

UNIVERSITY OF BAHRAIN



A STUDY OF GAUSS' HYPERGEOMETRIC POLYNOMIALS AND THEIR APPLICATIONS

BY

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ABSTRACT

In this thesis, we present a systematic study of Gauss' hypergeometric polynomials and their applications. We define the Gauss' hypergeometric polynomials by means of a generating function, a differential equation and an integral representation. Some integrals, orthogonal properties, semi-orthogonal properties and orthogonal symmetries of Gauss' hypergeometric polynomials and their particular cases are also presented. Some integrals and expansion formulae for Fox's H-function involving Gauss' hypergeometric polynomials are also given. We formulate and solve some partial differential equations related to the problems of heat conduction and other applied sciences, solutions of which lead to the Jacobi polynomials and the Legendre functions.

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INTRODUCTION

The main object of this work is to conduct a detailed and systematic study of the Gauss' hypergeometric polynomials:

$$F(-m, b; c; x) = \sum_{n=0}^m \frac{(-m)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad -\infty < x < \infty$$

The Gauss' hypergeometric polynomials are important in many areas of mathematical analysis and their applications. Partly this is a consequence of the fact that the following polynomials and functions are simply special cases of the hypergeometric polynomials:

$$\begin{aligned} P_n^{(a,b)}(x) &= \frac{(a+1)_n}{n!} {}_2F_1\left(-n, n+a+b+1; a+1; \frac{1-x}{2}\right) \\ &= (-1)^n \frac{(b+1)_n}{n!} {}_2F_1\left(-n, n+a+b+1; b+1; \frac{1+x}{2}\right) \\ &= \frac{(a+1)_n}{n!} \left(\frac{1+x}{2}\right)^n {}_2F_1\left(-n, -n-b; a+1; \frac{x-1}{x+1}\right) \\ &= \frac{(b+1)_n}{n!} \left(\frac{x-1}{2}\right)^n {}_2F_1\left(-n, -n-a; b+1; \frac{x+1}{x-1}\right) \end{aligned}$$

$$T_n(x) = {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right)$$

$$C_n^v(x) = \frac{(2v)_n}{n!} {}_2F_1\left(-n, n+2v; v + \frac{1}{2}; \frac{1-x}{2}\right)$$

$$P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right)$$

$$P_n^m(x) = \frac{1}{\Gamma(1-m)} \left(\frac{1+x}{1-x}\right)^{m/2} {}_2F_1\left(-n, n+1; 1-m; \frac{1-x}{2}\right)$$

$$U_n(x) = (n+1) {}_2F_1\left(-n, n+1; \frac{3}{2}; \frac{1-x}{2}\right)$$

In chapter I, using an elegant technique, we define the Gauss' hypergeometric polynomials by means of the following generating function:

$$e^t {}_1F_1(b; c; -xt) = \sum_{m=0}^{\infty} \frac{t^m}{m!} {}_2F_1(-m, b; c; x)$$

However this generating function may also be derived from result due to Bajpai [3, p. 17, (3.3)].

We further derive the generating functions of some classical polynomials and functions as particular cases of our generating function [6].

We also present the differential equation and integral representation of the Gauss' hypergeometric polynomials.

In chapter II, we establish some integrals involving Gauss' hypergeometric polynomials and present their particular cases, some of them are known and others are believed to be new. These integrals are of fundamental importance in the study of Gauss' hypergeometric polynomials and generalized hypergeometric functions. Some of these integrals have been employed in the subsequent chapters of this work.

In chapter III, we present three orthogonal properties of Gauss' hypergeometric polynomials and employ them to obtain the orthogonal properties of Jacobi polynomials, Legendre functions and some other polynomials.

In this chapter, we also present a direct method of proof [5,14] to establish the orthogonalities of the orthogonal polynomials. Our method of proof is much simpler and elegant than the traditional methods for establishing the orthogonalities of the orthogonal polynomials.

These orthogonality properties of Gauss' hypergeometric polynomials will be used to establish some expansion formulae in a subsequent chapters.

In chapter IV, we present three semi-orthogonal properties for Gauss' hypergeometric polynomials and employ them to derive orthogonal properties for the Jacobi polynomials and the Legendre functions. We have also derived results recently given by Bajpai [13, pp. 63-66, (2.1)].

This type of research has been recently developed and studied at our esteemed University [7, 9, 10, 11, 13, 15, 16].

In chapter V, we establish an orthogonal symmetry for the Jacobi polynomials over the interval $[-1, 1]$ with respect to the multiplier function $(1-x)^{a+v} (1+x)^b$, if $\operatorname{Re} a > -v-1$ and $\operatorname{Re} b > -1$. Also, we present an orthogonal symmetry for the Legendre functions over the interval $[-1, 1]$ with respect to the multiplier function $(1-x)^v$.

From our orthogonal symmetries, we derive orthogonal properties of the Jacobi polynomials and the Legendre functions, respectively. Some other particular cases of our orthogonal symmetries have been also obtained, which appears to be new.

In chapter VI, we evaluate three integrals involving Fox's H-function and Gauss' hypergeometric polynomials and employ them to establish three expansion formulae for Fox's H-function involving Gauss' hypergeometric polynomials. We obtain some particular cases of our integrals and expansion formulae, due to Bajpai [4,5].

In chapter VII, we formulate and solve three partial differential equations related to the problems of heat conduction and other applied sciences, solutions of which lead to the Jacobi polynomials and the Legendre functions, respectively. We also obtain particular solutions of our partial differential equations in terms of Fox's H-function and derive some special solutions.

CHAPTER I

GAUSS' HYPERGEOMETRIC POLYNOMIALS

1.1 Introduction

In this chapter, we define Gauss' hypergeometric polynomials by means of a generating function and obtain the generating functions of some classical orthogonal polynomials as particular cases. We also present the differential equation and integral representation of the Gauss' hypergeometric polynomials.

1.2 The Gauss' Hypergeometric Series $F(a, b; c; x)$

The series defined by:

$$1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{x^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \quad (1.2.1)$$

is called the hypergeometric series, where a , b and c are arbitrary parameters.

On writing the general term of (1.2.1) by $U_n(x)$ and applying the ratio test, we see that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{U_{n+1}(x)}{U_n(x)} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(a)_{n+1} (b)_{n+1} x^{n+1}}{(c)_{n+1} (n+1)!} \cdot \frac{(c)_n n!}{(a)_n (b)_n x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{(a+n)(b+n)}{(c+n)(n+1)} \right| \end{aligned}$$

Completing the limit process reveals that

$$\lim_{n \rightarrow \infty} \left| \frac{U_{n+1}(x)}{U_n(x)} \right| = |x| \quad (1.2.2)$$

provided none of a , b , or c is zero or a negative integer. Therefore, we conclude that the series (1.2.1) converges under these circumstances for all $|x| < 1$ and diverges for all $|x| > 1$. For $|x| = 1$, it can be shown [25, p.46] that a sufficient condition for convergence of the series is $c - a - b > 0$.

The function

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \quad (1.2.3)$$

is called the hypergeometric function. It is also commonly denoted by the symbol

$${}_2F_1(a, b; c; x) = F(a, b; c; x) \quad (1.2.4)$$

where the 2 and 1 denote to the number of numerator and denominator parameters, respectively, in its series representation. The semicolons separate the numerator parameters a and b (which are themselves separated by a comma), the denominator parameter c , and the argument x .

1.3 The Gauss' Hypergeometric Polynomials $F(-m, b; c; x)$

When c is zero or a negative integer, the series (1.2.3) generally does not exist, and hence the function $F(a, b; c; x)$ is not defined. However, when either a or b (or both) is zero or a negative integer, the series is finite and thus converges for all x . Therefore, if $a = -m$ ($m = 0, 1, 2, \dots$) then $(-m)_n = 0$ when $n > m$, and in this case (1.2.3) reduces to the hypergeometric polynomials defined by:

$$F(-m, b; c; x) = \sum_{n=0}^m \frac{(-m)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad -\infty < x < \infty \quad (1.3.1)$$

1.4 Relations to Other Polynomials

The Gauss' hypergeometric polynomials are important in many areas of mathematical analysis and their applications. Partly this is a consequence of the fact that so many polynomials and functions are simply special cases of the hypergeometric polynomials.

For example:

$$P_n^{(a,b)}(x) = \frac{(a+1)_n}{n!} {}_2F_1\left(-n, n+a+b+1; a+1; \frac{1-x}{2}\right) \quad (1.4.1)$$

$$= (-1)^n \frac{(b+1)_n}{n!} {}_2F_1\left(-n, n+a+b+1; b+1; \frac{1+x}{2}\right) \quad (1.4.2)$$

$$= \frac{(a+1)_n}{n!} \left(\frac{1+x}{2}\right)^n {}_2F_1\left(-n, -n-b; a+1; \frac{x-1}{x+1}\right) \quad (1.4.3)$$

$$= \frac{(b+1)_n}{n!} \left(\frac{x-1}{2}\right)^n {}_2F_1\left(-n, -n-a; b+1; \frac{x+1}{x-1}\right) \quad (1.4.4)$$

$$T_n(x) = {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right) \quad (1.4.5)$$

$$C_n^v(x) = \frac{(2v)_n}{n!} {}_2F_1\left(-n, n+2v; v+\frac{1}{2}; \frac{1-x}{2}\right) \quad (1.4.6)$$

$$P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right) \quad (1.4.7)$$

$$P_n^m(x) = \frac{1}{\Gamma(1-m)} \left(\frac{1+x}{1-x}\right)^{m/2} {}_2F_1\left(-n, n+1; 1-m; \frac{1-x}{2}\right) \quad (1.4.8)$$

$$U_n(x) = (n+1) {}_2F_1\left(-n, n+1; \frac{3}{2}; \frac{1-x}{2}\right) \quad (1.4.9)$$

1.5 The Generating Function of ${}_2F_1(-m, b; c; x)$

The generating function to be established is

$$e^t {}_1F_1(b; c; -xt) = \sum_{m=0}^{\infty} \frac{t^m}{m!} {}_2F_1(-m, b; c; x) \quad (1.5.1)$$

Proof:

By writing

$$\begin{aligned} e^t {}_1F_1(b; c; -xt) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} (-1)^k \frac{x^k t^k}{k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (b)_k}{(c)_k} \frac{x^k}{k!} \frac{t^{n+k}}{n!} \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{(-1)^k (b)_k}{(c)_k} \frac{x^k}{k!} \frac{t^m}{(m-k)!} \end{aligned} \quad (1.5.2)$$

where the last step follows from the index change $n=m-k$ [1, p. 13, (1.15)]

Equation (1.5.2) can be written as:

$$e^t {}_1F_1(b; c; -xt) = \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{(-1)^k (b)_k}{(c)_k} \frac{x^k}{k!} \frac{t^m}{\Gamma(1+m-k)} \quad (1.5.3)$$

Now using the modified form of [1, p.360] viz;

$$\Gamma(a+1-n) = \frac{(-1)^n \Gamma(a+1)}{(-a)_n}, \quad a \text{ is non integral} \quad (1.5.4)$$

in (1.5.3), we have

$$\begin{aligned} e^t {}_1F_1(b; c; -xt) &= \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{k=0}^m \frac{(-m)_k (b)_k}{(c)_k} \frac{x^k}{k!} \\ &= \sum_{m=0}^{\infty} \frac{t^m}{m!} {}_2F_1(-m, b; c; x) \end{aligned}$$

which proves (1.5.1).

1.6 Particular Cases of the Generating Function

The polynomials (1.3.1) lead to a great variety of polynomials as particular cases, some of these particular cases are given below:

(A) The generating functions for the Jacobi polynomials:

$$(i) \quad e^t {}_1F_1 \left(1+a+b+m; a+1; \frac{(x-1)t}{2} \right) = \sum_{m=0}^{\infty} \frac{t^m}{(a+1)_m} P_m^{(a, b)}(x) \quad (1.6.1)$$

$$(ii) \quad e^t {}_1F_1 \left(a+b+1; a+1; \frac{(x-1)t}{2} \right) = \sum_{m=0}^{\infty} \frac{t^m}{(a+1)_m} P_m^{(a, b-m)}(x) \quad (1.6.2)$$

$$(iii) \quad e^t {}_1F_1 \left(1+b+m; 1; \frac{(x-1)t}{2} \right) = \sum_{m=0}^{\infty} \frac{t^m}{m!} P_m^{(0, b)}(x) \quad (1.6.3)$$

which are special cases of (1.6.1).

Proof:

(i) In (1.5.1), setting $1+a+b+m$ for b , $a+1$ for c and $\frac{1-x}{2}$ for x , we have

$$e^t {}_1F_1 \left(1+a+b+m; a+1; \frac{-(1-x)t}{2} \right) = \sum_{m=0}^{\infty} \frac{t^m}{m!} {}_2F_1 \left(-m, 1+a+b+m; a+1; \frac{1-x}{2} \right)$$

Now with the help of a modified form of [19, p. 268] viz, (1.4.1); the generating function (1.6.1) is obtained.

(ii) In (1.5.1), substituting $1+a+b$ for b , $a+1$ for c and $\frac{1-x}{2}$ for x , we have

$$e^t {}_1F_1 \left(1+a+b; a+1; \frac{-(1-x)t}{2} \right) = \sum_{m=0}^{\infty} \frac{t^m}{m!} {}_2F_1 \left(-m, a+b+1; a+1; \frac{1-x}{2} \right)$$

Now with the help of a modified form of [18, p. 170, (16)] viz,

$P_n^{(a, b-n)}(x) = \frac{(a+1)_n}{n!} {}_2F_1\left(-n, a+b+1; a+1; \frac{1-x}{2}\right)$, the generating function (1.6.2) is obtained, which is a result due to Bajpai [6, p. 68, (3.25)].

(iii) In (1.6.1), putting $a=0$, the generating function (1.6.3) is obtained.

(B) The generating function for the Tchebichef polynomials:

$$(i) e^t {}_1F_1\left(m; \frac{1}{2}; \frac{(x-1)t}{2}\right) = \sum_{m=0}^{\infty} \frac{t^m}{m!} T_m(x) \quad (1.6.4)$$

$$(ii) e^t {}_1F_1\left(m+1; \frac{3}{2}; \frac{(x-1)t}{2}\right) = \sum_{m=0}^{\infty} \frac{t^m}{(m+1)!} U_m(x) \quad (1.6.5)$$

Proof:

(i) In (1.5.1), substituting m for b , $\frac{1}{2}$ for c and $\frac{1-x}{2}$ for x , we have

$$e^t {}_1F_1\left(m; \frac{1}{2}; \frac{(x-1)t}{2}\right) = \sum_{m=0}^{\infty} \frac{t^m}{m!} {}_2F_1\left(-m, m; \frac{1}{2}; \frac{1-x}{2}\right)$$

Now with the help of [19, p. 265] viz; (1.4.5), the generating function (1.6.4) is obtained.

(ii) In (1.5.1), setting $m+1$ for b , $\frac{3}{2}$ for c and $\frac{1-x}{2}$ for x , we have

$$e^t {}_1F_1\left(m+1; \frac{3}{2}; \frac{-(1-x)t}{2}\right) = \sum_{m=0}^{\infty} \frac{t^m}{m!} {}_2F_1\left(-m, m+1; \frac{3}{2}; \frac{1-x}{2}\right)$$

Now with the help of [19, p. 266] viz; (1.4.9), the generating function (1.6.5) is obtained.

(C) The generating function for the Gegenbauer polynomials

$$e^t {}_1F_1 \left(2\nu + m; \nu + \frac{1}{2}; \frac{(x-1)}{2} t \right) = \sum_{m=0}^{\infty} \frac{t^m}{(2\nu)_m} C_m^\nu(x) \quad (1.6.6)$$

Proof:

In (1.5.1), substituting $2\nu + m$ for b , $\nu + \frac{1}{2}$ for c and $\frac{1-x}{2}$ for x , we have

$$e^t {}_1F_1 \left(2\nu + m; \nu + \frac{1}{2}; \frac{(x-1)}{2} t \right) = \sum_{m=0}^{\infty} \frac{t^m}{m!} {}_2F_1 \left(-m, m + 2\nu; \nu + \frac{1}{2}; \frac{1-x}{2} \right)$$

Now with the help of [19, p. 267] viz; (1.4.6), the generating function (1.6.6) is obtained.

(D) The generating function for the Legendre function and polynomials

$$(i) e^t \left(\frac{1+x}{1-x} \right)^{n/2} \frac{1}{\Gamma(1-n)} {}_1F_1 \left(m+1; 1-n; \frac{(x-1)t}{2} \right) = \sum_{m=0}^{\infty} \frac{t^m}{m!} P_m^n(x) \quad (1.6.7)$$

$$(ii) e^t {}_1F_1 \left(m+1; 1; \frac{(x-1)}{2} t \right) = \sum_{m=0}^{\infty} \frac{t^m}{m!} P_m(x) \quad (1.6.8)$$

Proof:

(i) In (1.5.1), putting $b = 1+m$, $c = 1-n$ and $x = \frac{1-x}{2}$, we have

$$e^t {}_1F_1 \left(m+1; 1-n; \frac{-(1-x)}{2} t \right) = \sum_{m=0}^{\infty} \frac{t^m}{m!} {}_2F_1 \left(-m, m+1; 1-n; \frac{1-x}{2} \right)$$

Now with the help of a modified form of [17, p. 143,6] viz., (1.4.8), the generating function (1.6.7) is obtained.

(ii) In (1.5.1), putting $b=m+1$, $c=1$ and $x=\frac{1-x}{2}$, we have

$$e^t {}_1F_1\left(m+1; 1; \frac{-(1-x)}{2} t\right) = \sum_{m=0}^{\infty} \frac{t^m}{m!} {}_2F_1\left(-m, m+1; 1; \frac{1-x}{2}\right)$$

Now with the help of a modified form of [19, p.266] viz, (1.4.7), the generating function (1.6.8) is obtained.

1.7 Differential Equation

The linear Second order D.E

$$x(1-x)y'' + [c - (-m + b+1) x] y' + mby=0 \quad (1.7.1)$$

is the differential equation for the Gauss' hypergeometric polynomials

$$y = F(-m, b; c; x) \quad (1.7.2)$$

which has been derived from the hypergeometric equation of Gauss' [1, p. 365, (9.23)].

1.8 Integral Representation

The integral representation of the polynomials is given by:

$$F(-m, b; c; x) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^m dt \quad (1.8.1)$$

where $c > b > 0$

Proof:

Using Beta function relation [1. p. 364, (9.17)]:

$$B(n+b, c-b) = \int_0^1 t^{n+b-1} (1-t)^{c-b-1} dt, \quad c > b > 0 \quad (1.8.2)$$

we deduce ($n=0, 1, 2, \dots$)

$$\frac{(b)_n}{(c)_n} = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{n+b-1} (1-t)^{c-b-1} dt \quad (1.8.3)$$

The substitution of (1.8.3) into (1.3.1) yields

$$F(-m, b; c; x) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \sum_{n=0}^m \frac{(-m)_n}{n!} x^n \int_0^1 t^{n+b-1} (1-t)^{c-b-1} dt \quad (1.8.4)$$

Interchanging the intergration and summation, which is justified due to the convergence of the integral and series invovled in the process, we get (1.8.1).

CHAPTER II

SOME INTEGRALS INVOLVING GAUSS' HYPERGEOMETRIC POLYNOMIALS

2.1 Introduction

In this chapter, we establish some integrals involving Gauss' hypergeometric polynomials and present their particular cases, some of them are known and others are believed to be new.

2.2 The Integrals involving Gauss' Hypergeometric Polynomials

The integrals to be established are:

$$(i) \int_1^{\infty} x^{h-1} (x-1)^{b-c-n} {}_2F_1(-n, b; c; x) dx =$$

$$\frac{(-1)^n (c-h)_n \Gamma(c-b-h) \Gamma(1-c+b)}{(c)_n \Gamma(1-h)}, \quad n = 0, 1, 2, \dots \quad (2.2.1)$$

where $\text{Re}(b-c) > n-1$, $\text{Re}(c-b-h) > 0$, $\text{Re}(c-h) > -n$.

$$(ii) \int_0^{\infty} x^{h-1} (1+x)^{b-c-n} {}_2F_1(-n, b; c; -x) dx =$$

$$\frac{(c-h)_n \Gamma(h) \Gamma(c-b-h)}{(c)_n \Gamma(c-b)}, \quad n = 0, 1, 2, \dots \quad (2.2.2)$$

where $\text{Re } h > 0$, $\text{Re}(c-b-h) > 0$, $\text{Re}(h-c) < n$.

$$(iii) \int_0^1 x^{h-1} (1-x)^{b-c-n} {}_2F_1(-n, b; c; x) dx =$$

$$\frac{(c-h)_n \Gamma(h) \Gamma(1-c+b)}{(c)_n \Gamma(1-c+b+h)}, \quad n = 0, 1, 2, \dots \quad (2.2.3)$$

where $\operatorname{Re} h > 0$, $\operatorname{Re} (b-c) > n - 1$.

$$(iv) \int_1^\infty x^{h-1} (x-1)^{b-h-1} {}_2F_1(-n, b; c; x) dx =$$

$$\frac{(c-h)_n \Gamma(b-h) \Gamma(1-b)}{(c)_n \Gamma(1-h)}, \quad n = 0, 1, 2, \dots \quad (2.2.4)$$

where $\operatorname{Re}(b-h) > 0$, $\operatorname{Re} b < 1-n$.

$$(v) \int_0^\infty x^{h-1} (1+x)^{b-h-1} {}_2F_1(-n, b; c; -x) dx =$$

$$\frac{(c-h)_n \Gamma(h) \Gamma(1-b)}{(c)_n \Gamma(1-b+h)}, \quad n = 0, 1, 2, \dots \quad (2.2.5)$$

where $\operatorname{Re} h > 0$, $\operatorname{Re} b < 1-n$.

$$(vi) \int_0^1 x^{h-1} (1-x)^{b-h-1} {}_2F_1(-n, b; c; x) dx =$$

$$\frac{(c-h)_n \Gamma(h) \Gamma(b-h)}{(c)_n \Gamma(b)}, \quad n = 0, 1, 2, \dots \quad (2.2.6)$$

where $\operatorname{Re} h > 0$, $\operatorname{Re} (b-h) > 0$.

$$\begin{aligned}
\text{(vii)} \quad & \int_1^\infty x^{h+a-1} (x-1)^{b-c} {}_2F_1(-n, b+n; c; x) dx = \\
& \frac{(-1)^n (c-h-a)_n \Gamma(c-b-h-a-n) \Gamma(1-c+b+n)}{(c)_n \Gamma(1-h-a)}, \quad n = 0, 1, 2, \dots \quad (2.2.7)
\end{aligned}$$

where $\text{Re}(b-c) > -1$, $\text{Re}(c-b-a-h) > n$, $\text{Re}(h+a-c) < n$.

$$\begin{aligned}
\text{(viii)} \quad & \int_0^\infty x^{h+a-1} (1+x)^{b-c} {}_2F_1(-n, b+n; c; -x) dx = \\
& \frac{(c-h-a)_n \Gamma(h+a) \Gamma(c-b-h-a-n)}{(c)_n \Gamma(c-b-n)}, \quad n = 0, 1, 2, \dots \quad (2.2.8)
\end{aligned}$$

where $\text{Re}(h+a) > 0$, $\text{Re}(c-b-a-h) > n$, $\text{Re}(a+h-c) < n$.

$$\begin{aligned}
\text{(ix)} \quad & \int_0^1 x^{h+a-1} (1-x)^{b-c} {}_2F_1(-n, b+n; c; -x) dx = \\
& \frac{(c-h-a)_n \Gamma(h+a) \Gamma(1-c+b+n)}{(c)_n \Gamma(1-c+b+a+h+n)}, \quad n = 0, 1, 2, \dots \quad (2.2.9)
\end{aligned}$$

where $\text{Re}(h+a) > 0$, $\text{Re}(b-c) > -1$.

Proof :

(i) The integral (2.2.1) is established by expressing the hypergeometric polynomials in the integrand as its series representation (1.3.1), interchanging the order of integration and summation, (which is justified due to the convergence of the integral and series involved in the process), the left hand side of (2.2.1) can be written as

$$\sum_{r=0}^n \frac{(-n)_r (b)_r}{(c)_r r!} \int_1^\infty x^{h+r-1} (x-1)^{b-c-n} dx \quad (2.2.10)$$

Now evaluating the resulting integral with the help of the integral [19, p.201, (7)] :

$$\int_1^\infty x^{-\nu} (x-1)^{\omega-1} dx = \frac{\Gamma(\nu-\omega) \Gamma(\omega)}{\Gamma(\nu)} \quad , \operatorname{Re} \nu > \operatorname{Re} \omega > 0 \quad (2.2.11)$$

we get,

$$\frac{\Gamma(1+b-c-n) \Gamma(c-b-h-n)}{\Gamma(1-h)} {}_3F_2(-n, b, h; c; 1-c+b+h-n; 1) \quad (2.2.12)$$

It can easily be verified that the generalized hypergeometric series in (2.2.12) is Saalschutzian. Therefore, using Saalschutz's theorem [17, p. 188, (3)]:

$${}_3F_2(-n, a, b; c; 1-c+a+b-n; 1) = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n} \quad (2.2.13)$$

and the formula (1.5.4) , the right hand side of (2.2.1) is obtained .

(ii) The integral (2.2.2) is established on following the technique employed to establish (2.2.1) except instead if (2.2.11) , using the integral [19 , p.233, (8)] :

$$\int_0^\infty x^{\nu-1} (1+x)^{-\omega} dx = \frac{\Gamma(\nu) \Gamma(\omega-\nu)}{\Gamma(\omega)} \quad , \operatorname{Re} \omega > \operatorname{Re} \nu > 0 \quad (2.2.14)$$

(iii) The integral (2.2.3) is obtained on following the technique employed to establish (2.2.1) except instead of (2.2.11) , using the integral [17 , p.9 , (1) & (5)] :

$$\int_0^1 x^{p-1} (1-x)^{q-1} dx = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)} \quad , \operatorname{Re} p > 0 \quad , \operatorname{Re} q > 0 \quad (2.2.15)$$

(iv) The integral (2.2.4) is established on following the technique employed to establish (2.2.1) except instead of (2.2.11) , using the Vandermonde's theorem [1, p.368] :

$${}_2F_1(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n} \quad (2.2.16)$$

(v) The integral (2.2.5) is established on following the technique employed to establish (2.2.2) except instead of (2.2.11) , using the Vandermonde's theorem (2.2.16) .

(vi) The integral (2.2.6) is established on following the technique employed to establish (2.2.3) except instead of (2.2.11) , using the Vandermonde's theorem (2.2.16) .

(vii) The integral (2.2.7) is obtained by replacing h by $h+a$ and b by $b+n$ in (2.2.1).

(viii) The integral (2.2.8) is obtained by replacing h by $h+a$ and b by $b+n$ in (2.2.2) .

Note 1 :

Integrals (2.2.3), (2.2.6) and (2.2.9) are known results [19, pp. 398 - 399, (2), (3) & (7)] .

(ix) The integral (2.2.9) is obtained by replacing h by $h+a$ and b by $b+n$ in (2.2.3) .

2.3 Particular Cases

$$(1) \int_1^\infty (1+y)^{h-1} (y-1)^a P_n^{(a,b)}(y) dy = \frac{2^{h+a} \Gamma(-a-h-n) \Gamma(1+a+n) (1+b-h)_n}{n! \Gamma(1-h)} , \quad n=0, 1, 2, \dots \quad (2.3.1)$$

where $\operatorname{Re} a > -1$, $\operatorname{Re} (h+a) < -n$, $\operatorname{Re} (h-b) < 1 + n$.

$$(2) \int_1^\infty (1+y)^{h-1} (y-1)^{-1/2} T_n(y) dy =$$

$$(-1)^n \sqrt{\frac{\pi}{2}} \frac{2^n \Gamma\left(\frac{1}{2} - h + n\right)}{\Gamma(1-h) \left(h + \frac{1}{2}\right)_n}, \quad n = 0, 1, 2, \dots \quad (2.3.2)$$

where $\operatorname{Re} h < \frac{1}{2} - n$, $\operatorname{Re} h < \frac{1}{2} + n$.

$$(3) \int_1^\infty (1+y)^{h-1} (y-1)^{v-\frac{1}{2}} C_n^v(y) dy =$$

$$\frac{2^{h+v-\frac{1}{2}} (2v)_n \Gamma\left(v + \frac{1}{2}\right) \Gamma\left(\frac{1}{2} - v - h - n\right) \left(\frac{1}{2} + v - h\right)_n}{n! \Gamma(1-h)}, \quad n = 0, 1, 2, \dots \quad (2.3.3)$$

where $\operatorname{Re} v > -\frac{1}{2}$, $\operatorname{Re}(h+v) < \frac{1}{2} - n$, $\operatorname{Re}(h-v) < \frac{1}{2} + n$.

$$(4) \int_{-1}^1 (1-y)^{h-1} (1+y)^{a+b-h+n} P_n^{(a,b)}(y) dy =$$

$$\frac{2^{a+b+n} (1+a-h)_n \Gamma(h) \Gamma(1+a+b-h+n)}{n! \Gamma(1+a+b+n)}, \quad n = 0, 1, 2, \dots \quad (2.3.4)$$

where $\operatorname{Re} h > 0$, $\operatorname{Re}(h-a-b) < 1+n$.

$$(5) \int_{-1}^1 (1-y)^{h-1} (1+y)^{-1-h+n} T_n(y) dy =$$

$$\frac{2^{n-1} \left(\frac{1}{2} - h\right)_n \Gamma(h) \Gamma(-h+n)}{(1/2)_n \Gamma(n)}, \quad n = 1, 2, \dots \quad (2.3.5)$$

where $\operatorname{Re} h > 0$, $\operatorname{Re} h < n$.

$$(6) \int_{-1}^1 (1-y)^{h-1} (1+y)^{2v-1-h+n} C_n^v(y) dy =$$

$$\frac{2^{2v-1+n} \Gamma(h) \left(v + \frac{1}{2} - h\right)_n \Gamma(2v-h+n)}{n! \Gamma(2v) \left(v + \frac{1}{2}\right)_n}, \quad n = 0, 1, 2, \dots \quad (2.3.6)$$

where $\operatorname{Re} h > 0$, $\operatorname{Re} (h - 2v) < n$.

$$(7) \int_1^\infty (y-1)^{h-1+\frac{m}{2}} (1+y)^{m/2} P_n^m(y) dy =$$

$$\frac{(-1)^{-m/2} 2^{h+m} (1-m-h)_n \Gamma(h) \Gamma(-m-h-n)}{\Gamma(1-m+n) \Gamma(-m-n)}, \quad n = 0, 1, 2, \dots \quad (2.3.7)$$

where $\operatorname{Re} h > 0$, $\operatorname{Re} h < -m-n$, $\operatorname{Re} h < 1 - m + n$.

$$(8) \int_{-1}^1 (1-y)^{h-1} (1+y)^b P_n^{(a,b)}(y) dy =$$

$$\frac{2^{h+b} \Gamma(h) \Gamma(1+b+n) \Gamma(1+a-h+n)}{n! \Gamma(1+b+h+n) \Gamma(1+a-h)}, \quad n = 0, 1, 2, \dots \quad (2.3.8)$$

where $\operatorname{Re} h > 0$, $\operatorname{Re} b > -1$.

$$(9) \int_{-1}^1 (1-y)^{h-1} (1+y)^{1/2} T_n(y) dy =$$

$$\sqrt{\frac{\pi}{2}} \frac{2^h \Gamma(h) \left(\frac{1}{2} - h\right)_n}{\Gamma\left(\frac{1}{2} + h + n\right)}, \quad n = 0, 1, 2, \dots \quad (2.3.9)$$

where $\operatorname{Re} h > 0$.

$$\begin{aligned}
(10) \quad & \int_{-1}^1 (1-y)^{h-1} (1+y)^{v-\frac{1}{2}} C_n^v(y) dy = \\
& \frac{2^{h+v-\frac{1}{2}} (2v)_n \Gamma(h) \Gamma(1/2+v) (v+\frac{1}{2}-h)_n}{n! \Gamma(\frac{1}{2}+v+h+n)} , \quad n=0, 1, 2, \dots
\end{aligned} \tag{2.3.10}$$

where $\operatorname{Re} h > 0$, $\operatorname{Re} v > -1/2$.

$$\begin{aligned}
(11) \quad & \int_{-1}^1 (1-y)^{h-1+\frac{m}{2}} (1+y)^{m/2} P_n^m(y) dy = \\
& \frac{2^{h+m} (1-m-h)_n \Gamma(h) \Gamma(1+m+n)}{\Gamma(1-m+n) \Gamma(1+m+h+n)} , \quad n=0, 1, 2, \dots
\end{aligned} \tag{2.3.11}$$

where $\operatorname{Re} h > 0$, $m > -1$.

$$\begin{aligned}
(12) \quad & \int_1^\infty (1+y)^{h-1} (y-1)^{a+b-h+n} P_n^{(a,b)}(y) dy = \\
& \frac{2^{a+b+n} \Gamma(-a-b) \Gamma(1+a+b-h+n) (1+b-h)_n}{n! \Gamma(1-h) (1+a+b)_n} , \quad n=0, 1, 2, \dots
\end{aligned} \tag{2.3.12}$$

where $\operatorname{Re} (h-a-b) < 1+n$, $\operatorname{Re} (a+b) < -2n$.

$$\begin{aligned}
(13) \quad & \int_1^\infty (1+y)^{h-1} (y-1)^{2v-1-h+n} C_n^v(y) dy = \\
& \frac{2^{2v-1+n} \Gamma(1-2v) \Gamma(2v-h+n) (v+\frac{1}{2}-h)_n}{n! \Gamma(1-h) (v+\frac{1}{2})_n} , \quad n=0, 1, 2, \dots
\end{aligned} \tag{2.3.13}$$

where $\operatorname{Re}(2\nu) < 1 - 2n$, $\operatorname{Re}(h - 2\nu) < n$.

$$(14) \int_{-1}^1 (1-y)^{h-1+\frac{m}{2}} (1+y)^{n-h-\frac{m}{2}} P_n^m(y) dy =$$

$$\frac{2^n \Gamma(h) \Gamma(1-h+n) (1-m-h)_n}{n! \Gamma(1-m+n)}, \quad n = 0, 1, 2, \dots \quad (2.3.14)$$

where $\operatorname{Re} h > 0$, $\operatorname{Re} h < 1 + n$.

Proof :

(1) In (2.2.1), replacing b by $1+a+b+n$, c by $1+b$, x by $\frac{1+y}{2}$ and using the relation (1.4.2), we obtain (2.3.1).

(2) In (2.3.1), using the relation [19, p.265]:

$$T_n(x) = \frac{n!}{(1/2)_n} P_n^{(-1/2, -1/2)}(x) \quad (2.3.15)$$

we obtain (2.3.2).

(3) In (2.3.1), using the relation [19, p.267]:

$$C_n^\nu(x) = \frac{(2\nu)_n}{\left(\nu + \frac{1}{2}\right)_n} P_n^{\left(\nu - \frac{1}{2}, \nu - \frac{1}{2}\right)}(x) \quad (2.3.16)$$

we obtain (2.3.3).

(4) In (2.2.2), replacing b by $-n-b$, c by $1+a$, x by $\frac{1-y}{1+y}$ and using the relation (1.4.4), we obtain (2.3.4).

(5) In (2.3.4), using the relation (2.3.15), we obtain (2.3.5).

(6) In (2.3.4), using the relation (2.3.16), we obtain (2.3.6).

(7) The integral (2.3.7), replacing b by $n+1$, c by $1-m$, x by $\frac{1-y}{2}$ and using the relation (1.4.8), we obtain (2.3.7).

(8) In (2.2.3), replacing b by $1+a+b+n$, c by $1+a$, x by $\frac{1-y}{2}$ and using the relation (1.4.1), we obtain (2.3.8).

(9) In (2.3.8), using the relation (2.3.15), we obtain (2.3.9).

(10) In (2.3.8), using the relation (2.3.16), we obtain (2.3.10).

(11) In (2.2.3), following the technique employed to derive (2.3.3), we obtain (2.3.11).

(12) In (2.2.4), following the technique employed to derive (2.3.1), we obtain (2.3.12).

(13) In (2.3.12), using the relation (2.3.16), we obtain (2.3.13).

(14) In (2.2.6), following the technique employed to derive (2.3.5), we obtain (2.3.14).

Note 2 :

It is interesting to note that (2.3.8), (2.3.9) and (2.3.10) can also be obtained as particular cases of (2.2.5), also (2.3.4), (2.3.6) and (2.3.7) can be obtained as particular cases of (2.2.6).

Note 3 :

It is evident that if we set $h+a$ for h and $b+n$ for b in (2.2.1), (2.2.2) and (2.2.3), we get (2.2.7), (2.2.8) and (2.2.9), respectively .

The formulae (2.2.7), (2.2.8) and (2.2.9), have been employed to establish some interesting results in the subsequent chapters .

Note 4 :

(1) In (2.2.1), (2.2.2) and (2.2.3), replacing c by $1-a-n$, we obtain the following results due to Bajpai [12, pp. 188-189, (1.7) , (1.9) , (1.2)] :

$$(i) \int_1^{\infty} x^{h-1} (x-1)^{a+b-1} {}_2F_1 \left(-n, b; 1-a-n; x \right) dx = \frac{(-1)^n (1-a-h-n)_n \Gamma(1-a-b-h-n) \Gamma(a+b+n)}{(1-a-n)_n \Gamma(1-h)}$$

$$, n = 0, 1, 2, \dots \quad (2.3.17)$$

where $\operatorname{Re}(a+b) > 0$, $\operatorname{Re}(a+h) < 1$, $\operatorname{Re}(a+b+h) < 1-n$.

$$(ii) \int_0^{\infty} x^{h-1} (1+x)^{a+b-1} {}_2F_1 \left(-n, b; 1-a-n; -x \right) dx = \frac{(1-a-h-n)_n \Gamma(h) \Gamma(1-a-b-h-n)}{(1-a-n)_n \Gamma(1-a-b-n)}$$

$$, n = 0, 1, 2, \dots \quad (2.3.18)$$

where $\operatorname{Re} h > 0$, $\operatorname{Re}(a+h) < 1$, $\operatorname{Re}(a+b+h) < 1-n$.

$$(iii) \int_0^1 x^{h-1} (1-x)^{a+b-1} {}_2F_1(-n, b; 1-a-n; x) dx = \frac{\Gamma(1-a-n) \Gamma(h) \Gamma(a+b+n) \Gamma(1-a-h)}{\Gamma(1-a) \Gamma(1-a-n-h) \Gamma(a+b+h+n)}$$

$$, n = 0, 1, 2, \dots \quad (2.3.19)$$

where $\operatorname{Re} h > 0$, $\operatorname{Re}(a+b) > 0$.

(2) In (2.2.1), (2.2.2) and (2.2.3), replacing b by $1-b+n$ and c by $1-a-n$, we obtain the following results due to Bajpai [10, pp. 154-155, (1.7), (1.9) & (1.2)]:

$$(i) \int_1^\infty x^{h-1} (x-1)^{a-b+n} {}_2F_1(-n, 1-b+n; 1-a-n; x) dx = \frac{(-1)^n \Gamma(1-a-h) \Gamma(1-a-n) \Gamma(b-a-h-2n) \Gamma(1+a-b+2n)}{\Gamma(1-h) \Gamma(1-a) \Gamma(1-a-n-h)}, \quad n = 0, 1, 2, \dots \quad (2.3.20)$$

where $\operatorname{Re}(a-b) > -1$, $\operatorname{Re}(a-b+h) < 0$, $\operatorname{Re}(a+h) < 1$.

$$(ii) \int_0^\infty x^{h-1} (1+x)^{a-b+n} {}_2F_1(-n, 1-b+n; 1-a-n; -x) dx = \frac{\Gamma(h) \Gamma(1-a-h) \Gamma(1-a-n) \Gamma(b-a-h-2n)}{\Gamma(b-a-2n) \Gamma(1-a) \Gamma(1-a-n-h)}, \quad n = 0, 1, 2, \dots \quad (2.3.21)$$

where $\operatorname{Re} h > 0$, $\operatorname{Re}(a-b+h) < 0$, $\operatorname{Re}(a+h) < 1$.

$$(iii) \int_0^1 x^{h-1} (1-x)^{a-b+n} {}_2F_1(-n, 1-b+n; 1-a-n; x) dx = \frac{\Gamma(1-a-n) \Gamma(h) \Gamma(a-b+2n+1) \Gamma(1-a-h)}{\Gamma(1-a) \Gamma(1-a-h-n) \Gamma(a-b+h+2n+1)}, \quad n = 0, 1, 2, \dots \quad (2.3.22)$$

where $\operatorname{Re} h > 0$, $\operatorname{Re}(a-b) > -1$.

(3) In (2.2.2), replacing b by $1+a+b+n$, c by $1+a$, h by $w+1$, x by $\frac{y-1}{2}$ and using the relation (1.4.1), we obtain the following result due to Bajpai [5, p. 223, (2)]:

$$\int_1^\infty (y-1)^w (y+1)^b P_n^{(a,b)}(y) dy = \frac{2^{b+w+1} (-1)^n \Gamma(1+b+n) \Gamma(w+1) \Gamma(a-w+n)}{n! \Gamma(-b) \Gamma(1+b)} \frac{\Gamma(-1-b-w-n)}{\Gamma(a-w)}, \quad (2.3.23)$$

where $\text{Re } w > -1$, $\text{Re } (w+b) < -1-n$, $\text{Re } (a-w) > -n$.

Note 5 :

- (1) Integral (2.3.5) is a known result [19, p.271, (1)] with $h = 3/2$ and $-1 - h + n = d$.
- (2) Integral (2.3.6) is a known result [1, p. 280, (3)] with $h = v + 1/2$ and $2v - 1 - h + n = \beta$.
- (3) Integral (2.3.9) is a known result [19, p.271, (2)] with $\beta = 1/2$, $\alpha = h - 1$.
- (4) Integral (2.3.10) is a known result [19, p. 280, (3)] with $h = v + 1/2$ and $\beta = v - 1/2$.

CHAPTER III

ORTHOGONAL PROPERTIES OF GAUSS' HYPERGEOMETRIC POLYNOMIALS

3.1 Introduction

In this chapter, we present three orthogonal properties of Gauss' hypergeometric polynomials and employ them to obtain the orthogonal properties of Jacobi polynomials, Legendre functions and some other polynomials.

In this chapter, we also present a direct method of proof to establish the orthogonalities of the orthogonal polynomials. Our method of proof is much simpler and elegant than the traditional methods for establishing the orthogonalities of the orthogonal polynomials.

These orthogonality properties of Gauss' hypergeometric polynomials will be used to establish some expansion formulae in a subsequent chapters.

3.2 Orthogonal Properties

The orthogonal properties to be established are:

$$\begin{aligned}
 (i) \quad & \int_0^1 x^{c-1} (1-x)^{b-c} {}_2F_1(-m, b+m; c; x) {}_2F_1(-n, b+n; c; x) dx \\
 &= \frac{n! \Gamma(c) \Gamma(1-c+b+n)}{(c)_n (b+2n) \Gamma(b+n)} \delta_{m,n}
 \end{aligned} \tag{3.2.1}$$

where $\text{Re}(c) > 0$, $\text{Re}(b-c) > -1$.

$$(ii) \int_0^{\infty} x^{c-1} (1+x)^{b-c} {}_2F_1(-m, b+m; c; -x) {}_2F_1(-n, b+n; c; -x) dx \quad (3.2.2)$$

$$= \frac{n! \Gamma(c) \Gamma(1-c+b+n) \Gamma(1+b) \Gamma(-b)}{(c)_n (b+2n) \Gamma(b+n) \Gamma(1-c+b) \Gamma(c-b)} \delta_{m,n} \quad (3.2.2a)$$

$$= \frac{n! \Gamma(c) \Gamma(1-c+b+n)}{(c)_n (b+2n) \Gamma(b+n)} \frac{\sin \pi (c-b)}{\sin \pi (1+b)} \delta_{m,n} \quad (3.2.2b)$$

where $\text{Re}(c) > 0$, $(\text{Re } b < -m - n, \text{Re } b > -m - n)$, provided b is non-integral.

$$(iii) \int_1^{\infty} x^{c-1} (x-1)^{b-c} {}_2F_1(-m, b+m; c; x) {}_2F_1(-n, b+n; c; x) dx \quad (3.2.3)$$

$$= \frac{n! \Gamma(1-c+b+n) \Gamma(1+b) \Gamma(-b)}{(c)_n (b+2n) \Gamma(b+n) \Gamma(1-c)} \delta_{m,n} \quad (3.2.3a)$$

$$= \frac{n! \Gamma(1-c+b+n)}{(c)_n (b+2n) \Gamma(b+n) \Gamma(1-c)} \frac{\pi}{\sin \pi (1+b)} \delta_{m,n} \quad (3.2.3b)$$

where $\text{Re}(b-c) > -1$, $(\text{Re } b < -m - n, \text{Re } b > -m - n)$, provided b is non integral.

Proof:

(i) In (3.2.1), expressing the two hypergeometric polynomials in the integrand as their series representation (1.3.1), interchanging the order of integration and summation, which is justified due to the convergence of the integral and series involved in the process, the left hand side of (3.2.1) can be written as:

$$\sum_{r=0}^m \frac{(-m)_r (b+m)_r}{(c)_r r!} \sum_{u=0}^n \frac{(-n)_u (b+n)_u}{(c)_u u!} \int_0^1 x^{c+r+u-1} (1-x)^{b-c} dx \quad (3.2.4)$$

Now, evaluating the resulting integral with the help of the integral (2.2.15) and simplifying, we have

$$= \Gamma(1-c+b) \sum_{r=0}^m \frac{(-m)_r (b+m)_r}{(c)_r r!} \frac{\Gamma(c+r)}{\Gamma(1+b+r)} {}_3F_2(-n, b+n, c+r; c, 1+b+r; 1) \quad (3.2.5)$$

It can easily be verified that the generalized hypergeometric series (3.2.5) is Saalschutzian. Therefore, applying the Saalschutz's theorem (2.2.13) and simplifying using the formula (1.5.4) in (3.2.5), we get

$$= \frac{\Gamma(c) \Gamma(c-b) \Gamma(1-c+b) \Gamma(-b-n)}{(c)_n \Gamma(c-b-n) \Gamma(-b) \Gamma(1+b)} \sum_{r=0}^m \frac{(-m)_r (-r)_n (b+m)_r}{(1+b+n)_r r!} \quad (3.2.6)$$

If $r < n$, the numerator of (3.2.6) vanishes, and since r runs from 0 to m , it follows that (3.2.6) also vanishes, when $m < n$. By symmetry, it is evident that (3.2.6) also vanishes when $n < m$. Now, it is clear that for $m \neq n$ all terms of (3.2.6) vanish.

When $m = n$, using the standard result [1, p. 359, (9.3)]:

$$(-r)_n = \begin{cases} \frac{(-1)^n r!}{(r-n)!} & , \text{ if } 0 \leq n \leq r ; \\ 0 & , \text{ if } n > r \end{cases} \quad (3.2.7)$$

and simplifying (3.2.6), we obtain (3.2.1).

(ii) The result (3.2.2a) is established on following the technique employed to establish (3.2.1) except instead of using the integral (2.2.15), we use (2.2.14).

On applying the relation [17, p. 3, (6)]:

$$\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad (3.2.8)$$

to (3.2.2a), we obtain (3.2.2b).

(iii) The results (3.2.3a) and (3.2.3b) are established on following the technique employed to establish (3.2.2a) and (3.2.3b) except instead of using the integral (2.2.14), we use the integral (2.2.11).

3.3 Another Proof of the Orthogonal Properties of Gauss' Hypergeometric Polynomials

(i) In the integral (3.2.1) expressing one hypergeometric polynomials in the integrand as its series representation (1.3.1), interchanging the order of integration and summation, which is justified due to the convergence of the integral and series involved in the process, the left hand side of (3.2.1) can be written as:

$$= \sum_{r=0}^m \frac{(-m)_r (b+m)_r}{(c)_r r!} \int_0^1 x^{c+r-1} (1-x)^{b-c} {}_2F_1(-n, b+n; c; x) dx \quad (3.3.1)$$

Now, evaluating the resulting integral with the help of the integral (2.2.9) by replacing $h+a$ by $c+r$ and simplifying, we get

$$= \frac{\Gamma(c)}{(c)_n} \frac{\Gamma(1-c+b+n)}{\Gamma(1+b+n)} \sum_{r=0}^m \frac{(-m)_r (-r)_n (b+m)_r}{(1+b+r)_n r!} \quad (3.3.2)$$

If $r < n$, the numerator of (3.3.2) vanishes, and since r runs from 0 to m , it follows that (3.3.2) also vanishes, when $m < n$. By symmetry, it is evident that (3.3.2) also vanishes when $n < m$. Now, it is clear that for $m \neq n$ all terms of (3.3.2) vanish.

When $m = n$, using the standard result (3.2.7) and simplifying we obtain (3.2.1).

(ii) The result (3.2.2a) is established on following the above technique employed to establish (3.2.1) except instead of (2.2.9), using the integral (2.2.8). On applying the relation (3.2.8), we obtain (3.2.2b).

(iii) The formulae (3.2.3a) and (3.2.3b) are established on following the technique employed to establish (3.2.2a) and (3.2.2b) except instead of (2.2.8), using the integral (2.2.7).

3.4 Particular Cases

(i) In (3.2.1), replacing b by $1+a+b$, c by $1+a$ and x by $\frac{1-x}{2}$, we have

$$\int_{-1}^1 (1-x)^a (1+x)^b {}_2F_1\left(-m, 1+a+b+m; 1+a; \frac{1-x}{2}\right) {}_2F_1\left(-n, 1+a+b+n; 1+a; \frac{1-x}{2}\right) dx$$

$$= \frac{2^{a+b+1} n! \Gamma(1+a) \Gamma(1+b+n)}{(1+a)_n (1+a+b+2n) \Gamma(1+a+b+n)} \delta_{m,n} \quad (3.4.1)$$

Using the relation (1.4.1), we obtain the orthogonal property of Jacobi polynomials [19, p. 285, (5) and (9)].

(ii) In (3.2.1), replacing m by k , b by 1 , c by $1-m$ and x by $\frac{1-x}{2}$, we have

$$\begin{aligned} & \int_{-1}^1 (1-x)^{-m} (1+x)^m {}_2F_1\left(-k, k+1; 1-m; \frac{1-x}{2}\right) {}_2F_1\left(-n, 1+n; 1-m; \frac{1-x}{2}\right) dx \\ &= \frac{2}{(1+2n)} \frac{\Gamma(1-m) \Gamma(1+m+n)}{(1-m)_n} \delta_{k,n} \end{aligned} \quad (3.4.2)$$

Using the relation (1.4.8), we obtain the orthogonal property of Legendre functions [19, p. 279, (27) & (28)].

(iii) In (3.2.3), replacing b by $1+a+b$, c by $1+b$ and x by $\frac{1+x}{2}$, we have

$$\begin{aligned} & \int_{-1}^1 (x-1)^a (1+x)^b {}_2F_1\left(-m, 1+a+b+m; 1+b; \frac{1+x}{2}\right) {}_2F_1\left(-n, 1+a+b+n; 1+b; \frac{1+x}{2}\right) dx \\ &= \frac{2^{a+b+1} n! \Gamma(1+a+n) \Gamma(2+a+b) \Gamma(-1-a-b)}{(1+b)_n (1+a+b+2n) \Gamma(1+a+b+n) \Gamma(-b)} \end{aligned} \quad (3.4.3)$$

Using the relation (1.4.2) and the relation (3.2.7), we obtain the result due to Bajpai [5, pp. 224, (7) & (8)].

(iv) In (3.4.1), putting $a = -1/2$, $b = -1/2$ and using the relation (2.3.15), we obtain the orthogonal property of Tcheibyshev polynomials [19, p. 272, (8)].

(v) In (3.4.1), replacing a by $\nu - 1/2$, b by $\nu - 1/2$ and using the relation (2.3.16), we obtain the orthogonal property of Gegenbauer polynomials [19, p. 281, (8)].

(vi) In (3.4.1), putting $a = 0$, $b = 0$ and using the relation [19, p. 266]:

$$P_n(x) = P_n^{(0,0)}(x) \quad (3.4.4)$$

We obtain the orthogonal property of the Legendre polynomials [19, p. 277, (13) & (14)].

(vii) In (3.4.3), putting $a = \nu - 1/2$, $b = \nu - 1/2$ and using the relation (1.4.6), we obtain the orthogonal property for the Gegenbauer polynomials:

$$\int_1^\infty (x^2 - 1)^{\nu-1/2} C_m^\nu(x) C_n^\nu(x) dx \quad (3.4.5)$$

$$= \frac{2^{2\nu-1} (2\nu)_n \Gamma\left(\frac{1}{2} + \nu\right) \Gamma(-2\nu) \Gamma(2\nu + 1)}{n! (\nu + n) \Gamma(2\nu) \Gamma(1/2 - \nu)} \delta_{m,n} \quad (3.4.5a)$$

$$= \frac{2^{2\nu-1} (2\nu)_n \Gamma(1/2 + \nu)}{n! (\nu + n) \Gamma(2\nu) \Gamma(1/2 - \nu)} \frac{\pi}{\sin \pi (2\nu + 1)} \delta_{m,n} \quad (3.4.5b)$$

where $\operatorname{Re}(\nu) > -1/2$, $(\operatorname{Re}(2\nu) < -m-n, \operatorname{Re}(2\nu) > -m-n) \Rightarrow \operatorname{Re}(2\nu) \neq -2n$ when $m=n$.

Note:

It is interesting to note that the orthogonal property of Jacobi polynomials due to Bajpai [5, p. 224, (7) & (8)] can also be obtained if we replace b by $1+a+b$, c by $1+a$ and x by $\frac{x-1}{2}$ in (3.2.2).

CHAPTER IV

SEMI-ORTHOGONAL PROPERTIES FOR GAUSS' HYPERGEOMETRIC POLYNOMIALS

4.1 Introduction

In this chapter, we present three semi-orthogonal properties for Gauss' hypergeometric polynomials and employ them to derive orthogonal properties for the Jacobi polynomials and the Legendre functions. We have also derived results recently given by Bajpai [13, p. 63, (2.1)].

4.2 Semi-Orthogonal Properties

The semi-orthogonal properties to be established are:

$$(i) \quad \int_0^1 x^{c-1} (1-x)^{b-c-n} {}_2F_1(-m, b; c; x) {}_2F_1(-n, b; c; x) dx [\quad (4.2.1)$$

$$= 0 \quad , \text{ if } m < n \quad (4.2.1a)$$

$$= \frac{(b)_n \, n! \, \Gamma(c) \, \Gamma(1+b-c)}{(c)_n \, \Gamma(1+b+n)} \quad , \text{ if } m = n \quad (4.2.1b)$$

$$= \frac{(b)_n \, (n+1)! \, \Gamma(c) \, \Gamma(1+b-c)}{(c)_n \, \Gamma(2+b+n)} \quad , \text{ if } m = n + 1 \quad (4.2.1c)$$

where $\text{Re } c > 0, \text{Re } (b - c) > -1$.

$$(ii) \int_0^\infty x^{c-1} (1+x)^{b-c-n} {}_2F_1(-m, b; c; -x) {}_2F_1(-n, b; c; -x) dx \quad (4.2.2)$$

$$= 0 \quad , \text{ if } m < n \quad (4.2.2a)$$

$$= \frac{(b)_n n! \Gamma(c) \Gamma(-b) \Gamma(1+b)}{(c)_n \Gamma(1+b+n) \Gamma(c-b)} \quad (4.2.2b)$$

, if $m = n$

$$= \frac{(b)_n n! \Gamma(c)}{(c)_n \Gamma(1+b+n) \Gamma(c-b)} \frac{\pi}{\sin \pi (1+b)} \quad (4.2.2c)$$

$$= - \frac{(b)_n (n+1)! \Gamma(c) \Gamma(-1-b) \Gamma(2+b)}{(c)_n \Gamma(2+b+n) \Gamma(c-b)} \quad (4.2.2d)$$

, if $m = n + 1$

$$= - \frac{(b)_n (n+1)! \Gamma(c)}{(c)_n \Gamma(2+b+n) \Gamma(c-b)} \frac{\pi}{\sin \pi (2+b)} \quad (4.2.2e)$$

where $\text{Re } c > 0$, ($\text{Re } b < -m$, $\text{Re } b > -n$), provided b is nonintegral.

$$(iii) \int_1^\infty x^{c-1} (x-1)^{b-c-n} {}_2F_1(-m, b; c; x) {}_2F_1(-n, b; c; x) dx \quad (4.2.3)$$

$$= 0 \quad , \text{ if } m < n \quad (4.2.3a)$$

$$= (-1)^n \frac{(b)_n n! \Gamma(1+b-c) \Gamma(-b) \Gamma(1+b)}{(c)_n \Gamma(1+b+n) \Gamma(1-c)} \quad (4.2.3b)$$

, if $m = n$

$$= (-1)^n \frac{(b)_n n! \Gamma(1+b-c)}{(c)_n \Gamma(1+b+n) \Gamma(1-c)} \frac{\pi}{\sin \pi (1+b)} \quad (4.2.3c)$$

$$= (-1)^{n+1} \frac{(b)_n (n+1)! \Gamma(1+b-c) \Gamma(-1-b) \Gamma(2+b)}{(c)_n \Gamma(2+b+n) \Gamma(1-c)} \quad (4.2.3d)$$

, if $m = n + 1$

$$= (-1)^{n+1} \frac{(b)_n (n+1)! \Gamma(1+b-c)}{(c)_n \Gamma(2+b+n) \Gamma(1-c)} \frac{\pi}{\sin \pi (2+b)} \quad (4.2.3e)$$

where $\text{Re}(b-c) > n-1$, ($\text{Re } b < -m$, $\text{Re } b > -n$), provided b is nonintegral.

Proof:

(i) To prove (4.2.1), we write its left hand side in the form:

$$\sum_{r=0}^m \frac{(-m)_r (b)_r}{(c)_r r!} \int_0^1 x^{c+r-1} (1-x)^{b-c-n} {}_2F_1(-n, b; c; x) dx \quad (4.2.4)$$

On evaluating the integral in (4.2.4) with the help of (2.2.3), it reduces to the form

$$\sum_{r=0}^m \frac{(-m)_r (b)_r}{(c)_r r!} \frac{(-r)_n \Gamma(c+r) \Gamma(1-c+b)}{(c)_n \Gamma(1+b+r)} \quad (4.2.5)$$

If $r < n$, the numerator of (4.2.5) vanishes, and since r runs from 0 to m , it follows that (4.2.5) also vanishes, when $m < n$. Now, it is clear that for $m < n$ all terms of (4.2.5) vanish, which proves (4.2.1a).

When $m = n$, using the standard result (3.2.7) and simplifying, the right hand side of (4.2.1b) follows from (4.2.5).

In (4.2.5), putting $m = n + 1$, using (3.2.7) and adding the resulting two terms ($r = n$, $n+1$), we obtain (4.2.1c).

(ii) To prove (4.2.2), first we reduce its left hand side to the form similar to (4.2.4) then evaluate the last integral with the help of the integral (2.2.2) to obtain

$$\sum_{r=0}^m \frac{(-m)_r (b)_r}{(c)_r r!} \frac{(-1)^r (-r)_n \Gamma(c+r) \Gamma(-b-r)}{(c)_n \Gamma(c-b)} \quad (4.2.6)$$

We see that (4.2.6) is of the same form as (4.2.5). Therefore, for $m < n$ all terms of (4.2.6) vanish, which proves (4.2.2a).

When $m = n$, using the standard result (3.2.7) and simplifying, the right hand side of (4.2.2b) follows from (4.2.6).

On applying the relation (3.2.8) to (4.2.2b), we obtain (4.2.2c).

In (4.2.6), putting $m = n+1$, using (3.2.7) and adding the resulting two terms ($r = n, n+1$), we obtain (4.2.2d).

On applying the relation (3.2.8) to (4.2.2d), we obtain (4.2.2e).

(iii) To prove (4.2.3), first we reduce its left hand side to the form similar to (4.2.4) then evaluate the last integral with the help of the integral (2.2.1) to obtain

$$\sum_{r=0}^m \frac{(-m)_r (b)_r}{(c)_r r!} \frac{(-1)^n (-r)_n \Gamma(-b-r) \Gamma(1-c+b)}{(c)_n \Gamma(1-c-r)} \quad (4.2.7)$$

Since (4.2.7) is of the same form as (4.2.5). Therefore, for $m < n$ all terms of (4.2.7) vanish, which proves (4.2.3a).

When $m = n$, using the standard result (3.2.7) and simplifying, the result (4.2.3b) follows from (4.2.7).

On applying the relation (3.2.8) to (4.2.3b), we obtain (4.2.3c).

In (4.2.7), putting $m = n + 1$, using (3.2.7) and adding the resulting two terms ($r = n, n+1$), we obtain (4.2.3d).

On applying the relation (3.2.8) to (4.2.3d), we obtain (4.2.3e).

Note:

On continuing as above, we can find the values of the integrals (4.2.1), (4.2.2) and (4.2.3) for $m = n+2, n+3, n+4, \dots$

4.3 Particular Cases

(i) In (4.2.1), replacing x by $\frac{1-x}{2}$, we obtain

$$\int_{-1}^1 (1-x)^{c-1} (1+x)^{b-c-n} {}_2F_1\left(-m, b; c; \frac{1-x}{2}\right) {}_2F_1\left(-n, b; c; \frac{1-x}{2}\right) dx \quad (4.3.1)$$

$$= 0 \quad , \text{ if } m < n \quad (4.3.1a)$$

$$= \frac{2^{b-n} (b)_n n! \Gamma(c) \Gamma(1+b-c)}{(c)_n \Gamma(1+b+n)} \quad , \text{ if } m = n \quad (4.3.1b)$$

$$= \frac{2^{b-n} (b)_n (n+1)! \Gamma(c) \Gamma(1+b-c)}{(c)_n \Gamma(2+b+n)} \quad , \text{ if } m = n+1 \quad (4.3.1c)$$

valid under the conditions of (4.2.1).

(ii) In (4.2.2), replacing x by $\frac{x-1}{2}$, we obtain

$$\int_1^\infty (x-1)^{c-1} (x+1)^{b-c-n} {}_2F_1\left(-m, b; c; \frac{1-x}{2}\right) {}_2F_1\left(-n, b; c; \frac{1-x}{2}\right) dx \quad (4.3.2)$$

$$= 0 \quad , \text{ if } m < n \quad (4.3.2a)$$

$$= \frac{2^{b-n} (b)_n n! \Gamma(c) \Gamma(-b) \Gamma(1+b)}{(c)_n \Gamma(1+b+n) \Gamma(c-b)} \quad (4.3.2b)$$

, if $m = n$

$$= \frac{2^{b-n} (b)_n n! \Gamma(c)}{(c)_n \Gamma(1+b+n) \Gamma(c-b)} \frac{\pi}{\sin\pi(1+b)} \quad (4.3.2c)$$

$$= \frac{-2^{b-n} (b)_n (n+1)! \Gamma(c) \Gamma(-1-b) \Gamma(2+b)}{(c)_n \Gamma(2+b+n) \Gamma(c-b)} \quad (4.3.2d)$$

, if $m = n+1$

$$= \frac{-2^{b-n} (b)_n (n+1)! \Gamma(c)}{(c)_n \Gamma(2+b+n) \Gamma(c-b)} \frac{\pi}{\sin\pi(2+b)} \quad (4.3.2e)$$

valid under the conditions of (4.2.2).

(iii) In (4.2.3), replacing x by $\frac{x+1}{2}$, we get

$$\int_1^\infty (x+1)^{c-1} (x-1)^{b-c-n} {}_2F_1\left(-m, b; c; \frac{1+x}{2}\right) {}_2F_1\left(-n, b; c; \frac{1+x}{2}\right) dx \quad (4.3.3)$$

$$= 0 \quad , \text{ if } m < n \quad (4.3.3a)$$

$$= \frac{2^{b-n} (-1)^n (b)_n n! \Gamma(1-c+b) \Gamma(-b) \Gamma(1+b)}{(c)_n \Gamma(1+b+n) \Gamma(1-c)} \quad (4.3.3b)$$

, if $m = n$

$$= \frac{2^{b-n} (-1)^n (b)_n n! \Gamma(1-c+b)}{(c)_n \Gamma(1+b+n) \Gamma(1-c)} \frac{\pi}{\sin \pi (1+b)} \quad (4.3.3c)$$

$$= \frac{2^{b-n} (-1)^{n+1} (b)_n (n+1)! \Gamma(1-c+b) \Gamma(-1-b) \Gamma(2+b)}{(c)_n \Gamma(2+b+n) \Gamma(1-c)} \quad (4.3.3d)$$

, if $m = n + 1$

$$= \frac{2^{b-n} (-1)^{n+1} (b)_n (n+1)! \Gamma(1-c+b)}{(c)_n \Gamma(2+b+n) \Gamma(1-c)} \frac{\pi}{\sin \pi (2+b)} \quad (4.3.3e)$$

valid under the conditions of (4.2.3).

(iv) In (4.3.1), setting $1 + a + b + n$ for b , $1 + a$ for c and using the relation (1.4.1), we obtain the orthogonality property of the Jacobi polynomials [19, p. 285, (5) & (9)].

(v) In (4.3.1), putting $b = n+2v$ and $c = v + \frac{1}{2}$, and using the relation (1.4.6), we obtain the orthogonality property of the Gegenbauer polynomials [19, pp.281-282, (8) & (12)].

(vi) In (4.3.1), setting $b=n$, $c = \frac{1}{2}$ and using the relation (1.4.5), we obtain the orthogonality property of the Chebyshev polynomials of the first kind [19, p. 272, (8) & (9)].

(vii) In (4.3.1), setting $b= n + 1$, $c = 1$ and using the relation (1.4.7), we obtain the orthogonality property of the Legendre polynomials [19, p. 277, (13) & (14)].

(viii) In (4.3.1), putting $m = n$, then setting $b = n + 1$, $c = 1 - m$ and using the relation (1.4.8), it reduces to the orthogonality property of the Legendre functions [19, p. 279, (27)].

(ix) In (4.3.2), replacing $1 + a + b + n$ for b , $1 + a$ for c and using the relation (1.4.1), corresponding to (4.3.2a) and (4.3.2b), we obtain the orthogonality property of the Jacobi polynomials due to Bajpai [5, p. 224, (7) & (8)].

(x) In (4.3.2), replacing $n + 2v$ for b , $v + \frac{1}{2}$ for c and using the relation (1.4.6), corresponding to (4.3.2a) and (4.3.2b), we obtain the orthogonality property of the Gegenbauer polynomials (3.4.5).

(xi) In (4.3.3), replacing $1 + a + b + n$ for b and $1 + b$ for c and using the relation (1.4.2), corresponding to (4.3.3a) and (4.3.3b), we obtain the orthogonality property of the Jacobi polynomials due to Bajpai [5, p. 224, (7) & (8)].

(xii) In (4.3.3), putting $b = n + 2v$, $c = v + \frac{1}{2}$ and using the relation [19, p. 267]:

$$C_n^v(x) = (-1)^n \frac{(2v)_n}{n!} {}_2F_1\left(-n, n + 2v; v + \frac{1}{2}; \frac{1+x}{2}\right) = (-1)^n C_n^v(-x) \quad (4.3.4)$$

corresponding to (4.3.3a) and (4.3.3b), we get the orthogonality property of the Gegenbauer polynomials (3.4.5).

(xiii) In (4.2.2), replacing $-n - b$ for b , $1 + a$ for c , $\frac{1-x}{1+x}$ for x and using the relation (1.4.3), we obtain the following result recently given by Bajpai [13, p. 63, (2.1), (2.1a) & (2.1b)].

$$\int_{-1}^1 (1-x)^a (1+x)^{b+n-m-1} P_m^{(a,b)}(x) P_n^{(a,b)}(x) dx \quad (4.3.5)$$

$$= 0 \quad , \text{ if } m < n \quad (4.3.5a)$$

$$= \frac{2^{a+b}}{n!} \frac{\Gamma(1+a+n)}{b} \frac{\Gamma(1+b+n)}{\Gamma(1+a+b+n)} \quad , \text{ if } m = n \quad (4.3.5b)$$

Note:

Some other particular cases of our results may also be derived.

CHAPTER V

ORTHOGONAL SYMMETRIES OF THE JACOBI POLYNOMIALS AND LEGENDRE FUNCTIONS

5.1 Introduction

In this chapter, we establish an orthogonal symmetry for the Jacobi polynomials over the interval $[-1, 1]$ with respect to the multiplier function $(1-x)^{a+\vartheta} (1+x)^b$, if $\text{Re } a > -\vartheta - 1$ and $\text{Re } b > -1$. Also, we present an orthogonal symmetry for the Legendre functions over the interval $[-1, 1]$ with respect to the multiplier function $(1-x)^\vartheta$.

From our orthogonal symmetries, we derive orthogonal properties of the Jacobi polynomials and the Legendre functions, respectively. Some other particular cases of our orthogonal symmetries have been also obtained, which appears to be new.

5.2 Generalized Orthogonal Symmetry of the Jacobi Polynomials

The generalized orthogonal symmetry is

$$\int_{-1}^1 (1-x)^{a+\vartheta} (1+x)^b P_m^{(a,b)}(x) P_n^{(a,b)}(x) dx \quad (5.2.1)$$

$$= 0, \text{ if } n + \vartheta < m < n - \vartheta \quad (5.2.1a)$$

$$= \frac{(-1)^{-\vartheta} 2^{a+b+\vartheta+1} \Gamma(1+a+n) \Gamma(1+b+n) \Gamma(1+a+b-2\vartheta+2n)}{(n-\vartheta)! \Gamma(1+a+b-\vartheta+n) \Gamma(2+a+b+2n)}, \text{ if } m = n - \vartheta \quad (5.2.1b)$$

$$= \frac{(-1)^{-\vartheta} 2^{a+b+\vartheta+1} \Gamma(1+a+n) \Gamma(1+b+n) \Gamma(2+a+b-2\vartheta+2n)}{(n-\vartheta+1)! \Gamma(2+a+b-\vartheta+n) \Gamma(3+a+b+2n)} \times$$

$$\left[(n-\vartheta+1)(1+a-\vartheta+n)(2+a+b+2n) - (n+1)(1+a+n)(2+a+b-2\vartheta+2n) \right], \text{ if } m=n-\vartheta+1 \quad (5.2.1c)$$

$$= \frac{(-1)^{-\vartheta} 2^{a+b+\vartheta+1} \Gamma(a+\vartheta+n) \Gamma(b+\vartheta+n) \Gamma(a+b+2n)}{n! \Gamma(1+a+b+n) \Gamma(1+a+b+2\vartheta+2n)} \times$$

$$\left[n(a+n)(a+b+2\vartheta+2n) - (n+\vartheta)(a+\vartheta+n)(a+b+2n) \right], \text{ if } m=n+\vartheta-1 \quad (5.2.1d)$$

$$= \frac{(-1)^{-\vartheta} 2^{a+b+\vartheta+1} \Gamma(1+a+\vartheta+n) \Gamma(1+b+\vartheta+n) \Gamma(a+b+2n+1)}{n! \Gamma(a+b+1+n) \Gamma(2+a+b+2\vartheta+2n)}, \text{ if } m=n+\vartheta \quad (5.2.1e)$$

where $\text{Re}(a) > -1-\vartheta$, $\text{Re}(b) > -1$.

Proof:

In view of (1.4.1), the integral (5.2.1) can be written as

$$\begin{aligned} & \int_{-1}^1 (1-x)^{a+\vartheta} (1+x)^b P_m^{(a,b)}(x) P_n^{(a,b)}(x) dx \\ &= \sum_{r=0}^m \frac{(a+1)_m (-m)_r (1+a+b+m)_r}{m! (a+1)_r 2^r r!} \int_{-1}^1 (1-x)^{a+\vartheta+r} (1+x)^b P_n^{(a,b)}(x) dx \end{aligned} \quad (5.2.2)$$

On applying [19, p.284, (2)] :

$$\int_{-1}^1 (1-x)^\rho (1+x)^\beta P_n^{(\alpha,\beta)}(x) dx = \frac{2^{\beta+\rho+1} \Gamma(\rho+1) \Gamma(\beta+n+1) \Gamma(\alpha-\rho+n)}{n! \Gamma(\alpha-\rho) \Gamma(\beta+\rho+n+2)}$$

where $\text{Re } \rho > 0$, $\text{Re } \beta > 0$.

and simplifying (5.2.2), we get

and simplifying (5.2.2) , we get

$$= \frac{2^{a+b+\vartheta+1} (a+1)_m \Gamma(1+b+n) \Gamma(1+a+\vartheta)}{m! n! \Gamma(2+a+b+\vartheta+n)} \sum_{r=0}^m \frac{(-m)_r (1+a+\vartheta)_r (1+a+b+m)_r (-r-\vartheta)_n}{(1+a)_r (2+a+b+\vartheta+n)_r r!} \quad (5.2.3)$$

In view of the standard result (3.2.7), if $r < n - \vartheta$, the numerator of (5.2.3) vanishes, and since r runs from 0 to m , it follows that (5.2.3) also vanishes, when $m < n - \vartheta$. By Symmetry, it is evident that (5.2.3) also vanishes when $m > n + \vartheta$. Now, it is clear that for $n - \vartheta > m > n + \vartheta$ all terms of (5.2.3) vanish, which proves (5.2.1a).

In (5.2.3), setting $m = n - \vartheta$, using (3.2.7) and simplifying, we get

$$\begin{aligned} & \int_{-1}^1 (1-x)^{a+\vartheta} (1+x)^b P_{n-\vartheta}^{(a,b)}(x) P_n^{(a,b)}(x) dx \\ &= \frac{(-1)^{-\vartheta} 2^{a+b+\vartheta+1} \Gamma(1+a+n) \Gamma(1+b+n) \Gamma(1+a+b-2\vartheta+2n)}{(n-\vartheta)! \Gamma(1+a+b-\vartheta+n) \Gamma(2+a+b+2n)} \end{aligned} \quad (5.2.4)$$

which proves (5.2.1b).

In (5.2.3) , putting $m = n - \vartheta + 1$, using (3.2.7) and adding the resulting two terms ($r = n - \vartheta$, $n - \vartheta + 1$), we get

$$\begin{aligned} & \int_{-1}^1 (1-x)^{a+\vartheta} (1+x)^b P_{n-\vartheta+1}^{(a,b)}(x) P_n^{(a,b)}(x) dx \\ &= (-1)^{-\vartheta} 2^{a+b+\vartheta+1} \frac{\Gamma(1+a+n) \Gamma(1+b+n) \Gamma(2+a+b-2\vartheta+2n)}{(n-\vartheta+1)! \Gamma(2+a+b-\vartheta+n) \Gamma(3+a+b+2n)} \times \end{aligned}$$

$$\left[(n-\vartheta+1) (1+a-\vartheta+n) (2+a+b+2n) - (n+1) (1+a+n) (2+a+b-2\vartheta+2n) \right] \quad (5.2.5)$$

which proves (5.2.1c).

Note:

On continuing as above, we can find the values of the integral (5.2.1) for $m = n-\vartheta+2$, $n-\vartheta+3, \dots, n, \dots, n+\vartheta-3, n+\vartheta-2$.

In (5.2.5), substituting $n+\vartheta-1$ for n , we obtain

$$\begin{aligned} & \int_{-1}^1 (1-x)^{a+\vartheta} (1+x)^b P_{n+\vartheta-1}^{(a,b)}(x) P_n^{(a,b)}(x) dx \\ &= (-1)^{-\vartheta} 2^{a+b+\vartheta+1} \frac{\Gamma(a+\vartheta+n) \Gamma(b+\vartheta+n) \Gamma(a+b+n)}{n! \Gamma(1+a+b+n) \Gamma(1+a+b+2\vartheta+2n)} \times \\ & \left[n(a+n)(a+b+2\vartheta+2n) - (n+\vartheta)(a+\vartheta+n)(a+b+2n) \right] \end{aligned} \quad (5.2.6)$$

which proves (5.2.1d).

In (5.2.4), replacing n by $n+\vartheta$, we have

$$\begin{aligned} & \int_{-1}^1 (1-x)^{a+\vartheta} (1+x)^b P_{n+\vartheta}^{(a,b)}(x) P_n^{(a,b)}(x) dx \\ &= (-1)^{-\vartheta} 2^{a+b+\vartheta+1} \frac{\Gamma(1+a+\vartheta+n) \Gamma(1+b+\vartheta+n) \Gamma(a+b+2\vartheta+2n+1)}{n! \Gamma(a+b+1+n) \Gamma(2+a+b+2\vartheta+2n)} \end{aligned} \quad (5.2.7)$$

which proves (5.2.1e).

5.3 Another Proof of the Generalized Orthogonal Symmetry of the Jacobi Polynomials

In view of (1.4.1), the integral (5.2.1) can be written as

$$\int_{-1}^1 (1-x)^{a+\vartheta} (1+x)^b P_m^{(a,b)}(x) P_n^{(a,b)}(x) dx =$$

$$\frac{(1+a)_m (1+a)_n}{m! n!} \int_{-1}^1 (1-x)^{a+\vartheta} (1+x)^b {}_2F_1 \left(-m, m+a+b+1; 1+a; \frac{1-x}{2} \right) \times$$

$${}_2F_1 \left(-n, n+a+b+1; 1+a; \frac{1-x}{2} \right) dx$$

Now, expressing the hypergeometric function in the integrand as its series representation

(1.3.1), we have

$$= \frac{(1+a)_m (1+a)_n}{m! n!} \sum_{r=0}^m \frac{(-m)_r (m+a+b+1)_r}{(1+a)_r r! 2^r} \sum_{u=0}^n \frac{(-n)_u (n+a+b+1)_u}{(1+a)_u u! 2^u}$$

$$\times \int_{-1}^1 (1-x)^{a+\vartheta+r+u} (1+x)^b dx \quad (5.3.1)$$

Evaluating the last integral in (5.3.1) with the help of a modified form of the integral [17, p.9, (5)], viz.

$$\int_{-1}^1 (1-x)^a (1+x)^b dx = 2^{a+b+1} \frac{\Gamma(a+1) \Gamma(b+1)}{\Gamma(a+b+2)}, \text{ Re } a > -1, \text{ Re } b > -1 \quad (5.3.2)$$

then (5.3.1) becomes

$$= \frac{(1+a)_m (1+a)_n}{m! n!} \Gamma(b+1) 2^{a+b+\vartheta+1} \sum_{r=0}^m \frac{(-m)_r (1+a+b+m)_r}{(1+a)_r r!} \frac{\Gamma(a+\vartheta+r+1)}{\Gamma(a+b+\vartheta+r+2)} \times$$

$${}_3F_2 \left[-n, n+a+b+1, 1+a+\vartheta+r; 1+a, 2+a+b+\vartheta+r; 1 \right] \quad (5.3.3)$$

Now, applying Saalschutz's theorem (2.2.13) to (5.3.3), we get

$$= \frac{(1+a)_m}{m! \, n!} \Gamma(b+1) 2^{a+b+\vartheta+1} \sum_{r=0}^m \frac{(-m)_r \, (-r-\vartheta)_n \, (m+a+b+1)_r \, \Gamma(a+\vartheta+r+1) \, (-n-b)_n}{(1+a)_r \, r! \, \Gamma(a+b+\vartheta+r+2) \, (-n-a-b-\vartheta-r-1)_n} \quad (5.3.4)$$

In view of the standard result (3.2.7), if $r < n - \vartheta$, the numerator of (5.3.4) vanishes and since r runs from 0 to m , it follows that (5.3.4) also vanishes, when $m < n - \vartheta$. By symmetry, it is evident that (5.3.4) also vanishes when $m > n + \vartheta$. Now, it is clear that for $n - \vartheta > m > n + \vartheta$ all terms of (5.3.4) vanish, which proves (5.2.1a).

In (5.3.4), applying the technique of the previous proof we get (5.2.1b), (5.2.1c) and (5.2.1d).

5.4 Zero Orthogonal Symmetry of the Jacobi Polynomials

The orthogonality property of the Jacobi polynomials [19, p. 285, (5) & (9)] is

$$\int_{-1}^1 (1-x)^a (1+x)^b P_m^{(a,b)}(x) P_n^{(a,b)}(x) dx \quad (5.4.1)$$

$$= 0, \text{ if } m \neq n \quad (5.4.1a)$$

$$= \frac{2^{a+b+1} \Gamma(1+b+n) \Gamma(1+a+n)}{n! (1+a+b+2n) \Gamma(1+a+b+n)}, \text{ if } m = n \quad (5.4.1b)$$

where $\operatorname{Re}(a) > -1$, $\operatorname{Re}(b) > -1$.

New method of proof :

In (5.2.1) and (5.2.3), setting $\vartheta = 0$, we have

$$\int_{-1}^1 (1-x)^a (1+x)^b P_m^{(a,b)}(x) P_n^{(a,b)}(x) dx$$

$$= \frac{2^{a+b+1} (a+1)_m \Gamma(1+b+n) \Gamma(1+a)}{m! n! \Gamma(2+a+b+n)} \sum_{r=0}^m \frac{(-m)_r (-r)_n (1+a+b+m)_r}{(2+a+b+n)_r r!} \quad (5.4.2)$$

If $r < n$, the numerator of (5.4.2) vanishes, and since r runs from 0 to m , it follows that (5.4.2) also vanishes, when $m < n$. By symmetry, it is evident that (5.4.2) also vanishes when $m > n$. Now, it is clear that for $m \neq n$, all terms of (5.4.2) vanish, which proves (5.4.1a).

When $m = n$, using the standard result $(-n)_n = (-1)^n n!$, then (5.4.2) yields (5.4.1b).

The orthogonality property of the Jacobi polynomials can also be derived on setting $\vartheta = 0$ in (5.2.1), (5.2.1a) and (5.2.1b). Therefore we call it as zero orthogonal Symmetry of the Jacobi polynomials.

Note:

The technique given above, provides us a new and precise method to establish the orthogonality property of the Jacobi polynomials.

5.5 First Orthogonal Symmetry of the Jacobi Polynomials

The first orthogonal symmetry is

$$\int_{-1}^1 (1-x)^{a+1} (1+x)^b P_m^{(a,b)}(x) P_n^{(a,b)}(x) dx \quad (5.5.1)$$

$$= 0, \text{ if } m \neq n-1, n, n+1 \quad (n-1 > m > n+1) \quad (5.5.1a)$$

$$= \frac{-2^{a+b+2} \Gamma(1+a+n) \Gamma(1+b+n) \Gamma(a+b+2n-1)}{(n-1)! \Gamma(a+b+n) \Gamma(2+a+b+2n)}, \text{ if } m=n-1 \quad (5.5.1b)$$

$$= \frac{2^{a+b+2} \Gamma(1+a+n) \Gamma(1+b+n)}{n! \Gamma(1+a+b+n) \Gamma(1+a+b+2n)} \left[\frac{(n+1) (1+a+n)}{(2+a+b+2n)} - \frac{n (n+a)}{(a+b+2n)} \right], \text{ if } m = n \quad (5.5.1c)$$

$$= \frac{-2^{a+b+2} \Gamma(2+a+n) \Gamma(2+b+n) \Gamma(1+a+b+2n)}{n! \Gamma(1+a+b+n) \Gamma(4+a+b+2n)}, \text{ if } m = n+1 \quad (5.5.1d)$$

where $\text{Re}(a) > -2$, $\text{Re}(b) > -1$.

Proof:

In (5.2.1) and (5.2.3), setting $\vartheta = 1$, we have

$$\int_{-1}^1 (1-x)^{a+1} (1+x)^b P_m^{(a,b)}(x) P_n^{(a,b)}(x) dx$$

$$= \frac{2^{a+b+2} (a+1)_m \Gamma(1+b+n) \Gamma(2+a)}{m! n! \Gamma(3+a+b+n)} \sum_{r=0}^m \frac{(-m)_r (-r-1)_n (2+a)_r (1+a+b+m)_r}{(1+a)_r (3+a+b+n)_r r!} \quad (5.5.2)$$

It is evident from (5.2.3) that if $m \neq n-1, n, n+1$ all terms of (5.5.2) vanish, which proves (5.5.1a).

In (5.5.2), putting $m = n-1$, using (3.2.7) and simplifying, we obtain (5.5.1b).

In (5.5.2), putting $m = n$, using (3.2.7) and adding the resulting two terms ($r = n-1, n$), we get (5.5.1c).

In (5.5.2), putting $m = n+1$, using (3.2.7) and adding the resulting three terms ($r = n-1, n, n+1$), we obtain (5.5.1d). We can also derive (5.4.1d) on substituting $n+1$ for n in (5.5.1b).

The results (5.5.1a), (5.5.1b), (5.5.1c) and (5.5.1d) can also be derived by putting $\vartheta = 1$ in (5.2.1), (5.2.1a), (5.2.1b), (5.2.1c) and (5.2.1e).

5.6 Generalized Orthogonal Symmetry of the Legendre Functions

The generalized orthogonal symmetry is

$$\int_{-1}^1 (1-x)^{\vartheta} P_k^m(x) P_n^m(x) dx \quad (5.6.1)$$

$$= 0, \text{ if } n - \vartheta > k > n + \vartheta \quad (5.6.1a)$$

$$= (-1)^{\vartheta} 2^{\vartheta+1} \frac{n! (2n - 2\vartheta)! (n+m)!}{(2n+1)! (n-\vartheta)! (n-m-\vartheta)!}, \text{ if } k = n - \vartheta, m \leq n - \vartheta \quad (5.6.1b)$$

$$= (-1)^{\vartheta+1} 2^{\vartheta+2} \vartheta \frac{(n+1)! (2n - 2\vartheta + 1)! (n+m)!}{(2n+2)! (n-\vartheta)! (n-m-\vartheta+1)!}, \text{ if } k = n - \vartheta + 1, m \leq n - \vartheta + 1 \quad (5.6.1c)$$

$$= (-1)^{\vartheta+1} 2^{\vartheta+2} \vartheta \frac{(n+\vartheta)! (2n-1)! (n+m+\vartheta-1)!}{(2n+2\vartheta)! (n-1)! (n-m)!}, \text{ if } k = n + \vartheta - 1, m \leq n + \vartheta - 1 \quad (5.6.1d)$$

$$= (-1)^{\vartheta} 2^{\vartheta+1} \frac{(n+\vartheta)! (2n)! (n+m+\vartheta)!}{(2n+2\vartheta+1)! n! (n-m)!}, \text{ if } k = n + \vartheta, m \leq n + \vartheta \quad (5.6.1e)$$

Proof :

Expressing the Legendre functions in the integral (5.6.1) as hypergeometric functions (1.4.8),

we have

$$\begin{aligned} \int_{-1}^1 (1-x)^{\vartheta} P_k^m(x) P_n^m(x) dx &= \int_{-1}^1 \frac{(1-x)^{\vartheta}}{\{\Gamma(1-m)\}^2} \left(\frac{1+x}{1-x}\right)^m {}_2F_1(-k, k+1; 1-m; (1-x)/2) \\ &\quad \times {}_2F_1(-n, n+1; 1-m; (1-x)/2) dx \end{aligned} \quad (5.6.2)$$

Using the definition of the hypergeometric function (1.3.1), the right hand side of (5.6.2)

becomes

$$\int_{-1}^1 \frac{(1-x)^{\vartheta}}{\{\Gamma(1-m)\}^2} \left(\frac{1+x}{1-x}\right)^m \sum_{r=0}^k \frac{(-k)_r (k+1)_r}{(1-m)_r r!} \left(\frac{1-x}{2}\right)^r \sum_{u=0}^n \frac{(-n)_u (n+1)_u}{(1-m)_u u!} \left(\frac{1-x}{2}\right)^u dx \quad (5.6.3)$$

The order of the integral and the summations, can be interchanged (because of their absolute convergence) and we rewrite (5.6.3) as

$$\frac{1}{\{\Gamma(1-m)\}^2} \sum_{r=0}^k \frac{(-k)_r (k+1)_r}{(1-m)_r r! 2^r} \sum_{u=0}^n \frac{(-n)_u (n+1)_u}{(1-m)_u u! 2^u} \int_{-1}^1 (1+x)^m (1-x)^{r+u-m+\vartheta} dx \quad (5.6.4)$$

which, by using the Beta - integral (5.3.2), reduces to

$$\frac{2^{\vartheta+1}}{\{\Gamma(1-m)\}^2} \sum_{r=0}^k \frac{(-k)_r (k+1)_r}{(1-m)_r r!} \sum_{u=0}^n \frac{(-n)_u (n+1)_u}{(1-m)_u u!} \frac{\Gamma(m+1) \Gamma(r+u-m+\vartheta+1)}{\Gamma(r+u+\vartheta+2)} \quad (5.6.5)$$

Rewriting some of the gamma functions in (5.6.5) in Pochhammer's symbol, we have

$$\frac{2^{\vartheta+1}}{\{\Gamma(1-m)\}^2} \sum_{r=0}^k \frac{(-k)_r (k+1)_r}{(1-m)_r r!} \sum_{u=0}^n \frac{(-n)_u (n+1)_u}{(1-m)_u u!} \frac{\Gamma(m+1) (r-m+\vartheta+1)_u \Gamma(r-m+\vartheta+1)}{(r+\vartheta+2) \Gamma(r+\vartheta+2)} \quad (5.6.6)$$

The second summation in (5.6.6) can be replaced by the generalized hypergeometric function and we rewrite (5.6.6) as

$$\frac{2^{\vartheta+1} \Gamma(m+1)}{\{\Gamma(1-m)\}^2} \sum_{r=0}^k \frac{(-k)_r (k+1)_r \Gamma(r-m+\vartheta+1)}{(1-m)_r r! \Gamma(r+\vartheta+2)} {}_3F_2(-n, n+1, r-m+\vartheta+1; 1-m, r+\vartheta+2; 1) \quad (5.6.7)$$

We observe that the generalized hypergeometric series in (5.6.7) is Saalschutzian. Applying the Saalschutz's theorem (2.2.8) to (5.6.7), we obtain

$$\frac{2^{\vartheta+1} \Gamma(m+1)}{\{\Gamma(1-m)\}^2} \sum_{r=0}^k \frac{(-k)_r (-r-\vartheta)_n (k+1)_r \Gamma(r-m+\vartheta+1) (-m-n)_n}{(1-m)_r r! \Gamma(r+\vartheta+2) (1-m)_n (-n-r-\vartheta-1)_n} \quad (5.6.8)$$

In view of the Standard result (3.2.7), if $r < n - \vartheta$, the numerator of (5.6.8) vanishes, and since

r runs from 0 to k , it follows that (5.6.8) also vanishes when $k < n - \vartheta$. By symmetry, it is evident that (5.6.8) also vanishes when $k > n + \vartheta$. Now, it is clear that for $n - \vartheta > k > n + \vartheta$ all terms of (5.6.8) vanish, which proves (5.6.1a).

In (5.6.8), setting $k = n - \vartheta$, using (3.2.7) and simplifying, we have

$$\begin{aligned} & \int_{-1}^1 (1-x)^{\vartheta} P_{n-\vartheta}^m(x) P_n^m(x) dx \\ &= (-1)^{\vartheta} 2^{\vartheta+1} \frac{n! (2n-2\vartheta)! (n+m)!}{(2n+1)! (n-\vartheta)! (n-m-\vartheta)!}, \quad m \leq n - \vartheta \end{aligned} \quad (5.6.9)$$

which proves (5.6.1b).

In (5.6.8), putting $k = n - \vartheta + 1$, using (3.2.7) and adding the resulting two terms ($r = n - \vartheta$, $n - \vartheta + 1$), we get

$$\begin{aligned} & \int_{-1}^1 (1-x)^{\vartheta} P_{n-\vartheta+1}^m(x) P_n^m(x) dx \\ &= (-1)^{\vartheta+1} 2^{\vartheta+2} \vartheta \frac{(n+1)! (2n-2\vartheta+1)! (n+m)!}{(2n+2)! (n-\vartheta)! (n-m-\vartheta+1)!}, \quad m \leq n - \vartheta + 1 \end{aligned} \quad (5.6.10)$$

which proves (5.6.1c).

Note:

On continuing as above, we can find the values of the integral (5.6.1) for $k = n - \vartheta + 2, n - \vartheta + 3, \dots$, $n, \dots, n + \vartheta - 3, n + \vartheta - 2$.

In (5.6.10), substituting $n + \vartheta - 1$ for n , we obtain

$$\int_{-1}^1 (1-x)^{\vartheta} P_{n+\vartheta-1}^m(x) P_n^m(x) dx$$

$$= (-1)^{\vartheta+1} 2^{\vartheta+2} \vartheta \frac{(n+\vartheta)! (2\vartheta-1)! (n+m+\vartheta-1)!}{(2n+2\vartheta)! (n-1)! (n-m)!}, m \leq n + \vartheta - 1 \quad (5.6.11)$$

which proves (5.6.1d).

In (5.6.9.), replacing n by $n + \vartheta$, we have

$$\begin{aligned} & \int_{-1}^1 (1-x)^{\vartheta} P_{n+\vartheta}^m(x) P_n^m(x) dx \\ &= (-1)^{\vartheta} 2^{\vartheta+1} \frac{(n+\vartheta)! (2n)! (n+m+\vartheta)!}{(2n+2\vartheta+1)! n! (n-m)!}, m \leq n + \vartheta \end{aligned} \quad (5.6.12)$$

which proves (5.6.1e).

5.7 Zero Orthogonal Symmetry of the Legendre Functions

The orthogonality property of the Legendre functions [19, p. 279, (27) & (28)] is

$$\int_{-1}^1 P_k^m(x) P_n^m(x) dx \quad (5.7.1)$$

$$= 0, \text{ if } k \neq n \quad (5.7.1a)$$

$$= \frac{2(n+m)!}{(2n+1)(n-m)!}, \text{ if } k = n, m \leq n \quad (5.7.1b)$$

New method of proof :

In (5.6.1) and (5.6.8), setting $\vartheta = 0$, we have

$$\begin{aligned} & \int_{-1}^1 P_k^m(x) P_n^m(x) dx \\ &= \frac{2 \Gamma(m+1)}{\{\Gamma(1-m)\}^2} \sum_{r=0}^k \frac{(-k)_r (-r)_n (k+1)_r \Gamma(r-m+1) (-m-n)_n}{(1-m)_r r! \Gamma(r+2) (1-m)_n (-n-r-1)_n} \end{aligned} \quad (5.7.2)$$

If $r < n$, the numerator of (5.7.2) vanishes, and since r runs from 0 to k , it follows that (5.7.2) also vanishes, when $k < n$. By symmetry, it is evident that (5.7.2) also vanishes when $k > n$. Now, it is clear that for $k \neq n$, all terms of (5.7.2) vanish, which proves (5.7.1a).

When $k = n$, using the standard results $(-n)_n = (-1)^n n!$, then (5.7.2) yields (5.7.1b).

The orthogonality property of the Legendre functions can also be derived on setting $\vartheta = 0$ in (5.6.1), (5.6.1a) and (5.6.1b).

5.8 First Orthogonal Symmetry of the Legendre Functions

The first orthogonal symmetry is

$$\int_{-1}^1 (1-x) P_k^m(x) P_n^m(x) dx \quad (5.8.1)$$

$$= 0, \text{ if } k \neq n-1, n, n+1 \quad (n+1 < k < n-1) \quad (5.8.1a)$$

$$= \frac{2(n-m)(n+m)!}{(1-4n^2)(n-m)!}, \quad k = n-1, \quad n \geq m+1 \quad (5.8.1b)$$

$$= \frac{2(n+m)!}{(2n+1)(n-m)!}, \quad k = n, \quad n \geq m \quad (5.8.1c)$$

$$= \frac{2(n-m+1)(n+m+1)!}{[1-4(n+1)^2](n-m+1)!}, \quad k = n+1, \quad n \geq m-1 \quad (5.8.1d)$$

Proof:

In (5.6.1) and (5.6.8), setting $\vartheta = 1$, we get

$$\int_{-1}^1 (1-x) P_k^m(x) P_n^m(x) dx$$

$$= \frac{4 \Gamma(m+1)}{\left\{ \Gamma(1-m) \right\}^2} \sum_{r=0}^k \frac{(-k)_r (-r-1)_n (k+1)_r \Gamma(r-m+2) (-m-n)_n}{(1-m)_r r! \Gamma(r+3) (1-m)_n (-n-r-2)_n} \quad (5.8.2)$$

It is evident from (5.6.8) that if $k \neq n-1, n, n+1$ all terms of (5.8.2) vanish, which proves (5.8.1a).

In (5.8.2), putting $k = n-1$, using (3.2.7) and simplifying, we obtain (5.8.1b).

In (5.8.2), putting $k = n$, using (3.2.7) and adding the resulting two terms ($r = n-1, n$), we get (5.8.1c).

In (5.8.2), putting $k = n+1$, using (3.2.7) and adding the resulting three terms ($r = n-1, n, n+1$), we obtain (5.8.1d). We can also derive (5.8.1d) on substituting $n+1$ for n in (5.8.1b).

The results (5.8.1), (5.8.1a), (5.8.1b), (5.8.1c) and (5.8.1d) can also be derived on putting $\vartheta = 1$ in (5.6.1), (5.6.1a), (5.6.1b), (5.6.1c) and (5.6.1e).

CHAPTER VI

SOME INTEGRALS AND EXPANSION FORMULAE INVOLVING FOX'S H-FUNCTION AND GAUSS' HYPERGEOMETRIC POLYNOMIALS

6.1 Introduction

In this chapter, we evaluate three integrals involving Fox's H-function and Gauss' hypergeometric polynomials and employ them to establish three expansion formulae for Fox's H-function involving Gauss' hypergeometric polynomials.

We obtain some particular cases of our integrals and expansion formulae due to Bajpai [4,5].

The H-function introduced by [21, p.408], will be represented and defined as follows :

$$H_{p,q}^{m,n} \left[z \mid \begin{matrix} (a_1, e_1), \dots, (a_p, e_p) \\ (b_1, f_1), \dots, (b_q, f_q) \end{matrix} \right] = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - f_j s) \prod_{j=1}^n \Gamma(1 - a_j + e_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + f_j s) \prod_{j=n+1}^p \Gamma(a_j - e_j s)} z^s ds, \quad (6.1.1)$$

where an empty product is interpreted as 1, $0 \leq m \leq q$, $0 \leq n \leq p$; e 's and f 's are all positives; L is a suitable contour of Barnes type such that the poles of $\Gamma(b_j - f_j s)$, $j = 1, \dots, m$ lie on the right - hand side of the contour and those of $\Gamma(1 - a_j + e_j s)$, $j = 1, \dots, n$ lie on the left - hand side of the contour.

In what follows for sake of brevity :

$$\sum_{j=1}^p e_j - \sum_{j=1}^q f_j \equiv A, \quad \sum_{j=1}^n e_j - \sum_{j=n+1}^p e_j + \sum_{j=1}^m f_j - \sum_{j=m+1}^q f_j \equiv B \quad \text{and}$$

(a_p, e_p) represents $(a_1, e_1), \dots, (a_p, e_p)$.

6.2 Integrals Involving Fox's H-Function and Gauss' Hypergeometric Polynomials

The integrals to be established are

$$\begin{aligned} (i) & \int_1^{\infty} x^{h+a-1} (x-1)^{b-c} {}_2F_1(-r, b+r; c; x) H_{p,q}^{m,n} \left[z x^k \mid \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dx \\ &= (-1)^r \frac{\Gamma(1-c+b+r)}{(c)_r} H_{p+2,q+2}^{m+2,n} \left[z \mid \begin{matrix} (a_p, e_p), (1-a-h, k), (c-a-h, k) \\ (c-a-h+r, k), (c-b-a-h-r, k), (b_q, f_q) \end{matrix} \right] \end{aligned} \quad (6.2.1)$$

where k is a positive number, $A \leq 0$, $B > 0$, $\arg z < \frac{1}{2} B \pi$, $\operatorname{Re}(b-c) > -1$,

$$\operatorname{Re}(h-c+a+b) + k \max_{1 \leq j \leq n} \left[\operatorname{Re} \left(\frac{a_j - 1}{e_j} \right) \right] < -r, \quad \text{and}$$

$$\operatorname{Re}(h-c+a) + k \max_{1 \leq j \leq n} \left[\operatorname{Re} \left(\frac{a_j - 1}{e_j} \right) \right] < r.$$

$$\begin{aligned} (ii) & \int_0^{\infty} x^{h+a-1} (1+x)^{b-c} {}_2F_1(-r, b+r; c; -x) H_{p,q}^{m,n} \left[z x^k \mid \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dx \\ &= \frac{1}{(c)_r \Gamma(c-b-r)} H_{p+2,q+2}^{m+2,n+1} \left[z \mid \begin{matrix} (1-a-h, k), (a_p, e_p), (c-a-h, k) \\ (c-a-h+r, k), (c-b-a-h-r, k), (b_q, f_q) \end{matrix} \right] \end{aligned} \quad (6.2.2)$$

where k is a positive number, $A \leq 0$, $B > 0$, $|\arg z| < \frac{1}{2} B \pi$, $\operatorname{Re}(h+a) +$

$$k \min_{1 \leq j \leq m} \left[\operatorname{Re} \left(\frac{b_j}{f_j} \right) \right] > 0, \operatorname{Re}(h-c+a+b) + k \max_{1 \leq j \leq n} \left[\operatorname{Re} \left(\frac{a_j - 1}{e_j} \right) \right] < -r, \text{ and}$$

$$\operatorname{Re}(h-c+a) + k \max_{1 \leq j \leq n} \left[\operatorname{Re} \left(\frac{a_j - 1}{e_j} \right) \right] < r.$$

$$\begin{aligned} \text{(iii)} \quad & \int_0^1 x^{h+a-1} (1-x)^{b-c} {}_2F_1(-r, b+r; c; x) H_{p,q}^{m,n} \left[z x^k \mid \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dx \\ &= \frac{\Gamma(1-c+b+r)}{(c)_r} H_{p+2,q+2}^{m+1,n+1} \left[z \mid \begin{matrix} (1-a-h,k), (a_p, e_p), (c-a-h,k) \\ (c-a-h+r,k), (b_q, f_q), (c-a-b-h-r,k) \end{matrix} \right] \end{aligned} \quad (6.2.3)$$

where k is a positive number, $A \leq 0$, $B > 0$, $|\arg z| < \frac{1}{2} B \pi$, $\operatorname{Re}(h+a) +$

$$k \min_{1 \leq j \leq m} \left[\operatorname{Re} \left(\frac{b_j}{f_j} \right) \right] > 0, \text{ and } \operatorname{Re}(b-c) > -1.$$

Proof :

(i) The integral (6.2.1) is established by expressing the H-function in the integrand as a Mellin - Barnes type integral (6.1.1) and interchanging the order of the integration, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - f_j s) \prod_{j=1}^n \Gamma(1 - a_j + e_j s) z^s}{\prod_{j=m+1}^q \Gamma(1 - b_j + f_j s) \prod_{j=n+1}^p \Gamma(a_j - e_j s)} ds$$

$$\times \int_1^\infty x^{h+a+ks-1} (x-1)^{b-c} {}_2F_1(-r, b+r; c; x) dx \quad ds \quad (6.2.4)$$

Now, evaluating the inner - integral with the help of (2.2.7) , we have

$$\frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - f_j s) \prod_{j=1}^n \Gamma(1-a_j + e_j s) z^s}{\prod_{j=m+1}^q \Gamma(1-b_j + f_j s) \prod_{j=n+1}^p \Gamma(a_j - e_j s)} \times$$

$$\frac{(-1)^r \Gamma(1-c+b+r)}{(c)_r} \frac{\Gamma(c-h-a-ks+r) \Gamma(c-h-b-a-ks-r)}{\Gamma(c-h-ks-a) \Gamma(1-h-ks-a)} ds \quad (6.2.5)$$

On applying (6.1.1) , the integral (6.2.1) is established.

(ii) The integral (6.2.2) is established on following the technique employed to establish (6.2.1) except instead of (2.2.7), using the integral (2.2.8).

(iii) The integral (6.2.3) is established on following the technique employed to establish (6.2.1) except instead of (2.2.7), using the integral (2.2.9) .

6.3 Particular Cases

(i) In (6.2.2.), replacing b by $1+a+b$, c by $1+a$, z by $2^k z$, $h+a-1$ by w , x by $\frac{x-1}{2}$ and using the relation (1.4.1), we obtain the result due to Bajpai [5, p. 224, (6)] :

$$\begin{aligned}
& \int_1^{\infty} (x-1)^w (x+1)^b P_r^{(a,b)}(x) H_{p,q}^{m,n} \left[z (x-1)^k \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] dx \\
&= \frac{2^{(b+w+1)} (-1)^r \Gamma(1+b+r)}{r! \Gamma(-b) \Gamma(1+b)} H_{p+2, q+2}^{m+2, n+1} \left[z 2^k \left| \begin{matrix} (-w, k), (a_p, e_p), (a-w, k) \\ (a-w+r, k), (-1-b-w-r, k), (b_q, f_q) \end{matrix} \right. \right] \quad (6.3.1)
\end{aligned}$$

where k is a positive number, $A \leq 0, B > 0, |\arg z| < \frac{1}{2} B\pi$,

$$\operatorname{Re} w + k \min_{1 \leq j \leq m} \left[\operatorname{Re} \left(\frac{b_j}{f_j} \right) \right] > -1, \operatorname{Re} (w-a) + k \max_{1 \leq j \leq n} \operatorname{Re} \left(\frac{a_j - 1}{e_j} \right) < r,$$

$$\text{and } \operatorname{Re} (w+b) + r + k \max_{1 \leq j \leq n} \left[\operatorname{Re} \left(\frac{a_j - 1}{e_j} \right) \right] < -1.$$

(ii) In (6.2.3), replacing b by $1+a+b$, c by $1+a$, z by $2^k z$, $h+a-1$ by σ , x by $\frac{1-x}{2}$ and using the relation (1.4.1), we obtain the result due to Bajpai [4, p. 699, (2.3)]:

$$\begin{aligned}
& \int_{-1}^1 (1+x)^b (1-x)^\sigma P_r^{(a,b)}(x) H_{p,q}^{m,n} \left[z (1-x)^k \left| \begin{matrix} (a_p, e_p) \\ (p_q, f_q) \end{matrix} \right. \right] dx \\
&= \frac{2^{(b+\sigma+1)} \Gamma(1+b+r)}{r!} H_{p+2, q+2}^{m+1, n+1} \left[z 2^k \left| \begin{matrix} (-\sigma, k), (a_p, e_p), (a-\sigma, k) \\ (a-\sigma+r, k), (b_q, f_q), (-1-b-\sigma-r, k) \end{matrix} \right. \right] \quad (6.3.2)
\end{aligned}$$

where k is a positive number, $A \leq 0, B > 0, |\arg z| < \frac{1}{2} B\pi$, $\operatorname{Re} (b) > -1$, and

$$\operatorname{Re} \left(\sigma + k \min_{1 \leq j \leq m} \left(\frac{b_j}{f_j} \right) \right) > -1.$$

6.4 Some Expansion Formulae Involving the Fox's H-Function and the Gauss' Hypergeometric Polynomials

The expansion formulae to be established are

$$(i) \quad x^h H_{p,q}^{m,n} \left[z x^k \mid \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] = \frac{\Gamma(1-c)}{\Gamma(1+b)\Gamma(-b)} \sum_{r=0}^{\infty} \frac{(-1)^r (b+2r) \Gamma(b+r)}{r!} x$$

$${}_2F_1(-r, b+r; c; x) H_{p+2, q+2}^{m+2, n} \left[z \mid \begin{matrix} (a_p, e_p), (-h, k), (1-h-c, k) \\ (-h+r, k), (-h-b-r, k), (b_q, f_q) \end{matrix} \right], \quad (1 < x < \infty) \quad (6.4.1)$$

where k is a positive number, $A \leq 0$, $B > 0$, $|\arg z| < \frac{1}{2} B \pi$, $\operatorname{Re}(b-c) > -1$,

$$\operatorname{Re} \left(h+b+k \max_{1 \leq j \leq n} \left(\frac{a_j - 1}{e_j} \right) \right) < -r, \quad \operatorname{Re} \left(h+k \max_{1 \leq j \leq n} \left(\frac{a_j - 1}{e_j} \right) \right) < r, \quad \text{and} \quad \operatorname{Re} b \neq -2r.$$

$$(ii) \quad x^h H_{p,q}^{m,n} \left[z x^k \mid \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] = \frac{1}{\Gamma(c) \Gamma(1+b) \Gamma(-b)} \sum_{r=0}^{\infty} \frac{(-1)^r (b+2r) \Gamma(b+r)}{r!} x$$

$${}_2F_1(-r, b+r; c; -x) H_{p+2, q+2}^{m+2, n+1} \left[z \mid \begin{matrix} (1-h-c, k), (a_p, e_p), (-h, k) \\ (-h+r, k), (-h-b-r, k), (b_q, f_q) \end{matrix} \right], \quad 0 < x < \infty \quad (6.4.2)$$

where k is a positive number, $A \leq 0$, $B > 0$, $|\arg z| < \frac{1}{2} B \pi$,

$$\operatorname{Re} \left(c+h+k \min_{1 \leq j \leq m} \left(\frac{b_j}{f_j} \right) \right) > 0,$$

$$\operatorname{Re} \left(b+h+k \max_{1 \leq j \leq n} \left(\frac{a_j - 1}{e_j} \right) \right) < -r, \quad \operatorname{Re} \left(h+k \max_{1 \leq j \leq n} \left(\frac{a_j - 1}{e_j} \right) \right) < r, \quad \text{and} \quad \operatorname{Re} b \neq -2r.$$

$$(iii) \quad x^h H_{p,q}^{m,n} \left[z x^k \mid \begin{matrix} a_p, e_p \\ (b_q, f_q) \end{matrix} \right] = \frac{1}{\Gamma(c)} \sum_{r=0}^{\infty} \frac{(b+2r) \Gamma(b+r)}{r!} x$$

$${}_2F_1(-r, b+r; c; x) H_{p+2, q+2}^{m+1, n+1} \left[z \left| \begin{matrix} (1-h-c, k), (a_p, e_p), (-h, k) \\ (-h+r, k), (b_q, f_q), (-h-b-r, k) \end{matrix} \right. \right], \quad 0 < x < 1 \quad (6.4.3)$$

where k is a positive number, $A \leq 0$, $B > 0$, $|\arg z| < \frac{1}{2} B \pi$,

$$\operatorname{Re} \left(c + h + k \min_{1 \leq j \leq m} \left(\frac{b_j}{f_j} \right) \right) > 0, \quad \operatorname{Re}(b-c) > -1, \quad \text{and} \quad \operatorname{Re}(c) > 0.$$

Proof :

(i) To prove (6.4.1), let

$$\begin{aligned} f(x) &= x^h H_{p,q}^{m,n} \left[z x^k \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] \\ &= \sum_{r=0}^{\infty} A_r {}_2F_1(-r, b+r; c; x), \quad (1 < x < \infty) \end{aligned} \quad (6.4.4)$$

Equation (6.4.4) is valid since $f(x)$ is continuous and of bounded variation in the open interval $(1, \infty)$, when $h \geq 0$.

Multiplying both sides of (6.4.1) by $(x)^{c-1} (x-1)^{b-c} {}_2F_1(-s, b+s; c; x)$ and integrating with respect to x from 1 to ∞ , we get

$$\begin{aligned} &\int_1^{\infty} x^{c+h-1} (x-1)^{b-c} {}_2F_1(-s, b+s; c; x) H_{p,q}^{m,n} \left[z x^k \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] dx \\ &= \sum_{r=0}^{\infty} A_r \int_1^{\infty} x^{c-1} (x-1)^{b-c} {}_2F_1(-s, b+s; c; x) {}_2F_1(-r, b+r; c; x) dx \end{aligned}$$

Now, using the orthogonality property of Gauss' hypergeometric polynomials (3.2.3), we have

$$A_r = \frac{(c)_r (b+2r) \Gamma(b+r) \Gamma(1-c)}{r! \Gamma(1-c+b+r) \Gamma(1+b) \Gamma(-b)} \int_1^\infty x^{h+c-1} (x-1)^{b-c} {}_2F_1(-r, b+r; c; x) \times$$

$$H_{p,q}^{m,n} \left[zx^k \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] dx \quad (6.4.5)$$

On applying (6.2.1) to (6.4.5) , we obtain

$$A_r = \frac{(-1)^r (b+2r) \Gamma(b+r) \Gamma(1-c)}{r! \Gamma(1+b) \Gamma(-b)} H_{p+2,q+2}^{m+2,n} \left[z \left| \begin{matrix} (a_p, e_p), (-h, k), (1-c-h, k) \\ (-h+r, k), (-h-b-r, k), (b_q, f_q) \end{matrix} \right. \right] \quad (6.4.6)$$

From (6.4.4) and (6.4.6) the formula (6.4.1) follows .

(ii) To prove (6.4.2) , let

$$f(x) = x^h H_{p,q}^{m,n} \left[zx^k \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right]$$

$$= \sum_{r=0}^{\infty} B_r {}_2F_1(-r, b+r; c; -x) \quad , \quad (0 < x < \infty) \quad (6.4.7)$$

Equation (6.4.7) is valid since $f(x)$ is continuous and of bounded variation in the open interval $(0, \infty)$, when $h \geq 0$.

Multiplying both sides of (6.4.6) by $x^{c-1} (x+1)^{b-c} {}_2F_1(-s, b+s; c; -x)$ and integrating with respect to x from 0 to ∞ , we get

$$\int_0^\infty x^{c+h-1} (x+1)^{b-c} {}_2F_1(-s, b+s; c; -x) H_{p,q}^{m,n} \left[zx^k \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] dx$$

$$= \sum_{r=0}^{\infty} B_r \int_0^\infty x^{c-1} (x+1)^{b-c} {}_2F_1(-s, b+s; c; -x) {}_2F_1(-r, b+r; c; -x) dx$$

Now, using the orthogonality property of the Gauss' hypergeometric polynomials (3.2.2), we have

$$B_r = \frac{(c)_r (b+2r) \Gamma(b+r) \Gamma(1-c+b) \Gamma(c-b)}{r! \Gamma(c) \Gamma(1-c+b+r) \Gamma(1+b) \Gamma(-b)} \int_0^\infty x^{h+c-1} (x+1)^{b-c} {}_2F_1(-r, b+r; c; -x) \times$$

$$H_{p,q}^{m,n} \left[zX^k \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] dx \quad (6.4.8)$$

On applying (6.2.2) to (6.4.8), we obtain

$$B_r = \frac{(-1)^r (b+2r) \Gamma(b+r)}{r! \Gamma(c) \Gamma(1+b) \Gamma(-b)} H_{p+2,q+2}^{m+2,n+1} \left[z \left| \begin{matrix} (1-c-h, k), (a_p, e_p), (-h, k) \\ (-h+r, k), (-h-b-r, k), (b_q, f_q) \end{matrix} \right. \right] \quad (6.4.9)$$

From (6.4.7) and (6.4.9) the formula (6.4.2) follows.

(iii) To prove (6.4.3), let

$$f(x) = x^h H_{p,q}^{m,n} \left[zX^k \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right]$$

$$= \sum_{r=0}^{\infty} C_r {}_2F_1(-r, b+r; c; x) \quad , \quad (0 < x < 1) \quad (6.4.10)$$

Equation (6.4.10) is valid since $f(x)$ is continuous and of bounded variation in the open interval $(0, 1)$, when $h \geq 0$.

Multiplying both sides of (6.4.10) by $(x)^{c-1} (1-x)^{b-c} {}_2F_1(-s, b+s; c; x)$ and integrating with respect to x from 0 to 1, we get

$$\int_0^1 x^{c+h-1} (1-x)^{b-c} {}_2F_1(-s, b+s; c; x) H_{p,q}^{m,n} \left[zX^k \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] dx$$

$$= \sum_{r=0}^{\infty} C_r \int_0^1 x^{c-1} (1-x)^{b-c} {}_2F_1(-s, b+s; c; x) {}_2F_1(-r, b+r; c; x) dx$$

Now, using the orthogonality property of Gauss' hypergeometric polynomials (3.2.1), we have

$$C_r = \frac{(c)_r (b+2r) \Gamma(b+r)}{r! \Gamma(c) \Gamma(1-c+b+r)} \int_0^1 x^{h+c-1} (1-x)^{b-c} {}_2F_1(-r, b+r; c; x) \times$$

$$H_{p,q}^{m,n} \left[z x^k \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] dx \quad (6.4.11)$$

On applying (6.2.3), the above integral yields

$$C_r = \frac{(b+2r) \Gamma(b+r)}{r! \Gamma(c)} H_{p+2,q+2}^{m+1,n+1} \left[z \left| \begin{matrix} (1-h-c, k), (a_p, e_p), (-h, k) \\ (-h+r, k), (b_q, f_q), (-h-b-r, k) \end{matrix} \right. \right] \quad (6.4.12)$$

From (6.4.10) and (6.4.12) the formula (6.4.3) follows.

6.5 Particular Cases

(i) In (6.4.2), replacing b by $1+a+b$, c by $1+a$, z by $2^k z$, x by $\frac{x-1}{2}$ and using the relation (1.4.1), we obtain the result due to Bajpai [5, p.226, (14)]:

$$(x-1)^h H_{p,q}^{m,n} \left[z (x-1)^k \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] = 2^h \sum_{r=0}^{\infty} \frac{(-1)^r (1+a+b+2r)}{\Gamma(1+a+r) \Gamma(-1-a-b)} \frac{\Gamma(1+a+b+r)}{\Gamma(2+a+b)} \times$$

$$H_{p+2,q+2}^{m+2,n+1} \left[z 2^k \left| \begin{matrix} (-h-a, k), (a_p, e_p), (-h, k) \\ (-h+r, k), (-1-a-b-h-r, k), (b_q, f_q) \end{matrix} \right. \right] P_r^{(a,b)}(x), \quad 1 < x < \infty \quad (6.5.1)$$

where k is a positive number, $A \leq 0$, $B > 0$, $|\arg z| < \frac{1}{2} B\pi$,

$$\operatorname{Re} \left(h + k \min_{1 \leq j \leq m} \left(\frac{b_j}{f_j} \right) \right) > -1, \operatorname{Re} \left(h + b + k \max_{1 \leq j \leq n} \left(\frac{a_j - 1}{e_j} \right) \right) < -1, \operatorname{Re} b > -1, \text{ and } \operatorname{Re} (a+b) \neq -1 - 2r.$$

(ii) In (6.4.3), replacing b by $1+a+b$, c by $1+a$, z by $2^k z$, x by $\frac{1-x}{2}$ and using the relation (1.4.1), we obtain the result due to Bajpai [4, p.699, (3.2)] :

$$(1-x)^h H_{p,q}^{m,n} \left[z(1-x)^k \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] = 2^h \sum_{r=0}^{\infty} \frac{(1+a+b+2r) \Gamma(1+a+b+r)}{\Gamma(1+a+r)} \times$$

$$H_{p+2, q+2}^{m+1, n+1} \left[z2^k \left| \begin{matrix} (-a-h, k), (a_p, e_p), (-h, k) \\ (-h+r, k), (b_q, f_q), (-1-a-b-h-r, k) \end{matrix} \right. \right] P_r^{(a,b)}(x), \quad -1 < x < 1 \quad (6.5.2)$$

where k is a positive number, $A \leq 0$, $B > 0$, $|\arg z| < \frac{1}{2} B\pi$, $\operatorname{Re} a > -1$, $\operatorname{Re} b > -1$, $h \geq 0$, and $\operatorname{Re} \left(h + a + k \min_{1 \leq j \leq m} \left(\frac{b_j}{f_j} \right) \right) > 0$.

CHAPTER VII

PARTIAL DIFFERENTIAL EQUATIONS RELATED TO HEAT CONDUCTION AND OTHER APPLIED SCIENCES

7.1 Introduction

In this chapter, we formulate and solve three partial differential equations related to the problems of heat conduction and other applied sciences, solutions of which lead to the Jacobi polynomials and the Legendre functions respectively. We also obtain particular solutions of our partial differential equations in terms of Fox's H-function and derive some special solutions.

7.2 The First Partial Differential Equation

Consider the following partial differential equation:

$$\frac{\partial u}{\partial t} = k \left[(1-x^2) \frac{\partial^2 u}{\partial x^2} + (b-a-(2+a+b)x) \frac{\partial u}{\partial x} \right], \quad (-1 \leq x \leq 1) \quad (7.2.1)$$

The solutions of which lead to the Jacobi polynomials and they must satisfy the initial condition:

$$u(x, 0) = f(x),$$

and the boundary conditions :

$$\begin{aligned} u(-1, t) &= A \frac{(-b-r)_r}{r!} e^{-kr(r+a+b+1)t}, \\ u(1, t) &= A \frac{(1+a)_r}{r!} e^{-kr(r+a+b+1)t}, \end{aligned} \quad (7.2.2)$$

The equation (7.2.1) is related to some differential equations of heat conduction [20, p. 27, (3) and p. 148, (1)]. Let us consider [20, p. 27, (3)]:

$$\frac{\partial^2 u}{\partial x^2} - \frac{U}{k} \frac{\partial u}{\partial x} - \frac{1}{k} \frac{\partial u}{\partial t} = A u + B \quad (7.2.3)$$

where U , k , A and B are constants. However, cases in which thermal parameters, K , U etc. are functions of u are becoming of increasing importance, but have not yet been studied to any great extent [20, p. 28].

If in (7.2.3), we put $A=B=0$, set $k(1-x^2)$ for k and $-k(b-a-(2+a+b)x)$ for U , it reduces to the equation (7.2.1).

The same forms of equations arise in many other physical contexts [20, p. 28] and many of the solutions given in this chapter can be taken over with a simple change of notations.

7.3 The Solution of the First Partial Differential Equation in Terms of the Jacobi Polynomials

The solution to be obtained is

$$u(x,t) = \sum_{r=0}^{\infty} C_r e^{-kr(r+a+b+1)t} P_r^{(a,b)}(x) \quad , \quad k > 0 \quad , \quad (7.3.1)$$

where $P_r^{(a,b)}(x)$ are the Jacobi polynomials.

Proof:

Let us assume that (7.2.1) has a solution of the form:

$$u(x,t) = y(x) e^{-kr(r+a+b+1)t} \quad (7.3.2)$$

The substitution of $u(x,t)$ from (7.3.2) into (7.2.1) yields the following differential equation:

$$(1-x^2) y'' + [b-a-(2+a+b)x] y' + r(r+a+b+1)y = 0, \quad (7.3.3)$$

which is a Jacobi equation [1, p. 233, (5.123)] with solution $y = P_r^{(a,b)}(x)$. The other solutions of Jacobi equation are not appropriate in this problem.

Therefore the solution of (7.2.1) is of the form:

$$u(x,t) = e^{-kr(r+a+b+1)t} P_r^{(a,b)}(x) \quad (7.3.4)$$

On using the principle of superposition, a general solution of (7.2.1) is obtained as given by (7.3.1).

Putting $t = 0$ in (7.3.1), we have

$$f(x) = \sum_{r=0}^{\infty} C_r P_r^{(a,b)}(x) \quad (7.3.5)$$

Multiplying both sides of (7.3.5) by $(1-x)^a (1+x)^b P_u^{(a,b)}(x)$ and integrating with respect to x from -1 to 1 , and using the orthogonality property of Jacobi polynomials [19, p. 285, (5)], we get that

$$C_r = \frac{r! (a+b+2r+1) \Gamma(a+b+r+1)}{2^{a+b+1} \Gamma(a+r+1) \Gamma(b+r+1)} \int_{-1}^1 (1-x)^a (1+x)^b f(x) P_r^{(a,b)}(x) dx \quad (7.3.6)$$

where $\text{Re } a > -1$, $\text{Re } b > -1$.

7.4 Particular Solution of the First Partial Differential Equation in Terms of Fox's H-Function

If we take

$$f(x) = (1-x)^w H_{p,q}^{m,n} \left[z(1-x)^h \middle| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] \quad (7.4.1)$$

and substitute this value of $f(x)$ in (7.3.6), then using the following integral [4, p. 699, (2.2)]:

$$\begin{aligned} & \int_{-1}^1 (1+x)^b (1-x)^w P_v^{(a,b)} H_{p,q}^{m,n} \left[z(1-x)^h \middle| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dx \\ &= (-1)^v \frac{2^{b+w+1} \Gamma(1+v+b)}{v!} H_{p+2,q+2}^{m,n+2} \left[Z 2^h \middle| \begin{matrix} (-w,h), (a-w,h), (a_p, e_p) \\ (b_q, f_q), (a-w+v,h), (-1-b-w-v,h) \end{matrix} \right] \end{aligned} \quad (7.4.2)$$

where h is a positive number, $\sum_{j=1}^p e_j - \sum_{j=1}^q f_j = A \leq 0$,

$$\sum_{j=1}^n e_j - \sum_{j=n+1}^p e_j + \sum_{j=1}^m f_j - \sum_{j=m+1}^q f_j = B > 0, \quad |\arg z| < \frac{1}{2} B\pi, \quad \operatorname{Re} b > -1,$$

$$\operatorname{Re} \left(w+h \min_{1 \leq j \leq m} \left(\frac{b_j}{f_j} \right) \right) > -1,$$

to find the value of C_r , then the solution (7.3.1) reduces to

$$\begin{aligned} u(x,t) &= \sum_{r=0}^{\infty} e^{-kr(r+a+b+1)t} \frac{(-1)^r 2^w (a+b+2r+1) \Gamma(a+b+r+1)}{\Gamma(a+r+1)} P_r^{(a,b)}(x) \\ &\times H_{p+2,q+2}^{m,n+2} \left[Z 2^h \middle| \begin{matrix} (-a-w,h), (-w,h), (a_p, e_p) \\ (b_q, f_q), (-w+r,h), (-1-b-a-w-r,h) \end{matrix} \right] \end{aligned} \quad (7.4.3)$$

where h is a positive number, $A \leq 0$, $B > 0$, $|\arg z| < \frac{1}{2} B\pi$,

$$\operatorname{Re} \left(w+a+h \min_{1 \leq j \leq m} \left(\frac{b_j}{f_j} \right) \right) > -1, \text{ and } \operatorname{Re} \alpha > -1.$$

By virtue of the following identity, which is apparent from the definition of the H-function,

$$\begin{aligned} & H_{p+2,q+2}^{m,n+2} \left[Z \left| \begin{array}{l} (-a-w,h), (-w,h), (a_p, e_p) \\ (b_q, f_q), (-w+r,h), (-1-b-a-w-r,h) \end{array} \right. \right] \\ &= (-1)^r H_{p+2,q+2}^{m+1,n+1} \left[Z \left| \begin{array}{l} (-a-w,h), (a_p, e_p), (-w,h) \\ (-w+r,h), (b_q, f_q), (-1-b-a-w-r,h) \end{array} \right. \right] \end{aligned} \quad (7.4.4)$$

the formula (7.4.3), reduces to the form:

$$\begin{aligned} u(x,t) &= \sum_{r=0}^{\infty} e^{-kr(r+a+b+1)t} \frac{2^w (a+b+2r+1) \Gamma(a+b+r+1)}{\Gamma(a+r+1)} P_r^{(a,b)}(x) \\ &\times H_{p+2,q+2}^{m+1,n+1} \left[Z 2^h \left| \begin{array}{l} (-a-w,h), (a_p, e_p), (-w,h) \\ (-w+r,h), (b_q, f_q), (-1-b-a-w-r,h) \end{array} \right. \right] \end{aligned} \quad (7.4.5)$$

7.5 Special Solutions

The Fox's H-function is a generalization of Meijer's G-function [17, pp. 206-222].

The importance of the H-function lies largely from the possibility of expressing by means of the H-symbol a great many of special functions appearing in applied mathematics, physical sciences and statistics [23, pp. 145-159]. So that the solution given in section (7.4) becomes a master or key solution from which a very large number of solutions can be derived for Meijer's G-function, the generalized hypergeometric functions, Bessel, Legendre, Whittaker functions, their combinations and many other functions. However, we present below only two special cases of (7.4.5).

In (7.4.1) and (7.4.2), putting $m = q = 1$, $n = p = 0$, $b_1 = 0$, $f_1 = 1$, $h = 1$, and using [23, p. 151, (5)], viz.

$$H_{0,1}^{1,0} [z | (0,1)] = e^{-z} \quad (7.5.1)$$

and [23, p. 151, (1)], viz.

$$H_{p,q}^{1,p} \left[z \left| \begin{matrix} (a_p, 1) \\ (b_q, 1) \end{matrix} \right. \right] = \frac{\prod_{j=1}^p \Gamma(1+b_1-a_j)}{\prod_{j=1}^q \Gamma(1+b_1-b_j)} z^{b_1} \times {}_pF_{q-1} \left[\begin{matrix} 1+b_1-a_1, \dots, 1+b_1-a_p; -z \\ 1+b_1-b_2, \dots, 1+b_1-b_q \end{matrix} \right], \quad p \leq q \quad (7.5.2)$$

in (7.4.3), we obtain the following solution:

$$u(x,t) = \sum_{r=0}^{\infty} e^{-kr(r+a+b+1)t} \frac{(-1)^r 2^w (a+b+2r+1) \Gamma(a+b+r+1)}{\Gamma(a+r+1)} P_r^{(a,b)}(x) \\ \times \frac{\Gamma(1+a+w) \Gamma(1+w)}{\Gamma(1+w-r) \Gamma(2+b+a+w+r)} {}_2F_2 \left[\begin{matrix} 1+a+w, 1+w \\ 1+w-r, 2+b+a+w+r \end{matrix}; -2Z \right] \quad (7.5.3)$$

where $\text{Re}(w+a) > -1$.

$$f(x) = (1-x)^w e^{-Z(1-x)} \quad (7.5.4)$$

In (7.5.3), putting $z = 0$, we have

$$u(x,t) = \sum_{r=0}^{\infty} e^{-kr(r+a+b+1)t} \frac{(-1)^r 2^r (a+b+2r+1) \Gamma(a+b+r+1) \Gamma(1+a+w) \Gamma(1+w)}{\Gamma(a+r+1) \Gamma(1+w-r) \Gamma(2+b+a+w+r)} P_r^{(a,b)}(x), \quad (7.5.5)$$

where $\text{Re}(w+a) > -1$.

$$f(x) = (1-x)^w \quad (7.5.6)$$

7.6 The Second Partial Differential Equation

Consider the following partial differential equation:

$$\frac{\partial u}{\partial t} = (1-x^2) \frac{\partial^2 u}{\partial x^2} - 2x \frac{\partial u}{\partial x} - \left(\frac{w^2}{1-x^2} \right) u, \quad -1 < x < 1 \quad (7.6.1)$$

The solutions of which lead to the Legendre functions and they must satisfy the initial condition:

$$u(x,0) = f(x),$$

and the boundary conditions :

$$u(x,t) \rightarrow 0, \quad \text{if } x \rightarrow 1,$$

$$u(x,t) \rightarrow 0, \quad \text{if } x \rightarrow -1, \quad (7.6.2)$$

The equation (7.6.1) is related to some differential equations of heat conduction (7.2.3).

In (7.2.3.), putting $A = w^2$ and $B = 0$, setting $(1-x^2)$ for k , $(1-x^2)^{-2} u$ for u and $2x$ for U , it reduces to the equation (7.6.1).

7.7 The Solution of the Second Partial Differential Equation in Terms of the Legendre Functions

The solution to be obtained is

$$u(x,t) = \sum_{r=0}^{\infty} C_r e^{-r(r+1)t} P_r^w(x) \quad (7.7.1)$$

Proof:

Let us assume that (7.6.1) has a solution of the form:

$$u(x,t) = y(x) e^{-r(r+1)t} \quad (7.7.2)$$

The substitution of $u(x,t)$ from (7.7.2) into (7.6.1) yields the following differential equation:

$$(1-x^2) y'' - 2xy' + \left[r(r+1) - \frac{w^2}{1-x^2} \right] y = 0 \quad (7.7.3)$$

which is Legendre equation [1, p. 186, (4.92)], with solution $y = P_r^w(x)$. The other solutions of Legendre equation are not appropriate in this problem.

Therefore the solution of (7.6.1) is of the form:

$$u(x,t) = e^{-r(r+1)t} P_r^w(x) \quad (7.7.4)$$

In view of the principle of superposition, the general solutions of (7.6.1) are given by (7.7.1).

In (7.7.1), putting $t = 0$, we get

$$f(x) = \sum_{r=0}^{\infty} C_r P_r^w(x) \quad (7.7.5)$$

Multiplying both sides of (7.7.5) by $P_u^w(x)$, integrating with respect to x from -1 to 1 , then using the orthogonality properties of the Legendre function (5.7.1b), the Fourier Legendre coefficients are given by:

$$C_r = \frac{2r+1}{2} \frac{(r-w)!}{(r+w)!} \int_{-1}^1 P_r^w(x) f(x) dx \quad (7.7.6)$$

7.8 Particular Solution of the Second Partial Differential Equation in Terms of Fox's H-Function

(i) If we take

$$f(x) = (1-x^2)^{c-1} H_{p,q}^{m,n} \left[z(1-x^2)^h \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] \quad (7.8.1)$$

in (7.7.5), and use the following integral [26, pp. 325-332]:

$$\int_{-1}^1 (1-x^2)^{c-1} P_r^w(x) H_{p,q}^{m,n} \left[z(1-x^2)^h \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] dx$$

$$= \frac{\pi 2^w}{\Gamma\left(\frac{2+r-w}{2}\right) \Gamma\left(\frac{1-r-w}{2}\right)} H_{p+2, q+2}^{m, n+2} \left[z \left| \begin{matrix} \left(1-c \pm \frac{w}{2}, h\right), (a_p, e_p) \\ (b_q, f_q), \left(1-c+\frac{r}{2}, h\right), \left(-c-\frac{r}{2}, h\right) \end{matrix} \right. \right], \quad (7.8.2)$$

where $h > 0$, $A \leq 0$, $B > 0$,

$|\arg z| = \frac{1}{2} B \pi$, $\operatorname{Re} \left(c + h \min_{1 \leq j \leq m} (b_j/f_j) \right) > |\operatorname{Re} w|/2$, to find the value of C_r ,

then the solution (7.7.1) reduces to

$$u(x, t) = \sum_{r=0}^{\infty} e^{-r(r+1)t} (2r+1) \frac{(r-w)!}{(r+w)!} \frac{\pi 2^{w-1}}{\Gamma\left(\frac{2+r-w}{2}\right) \Gamma\left(\frac{1-r-w}{2}\right)} P_r^w(x) \\ \times H_{p+2, q+2}^{m, n+2} \left[z \left| \begin{matrix} \left(1-c-\frac{w}{2}, h\right), \left(1-c+\frac{w}{2}, h\right), (a_p, e_p) \\ (b_q, f_q), \left(1-c+\frac{r}{2}, h\right), \left(-c-\frac{r}{2}, h\right) \end{matrix} \right. \right] \quad (7.8.3)$$

where the conditions are the same as in (7.7.6) and (7.8.2).

(ii) If we take

$$f(x) = (1-x^2)^c H_{p, q}^{m, n} \left[z (1-x^2)^h \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right] \\ = \sum_{r=0}^{\infty} C_r P_r^w(x), \quad (-1 < x < 1) \quad (7.8.4)$$

Equation (7.8.4) is valid since $f(x)$ is continuous and of bounded variation in the open interval $(-1, 1)$, when $c \geq 0$.

Multiplying both sides of (7.8.4) by $(1-x^2)^{-1} P_r^v(x)$ and integrating with respect to x from -1 to 1 , we get

$$\int_{-1}^1 (1-x^2)^{c-1} P_r^w(x) H_{p,q}^{m,n} \left[z (1-x^2)^h \middle| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dx$$

$$= \sum_{r=0}^{\infty} C_r \int_{-1}^1 (1-x^2)^{-1} P_r^v(x) P_r^w(x) dx.$$

Now, using (7.8.2) and the orthogonality property of the Legendre functions [19, p. 279, (31)] :

$$\int_{-1}^1 (1-x^2)^{-1} P_n^m(x) P_n^k(x) dx = \begin{cases} 0 & (k \neq m) \\ \frac{(n+m)!}{m(n-m)!} & (k=m), m \leq n \end{cases} \quad (7.8.5)$$

we obtain the Fourier Legendre coefficients :

$$C_r = w \frac{(r-w)!}{(r+w)!} \frac{\pi 2^w}{\Gamma\left(\frac{2+r-w}{2}\right) \Gamma\left(\frac{1-r-w}{2}\right)} H_{p+2, q+2}^{m, n+2} \left[z \middle| \begin{matrix} (1-c \pm \frac{w}{2}, h), (a_p, e_p) \\ (b_q, f_q), (1-c + \frac{r}{2}, h), (-c - \frac{r}{2}, h) \end{matrix} \right] \quad (7.8.6)$$

where the conditions are the same as in (7.8.2) and (7.8.5).

So, the solution (7.7.1) reduces to

$$u(x, t) = \sum_{r=0}^{\infty} e^{-r(r+1)t} w \frac{(r-w)!}{(r+w)!} \frac{\pi 2^w}{\Gamma\left(\frac{2+r-w}{2}\right) \Gamma\left(\frac{1-r-w}{2}\right)} P_r^w(x)$$

$$\times H_{p+2, q+2}^{m, n+2} \left[z \middle| \begin{matrix} (1-c - \frac{w}{2}, h), (1-c + \frac{w}{2}, h), (a_p, e_p) \\ (b_q, f_q), (1-c + \frac{r}{2}, h), (-c - \frac{r}{2}, h) \end{matrix} \right] \quad (7.8.7)$$

valid under the conditions of (7.8.2) and (7.8.5).

The substitution of $u(x,t)$ from (7.10.2) into (7.9.1) yields the following differential equation :

$$(1-x^2)^2 y'' - 2x(1-x^2) y' + \left((1-x^2) w(w+1) - r^2 \right) y = 0, \quad (7.10.3)$$

which is Legendre equation [1, p. 186, (4.92)], with solution $y = P_w^r(x)$.

Therefore the solution of (7.10.1) is of the form :

$$u(x,t) = e^{r^2 t} P_w^r(x) \quad (7.10.4)$$

In view of the principle of superposition, the general solution of (7.9.1) is given by (7.10.1).

In (7.10.1), putting $t = 0$, we get

$$f(x) = \sum_{r=0}^{\infty} C_r P_w^r(x) \quad (7.10.5)$$

Multiplying both sides of (7.10.5) by $(1-x^2)^{-1} P_w^u(x)$, integrating with respect to x from -1 to 1 , then using the orthogonality properties of the Legendre functions (7.8.5), we obtain

$$C_r = r \frac{(w-r)!}{(w+r)!} \int_{-1}^1 (1-x^2)^{-1} f(x) P_w^r(x) dx \quad (7.10.6)$$

REFERENCES

- 1- Andrews, L.C. Special Functions of Mathematics for Engineers. Mc Graw-Hill, Inc., Singapore, 1992.
- 2- Bajpai, S.D. Some Integrals Involving Gauss' Hypergeometric function and Meijer's G-Function. Proc. Camb. Philos. Soc. 63, (1967), 1049-1053.
- 3- Bajpai, S.D. On Products of Generalized Hypergeometric Functions. Gazeta De Matematica, Lisboa 29, No. 109-112, (1968), 15-21.
- 4- Bajpai, S.D. On Some Results Involving Fox's H-Function and Jacobi Polynomials. Proc. Camb. Philos. Soc., 65, (1969), 697-701.
- 5- Bajpai, S.D. A New Orthogonality Property for the Jacobi Polynomials and its Applications. Nat, Acad. Sci. Lett., Vol. 15, India (1992), 223-226.
- 6- Bajpai, S.D. Generating Function for a New Class of Polynomials and its Applications. Bull. Math. Ass. India (1992), 59-71.
- 7- Bajpai, S.D. A New Integral Containing the Jacobi Polynomial. Acta Math. Vietnam, Vol. 18-2, (1993), 219-223.
- 8- Bajpai, S.D. Generating Functions and Orthogonality Relations of a Class of Hypergeometric Polynomials. Pan. American Math. J. Vol. 3, (1993), 95-102.
- 9- Bajpai, S.D. Generating Function and Semi-Orthogonal properties of a New Class of Polynomials. Rend. Mat. (7), Vol. (13), Roma (1993), 365-372.

- 10- Bajpai, S.D. Semi-Orthogonal Properties of a Class of Gauss' Hypergeometric Polynomials. *Le Mathematiche*, vol. XLVIII (1993), 153-158.
- 11- Bajpai, S.D. Al-Hawaj, A. Y. and Mishra, Sadhana. Generating Function and Semi-orthogonal Relations for a Class of Hypergeometric Function. *Honam Math. J.*, Vol. 16, Kwangju (1994), 47-58.
- 12- Bajpai, S.D. Generating Function and orthogonal Type Formulae of a Class of Hypergeometric Polynomials. *Soochow J. Math.*, Vol. 20, Taipei (1994), 187-192.
- 13- Bajpai, S.D. New Semi-Orthogonality Property for the Jacobi Polynomials and Finite Fourier Jacobi Expansion. *Mat. Bilten*, Skopje (1994), 63-66.
- 14- Bajpai, S.D. orthogonal Properties for a Typical Class of Hypergeometric Functions. *Period. Math. Hungar.*, Vol. 29(2), Hungaria (1994), 169-175.
- 15- Bajpai, S.D. and Arora, M. S. Semi-Orthogonality of a Class of the Gauss' Hypergeometric Polynomials. *Ann. Math. Blaise Pascal*, Vol. 1, (1994), 75-83.
- 16- Bajpai, S.D., Arora, M.S. and Mishra, Sadhana. Semi-Orthogonal Relations for a Class of Gauss' Hypergeometric Functions. *Integral Transforms and Special Functions*. Vol. 2 (1994), 83-90.
- 17- Erdelyi, A. et al. *Higher Transcendental Functions*, Vol. 1. Mc Graw-Hill, New York (1953).
- 18- Erdelyi, A. et al. *Higher Transcendental Functions*, Vol. 2. Mc Graw-Hill, New York (1953).

- 19- Erdelyi, A. et al. Tables of Integral Transforms, Vol. 2. Mc Graw-Hill, New York (1954).
- 20- Carslaw, H.S. and Jaeger, J.C. Conduction of Heat in Solids (2nd End). Clarendon Press, Oxford (1986).
- 21- Fox, C. The G and H-Functions as Symmetrical Fourier Kernels. Trans. Amer. Soc. 98 (1961), 395-429.
- 22- Luke, Y. L. The Special Functions and Their Approximations, Vol. 1, Academic Press, New York, (1969).
- 23- Mathai, A.M. and Saxena, R.K. The H-Function with Applications in Statistics and other Disciplines. Wiley Eastern Ltd., New Delhi (1978).
- 24- Mishra, Sadhana. Integrals Involving Jacobi Polynomials, Generalized Hypergeometric Series and Fox's H-Function, and Fourier-Jacobi Series for Products of Generalized Hypergeometric Functions. Proc. Nat. Acad. Sci, India (1991), 29-41.
- 25- Rainville, E.D. Special Functions, Mac Millan Co., New York (1960).
- 26- Singh, F. and Varma, R.C. Application of E-Operator to evaluate A definite Integral and its Application in Heat Conduction. J. Indian Math. Soc. (N.S.) 36, (1972), 325-332.