



**UNIVERSITY OF BAHRAIN
COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS**

**ASYMPTOTIC THEORY FOR A CLASS OF
FOURTH - ORDER
DIFFERENTIAL EQUATIONS**

BY

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**A Thesis submitted to the Department of Mathematics as
a Partial Fulfillment of the Requirements for the degree of**

MASTER OF SCIENCE

Under the supervision of Prof. A.S.A. Al-Hammadi

JUNE 2000



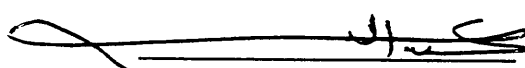
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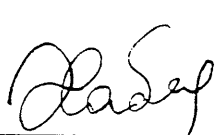
“Asymptotic Theory for a class of Fourth-Order Differential Equation”

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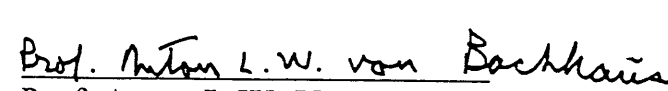
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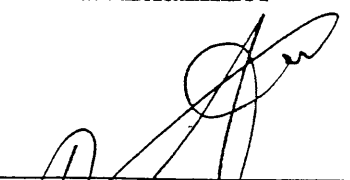
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ACKNOWLEDGMENT

I would like to express my deep appreciation to my advisor Prof. A.S.A Al-Hammadi for his excellent guidance and continuous encouragement to complete this research work. I would also like to thank the University of Bahrain for providing me with this opportunity. This thesis would have not been completed without moral support of my family specially my mother , to whom this piece of work is dedicated.

ABSTRACT

An asymptotic theory is developed for a class of fourth-order differential equations. Under a general condition on the coefficients of the differential equation we obtained the forms of the asymptotic solutions.

In this thesis we consider the asymptotic form of a fundamental set of solutions of the fourth-order differential equation (2.1.1) which is

$$(p_0(x) y''(x))'' + (p_1(x) y'(x))' - p_2(x) y(x) = 0$$

as $x \rightarrow \infty$. We shall consider the case where p_1 is small compared to p_0 and p_2 , in the sense that

$$p_1 = o\left\{(p_0 p_2)^{1/2}\right\}, \quad \text{as } x \rightarrow \infty$$

In this case all the solutions have a similar exponential factor as given in Theorem 2.3.1 in chapter 2 (§ 3).

In chapter 1 (§ 3), we introduce the fundamental theorem of Levinson, is that, subject to a dichotomy condition on A and to the smallness condition on R . Also, in chapter 1 (§ 4) we use the recent asymptotic theorem of Eastham which, although a consequence of the basic Levinson theorem, is in a form suitable for many applications. One of those applications is to differential equation of the second order is given in chapter 1 (§ 5).

The methods which we use for obtaining the solutions of (2.1.1) under the above case are given in chapter 2 . Also , in chapter 2 (§ 3) we use the asymptotic theorem of Eastham [7,8] to get the main asymptotic result .

Two examples of (2.1.1) are considered at the end of chapter 2 (§ 4) . Finally , in chapter 2 (§ 4) we discuss the relationship of our result to the asymptotic theory of fourth-order differential equations by Eastham result [5] , Al-Hammadi result [2] and Hinton result [10] .

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CHAPTER ONE

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CHAPTER ONE

ASYMPTOTICALLY DIAGONAL SYSTEMS

1.1 INTRODUCTION

We consider the system

$$Y'(x) = A(x)Y(x) \quad (1.1.1)$$

where x - interval to be $[a, \infty)$, $a \geq 0$ and $A(x)$ a square matrix defined for $x \in [a, \infty)$. Our aim is to obtain the asymptotic solution for (1.1.1) .

We have a basic question does for large x the solutions of (1.1.1) approximate to the known solutions to the simple system

$$Y'(x) = A_0(x)Y(x) \quad (1.1.2)$$

which can be solved explicitly .

We now introduce two simple systems (1.1.2) which form the basis for the theory in this chapter .

(i) The constant coefficient system

Here $A_0(x)$ is a constant matrix C , and (1.1.2) is

$$Y'(x) = C Y(x) \quad (1.1.3)$$

Let λ be an eigenvalue of C with corresponding eigenvector u . Then a solution of (1.1.3) is

$$Y(x) = u e^{\lambda x} \quad (1.1.4)$$

since here

$$Y'(x) = \lambda u e^{\lambda x} = C u e^{\lambda x} = C Y(x)$$

Here (1.1.4) is an explicit solution for (1.1.3). Now if C is an $n \times n$ matrix and if there are n linearly independent eigenvectors u_k corresponding to eigenvalues λ_k ($1 \leq k \leq n$), then (1.1.4) gives n linearly independent solutions

$$Y_k(x) = u_k e^{\lambda_k x} \quad (1.1.5)$$

of (1.1.3).

(ii) The diagonal system

Here $A_0(x)$ is a diagonal matrix $A(x)$, and (1.1.2) is

$$Y'(x) = A(x)Y(x) \quad (1.1.6)$$

We write $A(x) = \text{dg} (\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x))$

Then, denoting the components of $Y(x)$ by $y_j(x)$ ($1 \leq j \leq n$), we can split up (1.1.6) into the first-order equations

$$y'_j(x) = \lambda_j(x)y_j(x)$$

for each $y_j(x)$ separately. Hence

$$y_j(x) = c_j \exp \left(\int_a^x \lambda_j(t) dt \right)$$

where the c_j are arbitrary constants . If we now take any integer k ($1 \leq k \leq n$) and choose $c_k = 1$ and the other c_j zero , we obtain the solution

$$Y_k(x) = e_k \exp \left(\int_a^x \lambda_k(t) dt \right) \quad (1.1.7)$$

of (1.1.6) , where e_k is the coordinate vector whose k^{th} component is unity and other components zero . Again , as k varies from 1 to n , we have n explicitly linearly independent solutions of (1.1.6) .

Moving on now to systems (1.1.1) which approximate to (1.1.3) or (1.1.6) as $x \rightarrow \infty$, we consider

$$Y'(x) = \{ C + R(x) \} Y(x) \quad (1.1.8)$$

or
$$Y'(x) = \{ A(x) + R(x) \} Y(x) , \quad (1.1.9)$$

where $R(x)$ is an $n \times n$ matrix which is small as $x \rightarrow \infty$, in particular

$$\int_a^\infty |R(x)| dx < \infty \quad (1.1.10)$$

which means that each entry in $R(x)$ has an absolutely convergent infinite integral . under condition (1.1.10) and a condition on $A(x)$ as we shall see later , (1.1.9) has solutions $Y_k(x)$ ($1 \leq k \leq n$) such that

$$Y_k(x) = \{ e_k + o(1) \} \exp \left(\int_a^x \lambda_k(t) dt \right) , \quad x \rightarrow \infty \quad (1.1.11)$$

This theorem was proved by Levinson (1948) . Note that by simple transformation we can transform (1.1.8) to the form (1.1.9) . Thus , comparing (1.1.11) and (1.1.7) , we see that the Levinson theorem gives

conditions on R and A under which the solutions of (1.1.9) do approximate to those of (1.1.6) for large x .

We refer to a system (1.1.9) in which (1.1.10) holds as having the Levinson form.

1.2 NOTATION AND BASIC THEORY

There are several basic facts concerning linear algebra and linear differential systems which will be used on various occasions, and it is convenient to have these results available here for reference. We also introduce some notation that will be used throughout.

Notation For an $m \times n$ matrix $A = (a_{ij})$, we define the modulus $|A|$ by

$$|A| = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}| \quad (1.2.1)$$

Then, for two matrices whose product is defined, we have

$$|AB| \leq |A||B|$$

For a square matrix A , we denote by $dgA = (a_{11}, \dots, a_{nn})$ the diagonal matrix formed by the diagonal entries a_{ii} in A . We also denote by A^t and A^* the transposed and adjoint matrices defined by $A^t = (a_{ji})$ and $A^* = (\bar{a}_{ji})$.

For matrices whose entries are differentiable functions of x , we define

$$A'(x) = (a'_{ij}(x))$$

and similarly for the integral of $A(x)$ over an interval (a, b) . If $a < b$, we have immediately from (1.2.1)

$$\left| \int_a^b A(x) dx \right| \leq \int_a^b |A(x)| dx$$

If $A(x) \in L^p(a, \infty)$ and $A(x) = O(x^{-1})$ ($x \rightarrow \infty$) mean that every entry a_{ij} has the relevant property .

Linear algebra We shall often use the result that an $n \times n$ matrix A with n distinct eigenvalues λ_k can be expressed in the diagonal form

$$T^{-1}AT = A$$

where $A = dg(\lambda_1, \dots, \lambda_n)$

and T has the corresponding eigenvectors v_k as its columns . The computation of T^{-1} is generally complicated but , in certain cases , the form of T^{-1} can be obtained directly from the form of T . The simplest case is where A is hermitian and then , with normalized eigenvectors , we have $T^{-1} = T^*$.

Linear differential systems A solution matrix for (1.1.1) is an $n \times n$ matrix $\Phi(x)$ whose columns are solutions $Y_i(x)$ ($1 \leq i \leq n$) of (1.1.1). Then $\Phi(x)$ satisfies the corresponding matrix differential equation

$$\Phi'(x) = A(x) \Phi(x) \quad (1.2.2)$$

Further , $\Phi(x)$ is either singular $\forall x \in I$ or non-singular $\forall x \in I$, according as the $Y_i(x)$ are linearly dependent or linearly independent in I . In the latter case , $\Phi(x)$ is said to be a fundamental matrix for (1.1.1) . In this case any solution of (1.1.1) can be written as

$$Y(x) = \Phi(x) c$$

where c is a constant vector . If we also consider a second differential system

$$Z'(x) = \{ A(x) + B(x) \} Z(x) \quad (1.2.3)$$

then any solution $Z(x)$ can be related to given linearly independent solutions $Y_i(x)$ of (1.1.1) by the formula

$$Z(x) = \Phi(x)c + \Phi(x) \int_a^x \Phi^{-1}(t)B(t)Z(t) dt \quad (1.2.4)$$

where $a \in I$ and c is a constant vector . (1.2.4) is called the variation of parameters formula . A useful generalization of (1.2.4) is

$$Z(x) = \Phi(x)c + \Phi_1(x) \int_a^x \Phi^{-1}(t)B(t)Z(t) dt - \Phi_2(x) \int_x^b \Phi^{-1}(t)B(t)Z(t) dt \quad (1.2.5)$$

where a and b are given points in I , and the fundamental matrix $\Phi(x)$ is split up into any two separate solution matrices for (1.1.1) in the form

$$\Phi(x) = \Phi_1(x) + \Phi_2(x)$$

$$\text{Now let} \quad Z(x) = \Phi(x) W(x) \quad (1.2.6)$$

in (1.2.3) . The corresponding W -system is given by

$$\Phi W' + \Phi' W = (A + B)\Phi W$$

Using (1.2.2) , we obtain $W' = \Phi^{-1}B\Phi W$

Thus (1.2.6) remove the matrix $A(x)$ from (1.2.3) .

The determinant of Φ satisfies the Liouville-Jacobi formula

$$\det \Phi(x) = \det \Phi(a) \exp \left(\int_a^x \operatorname{tr} A(t) dt \right) \quad (1.2.7)$$

where $\operatorname{tr} A$ denotes then trace of A defined by

$$\operatorname{tr} A(x) = \sum_{i=1}^n a_{ii}(x)$$

1.3 THE LEVINSON'S THEOREM

In this section , we consider the linear differential system

$$Y'(x) = \{ A(x) + R(x) \} Y(x) \quad (1.3.1)$$

on an interval $[a, \infty)$, where Y is an n -component vector , A and R are $n \times n$ matrices and A is diagonal . The basic result in the asymptotic theory of (1.3.1) is the following theorem of Levinson in which $R(x)$ is required to be $L(a, \infty)$ and $A(x)$ satisfies a subsidiary condition , known as the Dichotomy condition . The following theorem was proved in [4,8] .

Theorem 1.3.1 (The Levinson theorem)

Let $A(x)$ be an $n \times n$ diagonal matrix ,

$$A(x) = \operatorname{dg} (\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x))$$

which satisfies the following two-part condition L .

L : For each pair of integers i and j in $[1, n](i \neq j)$ and for all x

and t such that $a \leq t \leq x < \infty$, either

$$(a) \quad \int_t^x \operatorname{re} \{ \lambda_i(s) - \lambda_j(s) \} ds \leq K_1 \quad (1.3.2)$$

or

$$(b) \quad \int_t^x \operatorname{re} \{ \lambda_i(s) - \lambda_j(s) \} ds \geq K_2 \quad (1.3.3)$$

where K_1 and K_2 are constants .

Also , let the $n \times n$ matrix $R(x)$ satisfy

$$\int_a^\infty |R(x)| dx < \infty \quad (1.3.4)$$

Then , as $x \rightarrow \infty$, the system (1.3.1) has solutions $Y_k(x) (1 \leq k \leq n)$ with the asymptotic form

$$Y_k(x) = \{ e_k + o(1) \} \exp \left(\int_a^x \lambda_k(t) dt \right) \quad (1.3.5)$$

where e_k denotes the coordinate vector with k^{th} component unity and other components zero , and the right-hand side of (1.3.5) is , apart from the $o(1)$ term , simply an exact solution of the diagonal system

$$Y'(x) = \Lambda(x)Y(x)$$

A system (1.3.1) in which (1.3.4) holds is said to have Levinson form .

The condition L is known as the Dichotomy Condition (D.C) . Also we mention a stronger form than L which is useful in applications .

L' : For each pair of integers i and j in $[1, n] (i \neq j)$ let

$$\operatorname{re} \{ \lambda_i(x) + \lambda_j(x) \} = f(x) + g(x) \quad (1.3.6)$$

where $f(x)$ does not change sign in $[a, \infty)$ (that is , either $f(x) \geq 0$ throughout $[a, \infty)$ or $f(x) \leq 0$ throughout $[a, \infty)$) and $g(x) \in L(a, \infty)$.

1.4 THE EASTHAM THEOREM

In this section we give the statement of the Eastham theorem on asymptotically diagonal systems which extends the application of Levinson's Theorem . The following theorem was proved in [8,7] .

Theorem 1.4.1 (The Eastham theorem)

Let $A(x)$ be an $n \times n$ diagonal matrix ,

$$A(x) = \text{dg} (\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x))$$

in which $\lambda_i(x) - \lambda_j(x)$ is nowhere zero in some interval $[a, \infty)$ for each pair of integers i and j in $[1, n]$ ($i \neq j$) . Let $R(x)$ be an $n \times n$ matrix such that

$$(i) \quad \left\{ \lambda_i(x) - \lambda_j(x) \right\}^{-1} R(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty \quad (1.4.1)$$

$$(ii) \quad \left(\left\{ \lambda_i(x) - \lambda_j(x) \right\}^{-1} R(x) \right)' \in L(a, \infty) \quad (1.4.2)$$

for each i and j in $[1, n]$ with $i \neq j$. Also , let the eigenvalues μ_k ($1 \leq k \leq n$) of $A(x) + R(x)$ satisfy the Dichotomy Condition a-b in Levinson's Theorem . Then , as $x \rightarrow \infty$, the system (1.3.1) has solutions $Y_k(x)$ ($1 \leq k \leq n$) with the asymptotic form

$$Y_k(x) = \{e_k + o(1)\} \exp \left(\int_a^x \mu_k(t) dt \right) \quad (1.4.3)$$

We emphasize that , in the statement of Theorem (1.4.1) , it is the μ_k rather than the λ_k which appear in (1.4.3) and which are obliged to satisfy the dichotomy condition . However , in practice , it may not be easy to calculate the μ_k and therefore it is useful to know that , if the additional condition

$$\left| \lambda_i(x) - \lambda_j(x) \right|^{-1} |R(x)| \in L(a, \infty) \quad (1.4.4)$$

holds for all $i \neq j$, μ_k can be replaced by the simple expression $\lambda_k + r_{kk}$ in the statement of the theorem . To see this we write

$$\mu_k = \lambda_k + r_{kk} + O \left(\max_{i \neq k} \frac{|R|^2}{|\lambda_i - \lambda_k|} \right) \quad (1.4.5)$$

The O-term here is $L(a, \infty)$ if (1.4.4) holds , and the required simplification follows . For future reference , then , we note that when the additional condition (1.4.4) holds , (1.4.3) can be replaced by

$$Y_k(x) = \{e_k + o(1)\} \exp \left(\int_a^x \{ \lambda_k(t) + r_{kk}(t) \} dt \right) \quad (1.4.6)$$

1.5 ASYMPTOTIC FORMULAE FOR SECOND-ORDER DIFFERENTIAL EQUATIONS

In this section we consider the second-order differential equation

$$y''(x) + \{k + r(x)\} y(x) = 0 \quad (a \leq x < \infty) \quad (1.5.1)$$

where k is real constant and

$$\int_a^\infty |r(x)| dx < \infty \quad (1.5.2)$$

The methods which we use for obtaining the solutions of (1.5.1) take advantage of recent developments in asymptotic theory and are much simpler. In particular we use the recent asymptotic theorem of Eastham [8,7] which, although a consequence of the basic Levinson's Theorem, is in a form more suitable for many applications.

We shall discuss the asymptotic solutions of (1.5.1) in two cases when $K > 0$ and $K < 0$. We begin now with $K < 0$.

Theorem 1.5.1 :

Let $k = -\alpha^2$, where $\alpha > 0$, and let (1.5.2) hold. Then (1.5.1) has solutions $y_1(x)$ and $y_2(x)$ such that, as $x \rightarrow \infty$,

$$y_1(x) \sim e^{\alpha x} \quad , \quad y_1'(x) \sim \alpha e^{\alpha x} \quad (1.5.3)$$

and

$$y_2(x) \sim e^{-\alpha x} \quad , \quad y_2'(x) \sim -\alpha e^{-\alpha x} \quad (1.5.4)$$

Proof :

Our aim is to write (1.5.1) as a first-order system and then to transform the system to Levinson form such that theorem (1.3.1) holds for that system .

Now define

$$y_1(x) = y(x) \quad , \quad y_2(x) = y'(x) \quad (1.5.5)$$

then

$$y_1'(x) = y_2(x) \quad , \quad y_2'(x) = -(k + r(x))y_1(x) \quad (1.5.6)$$

As in [8 chapter 2] , (1.5.1) can be written as a first- order system

$$Y'(x) = A_1(x)Y(x) \quad (1.5.7)$$

$$\text{i.e} \quad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(k + r) & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (1.5.8)$$

$$\text{or} \quad \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(k + r) & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (1.5.9)$$

i.e the first-component of Y is y and the second is y' .

Now we write

$$A_1(x) = A(x) + B(x)$$

$$\text{with} \quad A(x) = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix} \quad (1.5.10)$$

$$\text{and} \quad B(x) = \begin{pmatrix} 0 & 0 \\ -r & 0 \end{pmatrix} \quad (1.5.11)$$

i.e (1.5.8) can be written as

$$Y'(x) = \left\{ \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -r & 0 \end{pmatrix} \right\} Y(x) \quad (1.5.12)$$

Now (1.5.12) is not yet in the Levinson form . i.e we need to diagonalize the matrix A . Hence we require the eigenvalues λ_i and the eigenvectors v_i ($i = 1, 2$) .

$$\det(A - \lambda_i I) = 0 \quad \text{gives}$$

$$\lambda_i^2 + k = 0$$

$$\therefore \lambda_i = \pm \sqrt{-k} \quad (1.5.13)$$

Also , to find the eigenvectors v_i we consider

$$A v_i = \lambda_i v_i$$

$$\begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda_i \begin{pmatrix} u \\ v \end{pmatrix} \quad (1.5.14)$$

$$\therefore v = \lambda_i u, \quad -ku = \lambda_i v \quad (1.5.15)$$

$$\text{Define } u = 1, \quad v = \lambda_i \quad (1.5.16)$$

Hence we can define the matrix T of eigenvector by

$$T = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \quad (1.5.17)$$

Now we use the transformation

$$Y = TZ \quad (1.5.18)$$

To transform (1.5.12) into Levinson form where the constant matrix T is given by (1.5.17) .

$$Y' = TZ' \quad (1.5.19)$$

then by (1.5.18) , (1.5.19) and (1.5.12) gives

$$\begin{aligned} Z' &= (T^{-1}AT + T^{-1}BT)Z \\ Z' &= (A + R(x))Z \end{aligned} \quad (1.5.20)$$

$$\text{where} \quad A = T^{-1}AT = \text{dg}(\lambda_1, \lambda_2) \quad (1.5.21)$$

Now (1.5.20) is in the Levinson form , so first we require that

$$\int_a^\infty |R(x)| dx < \infty \quad \Leftrightarrow \quad \int_a^\infty |T^{-1}BT| dx < \infty$$

$$\Leftrightarrow \quad \int_a^\infty |r(x)| dx < \infty \quad \text{holds by (1.5.2).}$$

$$\text{Also ,} \quad \lambda_1 - \lambda_2 = 2\sqrt{-k}$$

$$\therefore \text{re}(\lambda_1 - \lambda_2) = 2\text{re}(\sqrt{-k}) \geq 0 \text{ or } \leq 0$$

So , the Dichotomy Condition hold because the λ_i are constants .

Hence , we can apply Levinson theorem to the system (1.5.20) , thus

$$Z_i(x) = \{e_i + o(1)\} \exp\left(\int_a^x \lambda_i(t) dt\right) \quad , \quad (i = 1, 2) \quad (1.5.22)$$

solution for (1.5.20) as $x \rightarrow \infty$, to obtain the solution of our original system (1.5.8) or (1.5.9) , we transform back by using (1.5.18)

$$\therefore Y_i(x) = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \{e_i + o(1)\} \exp\left(\int_a^x \lambda_i(t) dt\right), \quad (i = 1, 2) \quad (1.5.23)$$

so for $i = 1$, we obtain

$$Y_1(x) = \begin{pmatrix} y_1 \\ y'_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} o(1) \\ o(1) \end{pmatrix} \right\} \exp(\lambda_1(x - a))$$

$$Y_1(x) = \begin{pmatrix} y_1 \\ y'_1 \end{pmatrix} = \begin{pmatrix} 1 + o(1) \\ \lambda_1 + o(1) \end{pmatrix} \exp(\lambda_1(x - a))$$

after adjusting a constant multiple in y_1, y'_1 .

$$\begin{aligned} y_1(x) &= \{1 + o(1)\} \exp(\lambda_1 x) \\ &\quad (x \rightarrow \infty) \\ y'_1(x) &= \lambda_1 \{1 + o(1)\} \exp(\lambda_1 x) \end{aligned} \quad (1.5.24)$$

similarly for $i = 2$, we obtain

$$\begin{aligned} y_2(x) &= \{1 + o(1)\} \exp(\lambda_2 x) \\ &\quad (x \rightarrow \infty) \\ y'_2(x) &= \lambda_2 \{1 + o(1)\} \exp(\lambda_2 x) \end{aligned} \quad (1.5.25)$$

Now if $k < 0$, say $k = -\alpha^2$, $\therefore \lambda_j = \pm \sqrt{\alpha^2} = \pm \alpha \quad (j = 1, 2)$

$$\begin{aligned} \therefore y_j(x) &\sim \exp(\pm \alpha x) \\ &\quad (x \rightarrow \infty) \\ y'_j(x) &\sim \pm \alpha \exp(\pm \alpha x) \end{aligned} \quad (1.5.26)$$

Theorem 1.5.2 :

Let $k = \alpha^2$, where $\alpha > 0$, and let (1.5.2) hold . Then (1.5.1) has solutions $y_1(x)$ and $y_2(x)$ such that , as $x \rightarrow \infty$,

$$y_1(x) \sim e^{i\alpha x} \quad , \quad y_1'(x) \sim i\alpha e^{i\alpha x} \quad (1.5.27)$$

and

$$y_2(x) \sim e^{-i\alpha x} \quad , \quad y_2'(x) \sim -i\alpha e^{-i\alpha x} \quad (1.5.28)$$

Proof :

The method of proving this theorem is similar to the previous theorem (1.5.1). As in the previous theorem , we obtained that

$$\begin{aligned} \therefore \quad y_j(x) &\sim \exp(\lambda_j x) \\ &\quad (x \rightarrow \infty) \quad (j = 1, 2) \\ y_j'(x) &\sim \lambda_j \exp(\lambda_j x) \end{aligned}$$

if $k > 0$, say $k = \alpha^2$, $\therefore \lambda_j = \pm \sqrt{-\alpha^2} = \pm \alpha i$, $(j = 1, 2)$

$$\begin{aligned} \therefore \quad y_j(x) &\sim \exp(\pm \alpha i x) \\ &\quad (x \rightarrow \infty) \\ y_j'(x) &\sim \pm i\alpha \exp(\pm \alpha i x) \end{aligned} \quad (1.5.29)$$

CHAPTER TWO

ASYMPTOTIC FORMULAE OF FOURTH-ORDER DIFFERENTIAL EQUATIONS

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CHAPTER TWO

ASYMPTOTIC FORMULAE OF FOURTH-ORDER DIFFERENTIAL EQUATION

2.1 INTRODUCTION

In this chapter we consider the asymptotic form of a fundamental set of solution of the fourth- order differential equation

$$(p_0(x) y''(x))'' + (p_1(x) y'(x))' - p_2(x) y(x) = 0 \quad (2.1.1)$$

as $x \rightarrow \infty$, where x is the independent variable . The functions $p_i(x) (0 \leq i \leq 2)$ are defined on an interval $[a, \infty)$ and are not necessarily real- valued and are all nowhere zero in this interval .

Now , define

$$\begin{aligned} y_1 &= y & , & & y_1' &= y_2 \\ y_2 &= y' & , & & y_2' &= p_0^{-1} y_3 \\ y_3 &= p_0 y'' & , & & y_3' &= y_4 - p_1 y_2 \\ y_4 &= (p_0 y'')' + p_1 y' & , & & y_4' &= p_2 y_1 \end{aligned}$$

We write (2.1.1) in the standard way [9] as a first-order system

$$Y'(x) = A(x)Y(x) \quad (2.1.2)$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ p_0 y'' \\ (p_0 y'')' + p_1 y' \end{pmatrix}$$

and

$$A(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & p_0^{-1} & 0 \\ 0 & -p_1 & 0 & 1 \\ p_2 & 0 & 0 & 0 \end{pmatrix} \quad (2.1.3)$$

As in [1,2,3] , we need to express A in its diagonal form

$$T^{-1}AT = \Lambda \quad (2.1.4)$$

and we therefore require the eigenvalues λ_j and eigenvectors v_j ($1 \leq j \leq 4$) of A . First we require the characteristic equation. In order to find the characteristic equation put $\det(\lambda I - A) = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & p_0^{-1} & 0 \\ 0 & -p_1 & -\lambda & 1 \\ p_2 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda \begin{vmatrix} -\lambda & p_0^{-1} & 0 \\ -p_1 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & p_0^{-1} & 0 \\ 0 & -\lambda & 1 \\ p_2 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda \left\{ -\lambda \begin{vmatrix} -\lambda & p_o^{-1} \\ -p_1 & -\lambda \end{vmatrix} \right\} - 1 \cdot \left\{ p_2 \begin{vmatrix} p_o^{-1} & 0 \\ -\lambda & 1 \end{vmatrix} \right\} = 0$$

$$\Rightarrow \lambda^2 \left(\lambda^2 + p_1 p_o^{-1} \right) - p_2 \left(p_o^{-1} + 0 \right) = 0$$

$$\Rightarrow \lambda^4 + \frac{p_1}{p_o} \lambda^2 - \frac{p_2}{p_o} = 0$$

Therefore , the characteristic equation of A is given by

$$p_o \lambda^4 + p_1 \lambda^2 - p_2 = 0 \quad (2.1.5)$$

Now, consider $A v_j = \lambda v_j$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & p_o^{-1} & 0 \\ 0 & -p_1 & 0 & 1 \\ p_2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$$\Rightarrow v_2 = \lambda v_1$$

$$v_3 p_o^{-1} = \lambda v_2$$

$$-v_2 p_1 + v_4 = \lambda v_3$$

$$p_2 v_1 = \lambda v_4$$

Let $v_1 = 1$

$$v_2 = \lambda$$

$$v_3 = \lambda^2 p_o$$

$$v_4 = \frac{p_2}{\lambda} = p_2 \lambda^{-1}$$

Therefore , an eigenvector v_j of A corresponding to λ_j is

$$v_j = \begin{pmatrix} 1 & \lambda_j & \lambda_j^2 p_0 & p_2 \lambda_j^{-1} \end{pmatrix}^t \quad (2.1.6)$$

where the superscript t denotes the transpose . We assume at this stage that the λ_j are distinct , and we define the matrix T in (2.1.4) by

$$T = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix} \quad (2.1.7)$$

We have now to work out T^{-1} , which appears in (2.1.4) . To do this , we introduce the 4×4 matrix E whose elements e_{jk} are all zero except when $j + k = 5$ ($1 \leq j \leq 4$) , and these elements are unity . Then not only is E symmetric but so also is EA , where by symmetry we mean that the matrix coincides with its transpose , where

$$E = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (2.1.8)$$

Hence , given in the proof of [6 § 2(i) , 8 lemma 1.2.1] , the v_j have the orthogonality property

$$(E v_k)^t v_j = 0 \quad , \quad k \neq j \quad (2.1.9)$$

As in [1,2,3] , we define the scalars m_j , ($1 \leq j \leq 4$) by

$$m_j = (E v_j)^t v_j \quad (2.1.10)$$

and the row vectors

$$r_j = (E v_j)^t \quad (2.1.11)$$

$$\Rightarrow r_j = (E v_j)^t = \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \lambda_j \\ \lambda_j^2 p_0 \\ p_2 \lambda_j^{-1} \end{pmatrix} \right\}^t$$

$$\Rightarrow r_j = \begin{pmatrix} p_2 \lambda_j^{-1} \\ \lambda_j^2 p_0 \\ \lambda_j \\ 1 \end{pmatrix}^t = \begin{pmatrix} p_2 \lambda_j^{-1} & \lambda_j^2 p_0 & \lambda_j & 1 \end{pmatrix}$$

Then by [6 § 2], we can define T^{-1} by

$$T^{-1} = \begin{pmatrix} m_1^{-1} r_1 & m_2^{-1} r_2 & m_3^{-1} r_3 & m_4^{-1} r_4 \end{pmatrix}^t \quad (2.1.12)$$

and the

$$\begin{aligned} m_j &= \left[\frac{\partial}{\partial \lambda} \det(\lambda I - A) \right]_{\lambda = \lambda_j} \\ &= \left[4 p_0 \lambda^3 + 2 p_1 \lambda \right]_{\lambda = \lambda_j} \\ &= 4 p_0 \lambda_j^3 + 2 p_1 \lambda_j \end{aligned} \quad (2.1.13)$$

Now , we use the transformation

$$Y = TZ \quad (2.1.14)$$

to transform (2.1.2) into

$$TZ' + T'Z = ATZ$$

$$TZ' = ATZ - T'Z$$

$$Z' = T^{-1}ATZ - T^{-1}T'Z$$

$$Z' = (T^{-1}AT - T^{-1}T')Z$$

$$Z' = (\Lambda - T^{-1}T')Z \quad (2.1.15)$$

where $\Lambda = T^{-1}AT = \text{dg}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ (2.1.16)

Now if we write

$$T^{-1}T' = (t_{jk}), \quad (1 \leq j, k \leq 4)$$

then by (2.1.12) and (2.1.7)

$$t_{jk} = m_j^{-1} r_j v_k' \quad (2.1.17)$$

Hence, for the diagonal elements, we consider $k=j$, (2.1.17) gives

$$m_j t_{jj} = r_j v_j' \quad (2.1.18)$$

now by (2.1.10) $m_j = (E v_j)^t v_j$. Differentiating the m_j , we get

$$m_j' = 2r_j v_j' \quad (2.1.19)$$

Hence, by (2.1.18), (2.1.19) gives

$$t_{jj} = \frac{1}{m_j} r_j v_j'$$

$$\Rightarrow t_{jj} = \frac{1}{2} \frac{m_j'}{m_j} \quad , \quad (1 \leq j \leq 4) \quad (2.1.20)$$

Now by (2.1.6) and (2.1.11) , (2.1.17) gives , for $j \neq k$,
 $(1 \leq j , k \leq 4)$

$$t_{jk} = m_j^{-1} r_j v_k'$$

$\Rightarrow$

$$t_{jk} = m_j^{-1} \begin{pmatrix} -p_2 \lambda_j^{-1} & \lambda_j^2 p_0 & \lambda_j & 1 \end{pmatrix} \begin{pmatrix} 0 & \lambda_k' & (\lambda_k^2 p_0)' & (p_2 \lambda_k^{-1})' \end{pmatrix}^t$$

$$\therefore t_{jk} = m_j^{-1} \left(\lambda_j^2 p_0 \lambda_k' + \lambda_j (\lambda_k^2 p_0)' + (p_2 \lambda_k^{-1})' \right) , \quad (2.1.21)$$

Now we need to work out (2.1.20) and (2.1.21) in terms of p_j ,
 $(0 \leq j \leq 2)$ to determine the form (2.1.15) and then make progress towards
(2.1.1) .

2.2 THE SYSTEM $Z' = (A + R + S)Z$

In our analysis , it should be noted that all O and o terms are with respect to $x \rightarrow \infty$. We impose a basic condition on the coefficients as follows :

p_0 and p_2 are nowhere zero in some interval $[a, \infty)$,

$$p_1 = o \left\{ (p_0 p_2)^{1/2} \right\} , \quad \text{as } x \rightarrow \infty$$

and we write

$$\delta = \frac{p_1}{(p_0 p_2)^{1/2}} = o(1) \quad , \quad \text{as } x \rightarrow \infty \quad (2.2.1)$$

Now , as in [1,2,3] we can solve the characteristic equation (2.1.5) asymptotically as $x \rightarrow \infty$ using (2.2.1) to get the distinct eigenvalues λ_j as

$$\lambda_j = w_j \left(\frac{p_2}{p_0} \right)^{1/4} (1 + \delta_j) \quad , \quad (1 \leq j \leq 4) \quad (2.2.2)$$

where $\delta_j \rightarrow 0$ as $x \rightarrow \infty$ and

$$w_1 = 1 \quad , \quad w_2 = i \quad , \quad w_3 = -1 \quad , \quad w_4 = -i \quad (2.2.3)$$

$$\Rightarrow \quad \lambda_j \sim w_j \left(\frac{p_2}{p_0} \right)^{1/4} \quad \text{as } x \rightarrow \infty \quad (2.2.4)$$

Then we can solve (2.1.1) asymptotically as $x \rightarrow \infty$ using (2.2.2)

$$\Rightarrow \quad p_0 w_j^4 \frac{p_2}{p_0} (1 + \delta_j)^4 + p_1 w_j^2 \left(\frac{p_2}{p_0} \right)^{1/2} (1 + \delta_j)^2 - p_2 = 0$$

$$\Rightarrow \quad w_j^4 p_2 \left\{ 1 + 4\delta_j + O(\delta_j^2) \right\} + p_1 w_j^2 \left(\frac{p_2}{p_0} \right)^{1/2} \left\{ 1 + 2\delta_j + O(\delta_j^2) \right\} - p_2 = 0$$

since $w_j^4 = 1$

$$\Rightarrow \quad p_2 \left\{ 1 + 4\delta_j + O(\delta_j^2) \right\} + p_1 w_j^2 \left(\frac{p_2}{p_0} \right)^{1/2} \left\{ 1 + 2\delta_j + O(\delta_j^2) \right\} - p_2 = 0 \quad (2.2.5)$$

Divide (2.2.5) by p_2 and then solve for dominate term δ_j .

$$\Rightarrow \left\{1 + 4\delta_j + O(\delta_j^2)\right\} + w_j^2 \cdot \frac{p_1}{(p_0 p_2)^{1/2}} \cdot \left\{1 + 2\delta_j + O(\delta_j^2)\right\} - 1 = 0$$

$$\Rightarrow 1 + 4\delta_j + w_j^2 \delta (1 + 2\delta_j) - 1 = 0$$

$$\Rightarrow 4\delta_j + w_j^2 \delta + 2w_j^2 \delta \delta_j = 0$$

$$\Rightarrow 4\delta_j \left(1 + \frac{1}{2} w_j^2 \delta\right) = -w_j^2 \delta$$

$$\Rightarrow \delta_j = \frac{-1}{4} w_j^2 \delta \left(1 + \frac{1}{2} w_j^2 \delta\right)^{-1}$$

But

$$\begin{aligned} \left(1 + \frac{1}{2} w_j^2 \delta\right)^{-1} &= 1 - \frac{1}{2} w_j^2 \delta + \left(\frac{1}{2}\right)^2 w_j^4 \delta^2 - \left(\frac{1}{2}\right)^3 w_j^6 \delta^3 + \dots \\ &= \frac{1}{1 + \frac{1}{2} w_j^2 \delta} = \frac{1}{1 + o(1)} = O(1) \quad \text{as } x \rightarrow \infty \end{aligned}$$

$$\therefore \delta_j = \frac{-1}{4} w_j^2 \delta O(1) = O(\delta) \quad , \quad (1 \leq j \leq 4) \quad (2.2.6)$$

or we can write δ_j as
$$\delta_j = \frac{-1}{4} w_j^2 \delta \{1 + O(\delta)\}$$

$$= \frac{-1}{4} w_j^2 \delta + O(\delta^2)$$

$$= O(\delta)$$

In (2.1.20) , we require the form of m_j and m_j' as $x \rightarrow \infty$. First , we work out m_j , $(1 \leq j \leq 4)$ by (2.1.13) , (2.2.1) and (2.2.2)

$$\begin{aligned}
m_j &= 4p_o \lambda_j^3 + 2p_1 \lambda_j \\
&= 4p_o w_j^3 \left(\frac{p_2}{p_o} \right)^{3/4} (1 + \delta_j)^3 + 2p_1 w_j \left(\frac{p_2}{p_o} \right)^{1/4} (1 + \delta_j) \\
&= 4w_j^3 p_o^{1/4} p_2^{3/4} (1 + \delta_j)^3 + 2p_1 w_j \left(\frac{p_2}{p_o} \right)^{1/4} (1 + \delta_j) \\
&= 4w_j^3 p_o^{1/4} p_2^{3/4} (1 + \delta_j) \left\{ (1 + \delta_j)^2 + \frac{1}{2w_j^2} \cdot \frac{p_1}{(p_o p_2)^{1/2}} \right\} \\
&= 4w_j^3 p_o^{1/4} p_2^{3/4} (1 + \delta_j) \left\{ (1 + \delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\} \\
&= 4w_j^3 p_o^{1/4} p_2^{3/4} \{1 + O(\delta)\} \{1 + 2\delta_j + O(\delta^2) + O(\delta)\} \\
&= 4w_j^3 p_o^{1/4} p_2^{3/4} \{1 + O(\delta)\} \\
&= 4w_j^3 p_o^{1/4} p_2^{3/4} \{1 + O(\delta)\} \tag{2.2.7}
\end{aligned}$$

Also , by substituting (2.2.2) into (2.1.13) and differentiating , from previous steps we get that

$$m_j = 4w_j^3 p_o^{1/4} p_2^{3/4} (1 + \delta_j) \left\{ (1 + \delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}$$

$$\begin{aligned}
\therefore m_j' &= \left\{ (1+\delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\} \left\{ 4w_j^3 (1+\delta_j) p_2^{3/4} \cdot \frac{1}{4} \cdot p_o^{-3/4} \cdot p_o' \right. \\
&\quad \left. + 4w_j^3 (1+\delta_j) p_o^{1/4} \cdot \frac{3}{4} \cdot p_2^{-1/4} \cdot p_2' \right\} \\
&\quad + \left\{ (1+\delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\} 4w_j^3 \cdot p_2^{3/4} \cdot p_o^{1/4} \cdot \delta_j' \\
&\quad + 4w_j^3 \cdot p_2^{3/4} \cdot p_o^{1/4} \cdot (1+\delta_j) \left\{ 2(1+\delta_j) \delta_j' + \frac{1}{2} w_j^{-2} \delta' \right\} \\
m_j' &= 4w_j^3 p_2^{3/4} p_o^{1/4} (1+\delta_j) \left\{ \frac{1}{4} \frac{p_o'}{p_o} + \frac{3}{4} \frac{p_2'}{p_2} \right\} \left\{ (1+\delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\} \\
&\quad + O\left(p_2^{3/4} p_o^{1/4} \delta_j'\right) + O\left(p_2^{3/4} p_o^{1/4} \delta'\right) \quad (2.2.8)
\end{aligned}$$

At this stage we also require the following conditions

$$\delta \frac{p_o'}{p_o}, \quad \delta \frac{p_2'}{p_2} \quad \text{and} \quad \frac{p_1'}{(p_2 p_o)^{1/2}} \quad \text{are all} \quad L(a, \infty) \quad (2.2.9)$$

Further, by differentiating (2.2.1), we obtain

$$\begin{aligned}
\delta &= p_1 (p_2 p_o)^{-1/2} \\
\therefore \delta' &= p_1' (p_o p_2)^{-1/2} - \frac{1}{2} p_1 (p_o p_2)^{-3/2} \left\{ p_o p_2' + p_o' p_2 \right\} \\
&= \frac{p_1'}{(p_o p_2)^{1/2}} - \frac{1}{2} \cdot \frac{p_1 p_2'}{(p_o p_2)^{1/2} p_2} - \frac{1}{2} \cdot \frac{p_1 p_o'}{(p_o p_2)^{1/2} p_o} \\
&= \frac{p_1'}{(p_o p_2)^{1/2}} - \frac{1}{2} \delta \frac{p_2'}{p_2} - \frac{1}{2} \delta \frac{p_o'}{p_o} \quad (2.2.10)
\end{aligned}$$

$$\Rightarrow \delta' = O\left(\frac{p_1'}{(p_\circ p_2)^{1/2}}\right) + O\left(\delta \frac{p_\circ'}{p_\circ}\right) + O\left(\delta \frac{p_2'}{p_2}\right) \quad (2.2.11)$$

$$\text{Hence , by (2.2.9) we obtain that } \delta' \in L(a, \infty) \quad (2.2.12)$$

Now , substitute (2.2.2) into (2.1.5) in order to find δ_j' , we obtain that

$$(1 + \delta_j)^4 + w_j^2 \delta (1 + \delta_j)^2 - 1 = 0$$

By differentiating the above equation , we obtain

$$4(1 + \delta_j)^3 \delta_j' + 2w_j^2 \delta (1 + \delta_j) \delta_j' + w_j^2 \delta' (1 + \delta_j)^2 = 0$$

$$\delta_j' + \frac{1}{2} w_j^2 \cdot \frac{\delta \delta_j'}{(1 + \delta_j)^2} + \frac{1}{4} w_j^2 \cdot \frac{\delta'}{(1 + \delta_j)} = 0$$

$$\delta_j' \left\{ 1 + \frac{1}{2} w_j^2 \frac{\delta}{(1 + \delta_j)^2} \right\} = - \frac{1}{4} w_j^2 \frac{\delta'}{(1 + \delta_j)}$$

$$\delta_j' = - \frac{1}{4} w_j^2 \frac{\delta'}{(1 + \delta_j)} \left\{ 1 + \frac{1}{2} w_j^2 \frac{\delta}{(1 + \delta_j)^2} \right\}^{-1}$$

$$\therefore \delta_j' = O(\delta') \quad \text{as } x \rightarrow \infty \quad (2.2.13)$$

$$\text{since } \delta' \in L(a, \infty) , \text{ then } \delta_j' \in L(a, \infty) . \quad (2.2.14)$$

Now , for the diagonal elements t_{jj} , $1 \leq j \leq 4$, we use (2.1.20) ; hence , by (2.2.7) and (2.2.8) , (2.1.20) gives

$$t_{jj} = \frac{1}{2} \frac{m_j'}{m_j}$$

Then ,

$$\begin{aligned}
t_{jj} &= \frac{1}{2} \left\{ \frac{1}{4} \frac{p_o'}{p_o} + \frac{3}{4} \frac{p_2'}{p_2} + O(\delta) + O(\delta_j') \right\} \\
&= \frac{1}{2} \left\{ \frac{1}{4} \frac{p_o'}{p_o} + \frac{3}{4} \frac{p_2'}{p_2} \right\} + O(\delta') \\
&= \frac{1}{8} \left\{ \frac{p_o'}{p_o} + 3 \frac{p_2'}{p_2} \right\} + O(\delta') \\
&= \frac{1}{8} \frac{(p_o p_2^3)'}{p_o p_2^3} + O(\delta') \tag{2.2.15}
\end{aligned}$$

Now , for the non-diagonal elements t_{jk} , $1 \leq j, k \leq 4$ we consider (2.1.21) . Now , by (2.2.2) , (2.2.6) and (2.2.7) , we have

$$\begin{aligned}
m_j^{-1} &= 4^{-1} w_j^{-3} p_2^{-3/4} p_o^{-1/4} (1 + \delta_j)^{-1} \left\{ (1 + \delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
\lambda_j^2 &= w_j^2 \left(\frac{p_2}{p_o} \right)^{1/2} (1 + \delta_j)^2 = w_j^2 p_o^{-1/2} p_2^{1/2} (1 + \delta_j)^2 \\
\lambda_k' &= w_k (1 + \delta_k) \frac{1}{4} p_o^{-5/4} p_2^{1/4} p_o' + w_k (1 + \delta_k) \frac{1}{4} p_o^{-1/4} p_2^{-3/4} p_2' \\
&\quad + w_k p_o^{-1/4} p_2^{1/4} \delta_k' \\
\lambda_j^2 p_o &= w_j^2 p_o^{1/2} p_2^{1/2} (1 + \delta_j)^2 \\
\lambda_k' (p_o \lambda_j^2) &= \frac{-1}{4} w_j^2 w_k (1 + \delta_k) (1 + \delta_j)^2 p_o' p_2^{3/4} p_o^{-3/4} \\
&\quad + \frac{1}{4} w_j^2 w_k (1 + \delta_k) (1 + \delta_j)^2 p_2' p_2^{-1/4} p_o^{1/4} \\
&\quad + w_j^2 w_k p_2^{3/4} p_o^{1/4} (1 + \delta_j)^2 \delta_k'
\end{aligned}$$

$$\begin{aligned}
& \Rightarrow m_j^{-1} \lambda_k' (p_\circ \lambda_j^2) \cdot \\
& = \frac{-1}{16} w_j^{-1} w_k \frac{p_\circ'}{p_\circ} (1+\delta_k)(1+\delta_j) \left\{ (1+\delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
& \quad + \frac{1}{16} w_j^{-1} w_k \frac{p_2'}{p_2} (1+\delta_k)(1+\delta_j) \left\{ (1+\delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
& \quad + \frac{1}{4} w_j^{-1} w_k (1+\delta_j) \delta_k' \left\{ (1+\delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
& = \frac{1}{16} w_j^{-1} w_k (1+\delta_k)(1+\delta_j) \left\{ \frac{p_2'}{p_2} - \frac{p_\circ'}{p_\circ} \right\} \left\{ (1+\delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
& \quad + \frac{1}{4} w_j^{-1} w_k (1+\delta_j) \delta_k' \left\{ (1+\delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
& = \frac{1}{16} w_j^{-1} w_k \left\{ \frac{p_2'}{p_2} - \frac{p_\circ'}{p_\circ} \right\} \{1 + O(\delta)\} + O(\delta') \\
& \hspace{25em} (2.2.16)
\end{aligned}$$

Also ,

$$\begin{aligned}
\left(p_\circ \lambda_k^2 \right)' &= w_k^2 (1+\delta_k)^2 p_2^{1/2} \frac{1}{2} p_\circ^{-1/2} p_\circ' \\
&\quad + w_k^2 (1+\delta_k)^2 p_\circ^{1/2} \frac{1}{2} p_2^{-1/2} p_2' \\
&\quad + 2 w_k^2 p_2^{1/2} p_\circ^{1/2} (1+\delta_k) \delta_k'
\end{aligned}$$

$$\begin{aligned}
\lambda_j \left(p_{\circ} \lambda_k^2 \right)' &= \frac{1}{2} w_j w_k^2 (1 + \delta_k)^2 (1 + \delta_j) p_{\circ}' p_2^{3/4} p_{\circ}^{-3/4} \\
&\quad + \frac{1}{2} w_j w_k^2 (1 + \delta_k)^2 (1 + \delta_j) p_2' p_2^{-1/4} p_{\circ}^{1/4} \\
&\quad + 2 w_j w_k^2 p_2^{3/4} p_{\circ}^{1/4} (1 + \delta_k)^2 \delta_k' (1 + \delta_j)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow m_j^{-1} \lambda_j \left(p_{\circ} \lambda_k^2 \right)' &= \frac{1}{8} w_j^{-2} w_k^2 \frac{p_{\circ}'}{p_{\circ}} (1 + \delta_k)^2 \left\{ (1 + \delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
&\quad + \frac{1}{8} w_j^{-2} w_k^2 \frac{p_2'}{p_2} (1 + \delta_k)^2 \left\{ (1 + \delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
&\quad + \frac{1}{2} w_j^{-2} w_k^2 (1 + \delta_k) \delta_k' \left\{ (1 + \delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
&= \frac{1}{8} w_j^{-2} w_k^2 (1 + \delta_k)^2 \left\{ \frac{p_2'}{p_2} + \frac{p_{\circ}'}{p_{\circ}} \right\} \left\{ (1 + \delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
&\quad + \frac{1}{2} w_j^{-2} w_k^2 (1 + \delta_k) \delta_k' \left\{ (1 + \delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
&= \frac{1}{8} w_j^{-2} w_k^2 \left\{ \frac{p_2'}{p_2} + \frac{p_{\circ}'}{p_{\circ}} \right\} \{1 + O(\delta)\} + O(\delta') \\
\end{aligned} \tag{2.2.17}$$

Also ,

$$\begin{aligned}
\lambda_k^{-1} &= w_k^{-1} p_{\circ}^{1/4} p_2^{-1/4} (1 + \delta_k)^{-1} \\
\lambda_k^{-1} p_2 &= w_k^{-1} p_{\circ}^{1/4} p_2^{3/4} (1 + \delta_k)^{-1}
\end{aligned}$$

$$\begin{aligned}
& \left(p_2 \lambda_k^{-1} \right)' = w_k^{-1} (1 + \delta_k)^{-1} p_2^{3/4} \frac{1}{4} p_\circ^{-3/4} p_\circ' \\
& \quad + w_k^{-1} (1 + \delta_k)^{-1} p_\circ^{1/4} \frac{3}{4} p_2^{-1/4} p_2' \\
& \quad - w_k^{-1} p_2^{3/4} p_\circ^{1/4} (1 + \delta_k)^{-2} \delta_k' \\
\Rightarrow & m_j^{-1} \left(p_2 \lambda_k^{-1} \right)' \\
& = \frac{1}{16} w_j^{-3} w_k^{-1} \frac{p_\circ'}{p_\circ} (1 + \delta_k)^{-1} (1 + \delta_j)^{-1} \left\{ (1 + \delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
& \quad + \frac{3}{16} w_j^{-3} w_k^{-1} \frac{p_2'}{p_2} (1 + \delta_k)^{-1} (1 + \delta_j)^{-1} \left\{ (1 + \delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
& \quad - \frac{1}{4} w_j^{-3} w_k^{-1} (1 + \delta_k)^{-2} (1 + \delta_j)^{-1} \delta_k' \left\{ (1 + \delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
& = \frac{1}{16} w_j^{-3} w_k^{-1} (1 + \delta_k)^{-2} (1 + \delta_j)^{-1} \left\{ 3 \frac{p_2'}{p_2} + \frac{p_\circ'}{p_\circ} \right\} \left\{ (1 + \delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
& \quad - \frac{1}{4} w_j^{-3} w_k^{-1} (1 + \delta_k)^{-2} (1 + \delta_j)^{-1} \delta_k' \left\{ (1 + \delta_j)^2 + \frac{1}{2} w_j^{-2} \delta \right\}^{-1} \\
& = \frac{1}{16} w_j^{-3} w_k^{-1} \left\{ 3 \frac{p_2'}{p_2} + \frac{p_\circ'}{p_\circ} \right\} \left\{ 1 + O(\delta) \right\} + O(\delta')
\end{aligned} \tag{2.2.18}$$

similarly , as for t_{jj} , we can find t_{jk} , $j \neq k$, $1 \leq j$, $k \leq 4$, by using (2.2.16) , (2.2.17) , (2.2.18) and (2.1.21) gives

$$\begin{aligned}
t_{jk} &= m_j^{-1} \lambda_j^2 \left(p_{\circ} \lambda_k' \right) + m_j^{-1} \lambda_j \left(p_{\circ} \lambda_k^2 \right)' + m_j^{-1} \left(p_2 \lambda_k^{-1} \right)' \\
&= \frac{1}{16} w_j^{-1} w_k \left\{ \frac{p_2'}{p_2} - \frac{p_{\circ}'}{p_{\circ}} \right\} \{1 + O(\delta)\} + O(\delta') \\
&\quad + \frac{1}{8} w_j^{-2} w_k^2 \left\{ \frac{p_2'}{p_2} + \frac{p_{\circ}'}{p_{\circ}} \right\} \{1 + O(\delta)\} + O(\delta') \\
&\quad + \frac{1}{16} w_j^{-3} w_k^{-1} \left\{ 3 \frac{p_2'}{p_2} + \frac{p_{\circ}'}{p_{\circ}} \right\} \{1 + O(\delta)\} + O(\delta')
\end{aligned} \tag{2.2.19}$$

Hence , for $j=1$ and $k=2$, (2.2.19) gives

$$\begin{aligned}
t_{12} &= \{1 + O(\delta)\} \left\{ \begin{aligned} &\frac{1}{16} i \left\{ \frac{p_2'}{p_2} - \frac{p_{\circ}'}{p_{\circ}} \right\} - \frac{1}{8} \left\{ \frac{p_2'}{p_2} + \frac{p_{\circ}'}{p_{\circ}} \right\} \\ &\quad - \frac{1}{16} i \left\{ 3 \frac{p_2'}{p_2} + \frac{p_{\circ}'}{p_{\circ}} \right\} \end{aligned} \right\} + O(\delta') \\
&= \{1 + O(\delta)\} \left\{ -\frac{1}{8} (1+i) \left\{ \frac{p_2'}{p_2} + \frac{p_{\circ}'}{p_{\circ}} \right\} \right\} + O(\delta')
\end{aligned}$$

Also , put $j=1$, $k=3$ in (2.2.19) we get

$$\begin{aligned}
t_{13} &= \{1 + O(\delta)\} \left\{ \begin{aligned} &\frac{-1}{16} \left\{ \frac{p_2'}{p_2} - \frac{p_{\circ}'}{p_{\circ}} \right\} + \frac{1}{8} \left\{ \frac{p_2'}{p_2} + \frac{p_{\circ}'}{p_{\circ}} \right\} \\ &\quad - \frac{1}{16} \left\{ 3 \frac{p_2'}{p_2} + \frac{p_{\circ}'}{p_{\circ}} \right\} \end{aligned} \right\} + O(\delta') \\
&= \{1 + O(\delta)\} \left\{ -\frac{1}{8} \left\{ \frac{p_2'}{p_2} - \frac{p_{\circ}'}{p_{\circ}} \right\} \right\} + O(\delta')
\end{aligned}$$

Put $j = 1$, $k = 4$ in (2.2.19) we get

$$\begin{aligned}
 t_{14} &= \{1 + O(\delta)\} \left\{ \begin{aligned} &\frac{-1}{16} i \left\{ \frac{p_2'}{p_2} - \frac{p_o'}{p_o} \right\} - \frac{1}{8} \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \\ &\quad - \frac{1}{16} i \left\{ 3 \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \end{aligned} \right\} + O(\delta') \\
 &= \{1 + O(\delta)\} \left\{ \frac{1}{8} (i - 1) \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \right\} + O(\delta')
 \end{aligned}$$

Now , put $j = 2$, $k = 1$ in (2.2.19) we get

$$\begin{aligned}
 t_{21} &= \{1 + O(\delta)\} \left\{ \begin{aligned} &\frac{-1}{16} i \left\{ \frac{p_2'}{p_2} - \frac{p_o'}{p_o} \right\} - \frac{1}{8} \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \\ &\quad + \frac{1}{16} i \left\{ 3 \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \end{aligned} \right\} + O(\delta') \\
 &= \{1 + O(\delta)\} \left\{ \frac{1}{8} (i - 1) \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \right\} + O(\delta')
 \end{aligned}$$

Put $j = 2$, $k = 3$ in (2.2.19) we get

$$\begin{aligned}
 t_{23} &= \{1 + O(\delta)\} \left\{ \begin{aligned} &\frac{1}{16} i \left\{ \frac{p_2'}{p_2} - \frac{p_o'}{p_o} \right\} - \frac{1}{8} \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \\ &\quad - \frac{1}{16} i \left\{ 3 \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \end{aligned} \right\} + O(\delta') \\
 &= \{1 + O(\delta)\} \left\{ -\frac{1}{8} (1 + i) \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \right\} + O(\delta')
 \end{aligned}$$

Put $j = 2$, $k = 4$ in (2.2.19) we get

$$\begin{aligned}
 t_{24} &= \{1 + O(\delta)\} \left\{ \begin{aligned} &\frac{-1}{16} \left\{ \frac{p_2'}{p_2} - \frac{p_o'}{p_o} \right\} + \frac{1}{8} \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \\ &\quad - \frac{1}{16} \left\{ 3 \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \end{aligned} \right\} + O(\delta') \\
 &= \{1 + O(\delta)\} \left\{ -\frac{1}{8} \left\{ \frac{p_2'}{p_2} - \frac{p_o'}{p_o} \right\} \right\} + O(\delta')
 \end{aligned}$$

Also , put $j = 3$, $k = 1$ in (2.2.19) we get

$$\begin{aligned}
 t_{13} &= \{1 + O(\delta)\} \left\{ \begin{aligned} &\frac{-1}{16} \left\{ \frac{p_2'}{p_2} - \frac{p_o'}{p_o} \right\} + \frac{1}{8} \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \\ &\quad - \frac{1}{16} \left\{ 3 \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \end{aligned} \right\} + O(\delta') \\
 &= \{1 + O(\delta)\} \left\{ -\frac{1}{8} \left\{ \frac{p_2'}{p_2} - \frac{p_o'}{p_o} \right\} \right\} + O(\delta')
 \end{aligned}$$

Put $j = 3$, $k = 2$ in (2.2.19) we get

$$\begin{aligned}
 t_{32} &= \{1 + O(\delta)\} \left\{ \begin{aligned} &\frac{-1}{16} i \left\{ \frac{p_2'}{p_2} - \frac{p_o'}{p_o} \right\} - \frac{1}{8} \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \\ &\quad + \frac{1}{16} i \left\{ 3 \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \end{aligned} \right\} + O(\delta') \\
 &= \{1 + O(\delta)\} \left\{ \frac{1}{8} (i-1) \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \right\} + O(\delta')
 \end{aligned}$$

Put $j = 3$, $k = 4$ in (2.2.19) we get

$$\begin{aligned}
 t_{34} &= \{1 + O(\delta)\} \left\{ \begin{aligned} &\frac{1}{16} i \left\{ \frac{p_2'}{p_2} - \frac{p_o'}{p_o} \right\} - \frac{1}{8} \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \\ &- \frac{1}{16} i \left\{ 3 \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \end{aligned} \right\} + O(\delta') \\
 &= \{1 + O(\delta)\} \left\{ -\frac{1}{8} (1+i) \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \right\} + O(\delta')
 \end{aligned}$$

Finally , put $j = 4$, $k = 1$ in (2.2.19) we get

$$\begin{aligned}
 t_{41} &= \{1 + O(\delta)\} \left\{ \begin{aligned} &\frac{1}{16} i \left\{ \frac{p_2'}{p_2} - \frac{p_o'}{p_o} \right\} - \frac{1}{8} \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \\ &- \frac{1}{16} i \left\{ 3 \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \end{aligned} \right\} + O(\delta') \\
 &= \{1 + O(\delta)\} \left\{ -\frac{1}{8} (1+i) \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \right\} + O(\delta')
 \end{aligned}$$

Put $j = 4$, $k = 2$ in (2.2.19) we get

$$\begin{aligned}
 t_{42} &= \{1 + O(\delta)\} \left\{ \begin{aligned} &\frac{-1}{16} \left\{ \frac{p_2'}{p_2} - \frac{p_o'}{p_o} \right\} + \frac{1}{8} \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \\ &- \frac{1}{16} \left\{ 3 \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \end{aligned} \right\} + O(\delta') \\
 &= \{1 + O(\delta)\} \left\{ -\frac{1}{8} \left\{ \frac{p_2'}{p_2} - \frac{p_o'}{p_o} \right\} \right\} + O(\delta')
 \end{aligned}$$

Put $j = 4$, $k = 3$ in (2.2.19) we get

$$\begin{aligned}
 t_{43} &= \{1 + O(\delta)\} \left\{ \begin{aligned} &\frac{-1}{16} i \left\{ \frac{p_2'}{p_2} - \frac{p_o'}{p_o} \right\} - \frac{1}{8} \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \\ &+ \frac{1}{16} i \left\{ 3 \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \end{aligned} \right\} + O(\delta') \\
 &= \{1 + O(\delta)\} \left\{ \frac{1}{8} (i-1) \left\{ \frac{p_2'}{p_2} + \frac{p_o'}{p_o} \right\} \right\} + O(\delta')
 \end{aligned}$$

Now , we can write the system (2.1.15) as

$$\begin{aligned}
 Z' &= \left(A - T^{-1}T' \right) Z \\
 \Rightarrow Z' &= \left(A + R + S \right) Z
 \end{aligned} \tag{2.2.20}$$

where $R + S = -T^{-1}T'$,

$$R = \begin{pmatrix} \frac{-1}{8}\eta & \frac{1}{8}(i+1)\rho & \frac{1}{8}\sigma & \frac{-1}{8}(i-1)\rho \\ \frac{-1}{8}(i-1)\rho & \frac{-1}{8}\eta & \frac{1}{8}(i+1)\rho & \frac{1}{8}\sigma \\ \frac{1}{8}\sigma & \frac{-1}{8}(i-1)\rho & \frac{-1}{8}\eta & \frac{1}{8}(i+1)\rho \\ \frac{1}{8}(i+1)\rho & \frac{1}{8}\sigma & \frac{-1}{8}(i-1)\rho & \frac{-1}{8}\eta \end{pmatrix} \tag{2.2.21}$$

with

$$\eta = 3 \frac{p_2'}{p_2} + \frac{p_o'}{p_o} , \quad \rho = \frac{p_2'}{p_2} + \frac{p_o'}{p_o} , \quad \sigma = \frac{p_2'}{p_2} - \frac{p_o'}{p_o} \tag{2.2.22}$$

and S is $L[a, \infty)$ by (2.2.11) and (2.2.13) .

Now as [1,2,3] , we can apply the asymptotic theorem in [8] to (2.2.20) provided only that A and R satisfy the conditions in [8] which is in the asymptotic theorem. We have the following result for (2.1.1) .

2.3 THE ASYMPTOTIC RESULT

In this section the forms of the asymptotic solutions of (2.1.1) are given under general conditions on the coefficients for large x . We shall use the asymptotic theorem of Eastham to get our main result .

Theorem 2.3.1 :

Let the coefficients p_0 and p_2 be nowhere zero in $[a, \infty)$ and $C^{(2)}[a, \infty)$ with $p_1 \in C^{(1)}[a, \infty)$. Let (2.2.1) and (2.2.9) hold . Also , let

$$\frac{p_0'}{p_0} \left(\frac{p_0}{p_2} \right)^{1/4} \rightarrow 0 \quad \text{and} \quad \frac{p_2'}{p_2} \left(\frac{p_0}{p_2} \right)^{1/4} \rightarrow 0 \quad \text{as } x \rightarrow \infty \quad (2.3.1)$$

$$\begin{aligned} \text{and} \quad & \left(\frac{p_0}{p_2} \right)^{1/4} \left(\frac{p_0'}{p_0} \right)^2, \quad \left(\frac{p_0}{p_2} \right)^{1/4} \left(\frac{p_2'}{p_2} \right)^2 \\ & \left(\frac{p_0}{p_2} \right)^{1/4} \left(\frac{p_0''}{p_0} \right), \quad \left(\frac{p_0}{p_2} \right)^{1/4} \left(\frac{p_2''}{p_2} \right) \end{aligned} \quad (2.3.2)$$

are all $L[a, \infty)$.

$$\text{Also , let} \quad \operatorname{Re}(\lambda_i - \lambda_j) \quad (2.3.3)$$

have one sign in $[a, \infty)$ for each unequal pair (i, j) . Then (2.1.1) has

solutions y_k , $1 \leq k \leq 4$, such that

$$y_k(x) \sim p_o^{-1/8}(x) p_2^{-3/8}(x) \exp \left(\int_a^x \lambda_k(t) dt \right) , \quad \text{as } x \rightarrow \infty \quad (2.3.4)$$

Proof :

We apply the asymptotic theorem in [7 §2] to (2.2.20) . By (2.2.20) and (2.2.21) , we first require

$$(\lambda_i - \lambda_j)^{-1} R(x) \rightarrow 0 \quad (i \neq j) \quad (1 \leq i, j \leq 4) \quad \text{as } x \rightarrow \infty$$

$$\text{since } \lambda_i = w_i \left(\frac{p_2}{p_o} \right)^{1/4} (1 + \delta_i) \quad \text{and} \quad \lambda_j = w_j \left(\frac{p_2}{p_o} \right)^{1/4} (1 + \delta_j)$$

$$\Rightarrow \lambda_i - \lambda_j = w_i \left(\frac{p_2}{p_o} \right)^{1/4} (1 + \delta_i) - w_j \left(\frac{p_2}{p_o} \right)^{1/4} (1 + \delta_j)$$

$$= \left(\frac{p_2}{p_o} \right)^{1/4} [w_i (1 + \delta_i) - w_j (1 + \delta_j)]$$

$$= \left(\frac{p_2}{p_o} \right)^{1/4} [w_i + w_i \delta_i - w_j - w_j \delta_j]$$

$$= \left(\frac{p_2}{p_o} \right)^{1/4} [(w_i - w_j) + w_i \delta_i - w_j \delta_j]$$

$$= \left(\frac{p_2}{p_o} \right)^{1/4} (w_i - w_j) \left[1 + \frac{w_i}{w_i - w_j} \delta_i - \frac{w_j}{w_i - w_j} \delta_j \right]$$

$$= \left(\frac{p_2}{p_o} \right)^{1/4} (w_i - w_j) \{1 + o(1)\} \quad , \quad i \neq j$$

$$\Rightarrow \lambda_i - \lambda_j \sim \left(\frac{p_2}{p_o} \right)^{1/4} , \quad i \neq j \quad \text{as } x \rightarrow \infty$$

Therefore ,

$$\left(\frac{p_o}{p_2}\right)^{1/4} \left[\frac{-1}{8} \left(\frac{p_o'}{p_o} + 3 \frac{p_2'}{p_2} \right) \right] = \frac{-1}{8} \frac{p_o'}{p_o} \left(\frac{p_o}{p_2}\right)^{1/4} - \frac{3}{8} \frac{p_2'}{p_2} \left(\frac{p_o}{p_2}\right)^{1/4} = o(1)$$

as $x \rightarrow \infty$

since $\frac{p_o'}{p_o} \left(\frac{p_o}{p_2}\right)^{1/4} \rightarrow 0$ and $\frac{p_2'}{p_2} \left(\frac{p_o}{p_2}\right)^{1/4} \rightarrow 0$

and for our system this is true by (2.3.1) . We can do similarly for the other elements in the $(\lambda_i - \lambda_j)^{-1} R(x) \rightarrow 0$.

Hence $(\lambda_i - \lambda_j)^{-1} R(x) = o(1)$ as $x \rightarrow \infty$

i.e $R(x) = o(\lambda_i - \lambda_j)$ as $x \rightarrow \infty$

We also need that

$$\left\{ (\lambda_i - \lambda_j)^{-1} R(x) \right\}' \in L(a, \infty) , i \neq j$$

this is true when

$$\left\{ (\lambda_i - \lambda_j)^{-1} \frac{p_o'}{p_o} \right\}' \in L(a, \infty)$$

and $\left\{ (\lambda_i - \lambda_j)^{-1} \frac{p_2'}{p_2} \right\}' \in L(a, \infty)$ for $i \neq j$

Hence ,

$$\begin{aligned}
& \left\{ \left(\lambda_i - \lambda_j \right)^{-1} \frac{p_o'}{p_o} \right\}' = \left\{ \left(\frac{p_o}{p_2} \right)^{1/4} (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \left(\frac{p_o'}{p_o} \right) \right\}' \\
& = \left(\frac{p_o}{p_2} \right)^{1/4} (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \left(\frac{p_o p_o'' - p_o'^2}{p_o^2} \right) \\
& + \left(\frac{p_o}{p_2} \right)^{1/4} (w_i - w_j)^{-1} \left(\frac{p_o'}{p_o} \right) \left\{ - \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-2} \cdot \frac{1}{w_i - w_j} (w_i \delta_i' - w_j \delta_j') \right\} \\
& + \left(\frac{p_o'}{p_o} \right) (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \left\{ \frac{1}{4} \left(\frac{p_o}{p_2} \right)^{-3/4} \left(\frac{p_2 p_o' - p_o p_2'}{p_o^2} \right) \right\} \\
& = \left(\frac{p_o}{p_2} \right)^{1/4} (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \left(\frac{p_o''}{p_o} \right) \\
& - \left(\frac{p_o}{p_2} \right)^{1/4} (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \left(\frac{p_o'}{p_o} \right)^2 \\
& - \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_o'}{p_o} \right) (w_i - w_j)^{-2} (w_i \delta_i' - w_j \delta_j') \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-2} \\
& + \frac{1}{4} \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_o'}{p_o} \right)^2 (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \\
& - \frac{1}{4} \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_o' p_2'}{p_o p_2} \right) (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1}
\end{aligned}$$

$$\begin{aligned}
& \Rightarrow \left\{ \left(\lambda_i - \lambda_j \right)^{-1} \frac{p_{\circ}' }{p_{\circ}} \right\}' \\
& = \left(\frac{p_{\circ}}{p_2} \right)^{1/4} \left(\frac{p_{\circ}''}{p_{\circ}} \right) (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \\
& \quad - \frac{3}{4} \left(\frac{p_{\circ}}{p_2} \right)^{1/4} \left(\frac{p_{\circ}'}{p_{\circ}} \right)^2 (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \\
& \quad - \left(\frac{p_{\circ}}{p_2} \right)^{1/4} \left(\frac{p_{\circ}'}{p_{\circ}} \right) (w_i - w_j)^{-2} (w_i \delta_i' - w_j \delta_j') \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-2} \\
& \quad - \frac{1}{4} \left(\frac{p_{\circ}}{p_2} \right)^{1/4} \left(\frac{p_{\circ}' p_2'}{p_{\circ} p_2} \right) (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1}
\end{aligned}$$

Also ,

$$\begin{aligned}
\left\{ \left(\lambda_i - \lambda_j \right)^{-1} \cdot \frac{p_2'}{p_2} \right\}' & = \left\{ \left(\frac{p_{\circ}}{p_2} \right)^{1/4} (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \left(\frac{p_2'}{p_2} \right) \right\}' \\
& = \left(\frac{p_{\circ}}{p_2} \right)^{1/4} (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \left(\frac{p_2 p_2'' - p_2'^2}{p_2^2} \right) \\
& \quad + \left(\frac{p_{\circ}}{p_2} \right)^{1/4} (w_i - w_j)^{-1} \left(\frac{p_2'}{p_2} \right) \left\{ - \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-2} \cdot \frac{1}{w_i - w_j} (w_i \delta_i' - w_j \delta_j') \right\} \\
& \quad + \left(\frac{p_2'}{p_2} \right) (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \left\{ \frac{1}{4} \left(\frac{p_{\circ}}{p_2} \right)^{-3/4} \left(\frac{p_2 p_{\circ}' - p_{\circ} p_2'}{p_2^2} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& \Rightarrow \left\{ \left(\lambda_i - \lambda_j \right)^{-1} \frac{p_2'}{p_2} \right\}' \\
& = \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_2''}{p_2} \right) (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \\
& \quad - \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_2'}{p_2} \right)^2 (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \\
& \quad - \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_2'}{p_2} \right) (w_i - w_j)^{-2} (w_i \delta_i' - w_j \delta_j') \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-2} \\
& \quad + \frac{1}{4} \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_2'}{p_2} \right)^2 (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \\
& \quad - \frac{1}{4} \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_o' p_2'}{p_o p_2} \right) (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \\
& = \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_2''}{p_2} \right) (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \\
& \quad - \frac{3}{4} \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_2'}{p_2} \right)^2 (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1} \\
& \quad - \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_2'}{p_2} \right) (w_i - w_j)^{-2} (w_i \delta_i' - w_j \delta_j') \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-2} \\
& \quad - \frac{1}{4} \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_o' p_2'}{p_o p_2} \right) (w_i - w_j)^{-1} \left[1 + \frac{1}{w_i - w_j} (w_i \delta_i - w_j \delta_j) \right]^{-1}
\end{aligned}$$

since $\left(\frac{p_o}{p_2}\right)^{1/4} \left(\frac{p_o'}{p_o}\right)^2$, $\left(\frac{p_o}{p_2}\right)^{1/4} \left(\frac{p_2'}{p_2}\right)^2$

$$\left(\frac{p_o}{p_2}\right)^{1/4} \left(\frac{p_o''}{p_o}\right) \text{ and } \left(\frac{p_o}{p_2}\right)^{1/4} \left(\frac{p_2''}{p_2}\right) \text{ are all } L[a, \infty),$$

and this true by (2.3.2) , also δ_i' and δ_j' are all $L[a, \infty)$. Just we need to show that

$$\left(\frac{p_o}{p_2}\right)^{1/4} \left(\frac{p_o' p_2'}{p_o p_2}\right) \in L[a, \infty)$$

The Cauchy – Schwartz inequality

$$\int_a^x |fg(t)| dt \leq \left(\int_a^x |f(t)|^2 dt \right)^{1/2} \cdot \left(\int_a^x |g(t)|^2 dt \right)^{1/2} \quad (2.3.5)$$

Now let $f = \left(\frac{p_o}{p_2}\right)^{1/8} \left(\frac{p_o'}{p_o}\right)$ and $g = \left(\frac{p_o}{p_2}\right)^{1/8} \left(\frac{p_2'}{p_2}\right)$

Then

$$\int_a^x \left(\frac{p_o}{p_2}\right)^{1/4} \left(\frac{p_o' p_2'}{p_o p_2}\right) dx$$

$$\leq \left[\int_a^x \left(\frac{p_o}{p_2}\right)^{1/4} \left(\frac{p_o'}{p_o}\right)^2 dx \right]^{1/2} \cdot \left[\int_a^x \left(\frac{p_o}{p_2}\right)^{1/4} \left(\frac{p_2'}{p_2}\right)^2 dx \right]^{1/2}$$

since f^2 , $g^2 \in L[a, \infty)$ by (2.3.2) , so $f \cdot g \in L[a, \infty)$.

$$\text{i.e} \quad \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_o' p_2'}{p_o p_2} \right) \in L[a, \infty) \quad (2.3.6)$$

$$\text{Hence ,} \quad \left\{ \left(\lambda_i - \lambda_j \right)^{-1} \frac{p_o'}{p_o} \right\}' \in L(a, \infty)$$

$$\text{and} \quad \left\{ \left(\lambda_i - \lambda_j \right)^{-1} \frac{p_2'}{p_2} \right\}' \in L(a, \infty)$$

Therefore ,

$$\left\{ \left(\lambda_i - \lambda_j \right)^{-1} R(x) \right\}' \in L(a, \infty) \quad , \quad i \neq j$$

Also , we need to show that the eigenvalues $\mu_k(x)$, $(1 \leq k \leq 4)$ of $A(x) + R(x)$ satisfy the Dichotomy Condition in Levinson's theorem [7] .

To see this , write

$$\mu_k = \lambda_k + r_{kk} + O \left\{ \max_{i \neq j} \frac{|R|^2}{|\lambda_i - \lambda_j|} \right\} \quad (2.3.7)$$

$$\mu_k = \lambda_k + r_{kk} + O(h) \quad (2.3.8)$$

$$\text{where} \quad h = \max_{i \neq j} \frac{|R|^2}{|\lambda_i - \lambda_j|}$$

$$\text{since} \quad \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_o'}{p_o} \right)^2, \quad \left(\frac{p_o}{p_2} \right)^{1/4} \left(\frac{p_2'}{p_2} \right)^2$$

and $\left(\frac{p_o}{p_2}\right)^{1/4} \left(\frac{p_o' p_2'}{p_o p_2}\right)$ are all $L[a, \infty)$ by (2.3.2), (2.3.6).

Hence , $\frac{|R|^2}{|\lambda_i - \lambda_j|} \in L[a, \infty)$

Therefore , $h \in L[a, \infty)$ (2.3.9)

Now , we need to show that

$\text{Re}(\mu_j - \mu_k)$ does not change sign . ($j \neq k$)

$$\Rightarrow \text{Re}(\mu_j - \mu_k) = \text{Re}(\lambda_j - \lambda_k) + O\left\{\max_{i \neq j} \frac{|R|^2}{|\lambda_i - \lambda_j|}\right\}$$

so $\text{Re}(\lambda_j - \lambda_k)$ does not change sign by (2.3.3) and

$$\max_{i \neq j} \frac{|R|^2}{|\lambda_i - \lambda_j|} \in L[a, \infty) \quad \text{by (2.3.9) .}$$

Therefore , $\text{Re}(\mu_j - \mu_k)$ does not change sign , so the Dichotomy Condition true . Since (2.2.20) satisfies all conditions for the asymptotic result [7] it follows that as $x \rightarrow \infty$, (2.2.20) has four linearly independent solutions $Z_k(x)$, ($1 \leq k \leq 4$) , such that

$$Z_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t) dt\right)$$

where e_k is the coordinate vector with k^{th} component unit and other components zero .

Also , by (2.3.8) and(2.3.9) we get

$$\int_a^x \mu_k(t) dt = \int_a^x \lambda_k(t) dt + \int_a^x r_{kk}(t) dt + \ell + o(1)$$

$$\Rightarrow \exp\left(\int_a^x \mu_k(t) dt\right) = \exp\left(\int_a^x (\lambda_k(t) + r_{kk}(t)) dt\right) \cdot \exp \ell \cdot \exp o(1)$$

$$\Rightarrow \exp\left(\int_a^x \mu_k(t) dt\right) = \exp\left(\int_a^x (\lambda_k(t) + r_{kk}(t)) dt\right) \cdot e^\ell \cdot \{1 + o(1)\}$$

Hence ,

$$Z_k(x) = \{e_k + o(1)\} \exp\left\{\int_a^x (\lambda_k(t) + r_{kk}(t)) dt\right\} \cdot e^\ell \cdot \{1 + o(1)\}$$

as $x \rightarrow \infty$

To obtain the solution of our original system (2.1.2) we transform back by using (2.1.7) and (2.1.14) .

$$Y = TZ$$

$$\Rightarrow Y_k(x) = T\{e_k + o(1)\} \exp\left\{\int_a^x (\lambda_k(t) + r_{kk}(t)) dt\right\} \cdot e^\ell \cdot \{1 + o(1)\}$$

Therefore ,

$$Y_k(x) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \{e_k + o(1)\} \{1 + o(1)\} \cdot e^\ell \cdot \exp\left\{\int_a^x \lambda_k(t) dt\right\} \cdot \exp\left\{\int_a^x r_{kk}(t) dt\right\}$$

But

$$\begin{aligned}\int_a^x r_{kk}(t) dt &= \int_a^x \frac{-1}{8} \left(\frac{p_o'(t)}{p_o(t)} + 3 \frac{p_2'(t)}{p_2(t)} \right) dt = \int_a^x \frac{-1}{8} \frac{(p_o(t) p_2^3(t))'}{p_o(t) p_2^3(t)} dt \\ &= \ln(p_o(t) p_2^3(t))^{-1/8} + \ln(p_o(t) p_2^3(t))^{1/8}\end{aligned}$$

Then ,

$$\exp\left(\int_a^x r_{kk}(t) dt\right) = p_o^{-1/8}(x) p_2^{-3/8}(x) p_o^{1/8}(a) p_2^{3/8}(a)$$

Therefore , for $k = 1$

$$\begin{aligned}Y_1(x) &= T\{e_1 + o(1)\}\{1 + o(1)\} \cdot e^\ell \cdot \exp\left(\int_a^x \lambda_1(t) dt\right) \\ &\quad \cdot \{p_o^{-1/8}(x) p_2^{-3/8}(x) p_o^{1/8}(a) p_2^{3/8}(a)\}\end{aligned}$$

where

$$e_1 + o(1) = \begin{pmatrix} 1 + o(1) \\ o(1) \\ o(1) \\ o(1) \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 p_o & \lambda_2^2 p_o & \lambda_3^2 p_o & \lambda_4^2 p_o \\ \lambda_1^{-1} p_2 & \lambda_2^{-1} p_2 & \lambda_3^{-1} p_2 & \lambda_4^{-1} p_2 \end{pmatrix}$$

$\Rightarrow$

$$Y_1(x) = \begin{pmatrix} 1 + o(1) \\ \lambda_1 + o(1) \\ \lambda_1^2 p_\circ + o(1) \\ \lambda_1^{-1} p_2 + o(1) \end{pmatrix} \{1 + o(1)\} \cdot e^\ell \cdot \exp\left(\int_a^x \lambda_1(t) dt\right) \\ \cdot \left\{ p_\circ^{-1/8}(x) p_2^{-3/8}(x) p_\circ^{1/8}(a) p_2^{3/8}(a) \right\}$$

after adjusting a constant multiple in y_1 , we get

$$y_1(x) = \{1 + o(1)\} p_\circ^{-1/8}(x) p_2^{-3/8}(x) \exp\left(\int_a^x \lambda_1(t) dt\right)$$

$$\Rightarrow y_1(x) \sim p_\circ^{-1/8}(x) p_2^{-3/8}(x) \exp\left(\int_a^x \lambda_1(t) dt\right) \quad \text{as } x \rightarrow \infty$$

similar for $k = 2, 3, 4$.

Therefore , (2.1.1) has four linearly independent solutions y_k ,
($1 \leq k \leq 4$) such that

$$y_k(x) \sim p_\circ^{-1/8}(x) p_2^{-3/8}(x) \exp\left(\int_a^x \lambda_k(t) dt\right) , \quad \text{as } x \rightarrow \infty$$

The dichotomy condition (2.3.3)

We note that in (2.3.3) we have for convenience used a simplified form of the Levinson dichotomy condition [11,7§1] , as given in [1,2,3] the Theorem 2.3.1 also holds if (2.3.3) is generalized to

$$\operatorname{Re}(\lambda_j - \lambda_i) = f + g \quad (2.3.10)$$

where f has one sign in $[a, \infty)$ and g is $L[a, \infty)$ [7 §1] . Here we give case where (2.3.10) holds . we found that

$$\begin{aligned} \delta_j &= -\frac{1}{4} w_j^2 \delta (1 + O(\delta)) \\ &= -\frac{1}{4} w_j^2 \delta + O(\delta^2) \end{aligned}$$

Then , by (2.2.2) we obtain that

$$\lambda_j = w_j \left(\frac{p_2}{p_\circ} \right)^{1/4} \left(1 - \frac{1}{4} w_j^2 \delta + O(\delta^2) \right)$$

and
$$\lambda_k = w_k \left(\frac{p_2}{p_\circ} \right)^{1/4} \left(1 - \frac{1}{4} w_k^2 \delta + O(\delta^2) \right)$$

Now ,

$$\begin{aligned} \lambda_j - \lambda_k &= w_j \left(\frac{p_2}{p_\circ} \right)^{1/4} - \frac{1}{4} w_j^3 \left(\frac{p_2}{p_\circ} \right)^{1/4} \delta + O \left\{ \left(\frac{p_2}{p_\circ} \right)^{1/4} \delta^2 \right\} \\ &\quad - w_k \left(\frac{p_2}{p_\circ} \right)^{1/4} + \frac{1}{4} w_k^3 \left(\frac{p_2}{p_\circ} \right)^{1/4} \delta + O \left\{ \left(\frac{p_2}{p_\circ} \right)^{1/4} \delta^2 \right\} \end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \lambda_j - \lambda_k &= (w_j - w_k) \left(\frac{p_2}{p_\circ} \right)^{1/4} - \frac{1}{4} (w_j^3 - w_k^3) \left(\frac{p_2}{p_\circ} \right)^{1/4} \delta \\
&\quad + O \left\{ \left(\frac{p_2}{p_\circ} \right)^{1/4} \delta^2 \right\} \\
\Rightarrow \quad \lambda_j - \lambda_k &= \left(\frac{p_2}{p_\circ} \right)^{1/4} (w_j - w_k) \left\{ 1 - \frac{1}{4} (w_j^2 + w_j w_k + w_k^2) \delta \right\} \\
&\quad + O \left\{ \left(\frac{p_2}{p_\circ} \right)^{1/4} \delta^2 \right\}
\end{aligned}$$

Now , suppose that

$$f = \left(\frac{p_2}{p_\circ} \right)^{1/4} (w_j - w_k) \left\{ 1 - \frac{1}{4} (w_j^2 + w_j w_k + w_k^2) \delta \right\}$$

has one sign in $[a, \infty)$, and

$$\left(\frac{p_2}{p_\circ} \right)^{1/4} \delta^2 \in L[a, \infty) \quad (2.3.11)$$

with $\delta = \frac{p_1}{(p_\circ p_2)^{1/2}}$

Then it follows immediately from (2.3.1) and (2.3.9) that (2.3.10) is satisfied in the two following cases :

case 1 : $p_i, 0 \leq i \leq 2$ are real-valued functions .

case 2 : $p_i, 0 \leq i \leq 2$ are pure-imaginary functions .

Corollary 2.3.1 :

Let the coefficients p_0 and p_2 be nowhere zero in $[a, \infty)$ and $C^{(2)}[a, \infty)$ with $p_1 \in C^{(1)}[a, \infty)$. Let (2.2.1), (2.2.9), (2.3.1) - (2.3.3) hold. Also, let

$$\left(\frac{p_2}{p_0} \right)^{1/4} \delta^2 \in L[a, \infty) \quad (2.3.11)$$

where
$$\delta = \frac{p_1}{(p_0 p_2)^{1/2}} = o(1) \quad \text{as } x \rightarrow \infty$$

then

$$y_k(x) \sim p_0^{-1/8}(x) p_2^{-3/8}(x) \exp \left(\int_a^x \left(w_k \left(\frac{p_2}{p_0} \right)^{1/4} \left(1 - \frac{1}{4} w_k^2 \delta \right) \right) dt \right),$$

as $x \rightarrow \infty$

proof :

For a direct application of Theorem 2.3.1, we get that

$$y_k(x) = \{1 + o(1)\} p_0^{-1/8}(x) p_2^{-3/8}(x) \exp \left(\int_a^x \lambda_k(t) dt \right)$$

By (2.2.2) we obtain that

$$\lambda_k = w_k \left(\frac{p_2}{p_0} \right)^{1/4} (1 + \delta_k) \quad , \quad (1 \leq k \leq 4)$$

where $\delta_k = -\frac{1}{4}w_k^2\delta\{1+O(\delta)\}$

Hence , $\lambda_k = w_k\left(\frac{p_2}{p_\circ}\right)^{1/4} - \frac{1}{4}w_k^3\delta\left(\frac{p_2}{p_\circ}\right)^{1/4} + O\left\{\delta^2\left(\frac{p_2}{p_\circ}\right)^{1/4}\right\}$

By (2.3.11) we get that

$$\int_a^x \lambda_k(t) dt = \int_a^x \left(w_k\left(\frac{p_2}{p_\circ}\right)^{1/4} - \frac{1}{4}w_k^3\delta\left(\frac{p_2}{p_\circ}\right)^{1/4} \right) dt + \ell_1 + o(1)$$

Then ,

$$\exp\left(\int_a^x \lambda_k(t) dt\right) = \exp\left(\int_a^x \left(w_k\left(\frac{p_2}{p_\circ}\right)^{1/4} - \frac{1}{4}w_k^3\delta\left(\frac{p_2}{p_\circ}\right)^{1/4} \right) dt + \exp\ell_1 \cdot \{1+o(1)\}\right)$$

$$\exp\left(\int_a^x \lambda_k(t) dt\right) = \exp\left(\int_a^x \left(w_k\left(\frac{p_2}{p_\circ}\right)^{1/4} \left(1 - \frac{1}{4}w_k^2\delta\right) \right) dt + \exp\ell_1 \cdot \{1+o(1)\}\right)$$

Therefore ,

$$y_k(x) = \{1+o(1)\} p_\circ^{-1/8}(x) p_2^{-3/8}(x) \exp\left(\int_a^x \left(w_k\left(\frac{p_2}{p_\circ}\right)^{1/4} \left(1 - \frac{1}{4}w_k^2\delta\right) \right) dt + \exp\ell_1\right)$$

after adjusting a constant multiple in $y_k(x)$, we get

$$y_k(x) \sim p_\circ^{-1/8}(x) p_2^{-3/8}(x) \exp\left(\int_a^x \left(w_k\left(\frac{p_2}{p_\circ}\right)^{1/4} \left(1 - \frac{1}{4}w_k^2\delta\right) \right) dt \right)$$

as $x \rightarrow \infty$

2.4 CONCLUDING REMARKS

Remark 1 :

The Theorem 2.3.1 is illustrated by the familiar example where the coefficients are powers of x in the form

$$p_0(x) = c_1 x^\alpha, \quad p_1(x) = c_2 x^\beta, \quad p_2(x) = c_3 x^\gamma$$

where $\alpha, \beta, \gamma, c_i, (1 \leq i \leq 3)$ are real constants with $c_1 \neq 0$ and $c_3 \neq 0$.

Then :

(I) we required that $p_1 = o\left\{(p_0 p_2)^{1/2}\right\}$ as $x \rightarrow \infty$ holds.

So ,

$$\begin{aligned} \delta &= \frac{p_1}{(p_0 p_2)^{1/2}} = \frac{c_2 x^\beta}{(c_1 x^\alpha \cdot c_3 x^\gamma)^{1/2}} \\ &= \frac{c_2}{(c_1 c_3)^{1/2}} \cdot \frac{1}{x^{\frac{\alpha+\gamma}{2} - \beta}} \end{aligned}$$

Now $\delta \rightarrow 0$ as $x \rightarrow \infty$ if $\frac{\alpha+\gamma}{2} - \beta > 0$

$$\Rightarrow \alpha + \gamma - 2\beta > 0$$

Therefore , $\delta \rightarrow 0$ as $x \rightarrow \infty$ holds when $\alpha + \gamma - 2\beta > 0$.

(II) we need to show that the condition (2.2.9) holds

i.e $\delta \left(\frac{p'_0}{p_0} \right), \delta \left(\frac{p'_2}{p_2} \right)$ and $\frac{p'_1}{(p_0 p_2)^{1/2}}$ are all $L[a, \infty)$.

Now

$$\begin{aligned}\delta\left(\frac{p_{\circ}'}{p_{\circ}}\right) &= \frac{c_2}{(c_1 c_3)^{1/2}} \cdot \frac{1}{x^{\frac{\alpha+\gamma}{2}-\beta}} \cdot \left(\frac{c_1 \alpha x^{\alpha-1}}{c_1 x^{\alpha}}\right) \\ &= \frac{c_2 \cdot \alpha}{(c_1 c_3)^{1/2}} \cdot \frac{1}{x^{\frac{\alpha+\gamma}{2}-\beta+1}}\end{aligned}$$

$$\text{so, } \delta\left(\frac{p_{\circ}'}{p_{\circ}}\right) \in L[a, \infty) \quad \text{if } \frac{\alpha+\gamma}{2} - \beta + 1 > 1$$

$$\Rightarrow \alpha + \gamma - 2\beta > 0$$

also ,

$$\begin{aligned}\delta\left(\frac{p_2'}{p_2}\right) &= \frac{c_2}{(c_1 c_3)^{1/2}} \cdot \frac{1}{x^{\frac{\alpha+\gamma}{2}-\beta}} \cdot \left(\frac{c_3 \gamma x^{\gamma-1}}{c_3 x^{\gamma}}\right) \\ &= \frac{c_2 \cdot \gamma}{(c_1 c_3)^{1/2}} \cdot \frac{1}{x^{\frac{\alpha+\gamma}{2}-\beta+1}}\end{aligned}$$

$$\text{so, } \delta\left(\frac{p_2'}{p_2}\right) \in L[a, \infty) \quad \text{if } \frac{\alpha+\gamma}{2} - \beta + 1 > 1$$

$$\Rightarrow \alpha + \gamma - 2\beta > 0$$

and

$$\begin{aligned}\frac{p_1'}{(p_{\circ} p_2)^{1/2}} &= \frac{c_2 \beta x^{\beta-1}}{(c_1 c_3 x^{\alpha+\gamma})^{1/2}} \\ &= \frac{c_2 \beta}{(c_1 c_3)^{1/2}} \cdot \frac{1}{x^{\frac{\alpha+\gamma}{2}-\beta+1}}\end{aligned}$$

$$\text{so, } \frac{p_1'}{(p_0 p_2)^{1/2}} \in L[a, \infty) \text{ if } \frac{\alpha + \gamma}{2} - \beta + 1 > 1$$

$$\Rightarrow \alpha + \gamma - 2\beta > 0$$

Therefore, the condition (2.2.9) holds when $\alpha + \gamma - 2\beta > 0$.

(III) we require that the conditions (2.3.1) and (2.3.2) of Theorem 2.3.1 are all satisfy.

Now,

$$\begin{aligned} \left(\frac{p_0'}{p_0} \right) \left(\frac{p_0}{p_2} \right)^{1/4} &= \left(\frac{c_1 \alpha x^{\alpha-1}}{c_1 x^\alpha} \right) \cdot \left(\frac{c_1 x^\alpha}{c_3 x^\gamma} \right)^{1/4} \\ &= \left(\frac{c_1}{c_3} \right)^{1/4} \cdot \alpha \cdot \frac{1}{x^{\frac{\gamma-\alpha}{4}+1}} \end{aligned}$$

$$\text{so, } \left(\frac{p_0'}{p_0} \right) \left(\frac{p_0}{p_2} \right)^{1/4} \rightarrow 0 \text{ if } \frac{\gamma - \alpha}{4} + 1 > 0$$

$$\Rightarrow \gamma - \alpha + 4 > 0$$

and

$$\begin{aligned} \left(\frac{p_2'}{p_2} \right) \left(\frac{p_0}{p_2} \right)^{1/4} &= \left(\frac{c_3 \gamma x^{\alpha-1}}{c_3 x^\alpha} \right) \cdot \left(\frac{c_1 x^\alpha}{c_3 x^\gamma} \right)^{1/4} \\ &= \left(\frac{c_1}{c_3} \right)^{1/4} \cdot \gamma \cdot \frac{1}{x^{\frac{\gamma-\alpha}{4}+1}} \end{aligned}$$

$$\text{so , } \left(\frac{p_2'}{p_2} \right) \left(\frac{p_o}{p_2} \right)^{1/4} \rightarrow 0 \quad \text{if} \quad \frac{\gamma - \alpha}{4} + 1 > 0$$

$$\Rightarrow \gamma - \alpha + 4 > 0$$

Therefore , the condition (2.3.1) of Theorem 2.3.1 holds when $\gamma - \alpha + 4 > 0$.

For condition (2.3.2)

$$\begin{aligned} \left(\frac{p_o'}{p_o} \right)^2 \left(\frac{p_o}{p_2} \right)^{1/4} &= \left(\frac{c_1 \alpha x^{\alpha-1}}{c_1 x^\alpha} \right)^2 \cdot \left(\frac{c_1 x^\alpha}{c_3 x^\gamma} \right)^{1/4} \\ &= \left(\frac{c_1}{c_3} \right)^{1/4} \cdot \alpha^2 \cdot \frac{1}{x^{\frac{\gamma - \alpha}{4} + 2}} \end{aligned}$$

$$\text{so , } \left(\frac{p_o'}{p_o} \right)^2 \left(\frac{p_o}{p_2} \right)^{1/4} \in L[a, \infty) \quad \text{if} \quad \frac{\gamma - \alpha}{4} + 2 > 1$$

$$\Rightarrow \gamma - \alpha + 8 > 4$$

$$\Rightarrow \gamma - \alpha + 4 > 0$$

also ,

$$\begin{aligned} \left(\frac{p_2'}{p_2} \right)^2 \left(\frac{p_o}{p_2} \right)^{1/4} &= \left(\frac{c_3 \gamma x^{\gamma-1}}{c_3 x^\gamma} \right)^2 \cdot \left(\frac{c_1 x^\alpha}{c_3 x^\gamma} \right)^{1/4} \\ &= \left(\frac{c_1}{c_3} \right)^{1/4} \cdot \gamma^2 \cdot \frac{1}{x^{\frac{\gamma - \alpha}{4} + 2}} \end{aligned}$$

$$\text{so , } \left(\frac{p_2'}{p_2} \right)^2 \left(\frac{p_{\circ}}{p_2} \right)^{1/4} \in L[a, \infty) \quad \text{if } \frac{\gamma - \alpha}{4} + 2 > 1$$

$$\Rightarrow \gamma - \alpha + 8 > 4$$

$$\Rightarrow \gamma - \alpha + 4 > 0$$

also ,

$$\begin{aligned} \left(\frac{p_{\circ}''}{p_{\circ}} \right) \left(\frac{p_{\circ}}{p_2} \right)^{1/4} &= \left(\frac{c_1 \alpha (\alpha - 1) x^{\alpha - 2}}{c_1 x^{\alpha}} \right) \cdot \left(\frac{c_1 x^{\alpha}}{c_3 x^{\gamma}} \right)^{1/4} \\ &= \alpha (\alpha - 1) \left(\frac{c_1}{c_3} \right)^{1/4} \cdot \frac{1}{x^{\frac{\gamma - \alpha}{4} + 2}} \end{aligned}$$

$$\text{so , } \left(\frac{p_{\circ}''}{p_{\circ}} \right) \left(\frac{p_{\circ}}{p_2} \right)^{1/4} \in L[a, \infty) \quad \text{if } \frac{\gamma - \alpha}{4} + 2 > 1$$

$$\Rightarrow \gamma - \alpha + 4 > 0$$

and

$$\begin{aligned} \left(\frac{p_2''}{p_2} \right) \left(\frac{p_{\circ}}{p_2} \right)^{1/4} &= \left(\frac{c_3 \gamma (\gamma - 1) x^{\gamma - 2}}{c_3 x^{\gamma}} \right) \cdot \left(\frac{c_1 x^{\alpha}}{c_3 x^{\gamma}} \right)^{1/4} \\ &= \gamma (\gamma - 1) \left(\frac{c_1}{c_3} \right)^{1/4} \cdot \frac{1}{x^{\frac{\gamma - \alpha}{4} + 2}} \end{aligned}$$

$$\text{so , } \left(\frac{p_2''}{p_2} \right) \left(\frac{p_{\circ}}{p_2} \right)^{1/4} \in L[a, \infty) \quad \text{if } \frac{\gamma - \alpha}{4} + 2 > 1$$

$$\Rightarrow \gamma - \alpha + 4 > 0$$

Therefore , the condition (2.3.2) of Theorem 2.3.1 holds when $\gamma - \alpha + 4 > 0$.

(IV) Finally , we need that $\left(\frac{p_{\circ}}{p_2}\right)^{1/4} \delta^2 \in L[a, \infty)$.

So ,

$$\begin{aligned} \left(\frac{p_2}{p_{\circ}}\right)^{1/4} \delta^2 &= \left(\frac{c_3 x^{\gamma}}{c_1 x^{\alpha}}\right)^{1/4} \cdot \frac{c_2^2}{c_1 c_3} \cdot \frac{1}{x^{\alpha+\gamma-2\beta}} \\ &= \left(\frac{c_3}{c_1}\right)^{1/4} \cdot \frac{c_2^2}{c_1 c_3} \cdot \frac{1}{x^{\frac{5\alpha+3\gamma}{4}-2\beta}} \end{aligned}$$

$$\text{so , } \left(\frac{p_{\circ}}{p_2}\right)^{1/4} \delta^2 \in L[a, \infty) \quad \text{if} \quad \frac{5\alpha+3\gamma}{4} - 2\beta > 1$$

$$\Rightarrow 5\alpha + 3\gamma - 8\beta > 4$$

$$\Rightarrow 5\alpha + 3\gamma - 8\beta - 4 > 0$$

Therefore , $\left(\frac{p_{\circ}}{p_2}\right)^{1/4} \delta^2 \in L[a, \infty)$ holds when $5\alpha + 3\gamma - 8\beta - 4 > 0$.

Now it is easy to check that all the conditions of Theorem 2.3.1 , (2.2.1) , (2.2.9) and (2.3.11) are all hold under the set of conditions

$$\alpha + \gamma - 2\beta > 0 \tag{2.4.1}$$

$$\gamma - \alpha + 4 > 0 \tag{2.4.2}$$

$$5\alpha + 3\gamma - 8\beta - 4 > 0 \tag{2.4.3}$$

Now (2.4.1) is implied by (2.4.2) and (2.4.3) and hence we need only the latter two conditions (2.4.2) and (2.4.3) . We can for example take $\alpha = \gamma$

and $\alpha - \beta > \frac{1}{2}$.

Since the conditions (2.2.1) , (2.2.9) and (2.3.11) are held and all the conditions of Theorem 2.3.1 are satisfied , therefore by corollary 2.3.1 , (2.1.1) with the given coefficients has solutions $y_k(x)$, $(1 \leq k \leq 4)$ such that

$$y_k(x) = (1 + o(1)) \left(c_1 x^\alpha \right)^{-1/8} \cdot \left(c_3 x^\gamma \right)^{-3/8} \exp \left(\int_a^x \left(w_k \left(\frac{p_2}{p_0} \right)^{1/4} \left(1 - \frac{1}{4} w_k^2 \delta \right) \right) dt \right)$$

after adjusting a constant multiple in $y_k(x)$, we get that

$$y_k(x) \sim x^{-(\alpha + 3\gamma)/8} \cdot \exp \left\{ \int_a^x w_k \left(\frac{p_2}{p_0} \right)^{1/4} \left(1 - \frac{1}{4} w_k^2 \delta \right) dt \right\}$$

as $x \rightarrow \infty$

where δ as defined in (2.2.1) .

Remark 2 :

To give a quite different class of coefficients covered by our analysis , we consider

$$p_0(x) = c_1 x^\alpha e^{x^a} , \quad p_1(x) = c_2 x^\beta e^{2x^a} , \quad p_2(x) = c_3 x^\gamma e^{3x^b}$$

where $\alpha , \beta , \gamma , c_i$ ($1 \leq i \leq 3$) , a and b are real constants with $a \geq 0$, $b > a$, $c_1 \neq 0$ and $c_3 \neq 0$. Agin it is easy to check that the

conditions (2.2.1) , (2.2.9) , (2.3.11) and all the conditions of Theorem 2.3.1 are satisfied .

Remark 3 :

Al-Hammadi [2] considered the differential equation

$$(p_0 y'')'' + (p_1 y')' + \frac{1}{2} \sum_{j=0}^1 \left[\left(q_{2-j} \cdot y^{(j)} \right)^{(j+1)} + \left(q_{2-j} \cdot y^{(j)} \right)^{(j)} \right] - p_2 y = 0 \quad (2.4.4)$$

under general conditions on the coefficients p_i ($0 \leq i \leq 2$) and q_j , $j=1, 2$ for large x , he obtained the form of the solution

$$y_k(x) \sim p_0^{-1/8}(x) p_2^{-3/8}(x) \exp \left(\int_a^x \lambda_k(t) dt \right), \text{ as } x \rightarrow \infty$$

Now, in the case where $q_1 = q_2 = 0$, (2.4.4) reduces to (2.1.1), the asymptotic formula (2.3.4) agrees with the result of Al-Hammadi [2] .

Remark 4 :

Hinton [10] considered the differential equation

$$\left(r(x) y^{(m)}(x) \right)^{(\ell)} - q(x) y(x) = 0 \quad (2.4.5)$$

of order $n = m + \ell \geq 2$. He showed that, subject to certain conditions on $r(x)$ and $q(x)$, (2.4.5) has solutions

$$y_k \sim r^{-(n+1-2m)/2n}(x) \cdot q^{-(n-1)/2n}(x) \exp\left(w_k \int_a^x (q/r)^{1/n} dt\right)$$

as $x \rightarrow \infty$

In the case where $m = 2 = \ell$ and $n = 4$, (2.4.5) becomes

$$(r(x)y''(x))'' - q(x)y(x) = 0 \quad (2.4.6)$$

Hence, (2.4.6) has solutions in the form

$$y_k \sim r^{-1/8}(x) \cdot q^{-3/8}(x) \exp\left(w_k \int_a^x (q/r)^{1/4} dt\right), \text{ as } x \rightarrow \infty$$

Therefore, the asymptotic formula (2.3.4) with (2.2.4) agrees with the result of Hinton [10].

Remark 5 :

Eastham [5] considered the differential equation

$$\left(r_{n-1} \dots \left(r_2 (r_1 y')'\right)'\right)' - qy = 0 \quad (2.4.7)$$

He showed that, subject to certain conditions on $r_j(x)$ ($1 \leq j \leq n-1$)

and $q(x)$, (2.4.7) has solutions

$$y_k(x) \sim Q^{-(n-1)/2}(x) \cdot (r_1^{n-1} \cdot r_2^{n-2} \dots r_{n-2}^2 \cdot r_{n-1})^{-1/n}(x) \exp\left(w_k \int_a^x Q(t) dt\right)$$

where $Q = (q/(r_1 \dots r_{n-1}))^{1/n}$

as $x \rightarrow \infty$

In the case where $n = 4$ and $r_1 = r_3 = 1$, (2.4.7) becomes

$$(r_2 y'')'' - qy = 0 \quad (2.4.8)$$

Then, (2.4.8) has solutions in the form

$$y_k(x) \sim r^{-1/8}(x) \cdot q^{-3/8}(x) \exp \left(w_k \int_a^x Q(t) dt \right) \quad \text{where } Q = (q/r_2)^{1/4}$$

as $x \rightarrow \infty$

Therefore, the asymptotic formula (2.3.4) with (2.2.4) agrees with the result of Eastham [5].

Remark 6 :

For the familiar example of power coefficient which is in Remark 1, the result which we obtained here agrees with the result of Walker [12 p. 131 condition (ii)].

REFERENCES

REFERENCES

1. A.S.A. Al-Hammadi , *Asymptotic theory for general fourth-order differential equations* , Math. Nachr. 199 (1999) , 5-16.
2. A.S.A. Al-Hammadi , *Asymptotic formulae of Liouville-Green type for a general fourth-order differential equation* , Rocky Mountain , J. Math. (3) 28 (1998) , 801-812.
3. A.S.A. Al-Hammadi , *Asymptotic theory for a class of fourth-order differential equations* , Mathematika 43 (1996) , 198-208.
4. E.A. Coddington and N. Levinson , *Theory of ordinary differential equations* , McGraw-Hill , New York , 1955 .
5. M.S.P. Eastham , *Asymptotic formulae of Liouville-Green type for higher-order differential equations* , J. London Math. Soc. (2) 28 (1983) , 507-518.
6. M.S.P. Eastham , *On the eigenvectors for a class of matrices arising from quasi-derivatives* , Proc. Roy. Soc. Edinburgh Sect. A 97 (1984) , 73-78.
7. M.S.P. Eastham , *The Asymptotic solution of linear differential systems* , Mathematika 32 (1985) , 131-138.

8. M.S.P. Eastham , *The Asymptotic solution of linear differential systems , in Applications of the Levinson theorem* , Clarendon Press , Oxford , 1989.

9. W.N. Everitt and A. Zettl , *Generalized symmetric ordinary differential expressions I , The general theory* , Nieuw Arch. Wisk. (3) 27 (1979) , 363-397.

10. D.B.Hinton , *Asymptotic behaviour of solutions of $(ry^{(m)})^{(k)} + qy = 0$* , J. Differential Equations 4 (1968) , 590-596.

11. N.Levinson , *The asymptotic nature of solutions of linear differential equations* , Duke Math. J. 15(1948) , 111-126.

12. Philip W.Walker , *Asymptotica of the solutions to $[(ry'')'-y'] + qy = \sigma y$* , J. Differential Equations 9 (1971) , 108-132.

نظرية المقاربة لفئة المعادلات التفاضلية من الرتبة الرابعة

هذه الرسالة مكونة من بابين : الباب الاول مكون من خمسة فصول، الفصل الأول تم فيه تعريف النظام

$$Y'(x) = A(x)Y(x)$$

حيث x موجودة في الفترة $[a, \infty)$ ، $a \geq 0$ ، $A(x)$ هي مصفوفة مربعة معرفة لكل قيم x في الفترة $[a, \infty)$. و الهدف من ذكر النظام السابق هو أن نبين كيف تم الحصول على الحل المتقارب لهذا النظام . و تم ذكر طريقتين للوصول إلى هذا الهدف الطريقة الاولى هي طريقة نظام المعاملات الثابتة، و الطريقة الثانية هو النظام القطري .

أما في الفصل الثاني : فقد ذكرت النظريات الأساسية التي سوف تستخدم خلال هذه الرسالة كما تم تعريف الرموز التي سوف تستخدم أثناء ذلك .

و في الفصلين الثالث والرابع تم ذكر اهم نظريتين من نظريات التقارب و هما : نظرية Levinson ، و نظرية Eastham ، واللتين سوف يتم استخدامهما في الباب الثاني ، و تعتبر نظرية Eastham نتيجة لنظرية Levinson وهي النظرية الاساسية و التي تعتبر مناسبة جداً لكثير من التطبيقات و أسهل استخداماً من نظرية Levinson .

أما في الفصل الخامس فقد فرضنا المعادلة التفاضلية من الرتبة الثانية

$$Y''(x) + \{K + r(x)\}Y(x) = 0 \quad , \quad (a \leq x \leq \infty)$$

حيث K مقدار ثابت ، و تم إيجاد الحلول التقريبية لهذه المعادلة في حالتين الاولى في حالة $K > 0$ ، و الثانية في حالة $K < 0$ والحالتين يحققان الشرط

$$\int_a^\infty |r(x)| dx < \infty$$

أما الباب الثاني فهو مكون من أربعة فصول ، في الفصل الاول تم ذكر المعادلة الرئيسية لهذه الرسالة و

هي

$$(p_0 y'')'' + (p_1 y')' - p_2 = 0 \quad , \quad x \rightarrow \infty$$

و استعرضنا بالتفصيل الخطوات التي سوف يتم إتباعها لإيجاد الحلول الاربعة المتقاربة المستقلة خطياً عندما x تقترب من ما لا نهاية .

و في الفصلين الثاني و الثالث تم التعرض للشرط الاساسي ؛ لإيجاد الحلول المتقاربة للمعادلة السابقة و هو

$$p_1 = o\{(p \circ p_2)^{1/2}\} , \quad x \rightarrow \infty$$

و تم إيجاد الحلول التقاربية للمعادلة الرئيسية تحت شروط عامة على المعاملات لقيم x الكبيرة ، و قد وضعت تلك الشروط و الحلول على شكل نظرية وهى النتيجة الرئيسية في هذه الرسالة ، و لقد استخدمنا نظرية **Eastham** لإثبات هذه النتيجة .

و في الفصل الرابع تم عرض مثالين لتطبيق النتيجة الرئيسية التي تم التوصل إليها ، و في نهاية هذا الفصل تم عرض مقارنة بين النتيجة الرئيسية التي تم التوصل إليها و النتائج التي حصل عليها الآخرون في نفس المجال .