\}'$$

i.e.

$$\{(\lambda_1 - \lambda_2)^{-1}\theta\}' = O\left\{ \left( \frac{p}{r} \right)^{\frac{1}{2}} \theta \right\}' + O\left( \left( \frac{p}{r} \right)^{\frac{1}{2}} \theta \delta_1' \right) + O\left( \left( \frac{p}{r} \right)^{\frac{1}{2}} \theta \delta_2' \right). \quad (11.18)$$

We note that by (11.4), (9.2) and (9.4),  $\left( \frac{p}{r} \right)^{\frac{1}{2}} \theta = o(1)$  as  $x \rightarrow \infty$ .

Now

$$\left( \left( \frac{p}{r} \right)^{\frac{1}{2}} \theta \right)' = \left( \frac{p}{r} \right)^{\frac{1}{2}} \theta' + \theta \left( \left( \frac{p}{r} \right)^{\frac{1}{2}} \right)'. \quad (11.19)$$

Also by (11.4) we have that

$$\left( \frac{p}{r} \right)^{\frac{1}{2}} \theta' = \left( \frac{p}{r} \right)^{\frac{1}{2}} \left( \frac{p''}{p} - \frac{p'^2}{p^2} - \frac{1}{2} \frac{s''}{s} + \frac{1}{2} \frac{s'^2}{s^2} \right), \quad (11.20)$$

$$\theta \left( \left( \frac{p}{r} \right)^{\frac{1}{2}} \right)' = \frac{1}{2} \left( \frac{p}{r} \right)^{\frac{1}{2}} \left( \frac{p'^2}{p^2} - \frac{r'}{r} \frac{p'}{p} - \frac{1}{2} \frac{s'}{s} \frac{p'}{p} + \frac{1}{2} \frac{s'}{s} \frac{r'}{r} \right). \quad (11.21)$$

Hence by (11.20), (11.21) and (11.19)

$$\begin{aligned} \left( \left( \frac{p}{r} \right)^{\frac{1}{2}} \theta \right)' &= O\left( \frac{p''}{p} \left( \frac{p}{r} \right)^{\frac{1}{2}} \right) + O\left( \frac{s''}{s} \left( \frac{p}{r} \right)^{\frac{1}{2}} \right) + O\left( \frac{p'^2}{p^2} \left( \frac{p}{r} \right)^{\frac{1}{2}} \right) + O\left( \frac{s'^2}{s^2} \left( \frac{p}{r} \right)^{\frac{1}{2}} \right) \\ &\quad + O\left( \frac{r'}{r} \frac{p'}{p} \left( \frac{p}{r} \right)^{\frac{1}{2}} \right) + O\left( \frac{s'}{s} \frac{r'}{r} \left( \frac{p}{r} \right)^{\frac{1}{2}} \right). \end{aligned} \quad (11.22)$$

Thus, by (9.5) and remark 9.1,

$$\left( \left( \frac{p}{r} \right)^{\frac{1}{2}} \theta \right)' \quad \text{is } L(a, \infty). \quad (11.23)$$

Now by (11.23) and (9.35), all the terms on the right hand side of (11.18) are  $L(a, \infty)$ . Hence

$$\{(\lambda_1 - \lambda_2)^{-1} \theta\}', \quad \text{is } L(a, \infty). \quad (11.24)$$

We move now to  $\{(\lambda_3 - \lambda_1)^{-1} \eta\}'$ , then by (10.61)

$$\begin{aligned} \{(\lambda_3 - \lambda_1)^{-1} \eta\}' &= \left\{ \frac{s}{p} \eta (1 + \delta_3 - i\delta - i\delta\delta_1)^{-1} \right\}' \\ &= \left( \frac{s}{p} \eta \right)' (1 + \delta_3 - i\delta - i\delta\delta_1)^{-1} \\ &\quad - \frac{s}{p} \eta (1 + \delta_3 - i\delta - i\delta\delta_1)^{-2} \left\{ \delta_3' - i\delta'(1 + \delta_1) - i\delta\delta_1' \right\} \end{aligned}$$

i.e.

$$\{(\lambda_3 - \lambda_1)^{-1} \eta\}' = O\left\{ \left( \frac{s}{p} \eta \right)' \right\} + O\left( \frac{s}{p} \eta \delta_3' \right) + O\left( \frac{s}{p} \eta \delta' \right) + O\left( \frac{s}{p} \eta \delta \delta_1' \right). \quad (11.25)$$

We note that by (11.3), (9.2), (9.3) and (9.1)  $\frac{s}{p} \eta = o(1)$  as  $x \rightarrow \infty$ , and so  $\frac{s}{p} \eta \delta = o(1)$  as  $x \rightarrow \infty$ .

Now

$$\left( \frac{s}{p} \eta \right)' = \left( \frac{s}{p} \right)' \eta + \frac{s}{p} \eta'. \quad (11.26)$$

Now, by (11.3) and (9.14) we have that

$$\left( \frac{s}{p} \right)' \eta = \frac{1}{4} \delta \left( \frac{p}{r} \right)^{\frac{1}{2}} \left( \frac{p'}{p} \frac{s'}{s} - \frac{p'^2}{p^2} + \frac{r'}{r} \frac{s'}{s} - \frac{r'}{r} \frac{p'}{p} \right), \quad (11.27)$$

$$\frac{s}{p} \eta' = \frac{1}{4} \delta \left( \frac{p}{r} \right)^{\frac{1}{2}} \left( \frac{p''}{p} - \frac{p'^2}{p^2} + \frac{r''}{r} - \frac{r'^2}{r^2} \right). \quad (11.28)$$

Hence by (11.26), (11.27) and (11.28)

$$\begin{aligned} \left(\frac{s}{p}\eta\right)' &= o\left(\frac{p}{p}\left(\frac{p}{r}\right)^{\frac{1}{2}}\delta\right) + o\left(\frac{r}{r}\left(\frac{p}{r}\right)^{\frac{1}{2}}\delta\right) + o\left(\frac{r'^2}{r^2}\left(\frac{p}{r}\right)^{\frac{1}{2}}\delta\right) + o\left(\frac{p'^2}{p^2}\left(\frac{p}{r}\right)^{\frac{1}{2}}\delta\right) \\ &\quad + o\left(\frac{p'}{p}\frac{s'}{s}\left(\frac{p}{r}\right)^{\frac{1}{2}}\delta\right) + o\left(\frac{r'}{r}\frac{s'}{s}\left(\frac{p}{r}\right)^{\frac{1}{2}}\delta\right). \end{aligned}$$

Now by (9.5) and remark 9.1, all the terms on the right are  $L(a, \infty)$ . Thus

$$\left(\frac{s}{p}\eta\right)' \text{ is } L(a, \infty). \quad (11.29)$$

Also by (9.14)

$$\delta' = o(\delta \frac{p'}{p}) + o(\delta \frac{r'}{r}) + o(\delta \frac{s'}{s}). \quad (11.30)$$

Thus by remark 9.2

$$\delta'(x) \text{ is } L(a, \infty). \quad (11.31)$$

Now by (11.29), (11.31) and (9.35), all the terms on the right hand side of (11.25) are  $L(a, \infty)$ . Thus

$$\{(\lambda_3 - \lambda_1)^{-1}\eta\}' \text{ is } L(a, \infty). \quad (11.32)$$

Now, by (10.61)

$$\begin{aligned} \text{i.e. } \{(\lambda_3 - \lambda_1)^{-1}\theta\}' &= \left\{ \frac{s}{p} (1 + \delta_3 - i\delta - i\delta\delta_1)^{-1} \right\}' \\ \{(\lambda_3 - \lambda_1)^{-1}\theta\}' &= o\left(\left(\frac{s}{p}\theta\right)'\right) + o\left(\frac{s}{p}\theta\delta_3'\right) + o\left(\frac{s}{p}\theta\delta'\right) + o\left(\frac{s}{p}\theta\delta\delta_1'\right). \end{aligned} \quad (11.33)$$

We note that by (11.4), (9.2), (9.4) and (9.1)  $\frac{s}{p}\theta = o(1)$  as  $x \rightarrow \infty$ , and so  $\frac{s}{p}\theta\delta = o(1)$  as  $x \rightarrow \infty$ .

Now

$$\left(\frac{s}{p}\theta\right)' = \left(\frac{s}{p}\right)'\theta + \frac{s}{p}\theta'. \quad (11.34)$$

Now, by (11.4) and (9.14) we have that

$$\left(\frac{s}{p}\right)'\theta = \left(\frac{p}{r}\right)^{\frac{1}{2}} \delta \left( \frac{3}{2} \frac{s'}{s} \frac{p'}{p} - \frac{1}{2} \frac{s''^2}{s^2} - \frac{p'^2}{p^2} \right), \quad (11.35)$$

$$\frac{s}{p}\theta' = \left(\frac{p}{r}\right)^{\frac{1}{2}} \delta \left( \frac{p''}{p} - \frac{p'^2}{p^2} - \frac{1}{2} \frac{s''}{s} + \frac{1}{2} \frac{s'^2}{s^2} \right). \quad (11.36)$$

Hence by (11.36), (11.35) and (11.34)

$$\left(\frac{s}{p}\theta\right)' = O\left(\frac{p''}{p}\left(\frac{p}{r}\right)^{\frac{1}{2}}\delta\right) + O\left(\frac{s''}{s}\left(\frac{p}{r}\right)^{\frac{1}{2}}\delta\right) + O\left(\frac{p'^2}{p^2}\left(\frac{p}{r}\right)^{\frac{1}{2}}\delta\right) + O\left(\frac{s'}{s}\frac{p'}{p}\left(\frac{p}{r}\right)^{\frac{1}{2}}\delta\right).$$

Now by (9.5) and remark 9.1), all the terms on the right are  $L(a, \infty)$ . Thus

$$\left(\frac{s}{p}\theta\right)' \text{ is } L(a, \infty). \quad (11.37)$$

Hence by (11.37), (11.31) and (9.35), all the terms on the right hand side of (11.33) are  $L(a, \infty)$ . Thus

$$\{(\lambda_3 - \lambda_1)^{-1}\theta\}' \text{ is } L(a, \infty). \quad (11.38)$$

Finally the treatment of  $\{(\lambda_3 - \lambda_2)^{-1}\eta\}'$  and  $\{(\lambda_3 - \lambda_2)^{-1}\theta\}'$  being similar, and so by (10.74) we have that

$$\{(\lambda_3 - \lambda_2)^{-1}\eta\}' = O\left\{\left(\frac{s}{p}\eta\right)'\right\} + O\left(\frac{s}{p}\eta\delta'_3\right) + O\left(\frac{s}{p}\eta\delta'\right) + O\left(\frac{s}{p}\eta\delta\delta'_2\right),$$

$$(11.39)$$

$$\{(\lambda_3 - \lambda_2)^{-1} \theta\}' = O\left\{\left(\frac{s}{p} \theta\right)'\right\} + O\left(\frac{s}{p} \theta \delta_3'\right) + O\left(\frac{s}{p} \theta \delta'\right) + O\left(\frac{s}{p} \theta \delta \delta_2'\right). \quad (11.40)$$

Hence by (11.29), (11.31), (9.35) and (11.37)

$$\{(\lambda_3 - \lambda_2)^{-1} \eta\}', \{(\lambda_3 - \lambda_2)^{-1} \theta\}' \text{ are both } L(a, \infty) \quad (11.41)$$

Thus by previous work, (11.10) and (11.11) are both holds and consequently (11.9) also holds. Finally we need to show that the eigenvalues of  $\Lambda + R_1$  satisfy the Dichotomy condition I-II in Levinson's Theorem 1.1.

Let  $\mu_k(x)$  ( $1 \leq k \leq 3$ ) are the eigenvalues of  $\Lambda + R_1$ , then by (2.4) and (11.2) the characteristic equation of  $\Lambda + R_1$  is given by

$$\det\{\Lambda + R_1 - \mu I_3\} = \begin{vmatrix} \eta + \lambda_1 - \mu & -\eta & 0 \\ -\eta & \eta + \lambda_2 - \mu & 0 \\ -\frac{1}{2}\theta & -\frac{1}{2}\theta & \theta + \lambda_3 - \mu \end{vmatrix} = 0$$

i.e.

$$\{(\eta + \lambda_1 - \mu)(\eta + \lambda_2 - \mu) - \eta^2\}(\theta + \lambda_3 - \mu) = 0.$$

Hence the eigenvalues  $\mu_j$  ( $1 \leq j \leq 3$ ) of  $\Lambda + R_1$  are given by

$$(\eta + \lambda_1 - \mu)(\eta + \lambda_2 - \mu) - \eta^2 = 0 \quad (11.42)$$

and

$$\theta + \lambda_3 - \mu = 0. \quad (11.43)$$

Thus solving (11.42) for  $\mu_1$  and  $\mu_2$  we obtain

$$\mu_1 = \frac{(\lambda_1 + \lambda_2 + 2\eta)}{2} + \frac{1}{2}[(\lambda_1 - \lambda_2)^2 + 4\eta^2]^{\frac{1}{2}}, \quad (11.44)$$

$$\mu_2 = \frac{(\lambda_1 + \lambda_2 + 2\eta)}{2} - \frac{1}{2} [(\lambda_1 - \lambda_2)^2 + 4\eta^2]^{\frac{1}{2}}. \quad (11.45)$$

Also by (11.43)

$$\mu_3 = \theta + \lambda_3. \quad (11.46)$$

Now the Dichotomy condition I-II of Levinson's Theorem 1.1, holds if we show that (1.16) holds. i.e.

$$\operatorname{Re}\{\mu_i(x) - \mu_j(x)\} = f(x) + g(x), \quad (11.47)$$

where  $f(x)$  does not change sign in  $[a, \infty)$  and  $g(x)$  is  $L(a, \infty)$ .

First we need to show that (11.47) holds under Case A of Theorem 9.1.

Now by (11.44) and (11.45)

$$\mu_1 - \mu_2 = \lambda_1 - \lambda_2 + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.48)$$

Here we need to show that O-term on the right hand side of (11.48) is  $L(a, \infty)$ .

Now by (10.38) and (11.3)

$$(\lambda_1 - \lambda_2)^{-1} \eta^2 = O\left\{\left(\frac{p}{r}\right)^{\frac{1}{2}} \left(\frac{(pr)'}{pr}\right)^2\right\} = O\left(\frac{p^{\frac{1}{2}}}{p^2} \left(\frac{p}{r}\right)^{\frac{1}{2}}\right) + O\left(\frac{r^{\frac{1}{2}}}{r^2} \left(\frac{p}{r}\right)^{\frac{1}{2}}\right). \quad (11.49)$$

Hence by (9.5),

$$(\lambda_1 - \lambda_2)^{-1} \eta^2 \text{ is } L(a, \infty), \quad (11.50)$$

which makes the O-term on the right of (11.48) is  $L(a, \infty)$ .

Now by (9.14) and (10.39)

$$\lambda_1 - \lambda_2 = 2i (r/p)^{\frac{1}{2}} + O\left(\frac{s^2 r^{\frac{3}{2}}}{p^{\frac{1}{2}}}\right). \quad (11.51)$$

Hence by (11.51), (11.48) becomes

$$\mu_1 - \mu_2 = 2i (r/p)^{\frac{1}{2}} + O\left(\frac{s^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.52)$$

Here also the second O-term is  $L(a, \infty)$  by (9.5).

Since  $r, p$  are real-valued functions with  $r/p > 0$ , hence (11.52) gives

$$\operatorname{re}\{\mu_1 - \mu_2\} = O\left(\frac{s^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.53)$$

Now by (11.46) and (11.44)

$$\mu_3 - \mu_2 = \lambda_3 - \lambda_2 + \theta - \eta + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.54)$$

Now by (11.3) and (11.4)

$$\theta - \eta = \frac{3}{4} \frac{p'}{p} + \frac{1}{2} \frac{s'}{s} - \frac{1}{4} \frac{r'}{r}. \quad (11.55)$$

Also by (9.1), (9.2), (9.3) and (9.4)

$$\frac{p'}{p} = o\left(\frac{p}{s}\right), \quad \frac{s'}{s} = o\left(\frac{p}{s}\right), \quad \frac{r'}{r} = o\left(\frac{p}{s}\right) \quad \text{as } x \rightarrow \infty, \quad (11.56)$$

Now by (9.9), (9.10), (9.14), (9.21) and (9.24)

$$\lambda_3 - \lambda_2 = \frac{p}{s} + i (r/p)^{\frac{1}{2}} + \frac{3}{2} \frac{sr}{p^2} + O\left(\frac{s^2 r^{3/2}}{p^{7/2}}\right) \quad (11.57)$$

Again here the O-term is  $L(a, \infty)$  by (9.5). Hence by (11.55), (11.56) and (11.57), (11.54) becomes

$$\mu_3 - \mu_2 = \frac{p}{s} (1 + o(1)) + i (r/p)^{\frac{1}{2}} + \frac{3}{2} \frac{sr}{p^2} + O\left(\frac{s^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right)$$

since by (9.1)

$$\frac{sr}{p^2} = o\left(\frac{p}{s}\right) \quad \text{as } x \rightarrow \infty$$

Hence

$$\mu_3 - \mu_2 = \frac{p}{s} (1 + o(1)) + i (r/p)^{\frac{1}{2}} + O\left(\frac{s^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.58)$$

since case A holds, hence (11.58) gives

$$\operatorname{re}\{\mu_3 - \mu_2\} = \frac{p}{s} (1 + o(1)) + O\left(\frac{s^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.59)$$

Similarly

$$\operatorname{re}\{\mu_3 - \mu_1\} = \frac{p}{s} (1 + o(1)) + O\left(\frac{s^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.60)$$

Hence by (11.53), (11.59) and (11.60), (11.47) holds and consequently the Dichotomy condition of Levinson's Theorem 1.1 holds under case A of Theorem 9.1.

Next we need to show that (11.47) also holds under case B of Theorem 9.1. Since  $\frac{r}{p} < 0$ , hence by (11.52)

$$\operatorname{re}\{\mu_1 - \mu_2\} = -2 \frac{|r|^{\frac{1}{2}}}{|p|^{\frac{1}{2}}} + O\left(\frac{s^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.61)$$

Also by (11.58)

$$\operatorname{re}\{\mu_3 - \mu_2\} = \frac{p}{s} (1 + o(1)) - \frac{|r|^{\frac{1}{2}}}{|p|^{\frac{1}{2}}} + O\left(\frac{s^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right).$$

Thus by (9.1)

$$\operatorname{re}\{\mu_3 - \mu_1\} = \frac{p}{s} (1 + o(1)) + O\left(\frac{s^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.62)$$

Similarly

$$\operatorname{re}\{\mu_3 - \mu_1\} = \frac{p}{s} (1 + o(1)) + O\left(\frac{s^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.63)$$

Hence by (11.61), (11.62) and (11.63), (11.47) holds and consequently the Dichotomy condition of Levinson's Theorem 1.1 holds under case B of Theorem 9.1.

Now we need to show that (11.47) also holds under case C.

Since  $\frac{r}{p} > 0$  and  $s = iS$ , hence by (11.52)

$$\operatorname{re}\{\mu_1 - \mu_2\} = O\left(\frac{S^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.64)$$

Here the ~~terms on the right are~~  $L(a, \infty)$  by (9.5) and (11.50).

Now by (11.55) and (11.57), (11.54) becomes

$$\mu_3 - \mu_2 = \frac{p}{s} + i (r/p)^{1/2} + \frac{3}{2} \frac{sr}{p^2} + \frac{3}{4} \frac{p'}{p} + \frac{1}{2} \frac{s'}{s} - \frac{1}{4} \frac{r'}{r} + O\left(\frac{S^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.65)$$

Since  $s = iS$ , hence

$$\begin{aligned} \mu_3 - \mu_2 = & -i \frac{p}{S} + i (r/p)^{1/2} + \frac{3}{2} i \frac{Sr}{p^2} + \frac{3}{4} \frac{p'}{p} + \frac{1}{2} \frac{S'}{S} - \frac{1}{4} \frac{r'}{r} + O\left(\frac{S^2 r^{3/2}}{p^{7/2}}\right) \\ & + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \end{aligned}$$

since  $\frac{r}{p} > 0$  where  $p$ ,  $r$  and  $S$  are real-valued functions, hence

$$\operatorname{re}\{\mu_3 - \mu_2\} = \frac{3}{4} \frac{p'}{p} + \frac{1}{2} \frac{S'}{S} - \frac{1}{4} \frac{r'}{r} + O\left(\frac{S^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right)$$

i.e.

$$\operatorname{re}\{\mu_3 - \mu_2\} = \frac{R'}{R} + O\left(\frac{S^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right), \quad (11.66)$$

where

$$R = |p|^{3/4} |S|^{1/2} |r|^{-1/4}. \quad (11.67)$$

Thus by case C,  $R'$  does not change sign in  $[a, \infty)$ . Similarly

$$\operatorname{re}\{\mu_3 - \mu_1\} = \frac{R'}{R} + O\left(\frac{S^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.68)$$

Hence by (11.64), (11.66) and (11.68), (11.47) holds and consequently the Dichotomy condition of Levinson's Theorem 1.1 holds under case C.

Finally we need to show that (11.47) also holds under case D.

Since  $\frac{r}{p} < 0$  and  $s = iS$ , hence by (11.52)

$$\operatorname{re}\{\mu_1 - \mu_2\} = -2 \frac{|r|^{1/2}}{|p|^{1/2}} + O\left(\frac{S^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right), \quad (11.69)$$

since  $\frac{r}{p} < 0$ ,  $p, r$  and  $S$  are real-valued function, hence (11.65) gives

$$\operatorname{re}\{\mu_3 - \mu_2\} = -\frac{|r|^{1/2}}{|p|^{1/2}} + \frac{3}{4} \frac{p'}{p} + \frac{1}{2} \frac{S'}{S} - \frac{1}{4} \frac{r'}{r} + O\left(\frac{S^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right).$$

Now since  $\frac{r}{p} < 0$  in this case, hence by (9.2), (9.3) and (9.4)

$$\frac{p'}{p} = o\left(\frac{|r|^{1/2}}{|p|^{1/2}}\right), \quad \frac{r'}{r} = o\left(\frac{|r|^{1/2}}{|p|^{1/2}}\right), \quad \frac{S'}{S} = o\left(\frac{|r|^{1/2}}{|p|^{1/2}}\right) \quad \text{as } x \rightarrow \infty.$$

Thus

$$\operatorname{re}\{\mu_3 - \mu_2\} = -\frac{|r|^{1/2}}{|p|^{1/2}} (1 + o(1)) + O\left(\frac{S^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.70)$$

Similarly

$$\operatorname{re}\{\mu_3 - \mu_1\} = \frac{|r|^{1/2}}{|p|^{1/2}} (1 + o(1)) + O\left(\frac{S^2 r^{3/2}}{p^{7/2}}\right) + O\left(\frac{\eta^2}{\lambda_1 - \lambda_2}\right). \quad (11.71)$$

Hence by (11.69), (11.70) and (11.71), (11.47) holds and consequently the Dichotomy condition of Levinson's Theorem 1.1 holds under case D.

## 12. The Asymptotic Form of Solutions

Now (11.1) has been shown to have the form (2.3), hence Theorem 1.2, is applicable and it follows that as  $x \rightarrow \infty$ , (11.1) has three linearly independent solutions  $Z_k(x)$  such that

$$Z_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t) dt\right) \quad (1 \leq k \leq 3) \quad (12.1)$$

where  $e_k$  is the coordinate vector with  $k$ th component unity and other components zero. Now we need to obtain the three linearly independent solutions of our original system (2.2). Hence by (2.7), (2.2) has corresponding solutions,

$$Y_k(x) = T(x)Z_k(x) \quad (1 \leq k \leq 3). \quad (12.2)$$

Hence by (8.11) and (12.1), (12.2) gives as  $x \rightarrow \infty$ ,

$$Y_k = (m_1^{-1}v_1 \quad m_2^{-1}v_2 \quad m_3^{-1}v_3) \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t) dt\right) \quad (1 \leq k \leq 3). \quad (12.3)$$

where

$$v_j = \begin{pmatrix} 1 \\ s^{\frac{1}{2}}\lambda_j \\ r\lambda_j^{-1} \end{pmatrix} \quad (1 \leq j \leq 3).$$

Taking the first component on each side of (12.3), we obtain three solutions  $y_k(x)$  of (8.1) such that as  $x \rightarrow \infty$ ,

$$y_k(x) = \left\{ m_k^{-1}(x) + \sum_{r=1}^3 (m_r^{-1}(x)) \right\} \exp \left( \int_a^x \mu_k(t) dt \right) \quad (1 \leq k \leq 3) \quad (12.4)$$

Now taking  $k=1$  in (12.4), then as  $x \rightarrow \infty$

$$y_1 = \left\{ m_1^{-1}(x) + \sum_{r=1}^3 (m_r^{-1}(x)) \right\} \exp \left( \int_a^x \mu_1(t) dt \right). \quad (12.5)$$

By (10.4), (10.16) and (10.28) we have

$$m_1^{-1} \sim \frac{i}{2} (pr)^{-\frac{1}{2}}, \quad \text{as } x \rightarrow \infty \quad (12.6)$$

$$m_2^{-1} \sim -\frac{i}{2} (pr)^{-\frac{1}{2}}, \quad \text{as } x \rightarrow \infty \quad (12.7)$$

$$m_3^{-1} \sim \frac{s}{p}. \quad \text{as } x \rightarrow \infty \quad (12.8)$$

Thus by (12.7) and (12.6)

$$m_1 m_2^{-1} = -1 + o(1). \quad \text{as } x \rightarrow \infty. \quad (12.9)$$

Also by (12.8), (12.6) and (9.1)

$$m_1 m_3^{-1} = o(1). \quad \text{as } x \rightarrow \infty. \quad (12.10)$$

Hence by (12.9), (12.10) and (12.5)

$$y_1(x) \sim m_1^{-1}(x) \exp \left( \int_a^x \mu_1(t) dt \right) \quad \text{as } x \rightarrow \infty. \quad (12.11)$$

Similarly

$$y_2(x) \sim m_2^{-1}(x) \exp \left( \int_a^x \mu_2(t) dt \right) \quad \text{as } x \rightarrow \infty. \quad (12.12)$$

Now taking  $k=3$  in (12.4), then as  $x \rightarrow \infty$

$$y_3 = m_1^{-1} \left\{ m_1 m_3^{-1} + \sum_{r=1}^3 (m_1 m_r^{-1}) \right\} \exp \left( \int_a^x \mu_3(t) dt \right). \quad (12.13)$$

Hence by (12.9) and (12.10), (12.13) gives

$$y_3(x) = o \left\{ m_1^{-1}(x) \exp \left( \int_a^x \mu_3(t) dt \right) \right\} \quad \text{as } x \rightarrow \infty. \quad (12.14)$$

Now we have to work out (12.11), (12.12) and (12.14) in some detail in terms of  $s, p$  and  $r$  in order to determine the form (9.6) and (9.7).

Now by (11.44)

$$\mu_1 = \lambda_1 + \eta + O \left( \frac{\eta^2}{\lambda_1 - \lambda_2} \right). \quad (12.15)$$

Also by (11.45)

$$\mu_2 = \lambda_2 + \eta + O \left( \frac{\eta^2}{\lambda_1 - \lambda_2} \right). \quad (12.16)$$

Now by (9.8), (9.14) and (9.20)

$$\lambda_1 = i(r/p)^{\frac{1}{2}} - \frac{1}{2} \frac{sr}{p^2} + O \left( \frac{s^2 r^{3/2}}{p^{7/2}} \right). \quad (12.17)$$

Also by (9.9), (9.14) and (9.21)

$$\lambda_2 = -i(r/p)^{\frac{1}{2}} - \frac{1}{2} \frac{sr}{p^2} + O \left( \frac{s^2 r^{3/2}}{p^{7/2}} \right). \quad (12.18)$$

Hence by (11.3) and (12.17), (12.15) becomes

$$\mu_1 = i(r/p)^{\frac{1}{2}} - \frac{1}{2} \frac{sr}{p^2} + \frac{1}{4} \frac{(pr)'}{pr} + O \left( \frac{s^2 r^{3/2}}{p^{7/2}} \right) + O \left( \frac{\eta^2}{\lambda_1 - \lambda_2} \right). \quad (12.19)$$

Similarly by (11.3) and (12.19), (12.16) becomes

$$\mu_2 = -i (r/p)^{\frac{1}{2}} - \frac{1}{2} \frac{sr}{p^2} + \frac{1}{4} \frac{(pr)'}{pr} + o\left(\frac{s^2 r^{3/2}}{p^{7/2}}\right) + o\left(\frac{n^2}{\lambda_1 - \lambda_2}\right). \quad (12.20)$$

Thus by (12.6) and (12.19), (12.11) becomes after adjusting a constant multiple in  $y_1$ ,

$$y_1(x) \sim \{p(x)r(x)\}^{-1/4} \exp\left\{\int_a^x \left[i(r/p)^{\frac{1}{2}}(t) - \frac{1}{2} \frac{sr}{p^2}(t)\right] dt\right\}. \quad (12.21)$$

Similarly by (12.7) and (12.20), (12.12) becomes after adjusting a constant multiple in  $y_2$ ,

$$y_2(x) \sim \{p(x)r(x)\}^{-1/4} \exp\left\{\int_a^x \left[-i(r/p)^{\frac{1}{2}}(t) - \frac{1}{2} \frac{sr}{p^2}(t)\right] dt\right\} \quad \text{as } x \rightarrow \infty \quad (12.22)$$

Now by (9.10), (9.23) and (11.4), (11.46) gives

$$\mu_3 = \frac{p}{s} + \frac{sr}{p^2} + \frac{(p s^{-\frac{1}{2}})'}{p s^{-\frac{1}{2}}}. \quad (12.23)$$

Hence by (12.6) and (12.23), (12.14) gives

$$y_3(x) = o\left\{\left(\frac{p}{sr}\right)^{\frac{1}{2}} \exp\left[\int_a^x \frac{p}{s}(t) dt + \int_a^x \frac{sr}{p^2}(t) dt\right]\right\} \quad \text{as } x \rightarrow \infty \quad (12.24)$$

Now by previous work in §§9-12 we complete our proof of Theorem 9.1.

Now we shall give two examples on Theorem 9.1.

Example (1)

In the familiar case we consider

$$s(x) = x^\alpha, \quad p(x) = Bx^\beta, \quad r(x) = Cx^\gamma$$

where  $\alpha, \beta, \gamma, B(\neq 0)$  and  $C(\neq 0)$  are real constants, and  $(a \leq x < \infty)$  excluding  $a=0$ .

Now

$$\frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}} = \left(\frac{C}{B^3}\right)^{\frac{1}{2}} x^{\alpha + \frac{1}{2}(\gamma - 3\beta)}.$$

Hence (9.1) holds if

$$3\beta - 2\alpha - \gamma > 0. \quad (12.25)$$

Now

$$\begin{aligned} s' &= \alpha x^{\alpha-1}, \\ s'' &= \alpha(\alpha-1) x^{\alpha-2}, \\ s'^2 &= \alpha^2 x^{2(\alpha-1)}. \end{aligned}$$

Also

$$\begin{aligned} p' &= B \beta x^{\beta-1}, \\ p'' &= B \beta x^{\beta-2}, \\ p'^2 &= B^2 \beta^2 x^{2(\beta-1)}. \end{aligned}$$

And

$$r' = C \gamma x^{\gamma-1}, \quad r'' = C \gamma(\gamma-1) x^{\gamma-2}, \quad r'^2 = C^2 \gamma^2 x^{2(\gamma-1)}$$

Now

$$\begin{aligned} \left(\frac{p}{r}\right)^{\frac{1}{2}} &= \left(\frac{B}{C}\right)^{\frac{1}{2}} x^{\frac{1}{2}(\beta-\gamma)}, \\ \frac{p'}{p} \left(\frac{p}{r}\right)^{\frac{1}{2}} &= \left(\frac{B}{C}\right)^{\frac{1}{2}} x^{\frac{1}{2}(\beta-\gamma)-1} \end{aligned}$$

Thus (9.2) holds if

$$\gamma - \beta + 2 > 0. \quad (12.26)$$

Similarly (9.3) and (9.4) are all holds if (12.26) holds.

Also

$$\frac{r'^2}{r^2} \left( \frac{p}{r} \right)^{\frac{1}{2}} = \gamma^2 \left( \frac{B}{C} \right)^{\frac{1}{2}} x^{\frac{1}{2}(\beta-\gamma)} - 2$$

is  $L(a, \infty)$  if (12.26) holds.

Now

$$\frac{s''}{s} \left( \frac{p}{r} \right)^{\frac{1}{2}} = \left( \frac{B}{C} \right)^{\frac{1}{2}} \alpha(\alpha-1) x^{\frac{1}{2}(\beta-\gamma)} - 2$$

is  $L(a, \infty)$  if (12.26) holds.

Similarly

$\frac{p'^2}{p^2} \left( \frac{p}{r} \right)^{\frac{1}{2}}$ ,  $\frac{s'^2}{s^2} \left( \frac{p}{r} \right)^{\frac{1}{2}}$ ,  $\frac{p''}{p} \left( \frac{p}{r} \right)^{\frac{1}{2}}$  and  $\frac{r''}{r} \left( \frac{p}{r} \right)^{\frac{1}{2}}$  are all  $L(a, \infty)$  under condition (12.26).

Finally

$$\frac{s^2 r^{3/2}}{p^{7/2}} = C^{3/2} B^{-7/2} x^{2\alpha + \frac{1}{2}(3\gamma - 7\beta)} \quad \text{is } L(a, \infty) \text{ if}$$

$$7\beta - 4\alpha - 3\gamma - 2 > 0. \quad (12.27)$$

Hence putting together (12.25), (12.26) and (12.27), then conditions (9.1), (9.2), (9.3), (9.4) and (9.5) are all satisfied under a set of two conditions in  $\alpha, \beta$  and  $\gamma$  which is given by

$$\left. \begin{array}{l} \gamma - \beta + 2 > 0 \\ 7\beta - 4\alpha - 3\gamma - 2 > 0 \end{array} \right\}. \quad (12.28)$$

Also we mention that either case A or case B of Theorem 9.1, holds for this example.

Now (9.6) gives after adjusting a constant multiple in  $y_k$   
 $(k=1,2)$

$$y_k(x) \sim x^{-1/4(\beta+\gamma)} \exp \left\{ \pm i \left( \frac{C}{B} \right)^{1/2} \int_a^x t^{1/2(\gamma-\beta)} dt - \frac{1}{2} CB^{-2} \int_a^x t^{\alpha+\gamma-2\beta} dt \right\}.$$

$(k=1,2) \quad \text{as } x \rightarrow \infty$

Also (9.7) gives

$$y_3(x) = o \left\{ x^{1/2(\beta-\gamma-\alpha)} \exp \left( B \int_a^x t^{\beta-\alpha} dt + B^{-2} C \int_a^x t^{\alpha+\gamma-2\beta} dt \right) \right\}$$

as  $x \rightarrow \infty$ .

Example (2)

Here we give different class of coefficients covered by our analysis, we consider

$$s(x) = x^\alpha \exp x^b, \quad p(x) = Bx^\beta \exp 2x^d, \quad r(x) = Cx^\gamma \exp 4x^d$$

where  $\alpha, \beta, \gamma, B(\neq 0), C(\neq 0), b$  and  $d$  are real constants with  $b \geq 0$  and  $b < d$  and our interval  $(a \leq x < \infty)$  excluding  $a=0$ . Now (9.1) is given by

$$\frac{sr^{1/2}}{p^{3/2}} = \left( \frac{C}{B^3} \right)^{1/2} x^{\alpha+1/2\gamma-3/2\beta} \exp(x^b - x^d) = o(1)$$

as  $x \rightarrow \infty$ , because  $d > b$ .

Now

$$s' = \{\alpha + bx^b\} x^{\alpha-1} \exp x^b,$$

$$s'' = \{\alpha(\alpha-1) + b(2\alpha+b-1)x^b + b^2 x^{2b}\} x^{\alpha-2} \exp x^b.$$

Also

$$p' = \{\beta + 2d x^d\} Bx^{\beta-1} \exp 2x^d,$$

$$p'' = \{\beta(\beta-1) + 2d(2\beta + d-1)x^d + 4d^2 x^{2d}\} Bx^{\beta-2} \exp 2x^d.$$

Similarly

$$r' = \{\gamma + 4d x^d\} C x^{\gamma-1} \exp 4x^d,$$

$$r'' = \{\gamma(\gamma-1) + 4d(2\gamma + d-1)x^d + 16d^2 x^{2d}\} Cx^{\gamma-2} \exp 4x^d.$$

Hence

$$\frac{p'}{p} \left( \frac{p}{r} \right)^{\frac{1}{2}} = \left( \frac{B}{C} \right)^{\frac{1}{2}} x^{\frac{1}{2}(\beta-\gamma)-1} \{\beta + 2d x^d\} \exp -x^d.$$

Thus  $\frac{p'}{p} \left( \frac{p}{r} \right)^{\frac{1}{2}}$  tends to zero as  $x \rightarrow \infty$ . i.e. (9.2) holds.

Similarly (9.3) and (9.4) are both holds. Also

$$\begin{aligned} \frac{s''}{s} \left( \frac{p}{r} \right)^{\frac{1}{2}} &= \left( \frac{B}{C} \right)^{\frac{1}{2}} x^{\frac{1}{2}(\beta-\gamma) - 2} \left\{ \alpha(\alpha-1) + \right. \\ &\quad \left. + b(2\alpha + b-1)x^b + b^2 x^{2b} \right\} \exp -x^d. \end{aligned}$$

Thus  $\frac{s''}{s} \left( \frac{p}{r} \right)^{\frac{1}{2}}$  is  $L(a, \infty)$ .

Also

$$\begin{aligned} \frac{r''}{r} \left( \frac{p}{r} \right)^{\frac{1}{2}} &= \left( \frac{B}{C} \right)^{\frac{1}{2}} x^{\frac{1}{2}(\beta-\gamma) - 2} \left\{ \gamma(\gamma-1) + \right. \\ &\quad \left. + 4d(2\gamma + d-1)x^d + 16d^2 x^{2d} \right\} \exp -x^d. \end{aligned}$$

Thus  $\frac{r''}{r} \left( \frac{p}{r} \right)^{\frac{1}{2}}$  is  $L(a, \infty)$ .

Now similarly  $\frac{p''}{p} \left( \frac{p}{r} \right)^{\frac{1}{2}}$ ,  $\frac{s'^2}{s^2} \left( \frac{p}{r} \right)^{\frac{1}{2}}$ ,  $\frac{r'^2}{r^2} \left( \frac{p}{r} \right)^{\frac{1}{2}}$ ,  $\frac{p'^2}{p^2} \left( \frac{p}{r} \right)^{\frac{1}{2}}$  and  $\frac{s^2 r^{3/2}}{p^{7/2}}$  are all

$L(a, \infty)$ .

Also we mention that either case A or case B of Theorem 9.1, holds for this example. Since by previous work the conditions of Theorem 9.1, are all satisfied, hence (9.6) and (9.7) are both holds with  $s$ ,  $r$  and  $p$  as given in this example.

As general comments, we note that the main difference between the method which is handled for the differential equation (2.1) in §§2-7 and the method which is handled for the differential equation (8.1) in §§8-12 is that the  $\lambda_j(x)$  ( $1 \leq j \leq 3$ ) which is corresponding to equation (3.3) having same size as  $x \rightarrow \infty$  [i.e. (4.1) holds] under certain condition when (4.2) holds where  $s$  and  $r$  are nowhere zero in  $[a, \infty)$ , but the  $\lambda_j(x)$  which is corresponding to equation (8.4) having different size as  $x \rightarrow \infty$  [i.e. (9.12) and (9.28) both hold] under certain condition when (9.1) holds where the coefficients  $s$ ,  $p$  and  $r$  are nowhere zero in  $[a, \infty)$ . Of course it's possible to consider that the case for  $\lambda_j(x)$  of equation (8.4) having same size as  $x \rightarrow \infty$ , but subject to have  $\frac{p}{s^{2/3}r^{1/3}} \rightarrow 0$  as  $x \rightarrow \infty$ , where  $s$  and  $r$  are nowhere zero in  $[a, \infty)$ , then (8.4) can be solved for  $\lambda_j(x)$  such that

$$\lambda_j \sim \omega_j \left( \frac{r}{s} \right)^{1/3} \quad (1 \leq j \leq 3) \text{ as } x \rightarrow \infty$$

where  $\omega_1 = 1$ ,  $\omega_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ ,  $\omega_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ .

This situation can be handled similarly by our methods as done for the differential equation (2.1).

### 13. Asymptotic theory under Case (2), (the critical case)

In this section until §16 we discuss case (2) of the differential equation (8.1), this case will be given in the statement of Theorem 13.1, which is stated below. Of course case (2) is the critical case of the differential equation (8.1) as mentioned in the introduction section. As we shall see this critical case represents a borderline between situations where two solutions of (8.1) are small compared to algebraic functions and where two solutions are small compared to an exponential functions while the third solution having algebraic behaviour in both situations.

The proof of Theorem 13.1 will be given through subsequent sections, two examples on Theorem 13.1 are given in §16 and discussion of the form of the solutions for this case is given at the end of §16, also discussion about a critical case of the differential equation (2.1) and a second critical case of the differential equation (8.1) are given at the end of §16. First we shall state Theorem 13.1, and then some comments and the method in outline of this case are given directly after the statement of the Theorem 13.1.

#### Theorem 13.1

Let the coefficients  $p$ ,  $r$  and  $s$  in (8.1) be  $C^2[a, \infty)$  and nowhere zero in  $[a, \infty)$ . Let condition (9.1) of Theorem 9.1 hold, and Let

$$\frac{p'}{p} \delta, \frac{r'}{r} \delta, \frac{s'}{s} \delta \quad \text{be } L(a, \infty) \quad (13.1)$$

where  $\delta$  as given in (9.14), and let the critical case

$$\frac{(pr)'}{pr} = 4\sigma \frac{p}{s} (1+\phi), \quad (13.2)$$

$$\frac{(p s^{-\frac{1}{2}})'}{p s^{-\frac{1}{2}}} = \omega \frac{p}{s} (1+\psi) \quad (13.3)$$

hold, where  $\sigma$  and  $\omega$  are a non-zero constants such that

$$\omega \neq -1 \quad (13.4)$$

and

$$2\sigma - \omega \neq 1. \quad (13.5)$$

Also the functions  $\phi \rightarrow 0$  as  $x \rightarrow \infty$  and  $\psi \rightarrow 0$  as  $x \rightarrow \infty$ . Let

$$\left\{ \frac{s(pr)'}{p^2 r} \right\}' \text{ be } L(a, \infty), \quad (13.6)$$

$$\left\{ \frac{s^{3/2}(p s^{-\frac{1}{2}})'}{p^2} \right\}' \text{ be } L(a, \infty) \quad (13.7)$$

and

$$\frac{sr}{p^2} \text{ be } L(a, \infty). \quad (13.8)$$

Finally let **one** of the three following cases A, B and C hold.

(A)  $r$ ,  $p$  and  $s$  are real-valued functions.

(B)  $p(x) = i P(x)$  where  $P$  real-valued function and  $s$  real-valued function and  $r$  is the complex-valued is given by  $r(x) = \rho(x) e^{i\theta(x)}$ , where  $\rho$  and  $\theta$  are real-valued, such that

$$\lim_{x \rightarrow \infty} \frac{s}{p} \left\{ \frac{(P|r|)'}{P|r|} \right\} \neq 0.$$

and

$$\lim_{x \rightarrow \infty} \frac{s}{p} \left\{ \frac{(|r| s p^{-1})'}{|r| s p^{-1}} \right\} \neq 0.$$

(C)  $s(x) = iS(x)$  where  $S$  real-valued function and  $p$  real-valued function and  $r$  is the complex-valued function as given in case (B), such that

$$\lim_{x \rightarrow \infty} \frac{s}{p} \left\{ \frac{(p|r|)'}{p|r|} \right\} \neq 0$$

and

$$\lim_{x \rightarrow \infty} \frac{s}{p} \left\{ \frac{(|r| s p^{-1})'}{|r| s p^{-1}} \right\} \neq 0.$$

Then (8.1) has three linearly independent solutions  $y_k(x)$  such that

$$y_1(x) = o \left\{ (p(x)r(x))^{-\frac{1}{2}} \right\}, \quad \text{as } x \rightarrow \infty \quad (13.9)$$

$$y_2(x) \sim (p(x)r(x))^{-\frac{1}{2}} \exp \left[ 2\sigma \int_a^x \frac{p}{s} (1+\phi) dt \right], \quad \text{as } x \rightarrow \infty \quad (13.10)$$

$$y_3(x) = o \left\{ (p(x)r(x))^{-\frac{1}{2}} \exp \left[ \omega \int_a^x \frac{p}{s} \psi dt + (1+\omega) \int_a^x \frac{p}{s} dt \right] \right\}.$$

$$\text{as } x \rightarrow \infty \quad (13.11)$$

We note that as a comment in the above theorem, that by (13.2) the non-zero constant  $\sigma$  is given by

$$\sigma = \frac{1}{4} \lim_{x \rightarrow \infty} \frac{s}{p} \left\{ \frac{(pr)'}{pr} \right\}. \quad (13.12)$$

Also subject to (13.3), the non-zero constant  $\omega$  is given by

$$\omega = \lim_{x \rightarrow \infty} \frac{s}{p} \left\{ \frac{(p s^{-\frac{1}{2}})' }{p s^{-\frac{1}{2}}} \right\}. \quad (13.13)$$

Also by (13.2) and (13.3)

$$\phi' = \frac{1}{4\sigma} \left\{ \frac{s(pr)'}{p^2 r} \right\}, \quad (13.14)$$

and

$$\psi' = \frac{1}{\omega} \left\{ \frac{s^{3/2}(ps^{-\frac{1}{2}})'}{p^2} \right\}. \quad (13.15)$$

Hence subject to (13.6) and (13.7)

$$\phi' \quad \text{is } L(a, \infty), \quad (13.16)$$

and

$$\psi' \quad \text{is } L(a, \infty). \quad (13.17)$$

Also we note Theorem 13.1 covered two cases where some of our coefficients are complex-valued functions, in particular when  $p$  is pure imaginary and  $r$  is the complex-valued as given in case B, and the case C which  $s$  is pure imaginary and  $r$  complex-valued function.

Now we shall give the method in outline for this critical case, first we note subject to Theorem 13.1, the coefficients  $s$ ,  $r$  and  $p$  are nowhere zero in  $[a, \infty)$  and also condition (9.1) of Theorem 9.1, holds here, hence the characteristic equation (8.4) can be solved as given in (9.8) (9.9) and (9.10), thus some of the work which is done earlier for case (1) during the proof of Theorem 9.1 also holds for case (2), in particular from §9 equation (9.8) until §11 equation (11.4) where  $\delta'_k$  and  $R_2$  in this case are

$L(a, \infty)$  subject to (13.1). Hence our method for this case can be applied directly to the system (11.1), and so subject to (13.2) and (13.3), the non-zero elements of the matrix  $R_1$  which are given in (11.2), (11.3) and (11.4) having same size as  $\lambda_3$  as  $x \rightarrow \infty$ , thus Theorem 1.2 is not applied directly to the system (11.1) in this case because condition (i) of this theorem doesn't hold, and so the method here is to transform again the system (11.1) into the standard form

$$W' = (\Lambda_1 + M + T_1^{-1} R_1 T_1) W \quad (13.18)$$

where  $\Lambda_1$  is a diagonal matrix say

$$\Lambda_1(x) = \text{dg}(\tilde{\lambda}_1(x), \tilde{\lambda}_2(x), \tilde{\lambda}_3(x)) \quad (13.19)$$

such that  $T_1^{-1} R_1 T_1$  is  $L(a, \infty)$  and the matrix  $\Lambda_1 + M$  satisfy the hypotheses of Theorem 1.2, then Theorem 1.2 is applicable and it follows that (13.18) has solutions  $W_k$  ( $1 \leq k \leq 3$ ) with asymptotic forms

$$W_k(x) = \{e_k + o(1)\} \exp \left( \int_a^x \tilde{\mu}_k(t) dt \right) \quad (1 \leq k \leq 3) \quad (13.20)$$

as  $x \rightarrow \infty$ , where  $\tilde{\mu}_k$  ( $1 \leq k \leq 3$ ) are the eigenvalues of the matrix  $\Lambda_1 + M$ . Then the forms of the solutions of (8.1), are obtained by transforming back.

The first step in the transformation to (13.18), is by making use of (13.2), (13.3), (11.2), (11.3) and (11.4), we can write

$$\Lambda + R_1 = \mu_3(S_1 + S_2) \quad (13.21)$$

where  $S_1$  is a constant matrix and  $S_2(x) \rightarrow 0$  as  $x \rightarrow \infty$ , thus by (13.21) we can write (11.1) as

$$Z' = (\mu_3 S_1 + \mu_3 S_2 + R_2)Z. \quad (13.22)$$

The next step is to diagonalize the constant matrix  $S_1$ ; that, we define a suitable constant matrix  $T_1$  such that

$$T_1^{-1} S_1 T_1 = \text{dg}(\alpha_1, \alpha_2, \alpha_3) \quad (13.23)$$

say, the  $\alpha_k$  ( $1 \leq k \leq 3$ ) being the eigenvalues of the constant matrix  $S_1$ .

The substitution

$$Z = T_1 W \quad (13.24)$$

takes (13.22) into

$$W' = (\mu_3 T_1^{-1} S_1 T_1 + \mu_3 T_1^{-1} S_2 T_1 + T_1^{-1} R_2 T_1)W \quad (13.25)$$

such that

$$T_1^{-1} R_2 T_1 \text{ is } L(a, \infty).$$

Hence by taking

$$\Lambda_1 = \mu_3 T_1^{-1} S_1 T_1 = \text{dg}(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3),$$

$$M = \mu_3 T_1^{-1} S_2 T_1$$

then (13.25) is in the required form (13.18) by showing that

$\Lambda_1 + M$  satisfies Theorem 1.2.

Now we start our work in some detail by giving the proof of Theorem 13.1

#### Proof of Theorem 13.1

Now by Theorem 13.1 condition (9.1) of Theorem 9.1 holds, hence as explained above our method for this critical case can apply directly to the system (11.1), and so the system (11.1) holds with  $\Lambda(x)$  as given by (9.8), (9.9) and (9.10), and the matrix  $R_1$  as given by (11.2), (11.3) and (11.4) where  $\delta'_k$  ( $1 \leq k \leq 3$ ) and  $R_2$  in this case are both  $L(a, \infty)$  subject to (13.1).

Since Theorem 1.2 doesn't apply directly to this system under the critical case which is given by (13.2) and (13.3), hence it suffices to transform again the system (11.1) into the system (13.18).

The first step in the transformation to (13.18) is to write  $\Lambda + R_1$  as given in (13.21). Now since

$$\Lambda = \text{dg} (\lambda_1, \lambda_2, \lambda_3)$$

and (11.2) holds, hence

$$\Lambda + R_1 = \begin{pmatrix} \lambda_1 + \eta & -\eta & 0 \\ -\eta & \lambda_2 + \eta & 0 \\ -\frac{1}{2}\theta & -\frac{1}{2}\theta & \lambda_3 + \theta \end{pmatrix}.$$

Taking  $\lambda_3$  as a factor in the above matrix, then

$$\Lambda + R_1 = \lambda_3 \begin{bmatrix} \frac{\lambda_1}{\lambda_3} + \frac{\eta}{\lambda_3} & -\frac{\eta}{\lambda_3} & 0 \\ -\frac{\eta}{\lambda_3} & \frac{\lambda_2}{\lambda_3} + \frac{\eta}{\lambda_3} & 0 \\ -\frac{\theta}{2\lambda_3} & -\frac{\theta}{2\lambda_3} & 1 + \frac{\theta}{\lambda_3} \end{bmatrix} \quad (13.26)$$

Now our aim is to write (13.26) as given in (13.21), i.e. is to write the matrix which is on the right hand side of (13.26) as a sum of two matrix  $S_1$  and  $S_2$  such that  $S_1$  is a constant matrix and  $S_2(x) \rightarrow 0$  as  $x \rightarrow \infty$ .

Hence by (11.3) and (11.4)

$$\frac{\eta}{\lambda_3} = \frac{1}{4} \frac{(\text{pr})'}{\text{pr}} \cdot \frac{1}{\lambda_3}, \quad (13.27)$$

$$\frac{\theta}{\lambda_3} = \frac{(p s^{-\frac{1}{2}})'}{p s^{-\frac{1}{2}}} \cdot \frac{1}{\lambda_3}. \quad (13.28)$$

Now by (13.2), (13.27) becomes

$$\frac{\eta}{\lambda_3} = \sigma \frac{p}{s} (1+\phi) \cdot \frac{1}{\lambda_3}. \quad (13.29)$$

Hence by (9.10), (13.29) becomes

$$\frac{\eta}{\lambda_3} = \left\{ \sigma \frac{p}{s} (1+\phi) \right\} \left\{ \frac{s}{p} (1+\delta_3)^{-1} \right\} = \sigma (1+\phi) (1+\delta_3)^{-1}$$

i.e.

$$\frac{\eta}{\lambda_3} = \sigma \{1 + (\phi - \delta_3) \cdot (1+\delta_3)^{-1}\}. \quad (13.30)$$

Now since  $\phi \rightarrow 0$  as  $x \rightarrow \infty$  and  $\delta_3 \rightarrow 0$  as  $x \rightarrow \infty$ , hence

$$\frac{\eta}{\lambda_3} \rightarrow \sigma \quad \text{as } x \rightarrow \infty. \quad (13.31)$$

Similarly by (13.3), (13.28) becomes

$$\frac{\theta}{\lambda_3} = \omega \frac{p}{s} (1+\psi) \cdot \frac{1}{\lambda_3} \quad (13.32)$$

Now by (9.10), (13.32) becomes

$$\frac{\theta}{\lambda_3} = \{\omega \frac{p}{s} (1+\psi)\} \{\frac{s}{p} (1+\delta_3)^{-1}\} = \omega (1+\psi) (1+\delta_3)^{-1}$$

i.e.

$$\frac{\theta}{\lambda_3} = \omega \{1 + (\psi - \delta_3) (1+\delta_3)^{-1}\}. \quad (13.33)$$

Since  $\psi \rightarrow 0$  as  $x \rightarrow \infty$  and  $\delta_3 \rightarrow 0$  as  $x \rightarrow \infty$ , hence

$$\frac{\theta}{\lambda_3} \rightarrow \omega \quad \text{as } x \rightarrow \infty \quad (13.34)$$

Also by (9.8) and (9.10)

$$\frac{\lambda_1}{\lambda_3} = \{i (r/p)^{\frac{1}{2}} (1+\delta_1)\} \{\frac{s}{p} (1+\delta_3)^{-1}\} = i \frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}} (1+\delta_1) (1+\delta_3)^{-1}$$

i.e.

$$\frac{\lambda_1}{\lambda_3} = i \delta (1+\delta_1) (1+\delta_3)^{-1}, \quad (13.35)$$

and we make use of (9.14) in (13.35). Now since  $\delta \rightarrow 0$  as

$x \rightarrow \infty$ ,  $\delta_1 \rightarrow 0$  as  $x \rightarrow \infty$  and  $\delta_3 \rightarrow 0$  as  $x \rightarrow \infty$ , hence

$$\frac{\lambda_1}{\lambda_3} \rightarrow 0 \quad \text{as } x \rightarrow \infty \quad (13.36)$$

Similarly by (9.9), (9.10) and (9.14)

$$\frac{\lambda_2}{\lambda_3} = -i \delta (1+\delta_2) (1+\delta_3)^{-1}, \quad (13.37)$$

thus

$$\frac{\lambda_2}{\lambda_3} \rightarrow 0 \quad \text{as } x \rightarrow \infty. \quad (13.38)$$

Now by (13.30) and (13.33) we can write the matrix (13.25) as given in (13.21) by taking the constant matrix  $S_1$  to be

$$S_1 = \begin{pmatrix} \sigma & -\sigma & 0 \\ -\sigma & \sigma & 0 \\ -\frac{1}{2}\omega & -\frac{1}{2}\omega & 1+\omega \end{pmatrix}, \quad (13.39)$$

and the matrix  $S_2$  to be

$$S_2 = \begin{pmatrix} u_1 & u_2 & 0 \\ u_2 & u_3 & 0 \\ u_4 & u_4 & u_5 \end{pmatrix} \quad (13.40)$$

such that

$$\left. \begin{aligned} u_1 &= \frac{\lambda_1}{\lambda_3} + \rho_1 \\ u_2 &= -\rho_1 \\ u_3 &= \frac{\lambda_2}{\lambda_3} + \rho_1 \\ u_4 &= -\frac{1}{2} \rho_2 \\ u_5 &= \rho_2 \end{aligned} \right\} \quad (13.41)$$

where

$$\rho_1 = \sigma (1+\delta_3)^{-1} (\phi - \delta_3), \quad (13.42)$$

$$\rho_2 = \omega (1+\delta_3)^{-1} (\psi - \delta_3) \quad (13.43)$$

such that

$$\rho_1 \rightarrow 0, \quad \text{as } x \rightarrow \infty \quad (13.44)$$

$$\rho_2 \rightarrow 0 \quad \text{as } x \rightarrow \infty \quad (13.45)$$

because  $\phi \rightarrow 0$  as  $x \rightarrow \infty$ ,  $\psi \rightarrow 0$  as  $x \rightarrow \infty$  and  $\delta_3 \rightarrow 0$  as  $x \rightarrow \infty$ .

Hence by (13.36), (13.38), (13.44), (13.45) and (13.41) the matrix

$$S_2(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty. \quad (13.46)$$

The next step is to diagonalize the constant matrix  $S_1$ ; which is given by (13.39). Hence we require the eigenvalues  $\alpha_j$  ( $1 \leq j \leq 3$ ) of  $S_1$ . The characteristic equation of  $S_1$  is given by

$$\det(S_1 - \alpha I_3) = \begin{vmatrix} \sigma - \alpha & -\sigma & 0 \\ -\sigma & \sigma - \alpha & 0 \\ -\frac{1}{2}\omega & -\frac{1}{2}\omega & 1 + \omega - \alpha \end{vmatrix} = 0$$

i.e.

$$\alpha(\alpha - 2\sigma)(1 + \omega - \alpha) = 0 \quad (13.47)$$

Hence solving (13.47) for  $\alpha_j$  ( $1 \leq j \leq 3$ ) we obtain

$$\alpha_1 = 0, \quad \alpha_2 = 2\sigma, \quad \alpha_3 = 1 + \omega. \quad (13.48)$$

Now, it's obvious to see that  $\alpha_j$  ( $1 \leq j \leq 3$ ) are distinct subject to the hypotheses of Theorem 13.1 i.e. see (13.4) and (13.5)

Now by (13.39) and (13.48) we can define the suitable constant matrix  $T_1$  which satisfies the relation (13.23) by

$$T_1 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ \frac{\omega}{1+\omega} & 0 & 1 \end{pmatrix}. \quad (13.49)$$

Also subject to (13.48), (13.23) can be written

$$T_1^{-1} S_1 T_1 = \text{dg}(0, 2\sigma, 1+\omega) \quad (13.50)$$

Hence the substitution (13.24) takes (13.22) into the system (13.25).

14. The System  $W' = (\Lambda_1 + M + T_1^{-1} R_1 T_1) W$

Now taking

$$\Lambda_1 = \lambda_3 T_1^{-1} S_1 T_1 = \text{dg}(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3), \quad (14.1)$$

$$M = \lambda_3 T_1^{-1} S_2 T_1, \quad (14.2)$$

then the system (13.25) reduced to the system

$$W' = (\Lambda_1 + M + T_1^{-1} R_2 T_1) W. \quad (14.3)$$

Now since  $R_2$  is  $L(a, \infty)$  and  $T_1$  is a constant matrix, hence  $T_1^{-1} R_2 T_1$  is  $L(a, \infty)$ . Now our aim is to show that  $\Lambda_1 + M$  satisfies the hypotheses of Theorem 1.2.

Now by (13.50) and (14.1)

$$\Lambda_1 = \text{dg}(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3) = \text{dg}(0, 2\sigma\lambda_3, (1+\omega)\lambda_3). \quad (14.4)$$

Now first we need to show that the eigenvalues  $\tilde{\lambda}_j$  ( $1 \leq j \leq 3$ ) of  $\Lambda_1$  are distinct; since the  $\alpha_j$  which are given by (13.48) are distinct, hence  $\tilde{\lambda}_j$  are distinct.

Second we need to show that condition (i) of Theorem 1.2, holds i.e.

$$M(x) = o \left\{ \tilde{\lambda}_i(x) - \tilde{\lambda}_j(x) \right\} \quad \text{as } x \rightarrow \infty \quad (14.5)$$

for  $(i \neq j, 1 \leq i, j \leq 3)$ .

Now by (14.4)

$$\tilde{\lambda}_2 - \tilde{\lambda}_1 = 2\sigma\lambda_3, \quad (14.6)$$

$$\tilde{\lambda}_3 - \tilde{\lambda}_1 = (1+\omega)\lambda_3, \quad (14.7)$$

$$\tilde{\lambda}_2 - \tilde{\lambda}_3 = \{2\sigma - \omega - 1\} \lambda_3. \quad (14.8)$$

Thus by (14.6), (14.7) and (14.8), we can write

$$\tilde{\lambda}_i(x) - \tilde{\lambda}_j(x) = (\text{const.}) \lambda_3(x). \quad (i \neq j, 1 \leq i, j \leq 3) \quad (14.9)$$

Also we note that since  $T_1$  is a constant matrix and  $S_2(x) \rightarrow 0$  as  $x \rightarrow \infty$ , hence

$$(T_1^{-1} S_2 T_1) \rightarrow 0 \quad \text{as } x \rightarrow \infty \quad (14.10)$$

Now by (14.2) and (14.9), we have for  $(i \neq j, 1 \leq i, j \leq 3)$

$$\frac{M}{\tilde{\lambda}_i - \tilde{\lambda}_j} = (\text{const.}) \frac{(\lambda_3 T_1^{-1} S_2 T_1)}{\lambda_3} = (\text{const.}) (T_1^{-1} S_2 T_1) \rightarrow 0 \text{ as } x \rightarrow \infty \quad (14.11)$$

and we make use of (14.10). Thus by (14.11), (14.5) holds for  $(i \neq j, 1 \leq i, j \leq 3)$ .

Next we need to show (ii) holds i.e.

$$\{(\tilde{\lambda}_i - \tilde{\lambda}_j)^{-1} M\}' \quad \text{is } L(a, \infty) \quad (i \neq j, 1 \leq i, j \leq 3). \quad (14.12)$$

Now by (14.2) and (14.9), we have for  $(i \neq j, 1 \leq i, j \leq 3)$

$$\{(\tilde{\lambda}_i - \tilde{\lambda}_j)^{-1} M\}' = (\text{const.}) (T_1^{-1} S_2 T_1)' = (\text{const.}) T_1^{-1} S_2' T_1. \quad (14.13)$$

Thus subject to (14.13), (14.12) holds if we show that

$$S_2' \quad \text{is } (L(a, \infty)). \quad (14.14)$$

Now to show that (14.14) holds we make

use of (13.40) and (13.41). Differentiating (13.40) and taking the elements in  $S'_2$  individually, we begin with  $u'_1$ . Hence by (13.41)

$$u'_1 = \left( \frac{\lambda_1}{\lambda_3} \right)' + \rho'_1. \quad (14.15)$$

Now by (13.35)

$$\left( \frac{\lambda_1}{\lambda_3} \right)' = i \delta' (1+\delta_1) (1+\delta_3)^{-1} + i \delta'_1 \delta (1+\delta_3)^{-1} - i \delta (1+\delta_1) (1+\delta_3)^{-2} \delta'_3$$

i.e.

$$\left( \frac{\lambda_1}{\lambda_3} \right)' = O(\delta') + O(\delta'_1 \delta) + O(\delta'_3 \delta). \quad (14.16)$$

Now by (13.1) all the terms on the right of (11.30) are  $L(a, \infty)$ .

Thus

$$\delta'(x) \text{ is } L(a, \infty). \quad (14.17)$$

Also as mentioned in the beginning of the proof

$$\delta'_k(x) \text{ is } L(a, \infty) \quad (1 \leq k \leq 3). \quad (14.18)$$

subject to (13.1) in this case, where  $\delta'_k$  ( $1 \leq k \leq 3$ ) are given by (9.32), (9.33) and (9.34).

Hence by (14.17) and (14.18), all the terms on the right of (14.16) are  $L(a, \infty)$ , and so

$$\left( \frac{\lambda_1}{\lambda_3} \right)' \text{ is } L(a, \infty). \quad (14.19)$$

Also by (13.42)

$$\rho'_1 = -\sigma \delta'_3 (1+\delta_3)^{-1} \{1 + (\phi - \delta_3) (1+\delta_3)^{-1}\} + \sigma \phi' (1+\delta_3)^{-1},$$

i.e.

$$\rho_1' = O(\delta_3') + O(\phi'). \quad (14.20)$$

Thus ,by (13.16) and (14.18)

$$\rho_1' \text{ is } L(a, \infty). \quad (14.21)$$

Hence by (14.15), (14.19) and (14.21)

$$u_1' \text{ is } L(a, \infty). \quad (14.22)$$

Next we consider  $u_2'$ , then by (13.41)

$$u_2' = -\rho_1'.$$

Hence by (14.21)

$$u_2' \text{ is } L(a, \infty). \quad (14.23)$$

Now we consider  $u_3'$ , then by (13.41)

$$u_3' = \left( \frac{\lambda_2}{\lambda_3} \right)' + \rho_1'. \quad (14.24)$$

Now by (13.37)

$$\left( \frac{\lambda_2}{\lambda_3} \right)' = O(\delta') + O(\delta_2' \delta) + O(\delta_3' \delta). \quad (14.25)$$

Hence by (14.17) and (14.18), all the terms on the right of (14.25) are  $L(a, \infty)$ . Thus

$$\left( \frac{\lambda_2}{\lambda_3} \right)' \text{ is } L(a, \infty). \quad (14.26)$$

Hence making use of (14.21) and (14.26) in (14.24), then

$$u_3' \text{ is } L(a, \infty). \quad (14.27)$$

Finally we consider  $u'_5$  and  $u'_4$ , then by (13.41)

$$u'_5 = -2u'_4 = \rho'_2. \quad (14.28)$$

Now by (13.43)

$$\rho'_2 = O(\delta'_3) + O(\psi'). \quad (14.29)$$

So by (13.17) and (14.18)

$$\rho'_2 \text{ is } L(a, \infty). \quad (14.30)$$

Hence by (14.30) and (14.28)

$$u'_4 \text{ is } L(a, \infty), \quad (14.31)$$

and

$$u'_5 \text{ is } L(a, \infty). \quad (14.32)$$

Now by (14.22), (14.23), (14.27), (14.31) and (14.32), (14.14) holds which makes (14.12) holds.

Finally we need to show that the eigenvalues of  $\Lambda_1 + M$  satisfy the Dichotomy condition I-II in Levinson's Theorem 1.1.

Now since by (14.1) and (14.2)

$$\Lambda_1 + M = \lambda_3 (T_1^{-1} S_1 T_1 + T_1^{-1} S_2 T_1) = \lambda_3 T_1^{-1} (S_1 + S_2) T_1.$$

Thus from the definition of an eigenvalue, we can see that the eigenvalues of the matrix  $\Lambda_1 + M$  are equal to the eigenvalues of the matrix  $\lambda_3 (S_1 + S_2)$ . Hence if  $\tilde{\mu}_k(x)$  ( $1 \leq k \leq 3$ ) are the eigenvalues of the matrix  $\Lambda_1 + M$ , then subject to the above relation  $\tilde{\mu}_k$  can be obtained directly by considering the characteristic equation of the matrix  $\lambda_3 (S_1 + S_2)$ .

i.e.

$$\det(\Lambda_1 + M - \tilde{\mu} I_3) = \det(\lambda_3 s_1 + \lambda_3 s_2 - \tilde{\mu} I_3) = 0. \quad (14.33)$$

Hence by (13.39), (13.40) and (13.41), (14.33) gives

$$\left\{ [\lambda_3(\sigma + \rho_1) + \lambda_1 - \tilde{\mu}] [\lambda_3(\sigma + \rho_1) + \lambda_2 - \tilde{\mu}] - \lambda_3^2(\sigma + \rho_1)^2 \right\} \times \\ \times \left\{ \lambda_3(1 + \omega + \rho_2) - \tilde{\mu} \right\} = 0.$$

Hence the eigenvalues  $\tilde{\mu}_j$  ( $1 \leq j \leq 3$ ) of the matrix  $\Lambda_1 + M$  are given by

$$[\lambda_3(\sigma + \rho_1) + \lambda_1 - \tilde{\mu}] [\lambda_3(\sigma + \rho_1) + \lambda_2 - \tilde{\mu}] - \lambda_3^2(\sigma + \rho_1)^2 = 0 \quad (14.34)$$

and

$$\lambda_3(1 + \omega + \rho_2) - \tilde{\mu} = 0. \quad (14.35)$$

Hence solving (14.34) for  $\tilde{\mu}_1$  and  $\tilde{\mu}_2$  we obtain

$$\tilde{\mu}_1 = \frac{1}{2} (\lambda_1 + \lambda_2) + O \left\{ \frac{(\lambda_2 - \lambda_1)^2}{\lambda_3(\sigma + \rho_1)} \right\},$$

$$\tilde{\mu}_2 = \frac{1}{2} (\lambda_1 + \lambda_2) + 2\lambda_3(\sigma + \rho_1) + O \left\{ \frac{(\lambda_2 - \lambda_1)^2}{\lambda_3(\sigma + \rho_1)} \right\},$$

since

$$\frac{1}{\sigma + \rho_1} = \frac{1}{\sigma} (1 + \frac{\rho_1}{\sigma})^{-1} = O(1).$$

Hence

$$\tilde{\mu}_1 = \frac{1}{2} (\lambda_1 + \lambda_2) + O \left\{ \frac{(\lambda_2 - \lambda_1)^2}{\lambda_3} \right\}, \quad (14.36)$$

$$\tilde{\mu}_2 = \frac{1}{2} (\lambda_1 + \lambda_2) + 2\lambda_3(\sigma + \rho_1) + O \left\{ \frac{(\lambda_2 - \lambda_1)^2}{\lambda_3} \right\}. \quad (14.37)$$

Also by (14.35)

$$\tilde{\mu}_3 = \lambda_3(1 + \omega + \rho_2). \quad (14.38)$$

Now the Dichotomy condition I-II of Levinson's Theorem 1.1, holds if we show that

$$\operatorname{Re}\{\tilde{\mu}_i(x) - \tilde{\mu}_j(x)\} = f(x) + g(x), \quad (14.39)$$

where  $f(x)$  does not change sign in  $[a, \infty)$  and  $g(x)$  is  $L(a, \infty)$ .

Now by (14.36), (14.37) and (14.38)

$$\left. \begin{aligned} \tilde{\mu}_1 - \tilde{\mu}_2 &= -2\lambda_3(\sigma + \rho_1) + O\left\{\frac{(\lambda_2 - \lambda_1)^2}{\lambda_3}\right\}, \\ \tilde{\mu}_2 - \tilde{\mu}_3 &= \frac{1}{2}(\lambda_1 + \lambda_2) + 2\lambda_3(\sigma + \rho_1) - \lambda_3(1 + \omega + \rho_2) \\ &\quad + O\left\{\frac{(\lambda_2 - \lambda_1)^2}{\lambda_3}\right\}, \\ \tilde{\mu}_3 - \tilde{\mu}_1 &= \lambda_3(1 + \omega + \rho_2) - \frac{1}{2}(\lambda_1 + \lambda_2) + O\left\{\frac{(\lambda_2 - \lambda_1)^2}{\lambda_3}\right\}. \end{aligned} \right\} \quad (14.40)$$

Now we shall work out (14.40) in detail, and we shall see (14.39) holds in each case of Theorem 13.1.

Now by (9.10) and (13.42)

$$\lambda_3(\sigma + \rho_1) = \sigma \frac{p}{s}(1 + \phi). \quad (14.41)$$

Also by (9.8), (9.9) and (9.10)

$$\frac{(\lambda_2 - \lambda_1)^2}{\lambda_3} = -4 \frac{sr}{p^2} \left(1 + \frac{\delta_1}{2} + \frac{\delta_2}{2}\right) (1 + \delta_3)^{-1} = O(h) \quad (14.42)$$

where

$$h = \frac{sr}{p^2}. \quad (14.43)$$

Thus by (13.8)

$$h(x) \text{ is } L(a, \infty). \quad (14.44)$$

Now by (9.8), (9.9), (9.20) and (9.21)

$$\lambda_1 + \lambda_2 = i \left( \frac{r}{p} \right)^{\frac{1}{2}} \left\{ 1 - \frac{i\delta}{2-3i\delta} \right\} - i \left( \frac{r}{p} \right)^{\frac{1}{2}} \left\{ 1 - \frac{i\delta}{2+3i\delta} \right\}$$

i.e.

$$\lambda_1 + \lambda_2 = - (r/p)^{\frac{1}{2}} \delta \left\{ 1 + \frac{9}{4}\delta \right\}^{-1} = O(h), \quad (14.45)$$

and we make use of (9.14) and (14.43).

Now by (14.40), (14.41) and (14.42)

$$\tilde{\mu}_1 - \tilde{\mu}_2 = -2\sigma \frac{p}{s} (1+\phi) + O(h). \quad (14.46)$$

Also by (9.10), (9.24), (13.43) and (14.43)

$$\lambda_3 (1 + \omega + \rho_2) = \frac{p}{s} (1 + \omega + \omega\psi) + O(h). \quad (14.47)$$

Hence by (14.40), (14.41), (14.42), (14.45) and (14.47)

$$\tilde{\mu}_2 - \tilde{\mu}_3 = 2\sigma \frac{p}{s} (1+\phi) - \frac{p}{s} (1 + \omega + \omega\psi) + O(h). \quad (14.48)$$

Also by (14.40), (14.42), (14.45) and (14.47)

$$\tilde{\mu}_3 - \tilde{\mu}_1 = \frac{p}{s} (1 + \omega + \omega\psi) + O(h). \quad (14.49)$$

We note that the O-term in (14.46), (14.48), and (14.49) is

$L(a, \infty)$  subject to (14.44). Now first we need to show that

(14.39) holds under case A of Theorem 13.1. Since under case A  $r$ ,  $p$  and  $s$  are real-valued functions, hence the non-zero constants  $\sigma$  and  $\omega$  which are given in (13.2) and (13.3) are real constants, also  $\phi$  and  $\psi$  are real-valued. Hence by (14.46)

$$\operatorname{Re}(\tilde{\mu}_1 - \tilde{\mu}_2) = -2\sigma \frac{P}{S} (1 + o(1)) + O(h). \quad (14.50)$$

Also since (13.5) holds, hence by (14.48)

$$\operatorname{Re}(\tilde{\mu}_2 - \tilde{\mu}_3) = (2\sigma - \omega - 1) \frac{P}{S} \{1 + o(1)\} + O(h). \quad (14.51)$$

Also since (13.4) holds, hence by (14.49)

$$\operatorname{Re}(\tilde{\mu}_3 - \tilde{\mu}_1) = (1 + \omega) \frac{P}{S} \{1 + o(1)\} + O(h). \quad (14.52)$$

Thus by (14.50), (14.51) and (14.52), (14.39) holds under case A and consequently the Dichotomy condition of Levinson's Theorem 1.1 holds under case A of Theorem 13.1.

Next we need to show that (14.39) holds under case B of Theorem 13.1. Since under case B,  $p = iP$ , hence (14.46) gives

$$\tilde{\mu}_1 - \tilde{\mu}_2 = -2i \frac{P}{S} \sigma (1 + \phi) + O(h).$$

since  $P$  and  $S$  are real-valued, hence

$$\operatorname{Re}(\tilde{\mu}_1 - \tilde{\mu}_2) = 2 \frac{P}{S} \{ \operatorname{Im} \sigma + \operatorname{Im}(\sigma \phi) \} + O(h). \quad (14.53)$$

We note that

$$\operatorname{Im}(\sigma \phi) = \{ \operatorname{Re} \sigma \cdot \operatorname{Im} \phi + \operatorname{Re} \phi \cdot \operatorname{Im} \sigma \} \rightarrow 0 \quad \text{as } x \rightarrow \infty$$

because  $\sigma$  is a constant and  $\phi \rightarrow 0$  as  $x \rightarrow \infty$  (i.e.  $\operatorname{Re} \phi \rightarrow 0$  as  $x \rightarrow \infty$  and  $\operatorname{Im} \phi \rightarrow 0$  as  $x \rightarrow \infty$ ).

Now since  $p = iP$  and  $r = \rho e^{i\theta}$ , hence (13.2) gives

$$\frac{P'}{P} + \frac{|r|'}{|r|} + i\theta' = 4\sigma i \frac{P}{S} (1 + \phi)$$

and so

$$\sigma = -\frac{1}{4} \lim_{x \rightarrow \infty} \frac{S}{P} \left\{ i \frac{P'}{P} + i \frac{|r|'}{|r|} - \theta' \right\}$$

thus

$$\text{Im} \sigma = -\frac{1}{4} \lim_{x \rightarrow \infty} \frac{S}{P} \left\{ \frac{P|r|'}{P|r|} \right\} \neq 0 \quad (14.54)$$

subject to case (B).

Hence by (14.54) we can write (14.53) as

$$\text{Re}(\tilde{\mu}_1 - \tilde{\mu}_2) = 2 \frac{P}{S} \text{Im} \sigma \{1 + o(1)\} + O(h). \quad (14.55)$$

Now putting  $p = iP$  in (14.49), then

$$\tilde{\mu}_3 - \tilde{\mu}_1 = i \frac{P}{S} (1 + \omega + \omega\psi) + O(h)$$

hence

$$\text{Re}(\tilde{\mu}_3 - \tilde{\mu}_1) = -\frac{P}{S} (\text{Im} \omega + \text{Im} \omega\psi) + O(h). \quad (14.56)$$

We note that

$$\text{Im} \omega\psi = \{\text{Im} \omega \cdot \text{Re} \psi + \text{Im} \psi \cdot \text{Re} \omega\} \rightarrow 0 \quad \text{as } x \rightarrow \infty$$

because  $\omega$  is a constant and  $\psi \rightarrow 0$  as  $x \rightarrow \infty$  (i.e.  $\text{Re} \psi \rightarrow 0$  as  $x \rightarrow \infty$  and  $\text{Im} \psi \rightarrow 0$  as  $x \rightarrow \infty$ ).

Now since  $p = iP$ , hence (13.3) gives

$$\omega = i \lim_{x \rightarrow \infty} \frac{S}{P} \left\{ \frac{(P^{-1} S^{\frac{1}{2}})'}{P^{-1} S^{\frac{1}{2}}} \right\}. \quad (14.57)$$

Of course here  $\omega$  is pure imaginary, and since  $\omega$  is a non-zero constant, hence (14.57) gives

$$\operatorname{Im} \omega = \lim_{x \rightarrow \infty} \frac{S}{P} \left\{ \frac{(P^{-1} S^{\frac{1}{2}})' }{P^{-1} S^{\frac{1}{2}}} \right\} \neq 0 \quad (14.58)$$

Hence by (14.58) we can write (14.56) as

$$\operatorname{Re}(\tilde{\mu}_3 - \tilde{\mu}_1) = -\frac{P}{S} \operatorname{Im} \omega (1 + o(1)) + O(h). \quad (14.59)$$

Now putting  $p = iP$  in (14.48), then

$$\tilde{\mu}_2 - \tilde{\mu}_3 = 2\sigma i \frac{P}{S}(1+\phi) - i \frac{P}{S}(1+\omega+\omega\psi) + O(h).$$

Hence

$$\operatorname{Re}(\tilde{\mu}_2 - \tilde{\mu}_3) = \frac{P}{S} \{ \operatorname{Im} \omega - 2\operatorname{Im} \sigma + \operatorname{Im} \omega \psi - 2\operatorname{Im} \sigma \phi \} + O(h). \quad (14.60)$$

Now by (14.58) and (14.54)

$$\operatorname{Im} \omega - 2\operatorname{Im} \sigma = \lim_{x \rightarrow \infty} \frac{S}{P} \left\{ \frac{(P^{-1} S^{\frac{1}{2}})' }{P^{-1} S^{\frac{1}{2}}} \right\} + \frac{1}{2} \lim_{x \rightarrow \infty} \frac{S}{P} \left\{ \frac{(P|r|)' }{P|r|} \right\}$$

i.e.

$$\operatorname{Im} \omega - 2\operatorname{Im} \sigma = \frac{1}{2} \lim_{x \rightarrow \infty} \frac{S}{P} \left\{ \frac{(|r| P^{-1} S)' }{|r| P^{-1} S} \right\} \neq 0 \quad (14.61)$$

subject to case (B).

Hence by (14.61) we can write (14.60) as

$$\operatorname{Re}(\tilde{\mu}_2 - \tilde{\mu}_3) = (\operatorname{Im} \omega - 2\operatorname{Im} \sigma) \frac{P}{S} \{1 + o(1)\} + O(h). \quad (14.62)$$

Thus by (14.55), (14.59) and (14.62), (14.39) holds under case B and consequently the Dichotomy condition of Levinson's Theorem 1.1 holds under case B of Theorem 13.1.

Finally we consider case C. Since under case C,  $s = iS$ , where  $S$  real-valued, hence (14.46) gives

$$\operatorname{Re}(\tilde{\mu}_2 - \tilde{\mu}_1) = -2 \frac{p}{S} (\operatorname{Im}\sigma + \operatorname{Im}\sigma\phi) + O(h). \quad (14.63)$$

since  $s = iS$  and  $r = \rho e^{i\theta}$ , hence (13.2) gives

$$\frac{p'}{p} + \frac{|r|'}{|r|} + i\theta' = -4\sigma i \frac{p}{S} (1 + \phi)$$

i.e.

$$\sigma = \frac{1}{4} \lim_{x \rightarrow \infty} \frac{S}{p} \left\{ \frac{(p|r|)'}{p|r|} i - \theta' \right\}$$

thus

$$\operatorname{Im}\sigma = \frac{1}{4} \lim_{x \rightarrow \infty} \frac{S}{p} \left\{ \frac{(p|r|)'}{p|r|} \right\} \neq 0. \quad (14.64)$$

subject to case C. Hence by (14.64) we can write (14.63) as

$$\operatorname{Re}(\tilde{\mu}_2 - \tilde{\mu}_1) = -2 \frac{p}{S} \operatorname{Im}\sigma (1 + o(1)) + O(h). \quad (14.65)$$

Now putting  $s = iS$  in (14.49), then

$$\operatorname{Re}(\tilde{\mu}_3 - \tilde{\mu}_1) = \frac{p}{S} (\operatorname{Im}\omega + \operatorname{Im}\omega\psi) + O(h). \quad (14.66)$$

Now since  $s = iS$ , hence (13.3) gives

$$\omega = i \lim_{x \rightarrow \infty} \frac{S}{p} \left\{ \frac{(p S^{-\frac{1}{2}})'}{p S^{-\frac{1}{2}}} \right\} \quad (14.67)$$

Of course here  $\omega$  is pure imaginary, and since  $\omega$  is non-zero constant, hence (14.67) gives

$$\operatorname{Im}\omega = \lim_{x \rightarrow \infty} \frac{S}{p} \left\{ \frac{(p S^{-\frac{1}{2}})'}{p S^{-\frac{1}{2}}} \right\} \neq 0. \quad (14.68)$$

Thus by (14.68) we can write (14.66) as

$$\operatorname{Re}(\tilde{\mu}_2 - \tilde{\mu}_1) = \frac{p}{S} \operatorname{Im}\omega(1 + o(1)) + O(h). \quad (14.69)$$

Now putting  $s = iS$  in (14.48), then

$$\operatorname{Re}(\tilde{\mu}_2 - \tilde{\mu}_3) = \frac{p}{S} \{2\operatorname{Im}\sigma - \operatorname{Im}\omega + 2\operatorname{Im}\sigma\omega - \operatorname{Im}\omega\psi\} + O(h). \quad (14.70)$$

Now by (14.68) and (14.64)

$$2\operatorname{Im}\sigma - \operatorname{Im}\omega = \frac{1}{2} \lim_{x \rightarrow \infty} \frac{S}{p} \left\{ \frac{(p|r|)'}{p|r|} \right\} - \lim_{x \rightarrow \infty} \frac{S}{p} \left\{ \frac{(p S^{-\frac{1}{2}})'}{p S^{-\frac{1}{2}}} \right\}$$

i.e.

$$2\operatorname{Im}\sigma - \operatorname{Im}\omega = \frac{1}{2} \lim_{x \rightarrow \infty} \frac{S}{p} \left\{ \frac{(p^{-1}S r)'}{p^{-1}S r} \right\} \neq 0. \quad (14.71)$$

subject to case C, hence by (14.71) we can write (14.70) as

$$\operatorname{Re}(\tilde{\mu}_2 - \tilde{\mu}_3) = (2\operatorname{Im}\sigma - \operatorname{Im}\omega) \frac{p}{S} \{1 + o(1)\} + O(h). \quad (14.72)$$

Thus by (14.65), (14.69) and (14.72), (14.39) holds under case C and consequently the Dichotomy condition of Levinson's Theorem 1.1 holds under case C of Theorem 13.1.

## 15. The Form of Solutions

Now that (14.3) has been shown to have the form (13.18), hence Theorem 1.2, is applicable and it follows that as  $x \rightarrow \infty$ , (14.3) has three linearly independent solutions  $W_k$  such that

$$W_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \tilde{\mu}_k(t) dt\right) \quad (1 \leq k \leq 3) \quad (15.1)$$

where  $e_k$  is the coordinate vector with  $k$ th component unity and other components zero. Now we need to obtain the three linearly independent solutions of our original system (2.2). Hence by (2.7) and (13.24), (2.2) has corresponding solutions

$$Y_k(x) = T(x) T_1 W_k(x) \quad (1 \leq k \leq 3) \quad (15.2)$$

where  $T(x)$  as given by (8.11) and (8.10), also the matrix  $T_1$  is given by (13.49). Hence taking the first component on each side of (15.2), we obtain three solutions  $y_k(x)$  of (8.1) such that as  $x \rightarrow \infty$ ,

$$y_1(x) = \left\{ m_1^{-1}(x) + m_2^{-1}(x) + \frac{\omega}{1+\omega} m_3^{-1}(x) + \sum_{r=1}^3 o(m_r^{-1}(x)) \right\} \exp\left(\int_a^x \tilde{\mu}_1(t) dt\right), \quad (15.3)$$

$$y_2(x) = \left\{ -m_1^{-1}(x) + m_2^{-1}(x) + \sum_{r=1}^3 o(m_r^{-1}(x)) \right\} \exp\left(\int_a^x \tilde{\mu}_2(t) dt\right), \quad (15.4)$$

$$y_3(x) = \left\{ m_3^{-1}(x) + \sum_{r=1}^3 o(m_r^{-1}(x)) \right\} \exp\left(\int_a^x \tilde{\mu}_3(t) dt\right). \quad (15.5)$$

Now by (15.3)

$$Y_1 = m_1^{-1} \left\{ 1 + m_1 m_2^{-1} + \frac{\omega}{1+\omega} m_1 m_3^{-1} + \sum_{r=1}^3 o(m_1 m_r^{-1}) \right\} \exp \left( \int_a^x \tilde{\mu}_1(t) dt \right).$$

Hence by (12.9) and (12.10)

$$Y_1(x) = o \left\{ m_1^{-1}(x) \exp \left( \int_a^x \tilde{\mu}_1(t) dt \right) \right\} \quad \text{as } x \rightarrow \infty. \quad (15.6)$$

Also by (15.4)

$$Y_2 = m_2^{-1} \left\{ 1 - m_2 m_1^{-1} + \sum_{r=1}^3 o(m_2 m_r^{-1}) \right\} \exp \left( \int_a^x \tilde{\mu}_2(t) dt \right).$$

Now by (12.9)

$$m_2 m_1^{-1} = -1 + o(1). \quad \text{as } x \rightarrow \infty \quad (15.7)$$

Also by (12.8), (12.7) and (9.1)

$$m_2 m_3^{-1} = o(1). \quad \text{as } x \rightarrow \infty \quad (15.8)$$

Thus by (15.7) and (15.8)

$$Y_2(x) = 2m_2^{-1}(x) \{1 + o(1)\} \exp \left( \int_a^x \tilde{\mu}_2(t) dt \right) \\ \text{as } x \rightarrow \infty$$

and so after adjusting a constant multiple in  $y_2$  we obtain

$$Y_2(x) \sim m_2^{-1}(x) \exp \left( \int_a^x \tilde{\mu}_2(t) dt \right) \quad \text{as } x \rightarrow \infty \quad (15.9)$$

Finally by (15.5)

$$Y_3 = m_1^{-1} \left\{ m_1 m_3^{-1} + \sum_{r=1}^3 o(m_1 m_r^{-1}) \right\} \exp \left( \int_a^x \tilde{\mu}_3(t) dt \right)$$

Hence by (12.9) and (12.10)

$$Y_3(x) = o \left\{ m_1^{-1}(x) \exp \left( \int_a^x \tilde{\mu}_3(t) dt \right) \right\} \quad \text{as } x \rightarrow \infty \quad (15.10)$$

Now we have to work out (15.6), (15.9) and (15.10) in some detail in terms of  $s$ ,  $p$  and  $r$  in order to determine the forms (13.9), (13.10) and (13.11).

Now by (14.45) and (14.42), (14.36) becomes

$$\tilde{\mu}_1 = O(h). \quad (15.11)$$

Similarly by (14.45), (14.42) and (14.41), (14.37) becomes

$$\tilde{\mu}_3 = 2\sigma \frac{p}{s} (1+\phi) + O(h). \quad (15.12)$$

Also by (14.47), (14.38) becomes

$$\tilde{\mu}_3 = \frac{p}{s} (1+\omega+\omega\psi) + O(h). \quad (15.13)$$

Hence by (12.6), (14.44) and (15.11), (15.6) gives

$$y_1(x) = o \left\{ (p(x)r(x))^{-\frac{1}{2}} \right\} \text{ as } x \rightarrow \infty \quad (15.14)$$

Also by (12.7), (14.44) and (15.12), (15.9) gives after adjusting a constant multiple in  $y_2$

$$y_2(x) \sim (p(x)r(x))^{-\frac{1}{2}} \exp \left\{ 2\sigma \int_a^x \frac{p}{s} (1+\phi) dt \right\} \text{ as } x \rightarrow \infty \quad (15.15)$$

Finally by (12.6), (14.44) and (15.13), (15.10) gives

$$y_3(x) = o \left\{ (p(x)r(x))^{-\frac{1}{2}} \exp \left[ \omega \int_a^x \frac{p}{s} \psi dt + (1+\omega) \int_a^x \frac{p}{s} dt \right] \right\} \text{ as } x \rightarrow \infty \quad (15.16)$$

Now by previous work in §§13-15 we complete our proof of Theorem 13.1.

## Section 16: Examples and Discussion for Case (2)

Example (1)

In the familiar case we consider

$$s(x) = Ax^\alpha, \quad p(x) = Bx^\beta, \quad r(x) = Cx^\gamma$$

where  $\alpha, \beta, \gamma, A(\neq 0), B(\neq 0)$  and  $C(\neq 0)$  are real constants and our interval  $(a \leq x < \infty)$  excluding  $a = 0$ .

Now

$$\delta = \frac{sr}{p^{3/2}} = AC^{1/2} B^{-3/2} x^{\alpha + \frac{1}{2}\gamma - \frac{3}{2}\beta}$$

Hence (9.1) holds if

$$2\alpha + \gamma < 3\beta \quad (16.1)$$

Also

$$\delta \frac{p'}{p} = AC^{1/2} B^{-3/2} \beta x^{\alpha + \frac{1}{2}\gamma - \frac{3}{2}\beta - 1},$$

$$\delta \frac{r'}{r} = AC^{1/2} B^{-3/2} \gamma x^{\alpha + \frac{1}{2}\gamma - \frac{3}{2}\beta - 1},$$

$$\delta \frac{s'}{s} = AC^{1/2} B^{-3/2} \alpha x^{\alpha + \frac{1}{2}\gamma - \frac{3}{2}\beta - 1}.$$

Hence (13.1) holds if (16.1) holds.

Now

$$Pr = CB x^{\beta + \gamma},$$

$$\frac{P}{S} = A^{-1} B x^{\beta - \alpha},$$

$$P S^{-1/2} = A^{-1/2} B x^{\beta - \frac{1}{2}\alpha}.$$

Let

$$\beta + \gamma \neq 0 \quad \text{and} \quad \beta - \frac{1}{2}\alpha \neq 0, \quad (16.2)$$

hence the critical situation which is represented by (13.2) and (13.3) is given by

$$\alpha - \beta = 1. \quad (16.3)$$

Now the non-zero constant  $\sigma$  in (13.2) is given by

$$\sigma = \frac{1}{4} \frac{(\beta + \gamma)A}{B}. \quad (16.4)$$

Also the non-zero constant  $\omega$  in (13.3) is given by

$$\omega = \frac{A(\beta - \frac{1}{2}\alpha)}{B}. \quad (16.5)$$

Now by (16.3)

$$\beta - \frac{1}{2}\alpha = \frac{1}{2}(\beta - 1). \quad (16.6)$$

Hence the non-zero constant  $\omega$  which is given by (16.5) can be written as

$$\omega = \frac{A}{2B} (\beta - 1). \quad (16.7)$$

Now (13.4) is given by

$$\frac{B}{A} \neq \frac{1}{2}(1 - \beta). \quad (16.8)$$

Also (13.5) is given by

$$\frac{B}{A} \neq \frac{1}{2}(\gamma + 1). \quad (16.9)$$

Now  $\phi = 0$  in (13.2), then

$$\left\{ \frac{s(pr)'}{p^2 r} \right\}' = 0,$$

thus (13.6) obviously holds.

Also  $\psi = 0$  in (13.3), then

$$\left\{ \frac{s^{3/2} (ps^{-1/2})'}{p^2} \right\}' = 0,$$

thus (13.7) obviously holds.

Now

$$\frac{sr}{p^2} = ACB^{-2} x^{\alpha+\gamma-2\beta}.$$

Hence (13.8) holds if

$$2\beta - \alpha - \gamma > 1,$$

making use of (16.3) in the above inequality, then the above inequality reduced to (16.1). Hence by previous work conditions (13.1), (9.1) and (13.8) are all holds under (16.1), where our critical case is given by (16.2). Also we mention that case (A) holds for this example. Now (13.9) gives

$$y_1(x) = o \left\{ x^{-\frac{1}{2}(\beta+\gamma)} \right\} \quad \text{as } x \rightarrow \infty \quad (16.10)$$

Now since  $\phi = 0$  and (16.3) holds, hence

$$\int_a^x \frac{p}{s} (1+\phi) dt = A^{-1} B \int_a^x t^{-1} dt = A^{-1} B \ln x - A^{-1} B \ln a.$$

Hence after adjusting a constant multiple in  $y_2(x)$  and making use of (16.4), then (13.10) gives

$$y_2(x) \sim 1 \quad \text{as } x \rightarrow \infty \quad (16.11)$$

Also since (16.3) holds and  $\psi = 0$ , hence (13.11) gives

$$y_3(x) = o \left\{ x^{-\frac{1}{2}(\beta+\gamma)} \exp \left[ (1+\omega) A^{-1} B \int_a^x t^{-1} dt \right] \right\} \quad \text{as } x \rightarrow \infty$$

i.e.

$$y_3(x) = o \left\{ x^{-\frac{1}{2}(\gamma+1) + A^{-1}B} \right\} \quad \text{as } x \rightarrow \infty \quad (16.12)$$

and we make use of (16.7).

We note

$$-\frac{1}{2}(\gamma+1) + A^{-1}B \neq 0$$

subject to (16.9).

Example (2)

To give a quite different class of coefficients covered by our analysis, we consider

$$s(x) = Ax^\alpha \exp x^b, \quad p(x) = Bx^\beta \exp x^b, \quad r(x) = Cx^\gamma \exp \frac{1}{2}x^b$$

where  $\alpha, \beta, \gamma, A(\neq 0), B(\neq 0), C(\neq 0)$  and  $b(> 0)$  are real constants and our interval  $(a \leq x < \infty)$  excluding  $a = 0$ .

Now (9.1) is given by

$$\frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}} = AC^{\frac{1}{2}} B^{-\frac{3}{2}} x^{\alpha+\frac{1}{2}\gamma-\frac{3}{2}\beta} \exp -\frac{1}{4}x^b = o(1) \quad \text{as } x \rightarrow \infty.$$

Also

$$\delta \frac{p'}{p} = b A C^{\frac{1}{2}} B^{-\frac{3}{2}} x^{\alpha+\frac{1}{2}\gamma-\frac{3}{2}\beta+b-1} \left\{ 1 + \frac{\beta}{b} x^{-b} \right\} \exp -\frac{1}{4}x^b,$$

thus  $\delta \frac{p'}{p}$  is  $L(a, \infty)$ . Also  $\delta \frac{s'}{s}$  and  $\delta \frac{r'}{r}$  are  $L(a, \infty)$ . Hence (13.1) holds.

Now

$$pr = BC x^{\beta+\gamma} \exp \frac{3}{2} x^b$$

$$(pr)' = \left\{ BC(\beta+\gamma) x^{\beta+\gamma-1} + \frac{3}{2} b BC x^{\beta+\gamma+b-1} \right\} \exp \frac{3}{2} x^b.$$

Hence

$$\frac{(pr)'}{pr} = \frac{3}{2} b x^{b-1} \left\{ 1 + \frac{2}{3} b^{-1} (\beta+\gamma) x^{-b} \right\}$$

Also

$$ps^{-\frac{1}{2}} = A^{-\frac{1}{2}} B x^{\beta-\frac{1}{2}\alpha} \exp \frac{1}{2} x^b,$$

$$(ps^{-\frac{1}{2}})' = \left\{ BA^{-\frac{1}{2}} (\beta-\frac{1}{2}\alpha) x^{\beta-\frac{1}{2}\alpha-1} + \frac{1}{2} b BA^{-\frac{1}{2}} x^{\beta-\frac{1}{2}\alpha+b-1} \right\} \exp \frac{1}{2} x^b.$$

Hence

$$\frac{(ps^{-\frac{1}{2}})'}{ps^{-\frac{1}{2}}} = \frac{1}{2} b x^{b-1} \left\{ 1 + 2b^{-1} (\beta-\frac{1}{2}\alpha) x^{-b} \right\}$$

Also

$$\frac{p}{s} = A^{-1} B x^{\beta-\alpha}.$$

Thus our critical situation which is represented by (13.2) and (13.3) is given by

$$b-1 = \beta - \alpha. \quad (16.13)$$

Now the non-zero constant  $\omega$  in (13.2) is given by

$$\sigma = \frac{3}{8} \frac{bA}{B}. \quad (16.14)$$

Also the non-zero constant  $\omega$  in (13.3) is given by

$$\omega = \frac{1}{2} \frac{bA}{B}. \quad (16.15)$$

Hence (13.4) is given by

$$\frac{B}{A} \neq -\frac{1}{2}b. \quad (16.16)$$

Also (13.5) is given by

$$\frac{B}{A} \neq \frac{1}{4}b. \quad (16.17)$$

Now to obtain the function  $\phi(x)$ , we write (13.2) in full, hence (13.2) is given by

$$\frac{3}{2} b x^{b-1} \left\{ 1 + \frac{2}{3} b^{-1} (\beta + \gamma) x^{-b} \right\} = 4\sigma A^{-1} B x^{b-1} (1 + \phi).$$

Hence by (16.14)

$$\frac{3}{2} b x^{b-1} \left\{ 1 + \frac{2}{3} b^{-1} (\beta + \gamma) x^{-b} \right\} = \frac{3}{2} b x^{b-1} (1 + \phi). \quad (16.18)$$

Thus (16.18) gives

$$\phi(x) = \frac{2}{3} b^{-1} (\beta + \gamma) x^{-b}. \quad (16.19)$$

Of course here

$$\phi(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Similarly to obtain the function  $\psi(x)$ , we write (13.3) in full, hence (13.3) is given by

$$\frac{1}{2} b x^{b-1} \left\{ 1 + 2b^{-1} (\beta - \frac{1}{2}\alpha) x^{-b} \right\} = \frac{1}{2} b x^{b-1} (1 + \psi). \quad (16.20)$$

Thus (16.20) gives

$$\psi(x) = 2 b^{-1} (\beta - \frac{1}{2}\alpha) x^{-b}. \quad (16.21)$$

Of course

$$\psi(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Now differentiating (16.19), then

$$\phi'(x) = -\frac{2}{3}(\beta+\gamma)x^{-b-1}.$$

Hence  $\phi'$  is  $L(a, \infty)$  and so by (13.14), condition (13.6) holds.

Also differentiating (16.21), then

$$\psi'(x) = -2(\beta - \frac{1}{2}\alpha)x^{-b-1}.$$

Hence  $\psi'$  is  $L(a, \infty)$  and so by (13.15), condition (13.7) holds.

Also condition (13.8) holds and case (A) holds for this example.

Now (13.9) gives

$$y_1(x) = o\left\{x^{-\frac{1}{2}(\beta+\gamma)} \exp - \frac{3}{4}x^b\right\} \quad \text{as } x \rightarrow \infty \quad (16.22)$$

Also (13.10) is given by

$$\begin{aligned} y_2(x) \sim (BC)^{-\frac{1}{2}} x^{-\frac{1}{2}(\beta+\gamma)} \exp \left\{ -\frac{3}{4}x^b + 2\sigma A^{-1}B \int_a^x t^{b-1} dt \right. \\ \left. + \frac{4}{3}\sigma A^{-1}Bb^{-1}(\beta+\gamma) \int_a^x t^{-1} dt \right\} \\ \text{as } x \rightarrow \infty \end{aligned}$$

$$\begin{aligned} y_2(x) \sim (BC)^{-\frac{1}{2}} x^{-\frac{1}{2}(\beta+\gamma)} \exp \left\{ -\frac{3}{4}x^b + 2\sigma A^{-1}B x^b \right. \\ \left. + \frac{4}{3}\sigma A^{-1}Bb^{-1}(\beta+\gamma) \ln x + (\text{const.}) \right\} \\ \text{as } x \rightarrow \infty \end{aligned}$$

since  $\sigma = \frac{3}{8}b B^{-1}A$ , hence

$$y_2(x) \sim (BC)^{-\frac{1}{2}} x^{-\frac{1}{2}(\beta+\gamma)} \exp \left\{ \frac{1}{2}(\beta+\gamma) \ln x + (\text{const.}) \right\} \quad \text{as } x \rightarrow \infty$$

Thus after adjusting a constant multiple in  $y_2$ , we obtain

$$y_2(x) \sim 1 \quad \text{as } x \rightarrow \infty \quad (16.23)$$

Also (13.11) gives

$$y_3(x) = o \left\{ x^{\frac{1}{2}(\beta - \gamma - \alpha)} \exp \left[ \left( \frac{B}{Ab} - \frac{1}{4} \right) x^b \right] \right\} \quad \text{as } x \rightarrow \infty \quad (16.24)$$

We note  $\frac{B}{Ab} \neq \frac{1}{4}$ , subject to (16.17).

As a final general comment, we note that the relationship between the natures of the solutions  $y_j(x)$  ( $1 \leq j \leq 3$ ) of (8.1) in the above two examples, that two of the solutions are small compared to algebraic functions in terms of the coefficients while one solution has algebraic nature, this situation holds for example (1), but if the coefficients as given in example (2), then two of the solutions of (8.1) are small compared to an exponential functions while the other solution still has algebraic nature. The reason for both situations is the critical case which is given by (13.2) and (13.3). Thus this critical case represents a borderline between situations where two solutions of (8.1) are small compared to algebraic functions and where two solutions are small compared to an exponential functions while the third solution having algebraic behaviour in both situations. Also we mention that for the differential equation (2.1) in §2, since the eigenvalues  $\lambda_j(x)$  ( $1 \leq j \leq 3$ ) of (3.3) are having same size as  $x \rightarrow \infty$ , as given in (4.18), hence subject to (4.18) and (6.3) the Critical Case of the differential equation (2.1) is given by

$$\frac{(rs^{\frac{1}{2}})' }{rs^{\frac{1}{2}}} \sim (\text{const.}) \left( \frac{r}{s} \right)^{1/3}. \quad (16.25)$$

Thus if the coefficients  $r$ ,  $s$  and  $q$  as given in example (1) §7, then this critical case is given by  $\alpha - \gamma = 3$ , can be handled

similarly by our methods. We note for the differential equation (2.1) there is only one Critical Case which is given by (16.25) if  $\lambda_j(x)$  as given in (4.18). Now we return to our equation (8.1), and since the eigenvalues  $\lambda_j(x)$  ( $1 \leq j \leq 3$ ) of (8.4) are having different size as  $x \rightarrow \infty$ , as given in (9.25), (9.26) and (9.27), hence there are two critical cases for the differential equation (8.1) one which we covered in our previous work, and subject to (9.28), (11.3) and (11.4) the second critical case for (8.1) is given by

$$\frac{(pr)'}{pr} \sim (\text{const.}) \left(\frac{r}{p}\right)^{\frac{1}{2}}, \quad \text{as } x \rightarrow \infty$$

$$\frac{(ps^{-\frac{1}{2}})'}{ps^{-\frac{1}{2}}} \sim (\text{const.}) \left(\frac{r}{p}\right)^{\frac{1}{2}}, \quad \text{as } x \rightarrow \infty$$

Thus if the coefficients  $r$ ,  $s$  and  $p$  as given in example (1) §16, then this critical case is given by  $\beta - \gamma = 2$ , can be handled similarly by our methods.

## Bibliography

1. E.A. Coddington and N. Levinson, Theory of ordinary differential equations (McGraw-Hill, New York, 1955).
2. W.A. Coppel, Stability and Asymptotic Behaviour of Differential Equations (Heath, Boston, 1965).
3. Garrett Birkhoff and Gian-Carlo Rota, Ordinary differential equations (John Wiley & Sons. Inc. 1969).
4. Ruel V. Churchill, James W. Brown and Roger F. Verhey, Complex variables and applications. (McGraw-Hill, 1974).
5. Ruel V. Churchill and James W. Brown, Fourier series & Boundary value problems (McGraw-Hill, 1978).
6. Shisha, Inequalities (Academic Press, New York and London, 1967).
7. K.M. Agashe, Calculus-3 (published by R.A. Kulkarni, Pune, 1978).
8. George W. Pfeifer, 'Asymptotic Solutions of  $y''' + qy' + ry = 0$ ; J. of Differential Equations 11, (1972), 145-155.
9. W.A. Harris and D.A. Lutz, On the asymptotic integration of Linear differential systems, J. Math. Anal. Appl. 48 (1974), 1-16.
10. W.A. Harris and D.A. Lutz, Asymptotic integration of adiabatic oscillators, J. Math. Anal. Appl. 51 (1975), 76-93.
11. W.A. Harris and D.A. Lutz, A unified theory of asymptotic integration, J. Math. Anal. Appl. 57 (1977), 571-586.
12. M.S.P. Eastham, Asymptotic theory for a critical class of fourth-order differential equations. Proc. Roy. Soc. London 383A (1982), 465-476.
13. M.S.P. Eastham, Asymptotic formulae of Liouville-Green type for higher-order differential equations. J. London Math. Soc. (2), 28 (1983), 507-518.
14. M.S.P. Eastham, The asymptotic solution of Linear differential systems. Mathematika, 32 (1985), 131-138.
15. M.S.P. Eastham, Asymptotic theory and deficiency indices for differential equations of odd order, Proc. Roy. Soc. Edinburgh 90A (1981) 263-279.
16. M.S.P. Eastham and C.G.M. Grudniewicz, Asymptotic theory and deficiency indices for higher-order self-adjoint differential equations, J. London Math. Soc. (2), 24 (1981) 255-271.

17. M.S.P. Eastham, the deficiency index of a second-order differential system, J. London Math. Soc. (2), 23 (1981), 311-320.
18. M.S.P. Eastham, 'The Liouville-Green asymptotic theory for second-order differential equations: a new approach and some extensions', Ordinary differential equations and operators: a tribute to F.V. Atkinson, Lecture Notes in Mathematics 1032 (Springer, Berlin, 1983).
19. M.S.P. Eastham, A repeated transformation in the asymptotic solution of Linear differential systems, Proc. Roy. Soc. Edinburgh. (To be published).
20. A. Devinatz, The deficiency index of a certain class of ordinary self-adjoint differential operators, Adv. in Math. 8 (1972), 434-473.
21. A. Devinatz, The deficiency index of certain fourth-order ordinary self-adjoint differential operators, Quart. J. Math. (Oxford) (2) 23 (1972), 267-286.
22. P. Hartman, Ordinary Differential Equations, 2nd ed. (Boston: Birkhauser, 1982).
23. N. Levinson, The asymptotic nature of solutions of linear differential equations. Duke Math. J. 15 (1948), 111-126.
24. D.B. Hinton, Asymptotic behaviour of solutions of  $(ry^{(m)})^{(k)} \pm qy = 0$ . J. Differential Equations 4 (1968), 590-596.
25. M.V. Fedorjuk. Asymptotic behaviour of eigenvalues and eigenfunctions of one-dimensional singular differential operators. Soviet Math, 7 (1966), 929-932.
26. P.W. Walker. Asymptotic of the solutions to  $[(ry'')]' - py' l' + qy = 0y^*$ . 1971 a J. Diff. Equat. 9, 108-132.
27. P.W. Walker, Deficiency indices of fourth-order singular differential operators. 1972 b J. Diff. Equat. 9, 133-140.
28. P.W. Walker. Asymptotic for a class of fourth order differential equations. 1972 J. Diff. Equat. 11, 321-334.
29. M.A. Naimark, Linear differential operators (Harrap, 1968).
30. C.G. Grudniewicz, Asymptotic methods for fourth-order differential equations. Proc. Roy. Soc. Edinburgh, 87A (1980), 53-63.
31. T.T. Read, Higher order differential equations with small solutions. J. London Math. Soc. (2) 19 (1979) 107-122.