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ORDERED CONES AND THEIR DUALITY

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INTRODUCTION

The aim of this thesis is to outline an elementary theory of linear and sublinear functionals on an ordered cone. But ordered cones are taken in a slightly more general way than usual: they need not be imbeddable in a vector space. In consequence, we do not require a general cancellation law for the addition. Typical examples for the cones that we have in mind include $\overline{\mathfrak{R}} = \mathfrak{R} \cup \{+\infty\}$ with the usual algebraic operations (in particular $0 \cdot (+\infty) = 0$), $\overline{\mathfrak{R}}$ -valued functions on some topological space, measures on a given σ -algebra and, most prominently, families of convex subsets of a certain vector space. Cones are asymmetric structures. Accordingly, linear functionals on a cone should not be required to have values only in $\mathfrak{R}$. We choose $\overline{\mathfrak{R}}$ as their suitable range.

In various places the literature deals with linear functionals on cones that take values in $\mathfrak{R} \cup \{-\infty\}$ instead (cf. [8], [16]). The difference between those two points of view is essential and not just a question of reversing the order on $\mathfrak{R}$. In vector spaces both approaches coincide, as linear functional can take only finite values

there , but in applications for cones such as Measure Theory or Potential Theory , the value $+\infty$ arises more naturally . Moreover , allowing sublinear functionals with values in $\overline{\mathfrak{R}}$ in connection with Hahn-Banach type theorems leads to more satisfying results that include investigations of "hypolinear " functionals on vector spaces as in [2] .

We shall develop our theory for cones that carry a preorder . This is however no restriction to the generality of our approach , since equality may be considered as such a preorder . Much of the work in modern analysis consists of the investigation of inequalities . Accordingly , we combine the theory of linear functionals with order theoretic aspects in so far as we mostly deal with functionals being monotone with respect to suitable order relations .

The algebraic theory of an ordered cone was developed in [8] . In analysis one studies ordered cones from the topological point of view , taking into consideration the already existing algebraic structure. So we divided this thesis into two parts . The algebraic one , chapters I and II , and the topology part , chapter III .

In the first chapter we present the basic materials about ordered cones and linear functionals , we give our favorite versions of sandwich theorems . In the second chapter we obtain important information about extension and separation theorems and a Uniform Boundedness Principle for an ordered cone . In the third chapter , we will introduce a locally convex structure on our cones and study continuous operators between a locally convex cones . Much of the material was taken from the book by Keimel and Roth [13] .

Many of the results that we are going to present do not yet seem to be in the literature in this explicit form . Most of the material was sketched and provided to me by Dr.Roth . I furnished the details and exact formulations and completed the proofs .

CHAPTER I

THE SANDWICH THEOREMS

This chapter presents a brief introduction to the theory of ordered cones, beginning with definitions and basic properties followed by examples. After developing the basic properties of an ordered cone, in the second section we turn to the most profound results in this chapter which are the Sandwich Theorems (2.3) and (2.9). In the third section we turn to the finite dimensional Euclidean vector space $\mathfrak{R}^n$. We prove that every $\overline{\mathfrak{R}}$ -valued sublinear functional on $\mathfrak{R}^n$ dominates a linear one. For more detailed information on cones the reader is referred to [8] and [13].

1. ORDERED CONES

Ordered cones play a role in many branches of mathematics and its applications. The importance of several examples that can not be dealt with in the context of vector spaces suggests the independent concept of an ordered cone. We shall give the basic definitions and illustrate them with typical examples in this section, and discuss the

possibility of embeddability of an ordered cone into a vector space .
 More precisely , we show that a cone is embeddable in a vector space
 if and only if it satisfies the cancellation law .

As usual , we denote by $\mathbb{R}^+$ the set of all non-negative real
 numbers .

(1.1) Definition : A *cone* is a non-empty set P with
 functions

$$+ : P \times P \rightarrow P , \text{ and } \bullet : \mathbb{R}^+ \times P \rightarrow P ,$$

called addition and scalar multiplication , such that for all $x, y, z \in P$
 and $\alpha, \beta \in \mathbb{R}^+$ the following hold :

- (i) $x + y = y + x$
- (ii) $(x + y) + z = x + (y + z)$
- (iii) There exists a neutral element $0 \in P$ such that $0 + x = x$
- (iv) $\alpha(\beta x) = (\alpha\beta)x$
- (v) $(\alpha + \beta)x = \alpha x + \beta x$
- (vi) $\alpha(x + y) = \alpha x + \alpha y$
- (vii) $1 \bullet x = x$
- (viii) $0 \bullet x = 0$

A simple example given in [8] shows that (viii) is independent of (i) to (vii) , thus needs to be stated explicitly . It is easy to see that the neutral element $0 \in P$ is unique , and that $\alpha \cdot 0 = 0$ for all $\alpha \in \mathfrak{R}^+$. Indeed ; for the first statement , assume that both 0 and $0'$ are neutral elements for the addition . Then

$$0 + 0' = 0' \quad \text{and} \quad 0' + 0 = 0 \quad , \text{ hence } 0 = 0' .$$

The second statement follows from (viii) for $\alpha = 0$. For $\alpha > 0$ we have

$$x = \alpha \left(\alpha^{-1} x + 0 \right) = x + \alpha \cdot 0 \quad \text{for all } x \in P ,$$

hence $\alpha \cdot 0 = 0$ by the previously established uniticity of the neutral element .

(1.2) Definition : A non-empty subset Q of P is said to be *subcone* of P if ,

$$a + b \in Q , \text{ and } \alpha a \in Q \text{ for all } a, b \in Q \text{ and } \alpha \geq 0 .$$

Note that every subcone of P contains 0 . Consequently if Q is a cone in a real vector space ; that is , if Q is a non-empty subset of a real vector space E such that

$$a + b \in Q \text{ and } \alpha a \in Q \text{ for all } a, b \in Q \text{ and } \alpha \geq 0 ,$$

then Q is also a cone according to the Definition (1.1) .

Let P be a cone, then the intersection of any arbitrary family of subcones is again subcone; if the family is directed by " $\subseteq$ ", then the union of the members is also a subcone. Any finite linear combination of subcones is a subcone. If $A \subseteq P$, then the smallest subcone containing A is defined by $\text{cone}(A)$. Clearly, if $A \neq \emptyset$ then

$$\text{cone}(A) := \left\{ \sum_{i=1}^n \lambda_i a_i \mid \lambda_i \geq 0, a_i \in A, i = 1, 2, \dots, n \right\}$$

If $A \subseteq P$ is **convex**; that is, $\alpha a + (1 - \alpha) b \in A$ whenever $a, b \in A$ and $0 \leq \alpha \leq 1$, then $\text{cone}(A)$ has a simpler expression as shown in the following proposition.

(1.3) Proposition : *If A is non-empty convex subset of a cone P , then $\text{cone}(A) = \cup \{ \lambda A \mid \lambda \geq 0 \}$.*

Proof : Suppose that $Q := \cup \{ \lambda A \mid \lambda \geq 0 \}$. Then Q is a subcone containing A ; indeed, let $a, b \in Q$ and $\lambda \geq 0$, then $a = \lambda_1 a_1$, $b = \lambda_2 a_2$ for $\lambda_1, \lambda_2 \geq 0$ and $a_1, a_2 \in A$. Then $\lambda a = \lambda \lambda_1 a_1 \in Q$. If $\lambda_1 = \lambda_2 = 0$, then $a + b = 0 + 0 \in Q$. Otherwise, by the convexity of A we have

$$a + b = \lambda_1 a_1 + \lambda_2 a_2 = (\lambda_1 + \lambda_2) \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} a_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} a_2 \right) \in Q$$

Furthermore , if $\overline{Q}$ is a subcone containing A , then for all $a \in Q$, $a = \lambda a_1$ for $a_1 \in A$ and $\lambda \geq 0$ then $a = \lambda a_1 \in \overline{Q}$, and so $Q \subseteq \overline{Q}$. Therefore Q is the smallest subcone containing A ; that is , $\text{cone}(A) = Q$. $\square$

(1.4) Theorem : *A cone P can be embedded into a real vector space E if and only if the cancellation law holds ; that is , for all $a, b, c \in P$,*

$$a + c = b + c \quad \text{implies} \quad a = b .$$

Proof : Let P be a cone in a real vector space E , and assume for $a, b, c \in P$ that $a + c = b + c$. since $(-c) \in E$ then

$$a + c + (-c) = b + c + (-c) \quad \text{implies} \quad a = b .$$

Conversely , let P be a cone satisfying the cancellation law . On $P \times P$ define the relation $\sim$ by $(a,b) \sim (c,d)$ if and only if $a + d = b + c$ for $a, b, c, d \in P$, then $\sim$ is an equivalence relation . Indeed ; clearly $\sim$ is reflexive and symmetric . For transitivity let $(a,b) \sim (c,d)$ and $(c,d) \sim (\bar{a},\bar{b})$ then $a + d = b + c$ and $c + \bar{b} = d + \bar{a}$, hence $(a + d) + (\bar{b} + c) = (b + c) + (\bar{a} + d)$ and by the cancellation law $a + \bar{b} = b + \bar{a}$. Thus $(a,b) \sim (\bar{a},\bar{b})$.

Now we shall denote by $[P]$ the quotient space of $P \times P$ with respect to this equivalence relation. For $a, b \in P$, we denote by $\overline{(a,b)}$ the element of $[P]$ generated by (a,b) . Define the two functions

$$+ : [P] \times [P] \rightarrow [P] \quad \text{and} \quad \bullet : \mathfrak{R} \times [P] \rightarrow [P]$$

by $\overline{(a,b)} + \overline{(c,d)} = \overline{(a+c, b+d)}$ and

$$\lambda \bullet \overline{(a,b)} = \begin{cases} \overline{(\lambda a, \lambda b)} & \text{for } \lambda \geq 0 \\ \overline{(-\lambda b, -\lambda a)} & \text{for } \lambda < 0 \end{cases}$$

for all $a, b, c, d \in P$ and $\lambda \in \mathfrak{R}$. Then $+$ and $\bullet$ are well defined operations. By using the cancellation law we see that $0 = \overline{(a,a)}$ is the neutral element for the addition in $[P]$ and that $\overline{(b,a)}$ is the additive inverse of $\overline{(a,b)}$. Then $[P]$ is the required real vector space. Moreover the map $\psi : P \rightarrow [P]$ defined by

$$\psi(a) = \overline{(a,0)} \quad \text{for all } a \in P$$

is an injective map that preserves the algebraic operations, that is, an "embedding". Indeed; let $a, b \in P$ with $\psi(a) = \psi(b)$ then $\overline{(a,0)} = \overline{(b,0)}$ therefore $(a,0) \sim (b,0)$ thus $a = b$. $\square$

It is important to notice that cones are in general far from being embeddable in a vector space, as the cancellation law is not generally assumed.

(1.5) Example : Let $\overline{\mathfrak{R}} = \mathfrak{R} \cup \{+\infty\}$ with the usual extended operations ; that is , $x + (+\infty) = +\infty$ for $x \in \overline{\mathfrak{R}}$, $\lambda \cdot (+\infty) = +\infty$ for $\lambda > 0$ and $0 \cdot (+\infty) = 0$. Then $\overline{\mathfrak{R}}$ forms a cone without cancellation law .

(1.6) Example : Let A be a convex subset of a cone P . We denote by $\text{conv}(P)$ the set of all non-empty convex subsets of P , with the addition and scalar multiplication on $\text{conv}(P)$ defined as usual by

$$A + B = \{ a + b \mid a \in A, b \in B \} \quad \text{for } A, B \in \text{conv}(P) ,$$

$$\lambda A = \{ \lambda a \mid a \in A \} \quad \text{for } A \in \text{conv}(P) \text{ and } \lambda \geq 0 ,$$

it is easy to verify that $\text{conv}(P)$ is again a cone with the neutral element $\{0\}$. Convexity is required to show that

$$(\alpha + \beta) A = \alpha A + \beta A .$$

There is nothing to prove if $\alpha = 0$ or $\beta = 0$. Let $\alpha, \beta > 0$. It is clear that $(\alpha + \beta) A \subseteq \alpha A + \beta A$. To show the converse , consider an arbitrary element $c \in \alpha A + \beta A$; then there exist $a, b \in A$ such that ,

$$c = \alpha a + \beta b = (\alpha + \beta) \left(\frac{\alpha}{\alpha + \beta} a + \frac{\beta}{\alpha + \beta} b \right) \in (\alpha + \beta) A .$$

Note that every subcone Q of a cone P is convex and satisfies $Q + Q = Q$. In particular, the non-empty convex subsets of a real vector space form a cone in our sense which is far from being cancellative; for example, consider the vector space $\mathfrak{R}$ with the convex subsets $A = [0, 1]$, $B = [0, 2]$, $C = [0, +\infty)$ then

$$A + C = B + C \quad \text{but} \quad A \neq B.$$

(1.7) Example : An interesting subcone of $\text{conv}(P)$ is

$$K := \{ Q \subseteq P \mid Q \text{ is a subcone of } P \},$$

for this subcone the cancellation law is obviously also not valid.

(1.8) Example : let P be a cone and X any non-empty set. We denote by $F(X, P)$ the set of all P -valued functions on X , with the pointwise addition and scalar multiplication. Then $F(X, P)$ forms a cone, in general without cancellation law; for example, consider $f, g, h \in F(\mathfrak{R}, \overline{\mathfrak{R}})$ defined as follows: $f(x) = r < \infty$, $g(x) = s < \infty$ with $r \neq s$ and $h(x) = +\infty$ for all $x \in \mathfrak{R}$. Then $f + h = g + h$ but $f \neq g$. Moreover $F(X, P)$ satisfies the cancellation law if and only if P does. Indeed; suppose that $F(X, P)$ has the cancellation law, let $x, y, z \in P$ with $x + z = y + z$ then by defining the constant functions $f(a) = x$, $g(a) = y$, and

$h(a) = z$ for all $a \in X$ we get $x = y$. Conversely ; if P satisfies the cancellation law and if

$$f(x) + h(x) = g(x) + h(x)$$

for all $x \in X$ with $f, g, h \in F(X, P)$, then $f(x) = g(x)$.

(1.9) Example : Let X be a non-empty set . Consider $F(X, \mathfrak{R}^+)$ and define the scalar multiplication $\circ$ and the addition $\tilde{+}$ by :

$$\lambda \circ f = f^\lambda, \quad f \tilde{+} g = f \cdot g \quad \text{for all } f, g \in F(X, \mathfrak{R}^+) \text{ and } \lambda \geq 0$$

then , if we define $0^0 = 1$, $F(X, \mathfrak{R}^+)$ forms a cone under these operations . (the neutral element of the addition $\tilde{+}$ is the constant function 1) . Clearly this cone does not satisfy the cancellation law ; for example , let $f(x) = r$, $g(x) = s$ with $r \neq s$ and $h(x) = 0$ for all $x \in X$ then $f \tilde{+} h = g \tilde{+} h$ but $f \neq g$.

(1.10) Example : Let f be a real-(or extended - real)-valued function on a topological space E . If $\{x : f(x) > \alpha\}$ is open for every real α , f is said to be **lower semi-continuous** . If $\{x : f(x) < \alpha\}$ is open for every real α , f is said to be **upper semi-continuous** . Let P be the set of all $\overline{\mathfrak{R}}$ -valued lower semi-continuous functions on $[a, b] \subseteq \mathfrak{R}$, with the usual operations . Then P is a cone .

(1.11) Definition : A *preordered* cone is a cone P with a reflexive and transitive relation " $\leq$ " such that

$$x \leq y \text{ implies } x + z \leq y + z \text{ and } \alpha x \leq \alpha y$$

for all $x, y, z \in P$ and all $\alpha \geq 0$. If " $\leq$ " is in addition antisymmetric then P is called an *ordered cone*.

Note that equality is also an order in the sense of Definition (1.11). Thus, all our upcoming results about ordered cones apply to cones without an order structure as well. The " $\leq$ " then just has to be replaced by " $=$ " in all our assumption and results.

(1.12) Examples : Here we are going to list a few examples of preordered cones :

- (i) $\overline{\mathbb{R}}$ with the usual order.
- (ii) $\text{conv}(P)$ with the inclusion order, if P is a cone.
- (iii) $F(X, P)$ with the pointwise order, if P is an ordered cone and X is any set.
- (iv) Every cone P with its natural preorder defined by $x \leq y$ if $x + z = y$ for some $z \in P$.
- (v) The space of all $\overline{\mathbb{R}}$ -valued lower semi-continuous functions on a compact set in $\mathbb{R}^n$ with the pointwise ordering.

Convex sets in cones may look rather peculiar . For example in $\overline{\mathfrak{R}}$ all the two element sets $\{x, +\infty\}$ are convex . So this leads us to consider increasing or decreasing sets , or more generally convex sets that are also order convex . A subset A of a preordered cone P is **order convex** if and only if for $a, b \in A$ and $x \in P$,

$$a \leq x \leq b \quad \text{implies that} \quad x \in A .$$

(1.13) Definition : A subset A of a preordered cone P is called **decreasing** , if $a \in A$ and $x \leq a$ for some $x \in P$ implies $x \in A$. For a subset A of P we denote by

$$\downarrow A := \{ x \in P \mid x \leq a \text{ for some } a \in A \}$$

the decreasing subset generated by A . In a similar way one can define an **increasing** subset and $\uparrow A$, the increasing subset generated by A . We denote by **$Dconv(P)$** the set of all non-empty decreasing convex subsets of P .

(1.14) Remarks : (i) From the above definition one can check easily that each $\downarrow A$ and $\uparrow A$ both are convex , whenever A is convex . Indeed ; let $x, y \in \downarrow A$ and $0 \leq \lambda \leq 1$, then $x \leq a$ and $y \leq b$ for some $a, b \in A$, thus

$$\lambda x + (1-\lambda)y \leq \lambda a + (1-\lambda)b \in A .$$

Similarly we argue for $\uparrow A$.

(ii) For a decreasing convex subset A of an ordered cone P and $\lambda \geq 0$, the set λA is also decreasing and convex. But $A + B$ need not be decreasing, if A and B are. For example, consider the cone $P = \text{conv}(\mathbb{R}^2)$ with the inclusion order. Let

$$A := \{ C \in P \mid C \subseteq \{0\} \times \mathbb{R} \}$$

$$B := \{ C \in P \mid C \subseteq \mathbb{R} \times \{0\} \}$$

Clearly, A and B are convex and decreasing subsets of P .

But $A + B$ is not decreasing. Indeed; since

$$\mathbb{R}^2 = (\{0\} \times \mathbb{R}) + (\mathbb{R} \times \{0\})$$

then $\mathbb{R}^2 \in A + B$ but not every subset of $\mathbb{R}^2$ is in $A + B$,

for example $D := \{ (x, x) \mid x \in \mathbb{R} \} \notin A + B$.

According to our last remark we can not use the usual addition for $\text{Dconv}(P)$. We therefore modify the addition on $\text{Dconv}(P)$ and define :

$$A \oplus B = \downarrow(A+B) := \{ x \in P \mid x \leq a + b \text{ for some } a \in A, b \in B \}.$$

(1.15)Example : Let P be a cone. Then $\text{Dconv}(P)$ forms a cone with the above addition and the usual scalar multiplication. Moreover, it is an ordered cone with the inclusion as order; the set $\{0\}$ acts as the addition zero element.

(1.16)Theorem : Let P be a preordered cone . Define the map $f: P \rightarrow Dconv(P)$ by $f(x) = \downarrow\{x\}$ then

- (1) f is order preserving .
- (2) f is an embedding (i.e. one-to-one) if and only if the preorder of P is in fact an order .

Proof :

- (1) Let $x, y \in P$ with $x \leq y$, then $\downarrow\{x\} \subseteq \downarrow\{y\}$. Indeed ; let $a \in \downarrow\{x\}$ then $a \leq x \leq y$ thus $a \leq y$ hence $a \in \downarrow\{y\}$.
- (2) Let f be injective . Then the relation " $\leq$ " is antisymmetric .
Indeed ; let $x, y \in P$ with $x \leq y$ and $y \leq x$ then $\downarrow\{x\} \subseteq \downarrow\{y\}$ and $\downarrow\{y\} \subseteq \downarrow\{x\}$, hence $\downarrow\{x\} = \downarrow\{y\} = f(x) = f(y)$ thus $x = y$. Conversely , let $f(x) = f(y)$ then $\downarrow\{x\} = \downarrow\{y\}$ which implies that $x \leq y$ and $y \leq x$,but if " $\leq$ " is antisymmetric , then $x = y$. that is , f is injective .

(1.17) Example : Let E be a Hausdorff topological vector space . Recall that , $\overline{A}$ denotes the topological closure of a subset A of E , which is convex whenever A is . We denote by $Clconv(E)$ the set of all non-empty closed convex subsets of E . We define the addition by $A \oplus B = \overline{A + B}$, and the scalar multiplication to be as usual ; the set $\{0\}$ acts as the additive zero element . Then $Clconv(E)$ forms a cone,ordered by inclusion,without cancellation law (cf.(1.6)) .

(1.18) Example : Let E be a topological vector space , with the neighborhood system $\mathcal{V}$, denote by $Bconv(E)$ the set of all non-empty bounded closed convex subsets of E , with the operations as in Example(1.17). Then $Bconv(E)$ is an ordered cone satisfying the cancellation law . Indeed ; let $A, B, C \in Bconv(E)$ with $A \oplus C = B \oplus C$, that is $\overline{A + C} = \overline{B + C}$. We shall show that this implies $A = B$. Let $a \in A$, $c \in C$, and $U \in \mathcal{V}$. As $A + C \subseteq B + C + U$, $a + c \in b_1 + c_1 + U$ for $b_1 \in B$ and $c_1 \in C$. Also $2a + c \in b_1 + (a + c_1) + U$, and for $b_2 \in B$, $c_2 \in C$ we have $a + c_1 \in b_2 + c_2 + U$, hence

$$2a + c \in b_1 + b_2 + c_2 + 2U .$$

Repeating the argument , we obtain

$$na + c \in (b_1 + b_2 + \dots + b_n) + c_n + nU ,$$

thus

$$a + \frac{1}{n} c \in \frac{1}{n} (b_1 + b_2 + \dots + b_n) + \frac{1}{n} c_n + U .$$

But C is bounded , thus $\frac{1}{n} c$ and $\frac{1}{n} c_n$ converge to 0 , as

n tends to infinity . As B is convex , we have

$$\frac{1}{n} (b_1 + \dots + b_n) \in B ,$$

thus

$$a + \frac{1}{n} (c - c_n) \in B + U .$$

As n tends to infinity , we realize that

$$a \in \overline{B + U} \subseteq (B + U) + U = B + 2U$$

by the above . As this holds for all $a \in A$ and $U \in \mathcal{V}$ we get

$$A \subseteq \bigcap_{U \in \mathcal{V}} (B + U) = \overline{B} = B .$$

Similarly $B \subseteq A$.

(1.19) Example : Let E be a topological vector space , denote by $Cconv(E)$ the set of all non-empty compact convex subsets of E with the operations and order as in Example (1.6) . Then $Cconv(E)$ is an ordered cone and satisfies the cancellation law . In particular , if $E = \mathbb{R}^n$ then $Cconv(\mathbb{R}^n)$ is embeddable in the vector space $C(S)$ of all continuous functions on S , where S is the unit sphere in $\mathbb{R}^n$. (cf.[12]) .

2. SANDWICH THEOREMS

Sandwich theorems are versions of Hahn-Banach type theorems which serve as important tools in modern analysis, especially in functional analysis. Consequently we start this section with a sandwich theorem for an ordered cone. We show that between a sublinear and superlinear functional there is always a monotone linear functional provided that the sublinear and the superlinear functional comply with the order in an obvious way. Since no additional work is required we can immediately prove this result. We proceed to give an important theorem which is "The generalization Sandwich theorem" in which we have to use an extended superlinear and a sublinear functional with additional conditions. An illustrative example will be given at the end of this section.

Throughout this section, let $(P, \leq)$ be an ordered cone.

(2.1) Definition : (i) A map $p : P \rightarrow \overline{\mathfrak{R}}$ is said to be a *sublinear functional* on P if $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for all $x, y \in P$ and $\lambda \geq 0$.

- (ii) A map $q : P \rightarrow \overline{\mathfrak{R}}$ is said to be a *superlinear functional* if $q(x + y) \geq q(x) + q(y)$ and $q(\lambda x) = \lambda q(x)$ for all $x, y \in P$ and $\lambda \geq 0$.
- (iii) If $\mu : P \rightarrow \overline{\mathfrak{R}}$ is a sublinear and superlinear functional, then μ is a *linear functional*.
- (iv) A map $p : P \rightarrow \overline{\mathfrak{R}}$ is *monotone* if $x \leq y$ implies $p(x) \leq p(y)$ for $x, y \in P$.

(2.2)Remark : Let $q : P \rightarrow \overline{\mathfrak{R}}$ be a superlinear functional, then q is finite for all $x \in P$ such that $(-x) \in P$. Indeed; $0 = q(0) = q(x + (-x)) \geq q(x) + q(-x)$, hence both $q(x)$ and $q(-x)$ are finite. As any linear functional is also superlinear, the same holds for linear functionals.

Before we derive our Sandwich Theorem, we should mention that Fuchssteiner and Lusky (Theorem 1.2.5 [8]) formulate a sandwich theorem for functionals on ordered cones that take values in $\mathfrak{R} \cup \{-\infty\}$. Their theorem follows from our Theorem (2.3) just by reversing the order on P and replacing all functionals p, q , and μ with their negatives. But still our theorem is more general, as neither p nor q is required to be monotone.

(2.3)Theorem :(Sandwich Theorem): Let P be an ordered cone , $p : P \rightarrow \overline{\mathfrak{R}}$ a sublinear functional and $q : P \rightarrow \overline{\mathfrak{R}}$ a superlinear functional such that $q(x) \leq p(y)$ whenever $x \leq y$ for $x, y \in P$. Then there exists a monotone linear functional $\mu : P \rightarrow \overline{\mathfrak{R}}$ such that $q(x) \leq \mu(x) \leq p(x)$ for all $x \in P$.

Proof : Let

$$\Gamma := \{ v : P \rightarrow \overline{\mathfrak{R}} \mid v \text{ is a monotone sublinear functional}$$

$$\text{and } q(x) \leq v(x) \leq p(x) \text{ for all } x \in P \} .$$

Since $v(x) := \inf \{ p(y) \mid x \leq y \} \in \Gamma$, the collection Γ is not empty . It is easy to check that this functional v is a monotone sublinear functional and that $q \leq v \leq p$. We consider the pointwise order on Γ , that is , $\mu \leq v$ if $\mu(x) \leq v(x)$ for all $x \in P$, and we shall use Zorn's Lemma in order to show that Γ has a minimal element . Now consider a chain C in Γ , and define

$$v_0(x) := \inf \{ v(x) \mid v \in C \} \text{ for all } x \in P .$$

Then we have the following :

(i) v_0 is monotone : Let $x, y \in P$ with $x \leq y$, if $v_0(y) = +\infty$, then $v_0(x) \leq v_0(y)$, otherwise , let $\varepsilon \geq 0$ be given , then there exist $v \in C$ such that $v_0(y) + \varepsilon \geq v(y) \geq v(x) \geq v_0(x)$ thus $v_0(y) + \varepsilon \geq v_0(x)$ for all $\varepsilon \geq 0$, hence $v_0(x) \leq v_0(y)$.

(ii) Clearly , $q(x) \leq v_0(x) \leq p(x)$ holds for all $x \in P$.

(iii) v_0 is a sublinear functional :

(α) Clearly , $v_0(\lambda x) = \lambda v_0(x)$ for all $x \in P$ and $\lambda \geq 0$.

(β) Let $x, y \in P$, if either $v_0(x) = +\infty$ or $v_0(y) = +\infty$ then $v_0(x+y) \leq v_0(x) + v_0(y)$. Otherwise , let $\varepsilon \geq 0$ be given , there are $v_1, v_2 \in C$ (assume $v_2 \leq v_1$) such that

$$v_0(x) + \frac{\varepsilon}{2} \geq v_1(x) \quad \text{and} \quad v_0(y) + \frac{\varepsilon}{2} \geq v_2(y)$$

thus

$$\begin{aligned} v_0(x) + v_0(y) + \varepsilon &\geq v_1(x) + v_2(y) \geq v_2(x) + v_2(y) \\ &\geq v_2(x+y) \geq v_0(x+y) \end{aligned}$$

hence $v_0(x) + v_0(y) + \varepsilon \geq v_0(x+y)$ for all $\varepsilon \geq 0$, whence

$$v_0(x+y) \leq v_0(x) + v_0(y) .$$

(iv) It is obvious that $v_0 \leq v$ for all $v \in C$.

Therefore , every chain in Γ has a lower bound in Γ . By Zorn's lemma Γ has a minimal element , (say μ) . We shall show that μ is a linear functional . For an arbitrary fixed $x_0 \in P$ set $\alpha_0 := \sup \{q(x) - \mu(y) \mid x, y \in P, \mu(y) < \infty, x \leq x_0 + y\} \in \overline{\mathfrak{R}}$ we observe that for such x and y

$$\begin{aligned} q(x) - \mu(y) &\leq \mu(x) - \mu(y) \leq \mu(x_0 + y) - \mu(y) \\ &\leq \mu(x_0) + \mu(y) - \mu(y) = \mu(x_0) \end{aligned}$$

thus , $\alpha_0 \leq \mu(x_0)$ but $q(x_0) \leq \alpha_0$, hence

$$q(x_0) \leq \alpha_0 \leq \mu(x_0) .$$

Now we define a functional $\bar{\mu} : P \rightarrow \overline{\mathfrak{R}}$ by

$$\bar{\mu}(x) := \inf \{ \mu(y) + \lambda \alpha_0 \mid x \leq y + \lambda x_0, y \in P, \lambda \geq 0 \}$$

Then we observe the following :

(i) $\bar{\mu}(x) \leq \mu(x)$ for all $x \in P$.

(ii) $q(x) \leq \bar{\mu}(x) \leq p(x)$: Let $x, y \in P$ with $x \leq y + \lambda x_0$. We shall show that $q(x) \leq \mu(y) + \lambda \alpha_0$, but this is obvious if $\mu(y) = +\infty$ or $\lambda = 0$. If $\mu(y) < +\infty$ and $\lambda > 0$, we have

$$\frac{x}{\lambda} \leq \frac{y}{\lambda} + x_0 \quad \text{thus} \quad q\left(\frac{x}{\lambda}\right) - \mu\left(\frac{y}{\lambda}\right) \leq \alpha_0$$

hence $q(x) \leq \mu(y) + \lambda \alpha_0$.

(iii) $\bar{\mu}$ is monotone : obvious.

(iv) $\bar{\mu}$ is a sublinear functional :

(α) $\bar{\mu}(\lambda x) = \lambda \bar{\mu}(x)$ for all $x \in P$ and $\lambda \geq 0$.

(β) Let $x, y \in P$, if $\bar{\mu}(x) = +\infty$ or $\bar{\mu}(y) = +\infty$

then $\bar{\mu}(x + y) \leq \bar{\mu}(x) + \bar{\mu}(y)$ otherwise, let $\varepsilon \geq 0$ be given,

there exist $y_1, y_2 \in P$ and $\lambda_1, \lambda_2 \geq 0$ such that

$$\bar{\mu}(x) + \frac{\varepsilon}{2} \geq \mu(y_1) + \lambda_1 \alpha_0 \quad \text{and} \quad \bar{\mu}(y) + \frac{\varepsilon}{2} \geq \mu(y_2) + \lambda_2 \alpha_0$$

with $x \leq y_1 + \lambda_1 x_0$ and $y \leq y_2 + \lambda_2 x_0$ which implies that

$x + y \leq (y_1 + y_2) + (\lambda_1 + \lambda_2) x_0$ thus for all $\varepsilon \geq 0$ we have

$$\bar{\mu}(x) + \bar{\mu}(y) + \varepsilon \geq \mu(y_1 + y_2) + (\lambda_1 + \lambda_2) \alpha_0 \geq \bar{\mu}(x + y)$$

hence $\bar{\mu}(x + y) \leq \bar{\mu}(x) + \bar{\mu}(y)$.

Therefore $\bar{\mu} \in \Gamma$, and as $\bar{\mu} \leq \mu$ we conclude that $\bar{\mu} = \mu$ as μ is a minimal element in Γ . Thus

$$\mu(x_0) = \alpha_0 = \sup\{q(x) - \mu(y) \mid x, y \in P, \mu(y) < \infty, x \leq x_0 + y\}.$$

But the map $\bar{q} : P \rightarrow \bar{\mathfrak{R}}$ defined by $\bar{q}(x_0) = \alpha_0$ is a superlinear functional. Indeed;

(α) Clearly $\bar{q}(\lambda x_0) = \lambda \bar{q}(x_0)$ for all $x_0 \in P$ and $\lambda \geq 0$.

(β) Let $x_0, y_0 \in P$. For any choice of $x_1, x_2, y_1, y_2 \in P$ with $x_1 \leq x_0 + y_1$, $\mu(y_1) < \infty$ and $x_2 \leq y_0 + y_2$, $\mu(y_2) < \infty$ we have $(x_1 + x_2) \leq (x_0 + y_0) + (y_1 + y_2)$ with $\mu(y_1 + y_2) < \infty$. Hence

$$\begin{aligned} \bar{q}(x_0 + y_0) &\geq q(x_1 + x_2) - \mu(y_1 + y_2) \\ &\geq q(x_1) - \mu(y_1) + q(x_2) - \mu(y_2) \\ &\geq \bar{q}(x_0) + \bar{q}(y_0). \end{aligned}$$

Moreover, $\mu = \bar{q}$, thus μ is a linear functional. $\square$

(2.4) Remark : Note that the condition in Theorem (2.3) is fulfilled if we assume that $q(x) \leq p(x)$ for all $x \in P$ and if either functional q or p is monotone. And in the case of a vector space with the equality as an order we obtain the usual sandwich version of the Hahn-Banach theorem for vector spaces (cf. [6] II-272). (Usually, however, one requires the sublinear functional p to be finite, whereas q is allowed to attain $(-\infty)$).

(2.5) Corollary : Let $q : P \rightarrow \overline{\mathfrak{R}}$ be a monotone superlinear functional . For every $x_0 \in P$ there is a monotone linear functional $\mu : P \rightarrow \overline{\mathfrak{R}}$ such that $\mu(x_0) = q(x_0)$ and $q(x) \leq \mu(x)$ for all $x \in P$.

Proof : Let $p : P \rightarrow \overline{\mathfrak{R}}$ defined by

$$p(x) = \begin{cases} \lambda q(x_0) & \text{if } x = \lambda x_0 \text{ and } \lambda \geq 0 \\ +\infty & \text{elsewhere} \end{cases}$$

then p is a sublinear functional :

(α) Let $x \in P$ and $\alpha \geq 0$, if $\alpha = 0$ or $p(x) = +\infty$ then $p(\alpha x) = \alpha p(x)$, otherwise $\alpha x = \lambda x_0$ for $\lambda \geq 0$ and

$$p(\alpha x) = \lambda q(x_0) = \alpha \left(\frac{\lambda}{\alpha} q(x_0) \right) = \alpha p(x) .$$

(β) Let $x, y \in P$, if either $p(x) = +\infty$ or $p(y) = +\infty$ then $p(x+y) \leq p(x) + p(y)$. Otherwise , there exist $\lambda_1, \lambda_2 \geq 0$ such that $x = \lambda_1 x_0$ and $y = \lambda_2 x_0$ then $x+y = (\lambda_1 + \lambda_2) x_0$ and $p(x+y) = (\lambda_1 + \lambda_2) q(x_0) = \lambda_1 q(x_0) + \lambda_2 q(x_0) = p(x) + p(y)$.

Moreover , $q(x) \leq p(y)$ for all $x, y \in P$ with $x \leq y$. Indeed ; it is clear if $p(y) = +\infty$, otherwise $y = \lambda x_0$ for $\lambda \geq 0$, thus $q(x) \leq q(y) = \lambda q(x_0) = p(y)$. Applying Theorem (2.3) there exists a monotone linear functional $\mu : P \rightarrow \overline{\mathfrak{R}}$ with

$$q(x) \leq \mu(x) \leq p(x) \text{ for all } x \in P ,$$

but $p(x_0) = q(x_0)$, therefore $\mu(x_0) = q(x_0)$ □

Note that a corresponding result as in Corollary (2.5) is not available for a sublinear functional $p : P \rightarrow \overline{\mathfrak{R}}$ in general . (cf. Examples (2.16) , (2.17)) . In fact we shall have to impose further conditions on p (cf. Corollary (2.12)) .

(2.6) Definition : A map $q : P \rightarrow \mathfrak{R} \cup \{\pm\infty\}$ is called an *extended superlinear functional* , if

$$q(\lambda x) = \lambda q(x) \quad \text{for all } x \in P, \lambda \geq 0 \quad \text{and}$$

$$q(x + y) \geq q(x) + q(y) \quad \text{whenever } q(x), q(y) \in \overline{\mathfrak{R}} .$$

(2.7) Definition : A convex subset W with $0 \in W \subseteq P$ is called *absorbing* , if for every $x \in P$ there is $\lambda \geq 0$ such that $x \in \lambda W$; and *subabsorbing* , if for every $x \in P$ there is $w \in W$ and $\lambda \geq 0$ such that $\lambda w \leq x$.

(2.8) Lemma :(i) Let $q : P \rightarrow \overline{\mathfrak{R}}$ be a superlinear functional. Then for all $\lambda > 0$ the set $W := \{ w \in P \mid q(w) \geq -\lambda \}$ is an absorbing convex subset containing 0 .

(ii) If E is a vector space and $p : E \rightarrow \mathfrak{R}$ is a sublinear functional then for all $\lambda > 0$ the set $W := \{ w \in P \mid p(-w) \leq \lambda \}$ is an absorbing convex subset containing 0 .

Proof : (i) Clearly $0 \in W$. Let $w_1, w_2 \in W$ and $\alpha \in [0, 1]$ then $\alpha w_1 + (1-\alpha) w_2 \in W$ because

$$q(\alpha w_1 + (1-\alpha) w_2) \geq \alpha q(w_1) + (1-\alpha) q(w_2) \geq -\lambda .$$

For any $x \in P$, there exist $r \geq 0$ such that $q(x) \geq -r$ thus

$$x \in \left(\frac{r}{\lambda} \right) W .$$

(ii) Clearly $0 \in W$. Let $w_1, w_2 \in W$ and $\alpha \in [0, 1]$, then $\alpha w_1 + (1-\alpha) w_2 \in W$ because

$$\begin{aligned} p(-(\alpha w_1 + (1-\alpha) w_2)) &= p(-\alpha w_1 - (1-\alpha) w_2) \\ &\leq \alpha \lambda + (1-\alpha) \lambda = \lambda . \end{aligned}$$

For any $x \in E$, there exist $r \geq 0$ such that $p(-x) \leq r$ thus

$$x \in \left(\frac{r}{\lambda} \right) W . \quad \square$$

(2.9) Theorem: (Generalization of Sandwich Theorem) : Let $p : P \rightarrow \overline{\mathfrak{R}}$ be a sublinear functional , $q : P \rightarrow \mathfrak{R} \cup \{\pm \infty\}$ be an extended superlinear functional . The following are equivalent :

(i) There exists a monotone linear functional $\mu : P \rightarrow \overline{\mathfrak{R}}$ such that

$$q(x) \leq \mu(x) \leq p(x) \quad \text{for all } x \in P .$$

(ii) There exists an absorbing convex subset W with $0 \in W \subseteq P$, such that for all $w \in W$ and $x, y \in P$,

$$x + w \leq y \quad \text{implies} \quad q(x) \leq p(y) + 1 .$$

(iii) *There exists a convex subset W with $0 \in W \subseteq P$, such that*

(1) *For every $y \in P$ there are $x \in P$ with $q(x) > -\infty$,*

$w \in W$ and $\lambda \geq 0$ such that $x + \lambda w \leq y$.

(2) *For all $x, y \in P$ and $w \in W$,*

$x + w \leq y$ implies $q(x) \leq p(y) + 1$.

Moreover, if (iii) holds, then μ can be chosen such that

$\mu(w) \geq -1$ for all $w \in W$.

Proof : (i) $\Rightarrow$ (ii). Assume that $\mu : P \rightarrow \overline{\mathbb{R}}$ is a monotone linear functional such that $q(x) \leq \mu(x) \leq p(x)$ for all $x \in P$. Set $W := \{ w \in P : \mu(w) \geq -1 \}$. By Lemma (2.8) (i) W is an absorbing convex subset with $0 \in W \subseteq P$. Moreover, for $w \in W$ and $x, y \in P$ with $x + w \leq y$ we have

$$q(x) - 1 \leq \mu(x) - 1 \leq \mu(x) + \mu(w) = \mu(x + w) \leq \mu(y) \leq p(y)$$

(ii) $\Rightarrow$ (iii). We choose the same convex set W in (iii) as in (ii).

(1) Let $y \in P$, then there is $\lambda \geq 0$ and $w \in W$ such that $y = \lambda w$, then $0 + \lambda w \leq y$ for $0 \in P$, $w \in W$, and $\lambda \geq 0$. (2) Obvious.

(iii) $\Rightarrow$ (i). Let W as in (iii). For $y \in P$ set

$$\overline{q}(y) := \sup \{ q(x) - \lambda \mid x + \lambda w \leq y \text{ for } x \in P, w \in W, \lambda \geq 0 \}.$$

Then $\overline{q}(y) > -\infty$ for all $y \in P$. Indeed; let $y \in P$, by part (1) there are $x \in P$ with $q(x) > -\infty$, $w \in W$ and $\lambda \geq 0$ such that $x + \lambda w \leq y$, therefore $\overline{q}(y) > -\infty$.

Now the map $\bar{q} : P \rightarrow \bar{\mathfrak{R}}$ is a superlinear functional :

(α) Clearly $\bar{q}(\alpha x) = \alpha \bar{q}(x)$ for all $x \in P$ and $\lambda \geq 0$.

(β) Let $y_1, y_2 \in P$. For any choice of $x_1, x_2 \in P$ with $q(x_1) > -\infty$, $q(x_2) > -\infty$, $w_1, w_2 \in W$ and $\lambda_1, \lambda_2 \geq 0$ such that $x_1 + \lambda_1 w_1 \leq y_1$ and $x_2 + \lambda_2 w_2 \leq y_2$, we have $(x_1 + x_2) + (\lambda_1 + \lambda_2)w \leq (y_1 + y_2)$ with

$$w = \begin{cases} 0 & \text{if } \lambda_1 + \lambda_2 = 0 \\ \frac{\lambda_1}{\lambda_1 + \lambda_2} w_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} w_2 & \text{elsewhere} \end{cases},$$

which implies that $w \in W$ in any case.

$$\begin{aligned} \text{Hence } \bar{q}(y_1 + y_2) &\geq q(x_1 + x_2) - (\lambda_1 + \lambda_2) \\ &\geq (q(x_1) - \lambda_1) + (q(x_2) - \lambda_2) \end{aligned}$$

and $\bar{q}(y_1 + y_2) \geq \bar{q}(y_1) + \bar{q}(y_2)$. Moreover, for every

$y_1, y_2 \in P$ with $y_1 \leq y_2$, we have $\bar{q}(y_1) \leq p(y_2)$. Indeed;

let $x + \lambda w \leq y_1 \leq y_2$ for $x \in P$, $w \in W$ and $\lambda \geq 0$, it

is enough to show that $q(x) - \lambda \leq p(y_2)$. If $\lambda = 0$, then

$\varepsilon x + 0 \leq \varepsilon y_2$ for all $\varepsilon > 0$ and by part (2)

$$q(x) \leq p(y_2) + \frac{1}{\varepsilon} \quad \text{thus} \quad q(x) \leq p(y_2).$$

If $\lambda > 0$, then $\frac{x}{\lambda} + w \leq \frac{y_2}{\lambda}$, and again by using part (2)

$$q\left(\frac{x}{\lambda}\right) \leq p\left(\frac{y_2}{\lambda}\right) + 1 \quad \text{hence} \quad q(x) - \lambda \leq p(y_2)$$

thus $\bar{q}(y_1) \leq p(y_2)$. By Theorem (2.3) there is a monotone linear functional $\mu : P \rightarrow \bar{\mathfrak{R}}$ such that $\bar{q}(x) \leq \mu(x) \leq p(x)$ for all $x \in P$. As $q(x) \leq \bar{q}(x)$, $q(x) \leq \mu(x) \leq p(x)$. Also for all $w \in W$, $0 + 1w \leq w$ thus $\mu(w) \geq \bar{q}(w) \geq -1$. $\square$

(2.10) Remark : If $q : P \rightarrow \bar{\mathfrak{R}}$ in Theorem (2.9) is a superlinear functional ; that is , not extended , then the subset $W=\{0\}$ fulfills (1) in part (iii) of Theorem (2.9) , and condition (2) then means $q(x) \leq p(y)$ whenever $x \leq y$. Thus Theorem (2.9) generalizes the Sandwich Theorem (2.3) .

The following first corollary generalises a result for a vector spaces given in [2] .

(2.11)Corollary: Let $p : P \rightarrow \bar{\mathfrak{R}}$ be a sublinear functional .

The following are equivalent :

(i) *There is a monotone linear functional $\mu : P \rightarrow \bar{\mathfrak{R}}$ such that*

$$\mu(x) \leq p(x) \quad \text{for all } x \in P .$$

(ii) *There is a subabsorbing convex subset W with $0 \in W \subseteq P$, such that $p(x) \geq -1$ whenever $x \geq w$ for some $w \in W$.*

Proof : Follows directly from Theorem (2.9) (iii) with the extended superlinear functional

$$q(x) = \begin{cases} 0 & x = 0 \\ -\infty & \text{elsewhere} \end{cases} \quad \square$$

(2.12) Corollary : Let $p : P \rightarrow \overline{\mathbb{R}}$ be a monotone sublinear functional . Suppose there is an absorbing convex subset W with $0 \in W \subseteq P$ such that

$$p(x) \leq p(x + w) + 1 \quad \text{for all } x \in P \text{ and } w \in W .$$

Then for every extended superlinear functional $q : P \rightarrow \mathbb{R} \cup \{\pm \infty\}$ such that $q(x) \leq p(x)$ for all $x \in P$, there exists a monotone linear functional $\mu : P \rightarrow \overline{\mathbb{R}}$ such that

$$q(x) \leq \mu(x) \leq p(x) \quad \text{for all } x \in P .$$

In particular , for every $x_0 \in P$ there exists a monotone linear functional $\mu : P \rightarrow \overline{\mathbb{R}}$ such that $\mu(x_0) = p(x_0)$ and

$$\mu(x) \leq p(x) \quad \text{for all } x \in P .$$

Proof : Let $x, y \in P$ and $w \in W$ with $x + w \leq y$ then

$$q(x) \leq p(x) \leq p(x + w) + 1 \leq p(y) + 1 ,$$

thus by Theorem (2.9)(ii) we prove the first part . For the second part ,

let $x_0 \in P$, and define the map $q : P \rightarrow \mathbb{R} \cup \{\pm \infty\}$ by

$$q(x) := \begin{cases} \lambda p(x_0) & \text{if } x = \lambda x_0 \text{ and } \lambda \geq 0 \\ -\infty & \text{elsewhere} \end{cases}$$

then q is an extended superlinear functional :

(α) Clearly $q(\alpha x) = \alpha q(x)$ for all $x \in P$ and $\alpha \geq 0$.

(β) Let $x, y \in P$ with $q(x), q(y) \in \overline{\mathfrak{R}}$, then $x = \lambda_1 x_0$
 $y = \lambda_2 x_0$ for $\lambda_1, \lambda_2 \geq 0$ then $x + y = (\lambda_1 + \lambda_2) x_0$ and
 $q(x + y) = (\lambda_1 + \lambda_2) p(x_0) = q(x) + q(y)$. Moreover, for all $x \in P$,
 $q(x) \leq p(x)$. Thus by the first part there exists a monotone linear
functional $\mu : P \rightarrow \overline{\mathfrak{R}}$ such that

$$q(x) \leq \mu(x) \leq p(x) \quad \text{for all } x \in P,$$

but $q(x_0) = p(x_0)$ thus $\mu(x_0) = p(x_0)$ $\square$

The following is the classical Sandwich version of the Hahn -
Banach theorem for ordered vector spaces .

(2.13) Corollary : Let E be an ordered vector space and
 $p : E \rightarrow \mathfrak{R}$ monotone sublinear functional that assume only finite
values . Let $q : E \rightarrow \mathfrak{R} \cup \{-\infty\}$ be an extended superlinear
functional such that $q(x) \leq p(x)$ for all $x \in E$. Then there
exists a monotone linear functional $\mu : E \rightarrow \mathfrak{R}$ such that

$$q(x) \leq \mu(x) \leq p(x) \quad \text{for all } x \in E.$$

Proof : Consider the set $W := \{w \in E \mid p(-w) \leq 1\}$. Then
by Lemma(2.8)(ii) W is an absorbing convex subset with $0 \in W \subseteq E$.
Moreover for all $w \in W$ and $x \in E$ we have

$$p(x) \leq p(x + w) + p(-w) \leq p(x + w) + 1$$

Now applying Corollary (2.12) we get the result . $\square$

(2.14) Example : Let P be the ordered cone of $\overline{\mathfrak{R}}$ -valued lower semi-continuous functions on $[a, b]$ with the pointwise order. Let $\mu \neq 0$ be any positive linear functional on $C([a, b])$, that is, $\mu(f) \geq 0$ whenever, $f(t) \geq 0$ for every $t \in [a, b]$ then by Riesz Representation Theorem (see [6] I-40), μ coincides with a regular Borel measure on $C([a, b])$. For $f \in P$, set

$$p(f) := \inf \{ \mu(h) \mid h \in C([a, b]), f \leq h \} \in \overline{\mathfrak{R}}$$

$$q(f) := \sup \{ \mu(h) \mid h \in C([a, b]), h \leq f \} \in \overline{\mathfrak{R}}$$

then p is a monotone sublinear functional and q is a monotone superlinear functional with $q(f) \leq p(f)$ for all $f \in P$. Moreover, p and q coincide on $C([a, b])$, that is,

$$p(h) = \mu(h) = q(h) \quad \text{for every } h \in C([a, b]).$$

Also by Theorem(2.3) there is a monotone linear functional $\overline{\mu}: P \rightarrow \overline{\mathfrak{R}}$ such that

$$q(f) \leq \overline{\mu}(f) \leq p(f) \quad \text{for all } f \in P.$$

Since $C([a, b]) \subseteq P$ then we may say that $\overline{\mu}$ is an extension of μ .

(2.15) Example : let $(E, \|\cdot\|)$ be a normed linear space with B the unit ball of E , and let P be the cone of all non-empty convex subsets of E , endowed with the usual operations and inclusion as an order. For a linear functional $\mu: E \rightarrow \mathfrak{R}$ with $\|\mu\| \leq 1$, set

$$p(A) := \inf \{ \lambda \geq 0 \mid A \subseteq \lambda B \}, \text{ for } A \in P \text{ and}$$

$$q(A) := \sup \{ \mu(a) \mid a \in A \}, \text{ for } A \in P.$$

Then p is a sublinear functional and q is a monotone superlinear functional. Moreover, let $A \in P$ with $A \subseteq \lambda B$ for $\lambda \geq 0$.

If $a \in A$ then $\mu(a) \leq \| \mu \| \| a \|$ thus $\mu(a) \leq \lambda$, hence $q(A) \leq \lambda$ for all $\lambda \geq 0$ with $A \subseteq \lambda B$, thus $q(A) \leq p(A)$.

By Theorem (2.3) there is a monotone linear functional $\bar{\mu} : P \rightarrow \overline{\mathfrak{R}}$ such that

$$q(A) \leq \bar{\mu}(A) \leq p(A) \text{ for all } A \in P.$$

Moreover, $\bar{\mu}$ and μ coincide on the singleton subsets in P .

That is, $\bar{\mu}(\{a\}) = \mu(a)$ for all $a \in E$. Indeed; since

$$\mu(a) = q(\{a\}) \leq \bar{\mu}(\{a\}), \text{ and for all } a \in E$$

$$-\mu(a) = \mu(-a) = q(\{-a\}) \leq \bar{\mu}(\{-a\}) = -\bar{\mu}(\{a\}),$$

thus $\mu(a) \geq \bar{\mu}(\{a\})$, hence the result.

(2.16) Example : Let $E = c_{00}$, the vector space of the sequences of real numbers which are eventually 0, that is, $x = (x_i)_{i \in \mathbb{N}} \in c_{00}$ if and only if there exists N (depending on x) such that $x_i = 0$ for $i > N$, with the usual operations and pointwise order.

For $x = (x_i)_{i \in \mathbb{N}}$ set

$$p(x) := \begin{cases} 0 & \text{if } x = 0 \\ +\infty & \text{if at least one } x_i > 0 \\ n \sum_{i=1}^{\infty} x_i & \text{if all } x_i \leq 0 \text{ and } n = \max \{i \in \mathbb{N} \mid x_i \neq 0\} \end{cases}$$

then $p : E \rightarrow \overline{\mathbb{R}}$ is a monotone sublinear functional . Indeed ;

(α) Clearly $p(\lambda x) = \lambda p(x)$ for all $x \in E$ and $\lambda \geq 0$.

(β) Let $x, y \in E$, with $x = (x_i)_{i \in \mathbb{N}}$, $y = (y_i)_{i \in \mathbb{N}}$ if $x = 0$ or $y = 0$ or $p(x) = +\infty$ or $p(y) = +\infty$ then $p(x + y) \leq p(x) + p(y)$. Otherwise

$$p(x) = n \sum_{i=1}^{\infty} x_i \quad \text{and} \quad p(y) = m \sum_{i=1}^{\infty} y_i$$

for $n, m \in \mathbb{N}$, then by setting $N = \max \{n, m\}$, and as all x_i and y_i are non-positive we have

$$p(x) + p(y) = n \sum_{i=1}^{\infty} x_i + m \sum_{i=1}^{\infty} y_i \geq N \sum_{i=1}^{\infty} (x_i + y_i) = p(x + y)$$

(γ) Let $x, y \in E$, with $x = (x_i)_{i \in \mathbb{N}}$, $y = (y_i)_{i \in \mathbb{N}}$ such that $x_i \leq y_i$, if $y_i > 0$ for some $i \in \mathbb{N}$ then $p(x) \leq p(y)$.

Otherwise ,

$$p(x) = n \sum_{i=1}^{\infty} x_i \quad \text{and} \quad p(y) = m \sum_{i=1}^{\infty} y_i$$

with $n \geq m$ then

$$p(x) = n \sum_{i=1}^{\infty} x_i \leq m \sum_{i=1}^{\infty} y_i = p(y) .$$

We shall show that p does not fulfill the assumptions of Corollary (2.11) , that is , there is no linear functional $\mu : E \rightarrow \overline{\mathfrak{R}}$ such that $\mu(x) \leq p(x)$ for all $x \in E$. Assume on the contrary that there is such μ and set $\mu(e_i) = \alpha_i$, where e_i is the i -th unit vector over E . For any $n \in \mathbb{N}$ and $\lambda \geq 0$ set $x = -(\lambda e_1 + e_n)$. Then $p(x) = -n(\lambda + 1)$ and $\mu(x) = -(\lambda \alpha_1 + \alpha_n)$. Since $\mu(x) \leq p(x)$,

$$-(\lambda \alpha_1 + \alpha_n) \leq -n(\lambda + 1) \quad \text{or} \quad n(\lambda + 1) \leq (\lambda \alpha_1 + \alpha_n) ,$$

that is , $\alpha_n \geq n + \lambda(n - \alpha_1)$. Now let $n \in \mathbb{N}$ with $n \geq \alpha_1$ then our conclusion does not hold in general for all $\lambda > 0$, which is a contradiction . Thus there is no such linear functional μ on E .

The previous Example (2.16) shows that even on a vector space E for a non-finite sublinear functional $p : E \rightarrow \overline{\mathfrak{R}}$, there may not be a single linear functional $\mu : E \rightarrow \overline{\mathfrak{R}}$ dominated by p . Another example for this fact may be seen in [2] .

The next example will show that even for a finite sublinear functional p defined on a subcone of a vector space, there may not be a single linear functional μ dominated by p . (cf. Corollary (2.12)).

(2.17) Example : consider the vector space $E = c_{00}$, and the subcone $Q \subseteq c_{00}$ consisting of those sequences in c_{00} that have only non-positive components, with the usual operations and the pointwise order, define the map $p : Q \rightarrow \mathfrak{R}$ by

$$p(a) = \begin{cases} 0 & \text{if } a = 0 \\ n \sum_{i=1}^{\infty} a_i & \text{where } n = \{i \in \mathbb{N} \mid a_i \neq 0\} \end{cases}$$

then p is a finite monotone sublinear functional, but the same argument as in Example (2.16) shows that there is no linear functional μ dominated by p .

From the last two examples one can notice that the conditions in Corollaries (2.11) and (2.12) about the sublinear functional are essential. And later; precisely in section 4, we will give the sufficient and necessary conditions to get corresponding results about extensions of linear functionals defined on a subcone. (cf. Theorem (4.6)).

3 . SUBLINEAR FUNCTIONALS ON THE EUCLIDEAN VECTOR SPACE $\mathfrak{R}^n$.

In the previous section we have seen that not every $\overline{\mathfrak{R}}$ -valued sublinear functional dominates a linear functional in general , even on a vector space . However , in this section we will prove that for the finite dimensional Euclidean vector space $\mathfrak{R}^n$ such a statement holds true . Throughout this section we consider the equality as the order on $\mathfrak{R}^n$.

(3.1) Lemma: *Let $p : \mathfrak{R}^n \rightarrow \overline{\mathfrak{R}}$ be a sublinear functional . Then there exists an absorbing convex subset W with $0 \in W \subseteq \mathfrak{R}^n$ such that $p(w) \geq -1$ for all $w \in W$.*

Proof : Set $Q := \{ x \in \mathfrak{R}^n \mid p(x) < +\infty \}$, then Q is a subcone of $\mathfrak{R}^n$. Consider the subspace $E = Q - Q$ of $\mathfrak{R}^n$ with its basis $V := \{ e_1, e_2, \dots, e_m \}$, where $m \leq n$. Then for every $e_i \in V$ there exists $a_i, b_i \in Q$ such that

$$e_i = a_i - b_i \quad \text{for all } i = 1, 2, \dots, m .$$

As p is finite on Q , there exists $M \geq 0$ such that

$$M = \sup \{ p(a_i) , p(b_i) \mid i = 1, 2, \dots, m \} .$$

Set $a_0 = \sum_{i=1}^m (a_i + b_i) \in Q$, and choose $\alpha > 0$ such that

$$\alpha[p(a_0) - 4mM] \geq -1.$$

We extend the basis V of E to the basis $V' = V \cup \{e_{m+1}, \dots, e_n\}$ of $\mathfrak{R}^n$ and set

$$W := \left\{ \sum_{i=1}^n \lambda_i e_i \mid \lambda_i \in \mathfrak{R}, e_i \in V' \text{ with } \sum_{i=1}^n |\lambda_i| \leq \alpha \right\}.$$

Then W is clearly an absorbing convex subset of $\mathfrak{R}^n$ with $0 \in W \subseteq \mathfrak{R}^n$. We shall show that $p(w) \geq -1$ for all $w \in W$.

Let $w = \sum_{i=1}^n \lambda_i e_i$, if $\lambda_i \neq 0$ for at least one $i > m$ then

$w \notin E$ and $p(w) = +\infty > -1$. Otherwise, we compute

$$\begin{aligned} \alpha a_0 - w &= \alpha \sum_{i=1}^m (a_i + b_i) - \sum_{i=1}^m \lambda_i (a_i - b_i) \\ &= \sum_{i=1}^m (\alpha - \lambda_i) a_i + \sum_{i=1}^m (\alpha + \lambda_i) b_i. \end{aligned}$$

As all $|\lambda_i| \leq \alpha$ we have $\alpha a_0 - w \in Q$ and

$$\begin{aligned} p(\alpha a_0 - w) &\leq \sum_{i=1}^m (\alpha - \lambda_i) p(a_i) + \sum_{i=1}^m (\alpha + \lambda_i) p(b_i) \\ &\leq \sum_{i=1}^m 2\alpha M + \sum_{i=1}^m 2\alpha M = 4\alpha m M. \end{aligned}$$

As $p(\alpha a_0) \leq p(\alpha a_0 - w) + p(w)$, the latter implies that

$$\begin{aligned} p(w) &\geq p(\alpha a_0) - p(\alpha a_0 - w) \\ &\geq \alpha[p(a_0) - 4mM] \geq -1 \end{aligned}$$

as desired. $\square$

(3.2) Theorem : Let $p : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be a sublinear functional .
Then there exists a linear functional $\mu : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\mu(x) \leq p(x) \quad \text{for all } x \in \mathbb{R}^n .$$

Proof: Follows directly from Corollary (2.11) and Lemma (3.1) .

(3.3) Theorem : Let $p : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be a sublinear functional ,
and let $q : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ be an extended superlinear
functional such that $q(x) \leq p(x)$ for all $x \in \mathbb{R}^n$. The
following are equivalent :

(i) There exists a linear functional $\mu : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$q(x) \leq \mu(x) \leq p(x) \quad \text{for all } x \in \mathbb{R}^n .$$

(ii) For all $x \in \mathbb{R}^n$ $\inf \{ p(x+y) - q(y) \mid y \in \mathbb{R}^n \} > -\infty$.

Proof : (i) $\Rightarrow$ (ii) . If there is μ such that

$$q(x) \leq \mu(x) \leq p(x) \quad \text{for all } x \in \mathbb{R}^n .$$

Then

$$\mu(x) + q(y) \leq \mu(x+y) \leq p(x+y) \quad \text{for all } x, y \in \mathbb{R}^n ,$$

which implies that $\inf \{ p(x+y) - q(y) \mid y \in \mathbb{R}^n \} \geq \mu(x) > -\infty$.

(ii) $\Rightarrow$ (i) .If (ii) holds , then the map $\bar{p}: \mathfrak{R}^n \rightarrow \overline{\mathfrak{R}}$ defined by

$$\bar{p}(x) := \inf \{p(x+y) - q(y) \mid y \in \mathfrak{R}^n\}$$

is a sublinear functional . By Theorem (3.2) there is a linear functional

$$\mu: \mathfrak{R}^n \rightarrow \mathfrak{R} \quad \text{such that} \quad \mu(x) \leq \bar{p}(x) \quad \text{for all } x \in \mathfrak{R}^n .$$

Now let $x \in \mathfrak{R}^n$. As $\bar{p}(x) \leq p(x)$ we have $\mu(x) \leq p(x)$. Also

$$\mu(-x) \leq \bar{p}(-x) \leq p(-x+x) - q(x) , \text{ hence } q(x) \leq \mu(x) , \text{ thus}$$

$$q(x) \leq \mu(x) \leq p(x) \quad \text{for all } x \in \mathfrak{R}^n . \quad \square$$

One may notice that Theorem (3.2) can be obtained from Theorem (3.3) by setting $q(0) = 0$ and $q(x) = -\infty$ elsewhere .

Next we are going to show that the condition(ii) in Theorem(3.3) is essential .

(3.4)Example :Consider the vector space $\mathfrak{R}^2$, and set

$$p(x, y) = \begin{cases} 0 & \text{if } y > 0 \text{ or if } y = x = 0 \\ +\infty & \text{elsewhere} \end{cases}$$

and

$$q(x, y) = \begin{cases} x & \text{if } x \geq 0 \text{ and } y = 0 \\ -\infty & \text{elsewhere} \end{cases}$$

then p is a sublinear functional and q is an extended superlinear functional with $q(x, y) \leq p(x, y)$ for all $(x, y) \in \mathfrak{R}^2$ and it is easy to see that there is no linear functional $\mu: \mathfrak{R}^2 \rightarrow \mathfrak{R}$ such that

$$q(x, y) \leq \mu(x, y) \leq p(x, y) \quad \text{for all } (x, y) \in \mathfrak{R}^2.$$

In fact, that is because of the second condition of Theorem (3.3) does not hold, that is, for $x_0 = (0, 1)$ and $y = (n, 0)$ with $n \in \mathbf{N}$, we have $p(x_0 + y) = p(n, 1) = 0$ thus

$$p(x_0 + y) - q(y) = -n \quad \text{for all } n \in \mathbf{N}$$

which implies that

$$\inf \{p(x_0 + y) - q(y) \mid y \in \mathfrak{R}^2\} = -\infty.$$

CHAPTER II

EXTENSION AND SEPARATION THEOREMS

This chapter presents some of the most useful tools in functional analysis. In section four we introduce three extension theorems (4.2), (4.3), (4.6) for monotone linear functionals on ordered cone. In section five the separation theorems for convex subsets with additional conditions will be studied. In section six we will extend a version of the Hahn-Banach theorem on a vector space, due to W.Roth (see [22]), to a version for an ordered cone. (cf. (6.2)).

4. EXTENSION THEOREMS

Extension theorems for linear functionals on vector spaces are among the most fundamental tools in modern analysis. The same holds true for ordered cones, where linear functionals play also an important role. A skillful handling of linear functional turns out to be indispensable. Consequently, in this section we are going to introduce

three extension theorems (4.2) , (4.3) and the most important one is Theorem (4.6) which asserts conditions under which linear functionals on a subcone of an ordered cone may be extended to a linear functional on the full cone .

Throughout this section , let $(P, \leq)$ be an ordered cone .

(4.1) Definition : Let $A \subseteq P$ be a convex subset . A function $f : A \rightarrow \overline{\mathfrak{R}}$ is said to be a *convex function* if for all $a, b \in A$ and $0 \leq \lambda \leq 1$ we have

$$f(\lambda a + (1 - \lambda) b) \leq \lambda f(a) + (1 - \lambda) f(b) .$$

And a function $f : A \rightarrow \mathfrak{R} \cup \{\pm\infty\}$ is said to be a *concave function* if $f(\lambda a + (1 - \lambda) b) \geq \lambda f(a) + (1 - \lambda) f(b)$ for all $a, b \in A$ such that $f(a), f(b) > -\infty$ and $0 \leq \lambda \leq 1$.

(4.2) Theorem : Let $A \subseteq P$ be a convex subset , $q : P \rightarrow \overline{\mathfrak{R}}$ a monotone superlinear functional , and $f : A \rightarrow \overline{\mathfrak{R}}$ a convex function such that $q(a) \leq f(a)$ for all $a \in A$. Then there exists a monotone linear functional $\mu : P \rightarrow \overline{\mathfrak{R}}$ such that $q(x) \leq \mu(x)$ for all $x \in P$ and $\mu(a) \leq f(a)$ for all $a \in A$.

Proof : Let $p(x) := \inf \left\{ \frac{1}{\lambda} f(\lambda x) \mid \lambda > 0, \lambda x \in A \right\}$ and $p(0) = 0$ (As usual, we set $\inf \emptyset = +\infty$) .

We shall first show that $p(x) > -\infty$. More generally, let $x \leq y$.

We shall show that $-\infty < q(x) \leq p(y)$. If $p(y) = +\infty$ then $q(x) \leq p(y)$, otherwise for all $\lambda > 0$ such that $\lambda y \in A$, we have

$$q(\lambda y) \leq f(\lambda y) \quad \text{or} \quad q(y) \leq \frac{1}{\lambda} f(\lambda y)$$

which implies $q(y) \leq p(y)$ but $q(x) \leq q(y)$ thus $q(x) \leq p(y)$. It is straightforward to verify that $p: P \rightarrow \overline{\mathfrak{R}}$ is a sublinear functional. Following the Sandwich Theorem (2.3), there is a monotone linear functional $\mu: P \rightarrow \overline{\mathfrak{R}}$ such that

$$q(x) \leq \mu(x) \quad \text{for all } x \in P \quad \text{and}$$

$$\mu(a) \leq p(a) \leq f(a) \quad \text{for all } a \in A. \quad \square$$

(4.3) Theorem : Let $A \subseteq P$ be a convex subset, $p: P \rightarrow \overline{\mathfrak{R}}$ a sublinear functional. Let $f: A \rightarrow \mathfrak{R} \cup \{\pm\infty\}$ be a concave function such that $f(a) \leq p(a)$ for all $a \in A$. The following two properties are equivalent :

- (i) There exists a monotone linear functional $\mu: P \rightarrow \overline{\mathfrak{R}}$ such that $\mu(x) \leq p(x)$ for all $x \in P$ and $f(a) \leq \mu(a)$ for all $a \in A$.
- (ii) There exists an absorbing convex subset W with $0 \in W \subseteq P$ such that $f(a) \leq p(x) + \lambda$ whenever $a + \lambda w \leq x$ for all $a \in A$, $x \in P$, and $\lambda \geq 0$.

Proof: Let $q(x) := \sup \left\{ \frac{1}{\lambda} f(\lambda x) \mid \lambda > 0, \lambda x \in A \right\}$

and $q(0) = 0$ (As usual, we set $\sup \emptyset = -\infty$).

Then it is straightforward to check that $q: P \rightarrow \mathcal{R} \cup \{\pm\infty\}$ is an extended superlinear functional.

Now we want to apply Theorem (2.9), parts (i) and (ii) to verify the equivalence between (i) and (ii) above.

(i) $\Rightarrow$ (ii). Assume that the condition (i) of Theorem (4.3) holds, then for $x \in P$ and $\lambda > 0$ with $\lambda x \in A$ we have $f(\lambda x) \leq \mu(\lambda x)$ thus $q(x) \leq \mu(x) \leq p(x)$. By Theorem (2.9)(i) there exists an absorbing convex subset W with $0 \in W \subseteq P$ such that for all $w \in W$ and $x, y \in P$ with $x + w \leq y$ we have $q(x) \leq p(y) + 1$. Now let $a + \lambda w \leq x$ for $a \in A$, $x \in P$, $w \in W$, and $\lambda > 0$. Then

$$\frac{a}{\lambda} + w \leq \frac{x}{\lambda} \quad \text{and} \quad q\left(\frac{a}{\lambda}\right) \leq p\left(\frac{x}{\lambda}\right) + 1$$

or $q(a) \leq p(x) + \lambda$ but $f(a) \leq q(a)$, thus $f(a) \leq p(x) + \lambda$. (the case of $\lambda = 0$ is easy).

(ii) $\Rightarrow$ (i) . Let $x, y \in P$ and $w \in W$ with $x + w \leq y$.

We show that $q(x) \leq p(y) + 1$. For all $\lambda > 0$ with $\lambda x \in A$ we have $\lambda x + \lambda w \leq \lambda y$ thus $f(\lambda x) \leq p(\lambda y) + \lambda$ or

$$\frac{1}{\lambda} f(\lambda x) \leq p(y) + 1 \quad \text{then} \quad q(x) \leq p(y) + 1 .$$

By Theorem (2.9)(ii) there is a monotone linear functional $\mu : P \rightarrow \overline{\mathfrak{R}}$ such that $\mu(x) \leq p(x)$ for all $x \in P$ and

$$f(a) \leq q(x) \leq \mu(a) \quad \text{for all} \quad a \in A . \quad \square$$

The following generalization of the Hahn-Banach theorem for vector spaces was proved by Bonsall in 1955 (cf. [29] P.13). We are going to obtain it by using Theorem (4.3) .

(4.4) Theorem : (Bonsall) : Let Q be a subcone in a vector space E , $p : E \rightarrow \mathfrak{R}$ a sublinear functional , and suppose that $q : Q \rightarrow \mathfrak{R}$ is a superlinear functional such that

$$q(a) \leq p(a) \quad \text{for all} \quad a \in Q .$$

Then there is a linear functional $\mu : E \rightarrow \mathfrak{R}$ such that

$$\mu(x) \leq p(x) \quad \text{for all} \quad x \in E \quad \text{and}$$

$$q(a) \leq \mu(a) \quad \text{for all} \quad a \in Q .$$

Proof : Consider the equality as an order on E . Clearly, Q is convex and q is concave on Q . Let $W := \{ w \in E \mid p(-w) \leq 1 \}$ then by Lemma (2.8)(ii) W is an absorbing convex subset with $0 \in W \subseteq E$. Now let $a + \lambda w = x$ for $x \in E, a \in Q, w \in W$, and $\lambda \geq 0$, then $a = x - \lambda w$ and

$$q(a) = q(x - \lambda w) \leq p(x - \lambda w) \leq p(x) + \lambda p(-w) \leq p(x) + \lambda.$$

Applying Theorem (4.3) we obtain the result. $\square$

(4.5)Theorem : Let Q be a subcone of P and $p : Q \rightarrow \overline{\mathfrak{R}}$ a monotone sublinear functional. Suppose that there exists an absorbing convex subset W with $0 \in W \subseteq P$ such that for all $a, b \in Q$ and $w \in W$

$$a + w \leq b \quad \text{implies} \quad p(a) \leq p(b) + 1. \quad (*)$$

Then there exists a monotone sublinear functional $\bar{p} : P \rightarrow \overline{\mathfrak{R}}$ extending p such that for all $x, y \in P$ and $w \in W$

$$x + w \leq y \quad \text{implies} \quad \bar{p}(x) \leq \bar{p}(y) + 1 \quad (**)$$

Proof : Let $x \in P$, define a map $\bar{p} : P \rightarrow \overline{\mathfrak{R}}$ by

$$\bar{p}(x) := \inf \{ p(a) + \lambda \mid x + \lambda w \leq a \text{ for } a \in Q, w \in W, \lambda > 0 \}.$$

Let $x \in P$. As W is absorbing, there exist $\lambda_0 \geq 0, w_0 \in W$ such that $x = \lambda_0 w_0$, let $x + \lambda w \leq a$ for $a \in Q, w \in W, \lambda > 0$ then

$$x + \lambda w = \lambda_0 w_0 + \lambda w = 0 + (\lambda_0 + \lambda) \left(\frac{\lambda_0}{\lambda_0 + \lambda} w_0 + \frac{\lambda}{\lambda_0 + \lambda} w \right) \leq a$$

thus by (*) $0 \leq p(a) + (\lambda_0 + \lambda)$ or $p(a) + \lambda \geq -\lambda_0$ then $\bar{p}(x) \geq -\lambda_0 > -\infty$.

Then it is easy to verify that $\bar{p}$ is a monotone sublinear functional with $\bar{p}(a) = p(a)$ for all $a \in Q$. Indeed, let $a \in Q$. As $a + \lambda \cdot 0 \leq a$ for all $\lambda > 0$, we have $\bar{p}(a) \leq p(a) + \lambda$, thus $\bar{p}(a) \leq p(a)$. For the converse, let $b \in Q, w \in W$ and $\lambda > 0$ such that $a + \lambda w \leq b$. By condition (*) we have $p(a) \leq p(b) + \lambda$. This shows $p(a) \leq \bar{p}(a)$. It remains to show that $\bar{p}$ satisfies (* *). Let $x, y \in P$ and $w \in W$ with $x + w \leq y$. If $\bar{p}(y) = +\infty$ then $\bar{p}(x) \leq \bar{p}(y) + 1$, otherwise, for every $\varepsilon \geq 0$ there exist $a \in Q, w_0 \in W$, and $\lambda > 0$ such that $y + \lambda w_0 \leq a$ and $\bar{p}(y) + \varepsilon \geq p(a) + \lambda$. Now

$$x + w + \lambda w_0 = x + (\lambda + 1) \left(\frac{1}{\lambda + 1} w + \frac{\lambda}{\lambda + 1} w_0 \right) = x + (\lambda + 1) \bar{w} \leq a$$

for $\bar{w} \in W$ thus $\bar{p}(x) \leq p(a) + (\lambda + 1)$, and for every $\varepsilon \geq 0$ we have $\bar{p}(x) \leq \bar{p}(y) + \varepsilon + 1$ hence $\bar{p}(x) \leq \bar{p}(y) + 1$. $\square$

In the following extension theorem, which is our main result of this section we shall deal in a very general way with extensions for linear functional on a subcone of an ordered cone. Our result allows a wide range of special cases and is, like all our previous investigations, of interest even in a vector spaces, because we allow, other than in the corresponding classical results, the functionals p and q to take infinite values.

(4.6) Theorem : *Let Q_1 and Q_2 be subcones of P . Let $p : P \rightarrow \overline{\mathbb{R}}$ be a sublinear functional and let $q : P \rightarrow \mathbb{R} \cup \{\pm \infty\}$ be an extended superlinear functional. For a linear functionals $\mu_1 : Q_1 \rightarrow \overline{\mathbb{R}}$ and $\mu_2 : Q_2 \rightarrow \overline{\mathbb{R}}$ the following are equivalent :*

(i) *There exists a monotone linear functional $\bar{\mu} : P \rightarrow \overline{\mathbb{R}}$ such that*

$$q(x) \leq \bar{\mu}(x) \leq p(x) \text{ for all } x \in P,$$

$$\bar{\mu}(a) \geq \mu_1(a) \text{ for all } a \in Q_1, \text{ and}$$

$$\bar{\mu}(b) \leq \mu_2(b) \text{ for all } b \in Q_2.$$

(ii) *There exists an absorbing convex subset W with $0 \in W \subseteq P$, such that for all $x, y \in P$ with $q(x) > -\infty$, $a \in Q_1$, $b \in Q_2$, and $w \in W$:*

$$x + a + w \leq y + b \text{ implies } q(x) + \mu_1(a) \leq p(y) + \mu_2(b) + 1.$$

Moreover, if (ii) holds, then $\bar{\mu}$ can be chosen such that

$$\bar{\mu}(w) \geq -1 \text{ for all } w \in W.$$

Proof : (i) $\Rightarrow$ (ii) . Let $W := \{ w \in P : \bar{\mu}(w) \geq -1 \}$ then by Lemma (2.8)(i) W is an absorbing convex subset with $0 \in W \subseteq P$.
Now let $x, y \in P$ with $q(x) > -\infty$, $a \in Q_1$, $b \in Q_2$, and $w \in W$ with $x + a + w \leq y + b$ then

$$\begin{aligned}\bar{\mu}(x) + \bar{\mu}(a) + \bar{\mu}(w) &= \bar{\mu}(x + a + w) \\ &\leq \bar{\mu}(y + b) = \bar{\mu}(y) + \bar{\mu}(b)\end{aligned}$$

but we have $\bar{\mu}(x) + \bar{\mu}(a) + \bar{\mu}(w) \geq q(x) + \mu_1(a) - 1$ and $\bar{\mu}(y) + \bar{\mu}(b) \leq p(y) + \mu_2(b)$. It follows that

$$q(x) + \mu_1(a) \leq p(y) + \mu_2(b) - 1 .$$

(ii) $\Rightarrow$ (i) . For $x \in P$ set

$$\begin{aligned}\bar{p}(x) &:= \inf \{ p(y) + \mu_2(b) + \lambda \mid x + \lambda w \leq y + b , \\ &\quad \lambda \geq 0 , y \in P , w \in W , b \in Q_2 \} .\end{aligned}$$

$$\begin{aligned}\bar{q}(x) &:= \sup \{ q(y) + \mu_1(a) \mid y + a \leq x , \\ &\quad y \in P \text{ with } q(y) > -\infty , a \in Q_1 \} .\end{aligned}$$

Then we obtain :

(α) $\bar{p}(x) > -\infty$ for all $x \in P$. Let $x \in P$. As W is an absorbing subset , there exists $\lambda_0 > 0$ and $w_0 \in W$ such that $x = \lambda_0 w_0$. Let $x + \lambda w \leq y + b$ for $y \in P$, $w \in W$, $b \in Q_2$, and $\lambda \geq 0$ then

$$\begin{aligned}x + \lambda w &= \lambda_0 w_0 + \lambda w = (\lambda_0 + \lambda) \left(\frac{\lambda_0}{\lambda_0 + \lambda} w_0 + \frac{\lambda}{\lambda_0 + \lambda} w \right) \\ &= 0 + 0 + (\lambda_0 + \lambda) \bar{w} \leq y + b\end{aligned}$$

for $\bar{w} \in W$, thus by assumption $0 \leq p(y) + \mu_2(b) + (\lambda_0 + \lambda)$

or $p(y) + \mu_2(b) + \lambda \geq -\lambda_0$ then $\bar{p}(x) \geq -\lambda_0 > -\infty$.

(β) The map $\bar{p} : P \rightarrow \overline{\mathfrak{R}}$ is a sublinear functional : Indeed ,

($\beta 1$) Clearly $\bar{p}(\alpha x) = \alpha \bar{p}(x)$ holds for all $x \in P$ and $\alpha \geq 0$.

($\beta 2$) Let $x_1, x_2 \in P$, if $\bar{p}(x_1) = +\infty$ or $\bar{p}(x_2) = +\infty$ then $\bar{p}(x_1 + x_2) \leq \bar{p}(x_1) + \bar{p}(x_2)$ otherwise, let $\varepsilon \geq 0$ be given, then there exist $y_1, y_2 \in P$, $w_1, w_2 \in W$, $b_1, b_2 \in Q_2$, and $\lambda_1, \lambda_2 > 0$ such that

$$\bar{p}(x_1) + \frac{\varepsilon}{2} \geq p(y_1) + \mu_2(b_1) + \lambda_1$$

and

$$\bar{p}(x_2) + \frac{\varepsilon}{2} \geq p(y_2) + \mu_2(b_2) + \lambda_2,$$

such that $x_1 + \lambda_1 w_1 \leq y_1 + b_1$ and $x_2 + \lambda_2 w_2 \leq y_2 + b_2$

which implies that

$$(x_1 + x_2) + (\lambda_1 + \lambda_2) \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} w_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} w_2 \right) \leq (y_1 + y_2) + (b_1 + b_2)$$

thus $\bar{p}(x_1) + \bar{p}(x_2) + \varepsilon \geq p(y_1 + y_2) + \mu_2(b_1 + b_2) + (\lambda_1 + \lambda_2)$
 $\geq \bar{p}(x_1 + x_2).$

As this holds for all $\varepsilon \geq 0$, we get $\bar{p}(x_1) + \bar{p}(x_2) \geq \bar{p}(x_1 + x_2).$

(γ) Similarly, one may verify that $\bar{q} : P \rightarrow \mathfrak{R} \cup \{\pm \infty\}$ is an extended superlinear functional.

(δ) Let $x_1, x_2 \in P$ and $w_0 \in W$ such that $x_1 + w_0 \leq x_2$, we shall show that $\bar{q}(x_1) \leq \bar{p}(x_2) + 1$. Let $y_1, y_2 \in P$, with $q(y_1) > -\infty$, $a \in Q_1$, $b \in Q_2$, $w \in W$, and $\lambda \geq 0$ such that $y_1 + a \leq x_1$ and $x_2 + \lambda w \leq y_2 + b$ then

$$y_1 + a + w_0 + \lambda w \leq x_1 + w_0 + \lambda w \leq x_2 + \lambda w \leq y_2 + b$$

and $y_1 + a + (1 + \lambda)w' \leq y_2 + b$ for some $w' \in W$, and by assumption $q(y_1) + \mu_1(a) \leq p(y_2) + \mu_2(b) + (1 + \lambda)$, therefore $\bar{q}(x_1) \leq \bar{p}(x_2) + 1$. Now by Theorem (2.9)(ii) there exists a monotone linear functional $\bar{\mu}: P \rightarrow \bar{\mathcal{R}}$ such that $\bar{q}(x) \leq \bar{\mu}(x) \leq \bar{p}(x)$ for all $x \in P$. As $\bar{p}(x) \leq p(x)$ and $q(x) \leq \bar{q}(x)$ for all $x \in P$ we have $q(x) \leq \bar{q}(x) \leq \bar{\mu}(x) \leq \bar{p}(x) \leq p(x)$. Since

$$\bar{q}(a) \geq \mu_1(a) \text{ for all } a \in Q_1, \text{ and } \bar{p}(b) \leq \mu_2(b) \text{ for all } b \in Q_2$$

we have

$$\bar{\mu}(a) \geq \mu_1(a) \text{ for all } a \in Q_1, \text{ and } \bar{\mu}(b) \leq \mu_2(b) \text{ for all } b \in Q_2.$$

Moreover by Theorem (2.9), we can choose $\bar{\mu}$ such that $\bar{\mu}(w) \geq -1$ for all $w \in W$. $\square$

We shall consider a variety of special cases of this very general result. First we observe that for $Q_1 = Q_2 = \{0\}$ Theorem (4.6) reduces to the generalized sandwich theorem (2.9). If $Q_1 = Q_2$ and $\mu_1 = \mu_2$, then we obtain a criterion for the existence of an extension

of a given monotone linear functional on a subcone to a monotone linear functional on the whole cone P .

In our first corollary we shall formulate the case when q does not take the value $(-\infty)$.

(4.7) Corollary : *Let P, Q_1, Q_2, p, μ_1 , and μ_2 be as in Theorem (4.6), and let $q : P \rightarrow \overline{\mathbb{R}}$ be a superlinear functional. The following are equivalent :*

(i) *There exists a monotone linear functional $\bar{\mu} : P \rightarrow \overline{\mathbb{R}}$ such that*

$$q(x) \leq \bar{\mu}(x) \leq p(x) \quad \text{for all } x \in P,$$

$$\bar{\mu}(a) \geq \mu_1(a) \quad \text{for all } a \in Q_1, \text{ and}$$

$$\bar{\mu}(b) \leq \mu_2(b) \quad \text{for all } b \in Q_2.$$

(ii) *For all $x, y \in P, a \in Q_1$, and $b \in Q_2$:*

$$x + a \leq y + b \quad \text{implies} \quad q(x) + \mu_1(a) \leq p(y) + \mu_2(b).$$

Proof : (ii) is clearly necessary for (i). For the converse, assume that (ii) holds and let $W := \{w \in P : q(w) \geq -1\}$ then by Lemma (2.8)(i) W is an absorbing convex subset with $0 \in W \subseteq P$. Let $x, y \in P, a \in Q_1, b \in Q_2$, and $w \in W$ with $x + a + w \leq y + b$ then by assumption $q(x + w) + \mu_1(a) \leq p(y) + \mu_2(b)$ but we have $q(x) - 1 \leq q(x) + q(w) \leq q(x + w)$ which implies that

$$q(x) + \mu_1(a) \leq p(y) + \mu_2(b) + 1,$$

and Theorem (4.6) (ii) applies □

Also we can notice that the case $Q_1 = Q_2 = \{0\}$ in Corollary (4.7) reduces to the basic Sandwich Theorem (2.3) .

Next we consider the case that $q : P \rightarrow \mathfrak{R} \cup \{\pm \infty\}$ defined by $q(x) = -\infty$, for $x \neq 0$, $q(0) = 0$.

(4.8) Corollary : *Let P , Q_1 , Q_2 , p , μ_1 , and μ_2 be as in Theorem (4.6) . The following are equivalent :*

(i) *There exists a monotone linear functional $\bar{\mu} : P \rightarrow \overline{\mathfrak{R}}$ such that*

$$\bar{\mu}(x) \leq p(x) \quad \text{for all } x \in P ,$$

$$\bar{\mu}(a) \geq \mu_1(a) \quad \text{for all } a \in Q_1 , \text{ and}$$

$$\bar{\mu}(b) \leq \mu_2(b) \quad \text{for all } b \in Q_2 .$$

(ii) *There exist an absorbing convex subset W with $0 \in W \subseteq P$ such that for all $y \in P$, $a \in Q_1$, $b \in Q_2$, and $w \in W$:*

$$a + w \leq y + b \quad \text{implies} \quad \mu_1(a) \leq p(y) + \mu_2(b) + 1 .$$

Proof : Define the map $q : P \rightarrow \mathfrak{R} \cup \{\pm \infty\}$ by $q(x) = -\infty$, for $x \neq 0$, $q(0) = 0$. Let $x, y \in P$ with $q(x) > -\infty$ ($x = 0$) , $a \in Q_1$, $b \in Q_2$, and $w \in W$ with $x + a + w \leq y + b$ then

$$q(x) + \mu_1(a) \leq \mu_1(a) \leq p(y) + \mu_2(b) + 1 ,$$

and Theorem (4.6) applies . □

For $p : P \rightarrow \overline{\mathfrak{R}}$ defined by $p(x) = +\infty$ for $x \neq 0$, and $p(0) = 0$ we obtain from Theorem (4.6) :

(4.9) Corollary : *Let P, Q_1, Q_2, q, μ_1 , and μ_2 be as in Theorem (4.6). The following are equivalent :*

(i) *There exists a monotone linear functional $\bar{\mu} : P \rightarrow \overline{\mathfrak{R}}$ such that*

$$q(x) \leq \bar{\mu}(x) \quad \text{for all } x \in P,$$

$$\bar{\mu}(a) \geq \mu_1(a) \quad \text{for all } a \in Q_1, \text{ and}$$

$$\bar{\mu}(b) \leq \mu_2(b) \quad \text{for all } b \in Q_2.$$

(ii) *There exists an absorbing convex subset W with $0 \in W \subseteq P$ such that for all $x \in P$ with $q(x) > -\infty$, $a \in Q_1$, $b \in Q_2$, and $w \in W$:*

$$x + a + w \leq b \quad \text{implies} \quad q(x) + \mu_1(a) \leq \mu_2(b) + 1.$$

Proof : Define the map $p : P \rightarrow \overline{\mathfrak{R}}$ by $p(x) = +\infty$, for $x \neq 0$, $p(0) = 0$. Let $x, y \in P$ with $q(x) > -\infty$, $a \in Q_1$, $b \in Q_2$, and $w \in W$ with $x + a + w \leq y + b$ then

$$q(x) + \mu_1(a) \leq \mu_2(b) + 1 \leq p(y) + \mu_2(b) + 1. \quad \square$$

Combining Corollaries (4.7) and (4.9) yields :

(4.10) Corollary : Let P, Q_1, Q_2, μ_1 , and μ_2 be as in Theorem (4.6), and let $q : P \rightarrow \overline{\mathfrak{R}}$ be a superlinear functional. The following are equivalent :

(i) There exists a monotone linear functional $\bar{\mu} : P \rightarrow \overline{\mathfrak{R}}$ such that

$$q(x) \leq \bar{\mu}(x) \text{ for all } x \in P,$$

$$\bar{\mu}(a) \geq \mu_1(a) \text{ for all } a \in Q_1, \text{ and}$$

$$\bar{\mu}(b) \leq \mu_2(b) \text{ for all } b \in Q_2.$$

(ii) For all $x \in P, a \in Q_1$, and $b \in Q_2$:

$$x + a \leq b \text{ implies } q(x) + \mu_1(a) \leq \mu_2(b).$$

Proof : Define the map $p : P \rightarrow \overline{\mathfrak{R}}$ by $p(x) = +\infty$, for $x \neq 0$, $p(0) = 0$ and set $W := \{ w \in P : q(w) \geq -1 \}$ then by Lemma (2.8) W is an absorbing convex subset with $0 \in W \subseteq P$. Let $x, y \in P, a \in Q_1, b \in Q_2$, and $w \in W$ with $x + a + w \leq y + b$ then

$$q(x) - 1 + \mu_1(a) \leq q(x) + q(w) + \mu_1(a) \leq q(x + w) + \mu_1(a) \\ \leq p(y) + \mu_2(b). \quad \square$$

The combination of Corollaries (4.8) and (4.9) yields:

(4.11) Corollary : Let P, Q_1, Q_2, μ_1 , and μ_2 be as in Theorem (4.6). The following are equivalent :

(i) There exists a monotone linear functional $\bar{\mu} : P \rightarrow \overline{\mathfrak{R}}$ such that

$$\bar{\mu}(a) \geq \mu_1(a) \text{ for all } a \in Q_1, \text{ and}$$

$$\bar{\mu}(b) \leq \mu_2(b) \text{ for all } b \in Q_2.$$

(ii) *There exist an absorbing convex subset W with $0 \in W \subseteq P$ such that for all $a \in Q_1$, $b \in Q_2$, and $w \in W$:*

$$a + w \leq b \quad \text{implies} \quad \mu_1(a) \leq \mu_2(b) + 1.$$

Proof: Define the maps $p : P \rightarrow \overline{\mathfrak{R}}$ by $p(x) = +\infty$, for $x \neq 0$, $p(0) = 0$ and $q : P \rightarrow \mathfrak{R} \cup \{\pm\infty\}$ by $q(x) = -\infty$, for $x \neq 0$, $q(0) = 0$. Let $x, y \in P$ with $q(x) > -\infty$, $a \in Q_1$, $b \in Q_2$, and $w \in W$ with $x + a + w \leq y + b$ then

$$q(x) + \mu_1(a) \leq \mu_2(b) + 1 \leq p(y) + \mu_2(b) + 1 \quad \square$$

(4.12) Corollary : *Let E be an ordered vector space over $\mathfrak{R}$ and $Q \subseteq E$ be a subspace. Let $p : E \rightarrow \mathfrak{R}$ be a monotone sublinear functional, then for every linear functional $\mu : Q \rightarrow \mathfrak{R}$ with $\mu(a) \leq p(a)$ for all $a \in Q$, there exists a monotone linear functional $\bar{\mu} : E \rightarrow \mathfrak{R}$ such that*

$$\bar{\mu}(a) = \mu(a) \quad \text{for all } a \in Q, \text{ and}$$

$$\bar{\mu}(x) \leq p(x) \quad \text{for all } x \in E.$$

Proof : Let $W := \{w \in E : p(-w) \leq 1\}$ then by Lemma (2.8)(ii) W is an absorbing convex subset with $0 \in W \subseteq P$. Let $y \in E$, $a \in Q$, and $w \in W$ with $a + w \leq y$ then

$$\mu(a) \leq p(a) \leq p(a + w) + p(-w) \leq p(y) + 1,$$

thus by Corollary (4.8) with $Q_2 = \{0\}$ there exist a monotone linear functional $\bar{\mu} : E \rightarrow \mathfrak{R}$ such that $\bar{\mu}(x) \leq p(x)$ for all $x \in E$ and $\bar{\mu}(a) \geq \mu(a)$ for all $a \in Q$, but $\bar{\mu}(-a) \geq \mu(-a)$ which implies that $\bar{\mu}(a) = \mu(a)$ for all $a \in Q$. $\square$

Note that in Corollary (4.12) the functional μ was not explicitly required to be monotone on Q . We observe that the usual Hahn-Banach type theorem for vector space (see [17] P.62) follows directly from Corollary (4.12) just by replacing the order " $\leq$ " by the equality.

(4.13) Theorem : Let $\mu : P \rightarrow \overline{\mathfrak{R}}$ be a monotone linear functional, and let $p_1, p_2, \dots, p_n : P \rightarrow \overline{\mathfrak{R}}$ be sublinear functionals such that $p_i(x) \geq 0$ for all $x \in P$, $i = 1, 2, \dots, n$. Suppose that there exists an absorbing convex subset W with $0 \in W \subseteq P$ such that for all $x, y_i, z \in P$, and $w_i \in W$:

$$x + w_i \leq y_i + z \text{ implies } \mu(x) \leq \sup \{p_i(y_i)\} + \mu(z) + 1, i = 1, 2, \dots, n$$

then there exist monotone linear functionals $\mu_1, \mu_2, \dots, \mu_n : P \rightarrow \overline{\mathfrak{R}}$ such that

$$\mu_i(x) \leq p_i(x) \text{ for all } i = 1, 2, \dots, n \text{ and}$$

$$\mu(x) = \mu_1(x) + \mu_2(x) + \dots + \mu_n(x) \text{ for all } x \in P.$$

Proof : Set $\bar{P} := \{ (x_1, x_2, \dots, x_n) \mid x_i \in P, i=1,2,\dots,n \}$ and denote the diagonal by $\bar{Q} := \{ (a, a, \dots, a) \mid a \in P \}$. Then $\bar{P}$ is an ordered cone if the order relation and the operations are defined pointwise. Obviously $\bar{Q}$ is a subcone of $\bar{P}$. We define a monotone linear functional $\bar{\mu} : \bar{Q} \rightarrow \bar{\mathfrak{R}}$ by

$$\bar{\mu}(a, a, \dots, a) = \mu(a) \quad \text{for all } a \in P.$$

Let $\bar{W} := \{ (w_1, w_2, \dots, w_n) \mid w_i \in W, i=1,2,\dots,n \}$, then $\bar{W}$ is an absorbing convex subset with $0 \in \bar{W} \subseteq \bar{P}$. Let $\bar{p} : \bar{P} \rightarrow \bar{\mathfrak{R}}$ defined by

$$\bar{p}(x_1, x_2, \dots, x_n) := \sup \{ p_i(x_i) \mid x_i \in P, i=1,2,\dots,n \}$$

then $\bar{p}$ is a sublinear functional :

(α) Clearly $\bar{p}(\lambda(x_1, x_2, \dots, x_n)) = \lambda \bar{p}(x_1, x_2, \dots, x_n)$ for all $\lambda \geq 0$ and $(x_1, x_2, \dots, x_n) \in \bar{P}$.

(β) Let $\bar{x} = (x_1, x_2, \dots, x_n)$ and $\bar{y} = (y_1, y_2, \dots, y_n) \in \bar{P}$ then

$$\begin{aligned} \bar{p}(\bar{x} + \bar{y}) &= \bar{p}((x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)) \\ &= \sup \{ p_i(x_i + y_i) \mid (x_i + y_i) \in P, i=1,2,\dots,n \} \\ &\leq \sup \{ p_i(x_i) + p_i(y_i) \mid x_i, y_i \in P, i=1,2,\dots,n \} \\ &\leq \bar{p}(\bar{x}) + \bar{p}(\bar{y}). \end{aligned}$$

Furthermore, for $(y_1, y_2, \dots, y_n) \in \bar{P}$, $(a, a, \dots, a), (b, b, \dots, b) \in \bar{Q}$,

and $(w_1, w_2, \dots, w_n) \in \bar{W}$ with

$$(a, a, \dots, a) + (w_1, w_2, \dots, w_n) \leq (y_1, y_2, \dots, y_n) + (b, b, \dots, b),$$

we have $a + w_i \leq y_i + b$ for all $i = 1, 2, \dots, n$, thus by assumption

$\mu(a) \leq \sup\{p_i(y_i)\} + \mu(b) + 1$ for all $i = 1, 2, \dots, n$, therefore

$$\bar{\mu}(a, a, \dots, a) \leq \bar{p}(y_1, y_2, \dots, y_n) + \bar{\mu}(b, b, \dots, b) + 1.$$

Now by Corollary (4.8) with $Q_1 = Q_2 = \bar{Q}$ and $\mu_1 = \mu_2 = \bar{\mu}$,

there exist a monotone linear functional $\bar{\bar{\mu}} : \bar{P} \rightarrow \bar{\mathfrak{R}}$ such that

$$\bar{\bar{\mu}}(a, a, \dots, a) = \bar{\mu}(a, a, \dots, a) \quad \text{for all } a \in P,$$

and $\bar{\bar{\mu}} \leq \bar{p}$ on $\bar{P}$. Define the maps

$$\mu_i(x) = \bar{\bar{\mu}}(\Delta_i x) \quad \text{for all } x \in P \text{ and } i = 1, 2, \dots, n$$

where $\Delta_i x = (0, 0, \dots, 0, x, 0, \dots, 0, 0)$, that is, x is the i -th component of

$\Delta_i x$ and all other components are zero. Then μ_i are monotone

linear functionals and

$$\mu(x) = \bar{\mu}(x, \dots, x) = \bar{\bar{\mu}}(x, \dots, x) = \bar{\bar{\mu}}\left(\sum_{i=1}^n \Delta_i x\right) = \sum_{i=1}^n \bar{\bar{\mu}}(\Delta_i x) = \sum_{i=1}^n \mu_i(x)$$

for all $x \in P$ and $\mu_i(x) = \bar{\bar{\mu}}(\Delta_i x) \leq \bar{p}(\Delta_i x) = p_i(x)$. $\square$

(4.14) Example : Of course, our result contains the classical Hahn-Banach theorem for normed spaces. Let $(E, \|\cdot\|)$ be normed space, $Q \subseteq E$ be a subspace, let $\mu : Q \rightarrow \mathfrak{R}$ be bounded linear functional, that is, $|\mu(a)| \leq \lambda \|a\|$ for all $a \in Q$, for some real number λ . Then by considering the sublinear functional $p : E \rightarrow \mathfrak{R}$ defined by $p(x) = \|\mu\| \|x\|$, and applying Corollary (4.12) there exist a linear functional $\bar{\mu} : E \rightarrow \mathfrak{R}$ such that

$\bar{\mu}(a) = \mu(a)$ for all $a \in Q$, and $|\bar{\mu}(x)| \leq p(x) = \|\mu\| \|x\|$ for all $x \in E$, thus $\|\bar{\mu}\| \leq \|\mu\|$ and hence $\|\bar{\mu}\| = \|\mu\|$.

(4.15) Example : Let $(E, \leq)$ be an ordered vector space and let F be a *cofinal* linear subspace of E (that is, if for all $x \geq 0$ in E there is an $a \in F$ such that $a \geq x$). Let $\mu : F \rightarrow \mathfrak{R}$ be a positive linear functional (that is, $\mu(x) \geq 0$ whenever $x \geq 0$). Define the monotone sublinear functional $p : E \rightarrow \mathfrak{R}$ by

$$p(x) := \inf \{ \mu(a) \mid x \leq a, a \in F \}$$

then $\mu(a) \leq p(a)$ for all $a \in F$. Indeed, let $a, b \in F$ such that $a \leq b$ then $\mu(a) \leq \mu(b)$ and $\mu(a) \leq \inf \{ \mu(b) \mid a \leq b, b \in F \} = p(a)$.

Applying Corollary (4.12) there exists a monotone linear functional $\bar{\mu} : E \rightarrow \mathfrak{R}$ such that

$$\bar{\mu}(a) = \mu(a) \text{ for all } a \in F \text{ and } \bar{\mu}(x) \leq p(x) \text{ for all } x \in E.$$

Now let $x \in E$, if $x \geq 0$ then $-x \leq 0$ ($0 \in F$), $p(-x) \leq \mu(0) = 0$, and $-\bar{\mu}(x) = \bar{\mu}(-x) \leq p(-x) \leq 0$, therefore $\bar{\mu}$ is a positive linear functional.

The next example will show that a linear functional on a subspace of a vector space which is dominated by non-finite sublinear functional may not always be extended to the whole space. By this example we can say that the conditions in Corollary (4.8) are essential. For other examples see Examples (2.16) and (2.17).

(4.16) Example : Consider the subspace c_{00} of ℓ^∞ the vector space of all bounded sequence ; that is, $x=(x_i)_{i \in \mathbb{N}} \in \ell^\infty$ if and only if $\sup\{|x_i| \mid i \in \mathbb{N}\} < \infty$. Let $p : \ell^\infty \rightarrow \overline{\mathfrak{R}}$ be defined by

$$p(x) := \sum_{i=1}^{\infty} (\sup\{x_i, 0\}) \quad \text{for all } x = (x_i)_{i \in \mathbb{N}} \in \ell^\infty .$$

Then p is a sublinear functional and $p(x) \geq 0$ for all $x \in \ell^\infty$.

Let $\mu : c_{00} \rightarrow \mathfrak{R}$ defined by

$$\mu(a) = \sum_{i=1}^{\infty} a_i \quad \text{for all } a = (a_i)_{i \in \mathbb{N}} \in c_{00} .$$

Then μ is a linear functional with $\mu(a) \leq p(a)$ for all $a \in c_{00}$.

We claim that there is no linear functional that extends μ and is dominated by p . Assume on the contrary there is a linear functional $\bar{\mu} : \ell^\infty \rightarrow \mathfrak{R}$ such that $\bar{\mu}(a) = \mu(a)$ for all $a \in c_{00}$, and $\bar{\mu}(x) \leq p(x)$ for all $x \in \ell^\infty$. Now for every $x = (x_i)_{i \in \mathbb{N}} \in \ell^\infty$ with $x \leq 0$ we have $\bar{\mu}(x) \leq p(x) = 0$ thus for every $x \geq 0$, $\bar{\mu}(x) \geq 0$. Now consider $z = (1, 1, \dots) \in \ell^\infty$ then $z = z_1 + z_2$ where

$$z_1 = (\overbrace{1, \dots, 1}^n, 0, \dots, 0) , z_2 = (\overbrace{0, \dots, 0}^n, 1, \dots, 1) .$$

Thus $\bar{\mu}(z) = \bar{\mu}(z_1 + z_2) = \bar{\mu}(z_1) + \bar{\mu}(z_2) = n + \lambda$ where $\lambda \geq 0$ then $\bar{\mu}(z) \geq n$, and as this holds for all n , we get $\bar{\mu}(z) = +\infty$, that is a contradiction , as linear functional can take only finite values on a vector space .

5. SEPARATION THEOREMS

The results below are often referred to as separation theorems. Theorem (5.1) involves the separation of a convex subset and an absorbing convex increasing subset of an ordered cone by a monotone linear functional. The separation of two convex subsets with an additional condition is discussed in Theorem (5.2). Again we shall assume that P is an ordered cone throughout this section and recall the basic Definition (1.13).

(5.1) Theorem : *Let A, B be two non-empty disjoint convex subsets of P . If B is absorbing, $0 \in B$, and increasing, then there exists a monotone linear functional $\mu : P \rightarrow \overline{\mathfrak{R}}$ such that*

$$\mu(a) \leq -1 \leq \mu(b) \quad \text{for all } a \in A \text{ and } b \in B.$$

Proof : Set $p(x) := \inf \{ \lambda \geq 0 \mid x \in \lambda B \}$ for all $x \in P$. As B is absorbing, we have $p(x) \in [0, +\infty)$, and it is easy to see that p is a sublinear functional with $p(b) \leq 1$ for all $b \in B$. Let $x, y \in P$ with $x \leq y$ and $\lambda \geq 0$ such that $x \in \lambda B$, but B is increasing then $y \in \lambda B$ and $p(y) \leq p(x)$. Thus, the functional $q(x) = -p(x)$ is a monotone superlinear functional, and

$q(b) \geq -1$ for all $b \in B$. Now let $a \in A$. Then $p(a) \geq 1$ (otherwise $A \cap B \neq \emptyset$), hence for the convex functional $f: A \rightarrow \overline{\mathfrak{R}}$ defined by $f(a) = -1$ for all $a \in A$, we have $q(a) \leq f(a) = -1$ for all $a \in A$. Now we apply Theorem (4.2) with the convex subset A and the maps f and q . There exists a monotone linear functional $\mu: P \rightarrow \overline{\mathfrak{R}}$ such that

$$\mu(a) \leq -1 \leq \mu(b) \text{ for all } a \in A \text{ and } b \in B. \quad \square$$

(5.2)Theorem : Let A and B be two non-empty disjoint convex subsets of P . If B is decreasing, $0 \in B$ and if there exists an absorbing convex subset with $0 \in W \subseteq P$ such that with $D = \bigcup_{\lambda \geq 0} \lambda(A + W)$, (that is, the subcone generated by $A + W$), we have $(A + D) \cap B = \emptyset$, then there exists a monotone linear functional $\mu: P \rightarrow \overline{\mathfrak{R}}$ such that

$$\mu(b) \leq 1 \leq \mu(a) \text{ for all } a \in A \text{ and } b \in B.$$

Proof : Set $p(x) := \inf \{ \lambda \geq 0 \mid x \in \lambda B \}$ for all $x \in P$. Then p is a monotone sublinear functional with $p(a) \geq 1$ for all $a \in A$ (otherwise $A \cap B \neq \emptyset$). Now let $f \equiv 1$ on A . We shall show that $f(a) = 1 \leq p(x) + \lambda$ whenever $a + \lambda w \leq x$ for all $x \in P$, $a \in A$, $w \in W$ and $\lambda \geq 0$. As p is monotone, it is enough to show

$1 \leq p(a + \lambda w) + \lambda$. Assume on the contrary that $p(a + \lambda w) < 1 - \lambda$, since $p(x) \geq 0$ for all $x \in P$, $\lambda < 1$ and $a + \lambda w \in (1 - \lambda)B \subseteq P$, (B is convex and $0 \in B$). Thus

$$\frac{1}{1-\lambda} (a + \lambda w) = a + \frac{\lambda}{1-\lambda} (a + w) \in B$$

but

$$a + \frac{\lambda}{1-\lambda} (a + w) \in A + D$$

thus $B \cap (A + D) \neq \emptyset$ which is a contradiction. Applying Theorem

(4.3) there exists a monotone linear functional $\mu : P \rightarrow \overline{\mathfrak{R}}$ such that

$$\mu(b) \leq 1 \leq \mu(a) \quad \text{for all } a \in A \text{ and } b \in B. \quad \square$$

(5.3) Example : Consider the vector space $E = c_{00}$, with the pointwise order and let

$$A := \{ x = (x_i)_{i \in \mathbb{N}} \neq 0 \mid \text{the last non-zero element } x_i \text{ is negative} \}$$

$$B := \{ x = (x_i)_{i \in \mathbb{N}} \neq 0 \mid \text{the last non-zero element } x_i \text{ is positive} \} \cup \{0\}.$$

Then A and B are non-empty disjoint convex subsets of E ,

$0 \in B$ and B is increasing, but there is no linear functional μ

that separates A and B as in Theorem (5.1). Assume to the

contrary that there is such $\mu : E \rightarrow \mathfrak{R}$, and set $\mu(e_i) = \alpha_i \geq -1$.

Since $(e_i - \varepsilon e_k) \in A$ for all $\varepsilon > 0$ and $k > i$ then

$$\mu(e_i - \varepsilon e_k) = \alpha_i - \varepsilon \alpha_k \leq -1$$

thus $-1 \leq \alpha_i \leq \varepsilon \alpha_k - 1$ for all $\varepsilon > 0$ hence $\alpha_i = -1$, for

all $i \in \mathbb{N}$, but for $n \geq 3$, $b = (e_1 + e_2 + \dots + e_n) - e_{n-1} \in B$

and $\mu(b) \geq -1$ but $\mu(b) = -(n-1) \leq -2$, which is a contradiction. One can also notice that the assumption of Theorem (5.1) does not hold, that is; B is not absorbing. Indeed; B obviously does not absorb the elements of A .

(5.4)Example: Again consider the vector space $E = c_{00}$, and let
 $A := \{ x = (x_i)_{i \in \mathbb{N}} \neq 0 \mid \text{the last non-zero element } x_i \text{ is positive} \}$
 $B := \{ x = (x_i)_{i \in \mathbb{N}} \neq 0 \mid \text{the last non-zero element } x_i \text{ is negative} \} \cup \{0\}$.
Then A and B are non-empty disjoint convex subsets of E , $0 \in B$ and B is decreasing. A similar argument as in Example (5.3) shows that there is no monotone linear functional μ as in Theorem (5.2). Notice that the assumptions of Theorem (5.2) do not hold, that is; there is no any absorbing convex subset W with $0 \in W \subseteq c_{00}$ such that with $D = \bigcup_{\lambda \geq 0} \lambda(A + W)$, we have $(A + D) \cap B = \emptyset$. Indeed; assume that there is such W , since $-e_1 = (-1, 0, \dots, 0) \in c_{00}$ there is $\alpha > 0$ such that $(-\alpha e_1) \in W$.

Since $\frac{\alpha}{3} e_1 \in A$, $\frac{\alpha}{3} e_1 + (-\alpha e_1) = \frac{-2\alpha}{3} e_1 \in (A + W)$ and

$$\left(\frac{\alpha}{3} e_1 \right) + \left(\frac{\alpha}{3} e_1 + (-\alpha e_1) \right) = \frac{-\alpha}{3} e_1 \in A + D$$

but $\frac{-\alpha}{3} e_1 \in B$ thus $(A + D) \cap B \neq \emptyset$, therefore there is no such W .

(5.5) Example : Consider the subset A of c_{00} as in Example (5.4) . Let $0 \neq \mu : E \rightarrow \mathfrak{R}$ be a linear functional , then $\mu(e_i) \neq 0$ for some $i \in \mathbf{N}$, as $\lambda e_i + e_{i+1} \in A$ for all $\lambda \in \mathfrak{R}$, and $\mu(\lambda e_i + e_{i+1}) = \lambda \mu(e_i) + \mu(e_{i+1})$ may take on any value by choosing λ appropriately. On the other hand $\{0\} \cap A = \emptyset$, and they can not be separated by a non-zero linear functional .

From Examples (5.3) , (5.4) and (5.5) , we notice that the conditions on the convex subsets A and B are essential . It is however of interest , particularly in optimization theory , to know that arbitrary non-empty disjoint convex subsets in the n -dimensional Euclidean space $\mathfrak{R}^n$ can always be separated by a linear functional ([26] P.167) .

(5.6) Example : Let E be any vector space , A , B be non-empty disjoint convex subsets of E , and let B be absorbing with $0 \in B$. By using equality as an order and apply theorem (5.1) there exists a linear functional $v : E \rightarrow \mathfrak{R}$ such that

$$v(a) \leq -1 \leq v(b) \quad \text{for all } a \in A \text{ and } b \in B .$$

Moreover , by setting $\mu = -v$ we get

$$\mu(b) \leq 1 \leq \mu(a) \quad \text{for all } a \in A , b \in B .$$

6.A UNIFORM BOUNDEDNESS PRINCIPLE

In this section we give a version of the Hahn-Banach theorem for ordered cones which, though purely algebraic, implies the uniform boundedness principle and the open mapping theorems for normed spaces without using Baier's Category Theorem as the usual proof does. For vector spaces, this result was derived in [22].

Throughout this section, let $(P, \leq)$ be an ordered cone.

(6.1)Definition : Let U be a set of monotone linear functionals $\mu : P \rightarrow \overline{\mathfrak{R}}$. U is said to be *pointwise bounded* on a subset $A \subseteq P$ if for all $a \in A$ there exists $\lambda_a \geq 0$ such that $|\mu(a)| \leq \lambda_a$ for all $\mu \in U$, and U is said to be *uniformly bounded* on $A \subseteq P$ if there exists $\lambda \geq 0$ such that $|\mu(a)| \leq \lambda$ for all $a \in A$ and $\mu \in U$. A map $p : A \rightarrow \overline{\mathfrak{R}}$ is said to be *bounded* on A if and only if

$$\sup\{ |p(a)| \mid a \in A \} < \infty.$$

(6.2) Lemma : Let $A \subseteq P$ and let U be a set of monotone linear functional $\mu : P \rightarrow \overline{\mathbb{R}}$ such that U is pointwise bounded on P , but not uniformly bounded on A .

There are $\mu_i \in U$ and $\alpha_i \geq 0$ for $i = 0, 1, 2, \dots$, such that $\sum_{i=0}^{\infty} \alpha_i = 1$, and the monotone linear functional μ defined by

$$\mu(x) = \sum_{i=0}^{\infty} \alpha_i \mu_i(x) \quad \text{is unbounded on } A.$$

Proof : If U contains a linear functional that is unbounded on A , say μ , then by setting $\mu_i = \mu$, $\alpha_0 = 1$ and $\alpha_i = 0$ for $i \geq 1$, we get the result. Otherwise, if all linear functionals in U are bounded on A , define a map

$$p(x) := \sup \{ |\mu(x)| \mid \mu \in U \} \quad \text{for all } x \in P.$$

Then p is a sublinear functional on P and attains only finite values. By mathematical induction we select linear functionals $\mu_i \in U$ and $\alpha_i > 0$ as follows : We set $\alpha_1 = \frac{1}{2}$, and Since U is not uniformly bounded on A , we choose $x_1 \in A$ and $\mu_1 \in U$ such that $|\mu_1(x_1)| \geq 2$. Assume that it is true for $i = n - 1$, and we will show that is true for $i = n$ as well. Since, for all $i = 1, 2, \dots, n-1$, $p(x_i) \geq |\mu_i(x_i)| \geq 2^i$. For $n \geq 2$, we choose α_n such that

$$0 < \alpha_n \leq \left\{ \frac{1}{2^n}, \frac{1}{2^n p(x_i)} \mid i = 1, 2, \dots, n-1 \right\}.$$

Because all the linear functionals $\mu_1, \mu_2, \dots, \mu_{n-1}$ were supposed to be bounded on A but the set $\{\mu(a) \mid a \in A, \mu \in U\}$ is unbounded on A , we can find $\mu_n \in U$, $x_n \in A$ such that,

$$\left| \sum_{i=1}^{n-1} \alpha_i \mu_i(x_n) + \alpha_n \mu_n(x_n) \right| = \left| \sum_{i=1}^n \alpha_i \mu_i(x_n) \right| \geq n.$$

Finally, we set $\alpha_0 = 1 - \sum_{i=1}^{\infty} \alpha_i$ and choose any $\mu_0 \in U$.

Then the map $\mu(x) = \sum_{i=0}^{\infty} \alpha_i \mu_i(x)$ for all $x \in P$, defines a

linear functional, which we were looking for: Indeed, for every $x \in P$ the series $\sum_{i=0}^{\infty} \alpha_i \mu_i(x)$ is convergent, because

$$|\mu_i(x)| \leq p(x) < +\infty \quad \text{and} \quad |\alpha_i| \leq \frac{1}{2^i} \quad \text{for all } i = 0, 1, \dots$$

Furthermore, for every $n \in \mathbb{N}$, we compute that

$$\begin{aligned} |\mu(x_n)| &= \left| \sum_{i=0}^{\infty} \alpha_i \mu_i(x_n) \right| \\ &\geq \left| \sum_{i=1}^n \alpha_i \mu_i(x_n) \right| - \left| \sum_{i=n+1}^{\infty} \alpha_i \mu_i(x_n) \right| - |\mu_0(x_n)| \\ &\geq n - \sum_{i=n+1}^{\infty} \alpha_i p(x_n) - |\mu_0(x_n)| \\ &\geq n - \sum_{i=n+1}^{\infty} \frac{1}{2^i} - |\mu_0(x_n)| \\ &\geq n - 1 - |\mu_0(x_n)|. \end{aligned}$$

Now $\sup\{|\mu(x_n)| \mid n \in \mathbb{N}\} = +\infty$, because μ_0 was supposed to be bounded on A . This concludes the proof. $\square$

(6.3)Theorem : Let $p : P \rightarrow \overline{\mathbb{R}}$ be a monotone sublinear functional . Furthermore suppose that there exists an absorbing convex subset W with $0 \in W \subseteq P$ such that

$$p(x) \leq p(x + w) + 1 \quad \text{for all } x \in P, w \in W.$$

Let $A \subseteq P$. If p is unbounded on A , then there exists a monotone linear functional $\mu : P \rightarrow \overline{\mathbb{R}}$ such that

$$\mu(x) \leq p(x) \quad \text{for all } x \in P,$$

and μ is also unbounded on A .

Proof : The proof is concluded by the following three cases :

Case (1) : If $\inf \{ p(a) : a \in A \} = -\infty$ then the same holds for every linear functional μ that is dominated by p . By Corollary (2.12) there is at least one such μ , and hence our claim is obvious .

Case (2) : If there exist $a_0 \in A$ such that $p(a_0) = +\infty$, then again by Corollary (2.12) there is $\mu : P \rightarrow \overline{\mathbb{R}}$ such that $\mu(x) \leq p(x)$ for all $x \in P$ and $\mu(a_0) = p(a_0) = +\infty$. Thus μ is unbounded on A .

Case (3) : If $\sup \{ p(a) : a \in A \} = +\infty$ then given $n \in \mathbb{N}$, there exist $a_n \in A$ such that $p(a_n) \geq n$. By Corollary (2.12) we get a linear functional μ_n such that $\mu_n(x) \leq p(x)$ for all $x \in P$ with $\mu_n(a_n) = p(a_n) \geq n$ and $\mu_n(w) \geq -1$ for all $w \in W$. Let $Q := \text{cone}(A)$, then p is finite for all $b \in Q$.

Indeed ; since $b = \sum_{i=1}^n \lambda_i a_i$ for $\lambda_i \geq 0$ and $a_i \in A$ then

$$p(b) \leq \sum_{i=1}^n \lambda_i p(a_i) < +\infty .$$

Moreover , since W is absorbing , for all $b \in Q$ there exist $\lambda_b \geq 0$ and $w \in W$ such that $b = \lambda_b w$. Thus it follows from

$$\mu_n(b) = \mu_n(\lambda_b w) = \lambda_b \mu_n(w) \geq -\lambda_b$$

that

$$-\lambda_b \leq \mu_n(b) \leq p(b) \quad \text{for all } n \in \mathbb{N} .$$

So the set $U := \{ \mu_n \mid n \in \mathbb{N} \}$ of monotone linear functionals is pointwise bounded on Q and $\sup\{ \mu_n(a) : a \in A \} \geq n$.

Therefore U is not uniformly bounded on A . By Lemma (6.2)

there exists a monotone linear functional μ on Q which is unbounded on A and $\alpha_i \geq 0$ $i = 0, 1, \dots$ such that $\sum_{i=0}^{\infty} \alpha_i = 1$ and

$$\mu(b) = \sum_{i=0}^{\infty} \alpha_i \mu_i(b) \quad \text{for } b \in Q$$

thus

$$\mu(b) = \sum_{i=0}^{\infty} \alpha_i \mu_i(b) \leq \sum_{i=0}^{\infty} \alpha_i p(b) = p(b) \quad \text{for all } b \in Q .$$

Now for $y \in P$, $a \in Q$ and $w \in W$ with $a + w \leq y$ we have

$$\mu(a) \leq p(a) \leq p(a + w) + 1 \leq p(y) + 1 .$$

By Corollary (4.8) with $Q_1 = Q$, $Q_2 = \{0\}$, $\mu_1 = \mu$ and $\mu_2 = 0$ there exists a monotone linear functional $\bar{\mu} : P \rightarrow \overline{\mathfrak{R}}$ such that

$$\bar{\mu}(x) \leq p(x) \quad \text{for all } x \in P,$$

and $\mu(a) \leq \bar{\mu}(a)$ for all $a \in Q$. As μ is unbounded on A , $\bar{\mu}$ is also, and our proof is complete. $\square$

(6.4)Corollary : *Let p be a finite sublinear functional on a vector space E and let $A \subseteq E$. If p is unbounded on A , then there exists a linear functional μ such that*

$$\mu(x) \leq p(x) \quad \text{for all } x \in E$$

and μ is also unbounded on A .

Proof : Consider the subset $W := \{w \in E : p(-w) \leq 1\}$, then by Lemma (2.8)(ii), W is an absorbing convex subset of E with $0 \in W$. Furthermore, for all $x \in E$ and $w \in W$ we have $p(x) \leq p(x+w) + p(-w) \leq p(x+w) + 1$, whence by Theorem (6.3) we get the result. $\square$

Note that Corollary (6.4) recovers the basic version of the Hahn-Banach theorem for vector spaces, if we choose the subset $A = E$.

For a normed linear space E over $\mathfrak{R}$ we denote by E^* , its algebraic dual, that is, the vector space of all real-valued linear functional on E , and by E' , its normed dual, that is, the subspace of E^* of all continuous linear functional on E . Recall that a subset A of a normed space is ***norm-bounded***, if the set of the norms of its elements is bounded. That is, if and only if there exist $\lambda \geq 0$ such that $\|a\| \leq \lambda$ for all $a \in A$. A subset $A \subseteq E$ is said to be ***weakly bounded***, if for all $\mu \in E'$ there exist λ_μ such that

$$|\mu(x)| \leq \lambda_\mu \quad \text{for all } x \in A.$$

We know that one of the most important basic principles for normed spaces is "The uniform boundedness principle" and one of the most important consequence of it is the following : " A subset $A \subseteq E$ in a normed vector space, is weakly bounded if and only if A is norm-bounded ". And the usual proof of this theorem uses the uniform boundedness principle which involves the Baier Category theorem ([17] P.73) and ([27] P.170), but we introduce another proof which depends only on Corollary (6.4) as follows :

(6.5) Corollary : *Let E be a normed linear space and E' the normed dual of E , then $A \subseteq E$ is norm- bounded if and only if A is weakly bounded .*

Proof : Clearly every bounded set is weakly bounded , conversely , let A be weakly bounded and suppose that A is not bounded . Let $p : E \rightarrow \mathbb{R}$ be defined by $p(x) = \|x\|$ then p is an unbounded linear functional on A and by Corollary (6.4) there exists $\mu \in E'$ with $\mu(x) \leq p(x)$ for all $x \in E$, such that μ is not bounded on A . Thus A is not weakly bounded , which is a contradiction . $\square$

CHAPTER III

LOCALLY CONVEX CONES

This chapter contains our study of topologies on cones . The material has been arranged into three sections . In section seven we discuss the locally convex structure on our cones . Section eight deals with the properties of uniformly continuous operators . Finally, the ninth section deals with the dual cones and extension of linear functionals on a locally convex cone . The theory of locally convex cones that we are going to present was developed by Keimel and Roth in [13] .

7. DEFINITIONS AND BASIC CONCEPTS

In this section we will introduce three topologies on a given preordered cone ; the upper, lower and symmetric topologies . Before we define these topologies , we shall introduce some definitions and basic properties .

(7.1) Definition : Let P be a preordered cone . A subset V of P is called an *abstract 0-neighborhood system* , if the following properties holds :

- (1) $0 < v$ for all $v \in V$.
- (2) For all $u, v \in V$ there is $w \in V$ such that

$$w \leq u \text{ and } w \leq v .$$

- (3) $u + v \in V$ and $\alpha v \in V$ whenever $u, v \in V$ and $\alpha > 0$.

By $v > 0$ we mean that $v \geq 0$ but $v \not\leq 0$. One could say that V is a " subcone without zero directed towards 0 " . The elements

$$x + v, v \in V,$$

may be called *abstract neighborhood system* of $x \in P$.

(7.2) Definition : Let V be an abstract 0-neighborhood system . A subset $U \subseteq V$ is called a *base* of V , if for every $v \in V$ there is $u \in U$ and an $\alpha > 0$ such that $\alpha u \leq v$.

(7.3) Definition : A subset U of P is said to be an *abstract 0-neighborhood base* , if the following properties hold :

- (1) $0 < u$ for all $u \in U$.
- (2) For all $u, v \in U$ there is $w \in U$ and an $\alpha > 0$ such that

$$\alpha w \leq u \text{ and } \alpha w \leq v .$$

(7.4) Theorem : A subset U of P is a base for some abstract 0-neighborhood system if and only if U is an abstract 0-neighborhood base .

Proof : Let U be a base for an abstract 0-neighborhood system V . Then :

(1) As $U \subseteq V$, $0 < u$ for all $u \in U$.

(2) Let $u, v \in U$, then there is $\bar{w} \in V$ such that $\bar{w} \leq u$ and $\bar{w} \leq v$. But U is a base , then there is $w \in U$ and $\alpha > 0$ such that $\alpha w \leq \bar{w}$ which implies that $\alpha w \leq u$ and $\alpha w \leq v$.

Conversely , let U be an abstract 0-neighborhood base and let

$$V := \left\{ \sum_{i=1}^n \alpha_i u_i \mid \alpha_i > 0, u_i \in U, n \in \mathbf{N} \right\} .$$

Then $U \subseteq V$ and V satisfies the following :

(1) $0 < v$ for all $v \in V$.

(2) For $u = \sum_{i=1}^n \alpha_i u_i$ and $v = \sum_{i=1}^m \beta_i v_i$ there exists $w \in U$ and an $\alpha > 0$ such that $\alpha w \leq \alpha_1 u_1$ and $\alpha w \leq \beta_1 v_1$. Thus

$$\alpha w \leq \alpha_1 u_1 \leq \sum_{i=1}^n \alpha_i u_i = u \quad \text{and} \quad \alpha w \leq \beta_1 v_1 \leq \sum_{i=1}^m \beta_i v_i = v .$$

(3) For $u = \sum_{i=1}^n \alpha_i u_i$ and $v = \sum_{i=1}^m \beta_i v_i$ we have

$$u + v = \sum_{i=1}^n \alpha_i u_i + \sum_{i=1}^m \beta_i v_i \in V , \text{ and}$$

$$\alpha u = \sum_{i=1}^n (\alpha \alpha_i) u_i \in V \quad \text{for all } \alpha > 0 . \quad \square$$

(7.4) Definition : (Locally convex topologies) . Let P be a preordered cone with a 0-neighborhood system V . For $x \in P$ we define

$$v(x) := \{ y \in P \mid y \leq x + v \} , \quad v \in V$$

to be a neighborhood of x in the *upper topology* , and

$$x(v) := \{ y \in P \mid x \leq y + v \} , \quad v \in V$$

to be a neighborhood of x in the *lower topology* .

One easily verifies that these neighborhood systems define indeed topologies on P . The common refinement of these two topologies is called the *symmetric topology* on P .

(7.6) Remarks :

(1) Note that the $v(x)$ are decreasing convex subsets and the $x(v)$ are increasing convex subsets of P .

(2) The neighborhoods in the symmetric topology are both convex and order convex .

Thus , all of these three topologies merit to be called *locally convex* .

(3) Of course , the upper and the lower topologies are in general far from being Hausdorff .

(4) Since all of the three topologies are defined in terms of the preorder on P , we will not need to work with them explicitly. In fact, all the properties will be expressible by means of the ordering and the 0-neighborhoods alone.

We also consider subcones Q of P not necessarily containing V . They will be endowed with the topologies induced from P as follows :

(7.7) Definition : Let P be a cone with a 0-neighborhood system V and let Q be a subcone of P . For every $a \in Q$ we define the neighborhood bases for the upper and the lower topologies on Q by

$$v_Q(a) := \{b \in Q \mid b \leq a + v\}, \quad v \in V,$$

$$a_Q(v) := \{b \in Q \mid a \leq b + v\}, \quad v \in V$$

respectively.

(7.8) Definition : Let P be a cone with a 0-neighborhood system V . For $v \in V$, an element $x \in P$ is called **upper v -bounded**, if $x \leq \alpha v$ for some $\alpha > 0$, **lower v -bounded**, if $0 \leq x + \alpha v$ for some $\alpha > 0$, **v -bounded**, if it is both lower and upper v -bounded.

An element $x \in P$ is called **upper (lower) bounded**, if it is upper (lower) bounded for every $v \in V$.

The following is straightforward :

(7.9) Proposition : *Let P be a cone with a 0 -neighborhood system V . The set $B(v)$ of all upper v -bounded and likewise the set B of all upper bounded elements is a decreasing subcone in P .*

Now we are ready for our main definitions :

(7.10) Definition : let P be a cone with a 0 -neighborhood system V . The pair (P, V) is called a *full locally convex cone*, if every element of P is lower bounded.

In a full locally convex cone the bounded elements therefore coincide with the upper bounded ones.

(7.11) Definition : Let (P, V) be a full locally convex cone. A *locally convex cone* is a pair (Q, V) where Q is a subcone of P .

Of course, Q inherits the preorder and the topological structure of P . One can notice that we neglect to indicate P in the notation, that is because the elements of the form

$$a + v \quad \text{with } a \in Q \text{ and } v \in \{0\} \cup V$$

form a subcone of P already containing V and determine the neighborhoods in Q . But one has to keep in mind that Q in general does not contain the 0-neighborhood system.

If the locally convex cone Q is a vector space, that is, contains all negatives of its elements, then for all $a, b \in Q$ and $v \in V$

$$b \leq a + v \quad \text{if and only if} \quad b - a \leq v.$$

In this case all elements of Q are bounded, as they and their negatives are bounded below, and the neighborhoods of 0 with respect to the symmetric topology are given by

$$\begin{aligned} v_0 = v(0) \cap 0(v) &:= \{b \in Q \mid b \leq v \text{ and } 0 \leq b + v\} \\ &= \{b \in Q \mid b \leq v \text{ and } -b \leq v\}. \end{aligned}$$

They form the basis for a locally convex vector space topology on Q .

The following are our standard examples for locally convex cones.

(7.12) Example : The cone $\overline{\mathfrak{R}}$ will always be endowed with the abstract neighborhood system $V = \{\varepsilon \in \mathfrak{R} \mid \varepsilon > 0\}$. For $x \in \mathfrak{R}$ the upper and the lower neighborhoods are given by

$$\begin{aligned} \varepsilon(x) &:= \{y \in \overline{\mathfrak{R}} \mid y \leq x + \varepsilon\} = (-\infty, x + \varepsilon], \\ x(\varepsilon) &:= \{y \in \overline{\mathfrak{R}} \mid x \leq y + \varepsilon\} = [x - \varepsilon, +\infty] \end{aligned}$$

respectively , while for $x = +\infty$ the entire cone $\overline{\mathfrak{R}}$ is the only upper neighborhood , and $\{+\infty\}$ is the only one for the lower topology . The symmetric topology on $\overline{\mathfrak{R}}$ is the usual topology on $\mathfrak{R}$ with $\{+\infty\}$ as an isolated point .

(7.13) Example : Let E be a locally convex topological vector space with a 0-neighborhood base V consisting of convex neighborhoods V of 0 in E . Since $\text{conv}(E)$ contains V , is ordered by inclusion , and all its elements are bounded below , $(\text{conv}(E) , V)$ is a full locally convex cone . This induces three topologies on $\text{conv}(E)$ given by the respective neighborhood base for $A \in \text{conv}(E)$

in the upper topology $V(A) := \{ B \in \text{conv}(E) \mid B \subseteq A+V \} , V \in V$

in the lower topology $A(V) := \{ B \in \text{conv}(E) \mid A \subseteq B+V \} , V \in V$

in the symmetric topology $V(A) \cap A(V) , V \in V$.

Identifying elements of E with singleton sets , E is naturally embedded in $\text{conv}(E)$, thus (E, V) is a locally convex cone .

The preorder induced on E is just the equality , and all the three topologies on $\text{conv}(E)$ coincide with the given locally convex topology on E .

(7.14) Example : Let $(E, \leq)$ be an ordered locally convex topological vector space with 0-neighborhood base $\mathbf{V}$. For $A, B \in \text{conv}(E)$, we define $A \leq B$ if for every $a \in A$ there is some $b \in B$ such that $a \leq b$. Since $\text{conv}(E)$ contains $\mathbf{V}$, and all its elements are bounded below, we realize that $(\text{conv}(E), \mathbf{V})$ is a full locally convex cone. By considering the elements of E as singletons, $(E, \mathbf{V})$ is a locally convex cone. Note that the upper and lower neighborhoods of an element $x \in E$ are given by

$$V_E(x) := \{y \in E \mid \{y\} \leq \{x\} + V\}, \quad V \in \mathbf{V},$$

$$x_E(V) := \{y \in E \mid \{x\} \leq \{y\} + V\}, \quad V \in \mathbf{V},$$

respectively. In particular, the upper neighborhoods of $x \in E$ contain all elements smaller than x , the lower neighborhoods all larger ones. The symmetric topology on E consist of the 0-neighborhoods

$$\begin{aligned} V_0 &= V_E(0) \cap 0_E(V) = \{y \in E \mid \{y\} \leq V \text{ and } -\{y\} \leq V\} \\ &= \{y \in E \mid y \leq u \text{ and } -y \leq v \text{ for } u, v \in V\}, \quad V \in \mathbf{V}. \end{aligned}$$

If $V \in \mathbf{V}$ is order convex, then $V_0 = V$, that is, the symmetric topology on E coincides with the original one.

(7.15) Example : Let (P, V) be a full locally convex cone .
 If we identify the elements of V with singleton sets $\bar{v} = \{v\}$, then
 $\bar{V} = \{\bar{v} \mid v \in V\}$ is a subset of $\text{conv}(P)$, which can be preordered
 using the preorder of P . For $A, B \in \text{conv}(P)$, we define
 $A \leq B$ if for every $a \in A$ there is some $b \in B$ such that $a \leq b$.
 Since P is a full locally convex cone , all its elements are bounded
 below which implies that all elements of $\text{conv}(P)$ are bounded
 below , thus $(\text{conv}(P), \bar{V})$ is a full locally convex cone .

Furthermore , let (Q, V) be a locally convex subcone of some
 full locally convex cone (P, V) , then $\text{conv}(Q)$ is a subcone of
 $\text{conv}(P)$, thus $(\text{conv}(Q), \bar{V})$ is a locally convex cone . Generally , for
 every subcone $\bar{Q}$ of non-empty convex subset of Q , the pair
 $(\bar{Q}, \bar{V})$ is a locally convex cone .

We shall also consider the families $D\text{conv}(Q)$ and
 $D\text{Clconv}(Q)$ of decreasing convex subsets , respectively closed
 decreasing convex subsets of Q , where closure is meant with
 respect to the lower topology on Q . If we slightly modify the
 addition (c.f. Examples (1.15) and (1.17)) , both sets will become
 cones as well :

$$A \oplus B = \downarrow(A + B) \quad \text{for } A, B \in \text{Dconv}(Q)$$

$$A \overline{\oplus} B = \overline{(\downarrow(A + B))} \quad \text{for } A, B \in \text{DClconv}(Q)$$

where $\overline{(\downarrow(A + B))}$ denotes the closure with respect to the lower topology, that is,

$$\overline{A} = \{b \in Q \mid \text{for all } v \in V \text{ there is } a \in A \text{ such that } b \leq a + v\}$$

for $A \in \text{DClconv}(Q)$. (c.f. [13] Chapter I, Proposition (3.4)), with the preorder and the abstract neighborhood system induced by $\text{conv}(Q)$ (which coincide with the inclusion in the case of decreasing sets), both $(\text{Dconv}(Q), \overline{V})$ and $(\text{DClconv}(Q), \overline{V})$ then are locally convex cones.

(7.16) Example : Let $F(X, P)$ be the cone of P -valued functions on the set X , where (P, V) is a full locally convex cone. If we consider the pointwise order of $F(X, P)$ and identify the elements $v \in V$ with the constant functions

$$f(x) = v \quad \text{for all } x \in X,$$

then V is a subset of $F(X, P)$ and defines an abstract neighborhood system there. Of course, not all functions in $F(X, P)$ will be bounded below. So we have to restrict ourselves to the subcone $B_l(X, P)$ which consist of all elements that are bounded below, and $(B_l(X, P), V)$ then is a full locally convex cone.

Again if (Q, V) is a locally convex subcone of (P, V) , every subcone of $(B_l(X, Q), V)$ is seen to be a locally convex cone as well. If, in particular, X is a topological space, we can consider the following subcones of $F(X, Q)$:

The cone $C_u(X, Q)$ of functions continuous with respect to the upper topology on Q ,

The cone $C_l(X, Q)$ of functions continuous with respect to the lower topology on Q ,

The cone $C_s(X, Q)$ of functions continuous with respect to the symmetric topology on Q .

Their respective subcones of elements that are bounded below then are locally convex cones.

8. UNIFORMLY CONTINUOUS OPERATORS

In this section we develop the definition and basic properties of uniformly continuous operators between locally convex cones .

(8.1) Definition : For cones P and Q . A map $T : P \rightarrow Q$ is called a *linear operator* , if

$$T(x + y) = T(x) + T(y) \quad \text{for all } x, y \in P, \text{ and}$$

$$T(\alpha x) = \alpha T(x) \quad \text{for all } x \in P \text{ and } \alpha \geq 0 .$$

Note that this implies $T(0_P) = 0_Q$. A *linear functional* on P is a linear operator $\mu : P \rightarrow \overline{\mathfrak{R}}$

(8.2) Definition : Let (P, V) and (Q, W) be two locally convex cones . A mapping $T : P \rightarrow Q$ is called *continuous* at $x \in P$, if for every $w \in W$ there exists $v \in V$ such that

$$y \leq x + v \quad \text{implies} \quad T(y) \leq T(x) + w \quad \text{for all } y \in P .$$

We say that T is *continuous on* P if it is continuous at every $x \in P$.

Note that the continuity defined above is with respect to the upper topologies on P and Q , and in a similar way one can define the continuity with respect to the other topologies on P and Q .

(8.3) Definition : Let (P, V) and (Q, W) be two locally convex cones . A mapping $T : P \rightarrow Q$ is said to be ***uniformly continuous*** or ***u-continuous*** for short , if for every $w \in W$ one can find a $v \in V$ such that

$$x \leq y + v \text{ implies } T(x) \leq T(y) + w \text{ for all } x, y \in P .$$

uniform continuity is not just continuity . It is immediate from the definition that it implies and combines continuity with respect to the upper , lower and symmetric topologies on P and Q .

(8.4) Proposition : Let (P, V) be a full locally convex cone and (Q, W) be a locally convex cone . Then a monotone linear operator $T : P \rightarrow Q$ is *u-continuous* , if for every $w \in W$ there is a $v \in V$ such that $T(v) \leq w$.

Proof : Let $w \in W$ then there is $v \in V$ such that $T(v) \leq w$, and $x \leq y + v$ implies

$$T(x) \leq T(y + v) = T(y) + T(v) \leq T(y) + w \quad \square$$

The following is straightforward :

(8.5) Proposition : Let $T : P \rightarrow Q$ be a *u-continuous* map . For $x, y \in P$ if $x \leq y + v$ for all $v \in V$ then

$$T(x) \leq T(y) + w \text{ for all } w \in W .$$

Since for the original preorder $\leq$ on P ,

$$x \leq y \text{ implies } x \leq y + v \text{ for all } v \in V$$

we conclude that, for a u -continuous operator $T : P \rightarrow Q$

$$x \leq y \text{ implies } T(x) \leq T(y) + w \text{ for all } w \in W.$$

Note that the last inequality does not necessarily imply that $T(x) \leq T(y)$.

(8.6) Examples :

(a) The addition $(x, y) \rightarrow x + y$ is a u -continuous operator from $P \times P$ into P ($P \times P$ is endowed with the neighborhood system $V \times V$).

Indeed, given $v \in V$, we choose $w = v/2$, then we obtain that

$$(x, y) \leq (a, b) + (w, w) \text{ implies } x + y \leq a + b + v.$$

(b) For a fixed $\lambda \geq 0$, the operator $x \rightarrow \lambda x : P \rightarrow P$ is u -continuous.

(c) For every bounded element $x \geq 0$, the map $\lambda \rightarrow \lambda x : \mathfrak{R}^+ \rightarrow P$ is u -continuous. Indeed, given $v \in V$. As x is bounded, there is $\varepsilon > 0$ such that $x \leq \varepsilon v$. Then

$$\alpha \leq \beta + \frac{1}{\varepsilon} \text{ and } x \geq 0 \text{ implies } \alpha x \leq \left(\beta + \frac{1}{\varepsilon} \right) x \leq \beta x + v.$$

(d) As an immediate consequence of (a) and (c) we obtain : For a finitely many bounded positive elements $x_1, x_2, \dots, x_n \in P$, the map

$$(\lambda_1, \lambda_2, \dots, \lambda_n) \rightarrow \sum_{i=1}^n \lambda_i x_i : (\mathfrak{R}^+)^n \rightarrow P$$

is u -continuous.

(e) Let E and F be two locally convex topological vector spaces and let $T : E \rightarrow F$ be a continuous linear operator. We extend T to non-empty convex subsets by defining

$$\bar{T}(A) := \{ T(a) \mid a \in A \} : \text{conv}(E) \rightarrow \text{conv}(F) .$$

Thus $\bar{T}$ is a u -continuous linear operator. Indeed, let W be a convex neighborhood of 0 in F . As T is continuous at 0 there exist a convex neighborhood V of 0 in E such that $T(V) \subseteq W$, then

$$A \subseteq B + V \text{ implies } \bar{T}(A) \subseteq \bar{T}(B + V) \subseteq \bar{T}(B) + W .$$

The same procedure can be applied more generally to extend any u -continuous operator $T : P \rightarrow Q$ between arbitrary locally convex cones to a u -continuous linear operator

$$\bar{T} : (\text{conv}(P), \bar{V}) \rightarrow (\text{conv}(Q), \bar{W})$$

[for their algebraic and locally convex structure, see Example(7.15)].

Extension to the cones $D\text{conv}(P)$ and $DCI\text{conv}(P)$, however, requires a slight modification in the definition of $\bar{T}$ (c.f. [13] II-1).

(8.7) Remark : For two linear operators S and T from P into Q and for $\lambda \geq 0$, the sum $S + T$ and λT are also linear. If both S and T are u -continuous, so are $S + T$ and λT . Thus, the u -continuous linear operators from P into Q form a cone $L(P, Q)$.

9. EXTENSIONS OF U -CONTINUOUS LINEAR FUNCTIONALS ON A LOCALLY CONVEX CONES .

In this section we shall study the dual cone of a locally convex cone and give some versions of Hahn-Banach type theorems for uniformly continuous sublinear and linear functionals .

According to the Definition (8.3) , a sublinear functional $p : P \rightarrow \overline{\mathfrak{R}}$ on a locally convex cone (P, V) is called *u -continuous* , if there is a neighborhood $v \in V$ such that

$$x \leq y + v \quad \text{implies} \quad p(x) \leq p(y) + 1 .$$

Every u -continuous sublinear functional is monotone by Proposition (8.5) . The u -continuous linear functionals on P form again a cone denoted by P^* and called the *dual cone* of P . We shall endow P^* with the topology $w(P^*, P)$ of pointwise convergence of the elements of P , considered as functions on P^* with values in $\overline{\mathfrak{R}}$ with its usual topology .

If (P, V) is a full locally convex cone and p is a monotone sublinear functional on P , then the definition of u -continuity becomes particularly simple : p will be u -continuous if and only if there is a $v \in V$ such that $p(v) \leq 1$.

(9.1) Definition : Let (P, V) be a locally convex cone . The *polar of v* is defined to be the set v_P^0 of all linear functionals μ on P satisfying

$$x \leq y + v \quad \text{implies} \quad \mu(x) \leq \mu(y) + 1 \quad \text{for all } x, y \in P$$

that is ,

$$v_P^0 := \left\{ \mu \in P^* \mid x \leq y + v \quad \text{implies} \quad \mu(x) \leq \mu(y) + 1 \right\} .$$

If P is a full locally convex cone , then v_P^0 becomes the set of all monotone linear functionals on P such that $\mu(v) \leq 1$.

For the sake of simplicity , we shall develop our next theorems for full locally convex cones but they apply to locally convex cone , in general , because every such cone may (by definition) be embedded into a full cone .

(9.2) Lemma : Let Q be a subcone of the locally convex cone (P, V) . Every u -continuous sublinear functional on Q can be extended to a u -continuous sublinear functional on P ; more precisely : let p be a sublinear functional on Q and $v \in V$ such that

$$a \leq b + v \quad \text{implies} \quad p(a) \leq p(b) + 1 \quad , \quad \text{whenever } a, b \in Q$$

then there is a sublinear functional $\bar{p}$ on P extending p such that

$$x \leq y + v \quad \text{implies} \quad \bar{p}(x) \leq \bar{p}(y) + 1 \quad , \quad \text{whenever } x, y \in P .$$

Proof : For every $x \in P$ we define

$$\bar{p}(x) := \inf \{ p(a) + \lambda \mid x \leq a + \lambda v \text{ for some } a \in Q \text{ and } \lambda > 0 \} .$$

Then clearly $\bar{p}$ is a sublinear functional and $\bar{p}(a) = p(a)$ for all $a \in Q$. Indeed ; Let $a \in Q$. Clearly $\bar{p}(a) \leq p(a)$. For the converse inequality consider any α such that $\bar{p}(a) \leq \alpha$. Then there is $b \in Q$ and $\lambda > 0$ such that $a \leq b + \lambda v$ and $p(b) + \lambda \leq \alpha$. By assumption

$$a \leq b + \lambda v \text{ implies } p(a) \leq p(b) + \lambda \leq \alpha .$$

As this holds for all $\alpha > \bar{p}(a)$, we infer $p(a) \leq \bar{p}(a)$.

Furthermore, $\bar{p}(x) \leq \bar{p}(y) + 1$ for all $x, y \in P$ with $x \leq y + v$.

This is clear if $\bar{p}(y) = +\infty$, otherwise, suppose that $\bar{p}(y) < +\infty$.

For every $\alpha > \bar{p}(y)$ we may find $a \in Q$ and $\lambda > 0$ such that

$$y \leq a + \lambda v \text{ and } p(a) + \lambda \leq \alpha . \text{ Thus } x \leq y + v \leq a + (\lambda + 1)v$$

and $\bar{p}(x) \leq p(a) + (\lambda + 1) \leq \alpha + 1$. As this holds for every

$\alpha \geq \bar{p}(y)$, we obtain $\bar{p}(x) \leq \bar{p}(y) + 1$. $\square$

The following is the topological version of our Sandwich Theorem (2.9). It is the main result of this chapter.

(9.3) Theorem : Let (P, V) be a full locally convex cone .

Let $p : P \rightarrow \overline{\mathbb{R}}$ be a sublinear functional , $q : P \rightarrow \mathbb{R} \cup \{\pm\infty\}$ an extended superlinear functional . Let $v \in V$. The following are equivalent :

(i) There is a linear functional $\mu \in v_P^0$ such that

$$q(x) \leq \mu(x) \leq p(x) \quad \text{for all } x \in P .$$

(ii) $q(x) \leq p(y) + 1$ whenever $x \leq y + v$.

Proof :

(i) $\Rightarrow$ (ii) : Let $\mu \in v_P^0$. For every $x, y \in P$ with $x \leq y + v$ we have $\mu(x) \leq \mu(y) + 1$. Therefore

$$q(x) \leq \mu(x) \leq \mu(y) + 1 \leq p(y) + 1 .$$

(ii) $\Rightarrow$ (i) : For every $x \in P$ we define

$$\bar{p}(x) := \inf \{ p(y) + \lambda \mid x \leq y + \lambda v \text{ for some } y \in P \text{ and } \lambda > 0 \}$$

then $\bar{p}$ is a sublinear functional and $\bar{p}(x) \leq p(x)$. Now set

$$W := \{ w \in P \mid 0 \leq w + v \} .$$

Since every element of a locally convex cone is supposed to be bounded below , W is an absorbing convex subset of P with $0 \in W \subseteq P$. Now let $x, y \in P$ with $x \leq y + v$. We shall show that $q(x) \leq \bar{p}(y) + 1$. Let $y \leq z + \lambda v$ for $z \in P$ and $\lambda > 0$ then $x \leq z + (\lambda + 1) v$. Condition (ii) implies

$$q(x) \leq (p(z) + \lambda) + 1 \leq \bar{p}(y) + 1 .$$

Furthermore, let $x, y \in P$ and $w \in W$ with $x + w \leq y$. As $0 \leq w + v$, this implies $x \leq x + w + v \leq y + v$. The above argument implies $q(x) \leq \bar{p}(y) + 1$. Thus by Theorem (2.9) there is a monotone linear functional $\mu: P \rightarrow \overline{\mathfrak{R}}$ such that

$$q(x) \leq \mu(x) \leq \bar{p}(x) \leq p(x) \quad \text{for all } x \in P.$$

As $v \leq 0 + 1 \cdot v$ we have $\bar{p}(v) \leq 1$ which implies $\mu(v) \leq 1$. $\square$

As an immediate consequence of Theorem (9.3) we obtain :

(9.4) Corollary: *Let (P, V) be a full locally convex cone. Let $p: P \rightarrow \overline{\mathfrak{R}}$ be a sublinear functional, $q: P \rightarrow \mathfrak{R} \cup \{\pm\infty\}$ an extended superlinear functional with $q(x) \leq p(x)$ for all $x \in P$. If p is u -continuous then there is a u -continuous linear functional $\mu: P \rightarrow \overline{\mathfrak{R}}$ such that*

$$q(x) \leq \mu(x) \leq p(x) \quad \text{for all } x \in P ;$$

more precisely : if for some $v \in V$, the functional p satisfies :

$$x \leq y + v \quad \text{implies} \quad p(x) \leq p(y) + 1, \quad \text{whenever } x, y \in P,$$

then there is a linear functional $\mu \in v_P^0$ such that $q \leq \mu \leq p$ on P .

(9.5) Corollary : *Let Q be a subcone of a full locally convex cone (P, V) . Then every u -continuous linear functional on Q can be extended to a u -continuous linear functional on P ; more precisely : For every $\mu \in v_Q^0$ there is a $\bar{\mu} \in v_P^0$ such that*

$$\bar{\mu}(a) = \mu(a) \quad \text{for all } a \in Q.$$

Proof : Let $v \in V$. By Lemma (9.2) we may extend μ to a sublinear functional p on P which satisfies

$$x \leq y + v \text{ implies } p(x) \leq p(y) + 1, \text{ whenever } x, y \in P.$$

We define an extended superlinear functional $q: P \rightarrow \mathfrak{R} \cup \{\pm\infty\}$ by

$$q(x) = \begin{cases} \mu(a) & a \in Q \\ -\infty & \text{elsewhere} \end{cases}.$$

By Corollary (9.4) we find a linear functional $\bar{\mu} \in v_P^0$ such that

$$q(x) \leq \bar{\mu}(x) \leq p(x) \text{ for all } x \in P.$$

As $q(a) = \mu(a) = p(a)$ for all $a \in Q$ the functional $\bar{\mu}$ extends μ . □

Now let us investigate the dual cone of some examples from the previous section (for the other examples see [13] II-2) :

For all $\rho \geq 0$, clearly $\overline{\mathfrak{R}}^*$ contains the maps

$$x \rightarrow \rho x, \quad x \in \overline{\mathfrak{R}}.$$

(recall that $0 \cdot \infty = 0$) . If we add the mapping 0^* where

$$0^*(x) = \begin{cases} 0 & x \in \mathfrak{R} \\ +\infty & x = +\infty \end{cases}$$

then we have all of $\overline{\mathfrak{R}}^*$.

The dual cone of the locally convex ordered vector space $(E, \leq)$ (c.f. Example (7.14)) consists exactly of the positive functionals in the dual space of E .

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INDEX OF SYMBOLS

The following list contains the basic symbols which are used in more than one or two section .

$ x $	absolute value of x .
$\overline{A}$	closure of A .
$\uparrow A$	The increasing subset generated by A .
$\downarrow A$	The decreasing subset generated by A .
$B\text{conv}(E)$	The set of all non-empty bounded closed convex subsets of a topological vector space E .
$B_l(X, P)$	The subcone of the cone $F(X, P)$ of all elements that are bounded below .
$\text{cone}(A)$	The smallest subcone containing A .
$\text{conv}(P)$	The set of all non-empty convex subsets of P .
$C\text{conv}(E)$	The set of all non-empty compact convex subsets of a topological vector space E .
$Cl\text{conv}(E)$	The set of all non-empty closed convex subsets of a topological vector space E .
$C[a, b]$	The set of all real-valued continuous functions on $[a, b]$.
c_{00}	The space of all ultimately zero sequences .
$D\text{conv}(P)$	The set of all non-empty decreasing convex subsets of P .

$\text{DCIconv}(P)$	The set of all non-empty decreasing closed convex subsets of P .
$\emptyset$	The empty set .
$F(X,P)$	The set of all P -valued functions on X .
(P,V)	The full locally convex cone with its abstract neighborhood system V .
$\inf A$	The infimum or greatest lower bound of A .
l^∞	The space of bounded sequences .
$\mathbb{N}$	The set of natural numbers .
$\ x\ $	The norm of x .
$v(x)$	The upper neighborhood of the element x .
$x(v)$	The lower neighborhood of the element x .
v_P^0	The polar of a neighborhood v in (P,V) .
v^0	The neighborhood of 0 with respect to the symmetric topology .
P^*	The cone of all u -continuous linear functionals on a locally convex cone P .
$\mathbb{R}$	The set of real numbers .
$\overline{\mathbb{R}}$	The extended real numbers , that is $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.
$\mathbb{R}^+$	The set of non-negative real numbers .
$\mathbb{R}^n$	The Euclidean n -space .
$\sup A$	The supremum or least upper bound of A .