

UNIVERSITY OF BAHRAIN

College of Science

Department of Mathematics



Fourier Harmonics of Non-Autonomous Bloch Equations

A Thesis submitted in Partial Fulfillment of the Requirements for the
Master Degree in Mathematics

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The Discussion Committee met on Thursday, 5/11/2015 (day and date) in Hall no. Bldg. 541 (hall/room no.) 200. Upon discussing the student whose details are stated above in public, the discussion committee recommends the following:

The student awarded grade A In the thesis subject MATHS 599

- ☐ Without making any changes to the thesis
☒ After making the modifications attached with this decision

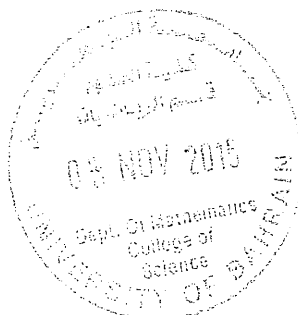
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Thesis Modifications Approval Form

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We would like to report that the student has carried out all the required modifications made by the discussion committee. Upon this, we recommend awarding the student grade A in thesis course number MATHS 399 based on the Discussion Committee recommendation.

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ABSTRACT

A mathematical model of non-autonomous ordinary differential equations (Bloch equations) with periodic coefficients is treated via two approaches of Fourier Series (FS) decomposition ansatz, namely, Truncated and Adiabatic FS.

These Bloch equations, within the field of theoretical quantum optics, model the coupling of a single atom of two-energy levels with a non-classical source of light (squeezed light, of reduced quantum fluctuation) where the atomic transition from lower to upper state or vice-versa takes place indirectly via two-way transitions.

The specific investigation, analytically and computationally, is concerned with physical quantities characteristic of the atomic medium, namely, absorption-dispersion line shapes and scattered intensity (up to 1st harmonics). One *major* result of the investigation is that, the (quantum) interference between the two indirect transitions *induces* a zero-absorption with finite dispersion at resonance (i.e. transparent medium).

DEDICATION

To My Dear Parents..

Lovely Husband, Hussain..

Sweet Child, Haidar Ali..

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CHAPTER 1

INTRODUCTION

- 1.1 Statement of the Problem
- 1.2 Background and History
- 1.3 Justification of the Problem
- 1.4 Theoretical Framework
- 1.5 Methods and Procedures
- 1.6 Proposed Chapters Outline

CHAPTER 1

INTRODUCTION

1.1 Statement of the Problem

The work in this thesis is concerned with an investigation of a mathematical model of a set of non-autonomous linear differential equations (Bloch equations) (Hassan and Ali 2001). In other words, the coefficients of the Bloch equations are explicitly functions of the independent variable (t), and particularly of harmonic forms, $e^{\pm i\omega t}$, with (ω) is a frequency parameter.

These Bloch equations (Hassan and Ali 2001), within the field of quantum optics, represent the interaction of a single atom of two-energy levels with a non-classical radiation source (of reduced quantum fluctuations), called a squeezed vacuum (SV) field, where the transition within the atomic system takes place from lower to upper state or vice-versa indirectly, via a 2-photon transition. Specifically, we analyze the atomic absorption and dispersion functions and intensity of scattered radiation due to this interaction.

We adopt two Fourier series (FS) decomposition approaches. One is based on truncation up-to 1st harmonics in ($n\omega$); $n = 0, \pm 1$. The other is based on adiabatic elimination of relevant variables prior to FS decomposition of other variables. We particularly examine the

fundamental ($n = 0$) and the first ($n = \pm 1$) harmonic components analytically and computationally. Physical interpretation is given in comparison to other one-photon transition models (Hassan et al 1995).

1.2 Background and History

The field of quantum optics or non-linear optics is concerned with the study of the interaction of an atomic (or molecular) system with various sources of radiation. In addition to the known sources of "thermal" and "laser" radiation (e.g. Agarwal 2013, Gerry and Knight 2005), the "squeezed vacuum state" of radiation field, as extensively studied and produced in the Laboratories (Agarwal 2013, Garrison and Chiao 2014), offers the opportunity of more precise measurements of atomic system parameters and detection of weak gravitational waves.

The proposed mathematical model represents the dynamical behavior of a single 2-level atom in the presence of a squeezed vacuum radiation, where the atomic transition occurs via a 2-photon transition. Due to the nature of this transition, the model system of 1st order ordinary differential equations (ODEs) called Bloch equations, have harmonic coefficients dependent on time (t). In general, these Bloch equations of harmonic coefficients have no simple closed form of solutions (e.g. Hassan et al 1995, Awojoyogbe 2004) - see Chapter 2 for more details. According to Floquet theorem (Hale 1996, Kelley and Peterson 2010) - or Bloch wave theory (Zwillinger 1989, Markos and Soukoulis 2008), their solution contain all harmonics (as FS) of the concerned frequency (ω).

It should be noted that, for other non-autonomous linear systems of ODEs, previous treatments were mainly adopting the following four approaches:

- (i) Continued fractions based on FS expansion for all variables as components of all harmonics ($n\omega$); $n = 0, \pm 1, \dots$. This leads to 3-term recurrence relations and hence treated numerically (Hassan et al 1995).
- (ii) Secular approximation, which involves a strong (external) parameter limit (Joshi and Hassan 1997, Hassan et al 2003).
- (iii) Perturbative approach, which involves a weak (external) parameter limit (Hassan et al 2003).
- (iv) Iterative approach for large harmonic detuning parameter $|\omega - \omega_0| \gg 1$ (ω_0 is a reference frequency), with all other parameters kept arbitrary (Batarfi et al 2010, Hassan and Al-Harbey 2012, Hassan and Al-Harbey 2013).

The proposed subject in the thesis deals with an analytical examination of the non-autonomous model of Bloch equations derived in (Hassan and Ali 2001) via FS decomposition (up to fundamental and 1st harmonic components) for the intensity and absorption-dispersion atomic variables. Analytical expressions are derived and will be investigated computationally for various system parameters.

1.3 Justification of the Problem

For a combined system of (matter + radiation), the study of atomic variables, namely the scattered intensity and absorption-dispersion polarization functions is one way to gain information about how the matter

system responds to a particular state of radiation environment. In particular, some implications of such a study are:

- (i) "Lasing action" without external pumping of the atomic system, can be achieved, in principle, with the use of the "squeezed vacuum" radiation field (Hassan et al 1997, Hassan et al 2003, Hassan et al 2005).
- (ii) "Slowing and freezing of light" (Hau 2001) may be achieved through a manipulation of the absorption-dispersion functions ((Hassan et al 2003, Hassan and EL-Deberkey 2005); and refs. therein).

The proposed model of non-autonomous Bloch equations offers the possibility to achieve zero-absorption with finite dispersion with the fundamental component, in addition to amplified (negative) absorption associated with the 1st harmonic components of the atomic polarization components.

1.4 Theoretical Framework

The proposed system of non-autonomous model of Bloch equations (Hassan and Ali 2001) is to be investigated within FS decomposition with two specific aspects:

- (i) Intensity harmonics lineshape of scattered radiation.
- (ii) The absorption-dispersion atomic behavior and the search for regions of zero or negative absorption accompanied by finite dispersion. This is directly related to a propagating pulse in the medium without gain or loss of information with controllable speed.

1.5 Methods and Procedures

Within the mathematical framework of FS decomposition, the calculation steps of dealing with the non-autonomous model of Bloch equations (Hassan and Ali 2001) are spelled as follows:

- (i) The use of FS decomposition to solve the model DEs in the steady state. Specifically, we use two approaches. The first approach is the truncated FS expansion for all atomic variables (atomic inversion and polarization components), up to 1st harmonic components in $e^{in\omega t}$ ($n = 0, \pm 1$), with (ω) the concerned frequency (or frequency mis-match parameter). The other approach uses adiabatic elimination of some of the atomic variables to derive a single 1st order DE for one of the variables, which is then treated by FS decomposition. Comparison of the two approaches is made.
- (ii) Based on the linear independence of these harmonics, we equate the coefficients (which are functions of the system parameters) of equal harmonics on both sides of the equations.
- (iii) In the steady state regime, this leads to a system of linear algebraic equations for the envelope harmonics.
- (iv) This in turn, is treated analytically and computationally to examine the behavior of these envelope functions against various (controllable) system parameters.

1.6 Proposed Chapters Outline

In Chapter (2), (Literature Review), we present the model Bloch equations and the physical quantities they represent. A brief reference to

Floquet's theorem, linear independence of functions and Fourier series are presented.

In Chapter (3), (Methodology: Fourier Series Decomposition), we use both FS decompositions (truncated and adiabatic elimination) to solve the model Bloch equations and derive a system of algebraic equations for the harmonic envelopes, in the steady state regime. Three cases of transition models are classified with the corresponding algebraic equations, namely, the cascade, the Λ - and V- models.

In Chapter (4), (Results: Harmonic Envelopes), we computationally investigate the analytical results derived in Chapter (3) for the harmonic envelope functions against various controllable system parameters, in the three cases of transition models. Analytical expressions for the harmonic envelope functions are derived in some special cases. Comparison with the one-photon transition case (Hassan et al 1995) is made.

CHAPTER 2

LITERATURE REVIEW

- 2.1 The Model Equations
- 2.2 Floquet's Theorem
- 2.3 Linear Independence of Functions
- 2.4 Fourier Series

CHAPTER 2

Literature Review

Investigation of the suggested Bloch equations (Hassan and Ali 2001) modeling the interaction of a single 2-level atom with squeezed vacuum radiation via 2-photon transition processes is motivated by two main factors:

- (i) Availability of such squeezed vacuum (SV) radiation in the Laboratories (c.f. Agarwal 2013, Gerry and Knight 2005).
- (ii) The manipulation of many control parameters in the suggested model Bloch equations (Hassan and Ali 2001) that enhance the possibility of finding regions of zero or negative - absorption with finite atomic dispersion. This is directly related to probing the suggested model with a propagating weak pulse within the medium without any loss or absorption of energy and with controllable speed of propagation (c.f. Hassan and Al-Harbey 2012, Hassan et al 2005, Hau 2001).

In general, Bloch equations with variable coefficients arise in many branches of science, e.g., physics, fluid mechanics, chemistry, ... They are either solved analytically (exactly or perturbatively) for special forms of time-dependent coefficients or under preset conditions with involved computation of the derived solutions in some cases. For example:

In *optics*, the Bloch equations together with Maxwell equation (Maxwell-Bloch equations) modelling the photon-echo efficiency of the multiple pulse sequences in optically thick media (Tsang et al 2003) have

been solved analytically in the two cases of energetically weak or temporarily brief pulses.

The model Bloch equations for 2-level atom driven by amplitude and frequency-modulated fields (Prants and Yakupova 1990) have been solved using group theoretic approach in the case of no relaxation and in special cases of envelope amplitude fields and frequency modulations. The inclusion of damping constants (γ) has been treated only in a small time limit ($\gamma t < 1$).

In *nuclear magnetic resonance* (NMR), the model Bloch equations modelling the relaxing magnetic moment in the presence of an external oscillating magnetic field have been solved subject to a high static field near resonance (Leyte 1990).

In biomedical *fluid dynamics*, the model Bloch equation describing the hemodynamics, functions of blood flow system, (Awojoyogbe and Boubaker 2009) have their proposed solutions in terms of polynomials and further treated numerically. Furthermore, some analytical solutions for the Bloch equations were derived using translational mechanical analysis (Awojoyogbe 2004), but their computations required laborious work!

Investigation of the model Bloch equations (Hassan and Ali 2001) have not been done previously. The advantage of the present method of the FS decomposition to solve the non-autonomous model Bloch equations (Hassan and Ali 2001) is that it represents the oscillatory wave solutions of the Bloch variables into component waves of different frequencies and amplitudes, i.e. harmonic components. This leads to the identification

(manipulation) of different frequency harmonics separately out of the whole harmonics (spectrum).

2.1 The Model Equations

Within the framework of quantum mechanics, energy of both radiation fields and atomic systems are discrete (quantized). Radiation fields have infinite number of energy levels, while atomic systems have finite number of energy levels depending on their structure (e.g. Gerry and Knight 2005). When radiation fields interact with an atomic system, it affects its energy (called atomic inversion) and it induces an atomic field (called atomic polarization) and consequently a scattered (or emitted) radiation is born out of this interaction.

The simplest atomic structure is the two-energy level atomic system with upper and lower energies E_b and E_a , respectively. According to quantum mechanics, energy is proportional to the atomic frequency: $E_b = \hbar\omega_b, E_a = \hbar\omega_a$ ($\omega_b > \omega_a$) with $\hbar = 1.0545717 \dots \times 10^{-34} \text{ Js}$ is the universal (reduced) Planck's constant.

On the other hand, the case of a laser radiation field with reduced (or squeezed) quantum fluctuations, called squeezed vacuum (SV) field, represents an advanced step (both theoretically and experimentally) towards precision measurements with microscopic systems in the field of nonlinear quantum optics (Gerry and Knight 2005).

The model (Bloch) equations for the averaged atomic variables, representing the interaction of a two-energy level atomic system of

energies E_b, E_a ($E_b > E_a$) with SV field, where the transition between the two levels E_b, E_a occurs via an intermediate set of dense levels $\{E_j\}$, are of the form (Hassan and Ali 2001),

$$\begin{aligned}\dot{w} = & - \left[\gamma_1^+ + \frac{\gamma_2^+ M}{2} (e^{2i\omega_p t} + e^{-2i\omega_p t}) \right] w \\ & - \left[\gamma_1^- + \frac{\gamma_2^- M}{2} (e^{2i\omega_p t} + e^{-2i\omega_p t}) \right] \\ & + \gamma_3^- (r e^{-2i\omega_p t} + s e^{2i\omega_p t}) + \frac{1}{2} \gamma_4 M (r + s)\end{aligned}\tag{2.1a}$$

$$\begin{aligned}\dot{r} = & - [\gamma_1^+ + i\Delta - M(\lambda_1 e^{2i\omega_p t} + \lambda_2 e^{-2i\omega_p t})] r \\ & - (\gamma_3^- w + \gamma_3^+) e^{2i\omega_p t} - M(\lambda_3^- w + \lambda_3^+) \\ = & (\dot{s})^*\end{aligned}\tag{2.1b}$$

where $\dot{w} = \frac{dw}{dt}$, etc. is the total derivative with respect to time (t) and $(\)^*$ means the complex conjugate.

The notations in eqs (2.1) are: w is the (real) averaged atomic inversion, $r = s^*$ is the (complex) averaged induced atomic polarization. The SV radiation field parameters N and (real) $M = \sqrt{N(N+1)}$ are the measures of averaged photon number and degree of squeezing (e.g. Gerry and Knight 2005, Agarwal 2013). The special case of zero degree of squeezing ($M = 0$) represents a usual thermal radiation source. The frequency detuning parameter $\Delta = \omega_{ba} - 2\omega_p$ where $\omega_{ba} = \omega_b - \omega_a$ is the atomic frequency difference and ω_p is the central frequency of the SV field.

The various damping rates and constants in eqs (2.1) are given explicitly as follows (Hassan and Ali 2001):

$$\begin{aligned}
\gamma_1^\pm &= \frac{1}{2} \sum_j (\Gamma_{bj} \pm \Gamma_{aj}), \\
\gamma_2^\pm &= \frac{1}{2} \sum_j \{ (\gamma_{bj} \theta(\omega_{bj}) + \gamma_{jb} \theta(\omega_{jb})) \pm (\text{same with } b \rightarrow a) \}, \\
\gamma_3^\pm &= \frac{1}{2} \sum_j (\Gamma_{bj}^{ab} \pm \Gamma_{aj}^{ab}), \\
\gamma_4 &= \sum_j \{ (\gamma_{aj}^{ab} \theta(\omega_{aj}) + \gamma_{ja}^{ab} \theta(\omega_{ja})) - (\text{same with } a \rightarrow b) \}; \\
\Gamma_{ja} &= \gamma_{ja} \theta(\omega_{ja})(1 + N) + \gamma_{aj} \theta(\omega_{aj})N; \quad \text{etc.} \\
\Gamma_{aj}^{ab} &= \gamma_{aj}^{ab} \theta(\omega_{aj})(1 + N) + \gamma_{ja}^{ab} \theta(\omega_{ja})N, \\
\lambda_1 &= \frac{1}{2} \sum_j [\gamma_{bj} \theta(\omega_{bj}) + \gamma_{ja} \theta(\omega_{ja})], \\
\lambda_2 &= \lambda_1(a \leftrightarrow b), \\
\lambda_3^\pm &= \frac{1}{2} \sum_j [\gamma_{ja}^{ab} \theta(\omega_{ja}) \pm \gamma_{bj}^{ab} \theta(\omega_{bj})] \tag{2.2}
\end{aligned}$$

where γ_{bj} is the decay rate from the state (E_b) to one of the states $\{E_j\}$, etc., and γ_{aj}^{ab} represents a (quantum) interference effect between the two transition channels $E_a \rightarrow E_j$ and $E_b \rightarrow E_j$ with $\omega_{bj} = \omega_b - \omega_j$, etc. The step function $\theta(x)$ is defined by,

$$\theta(x) = \begin{cases} 1 & , \quad x > 0 \\ 0 & , \quad x < 0 \end{cases} \tag{2.3}$$

and its role is determined by the configuration of the intermediate levels $\{E_j\}$ (see Chapter 3).

The physical interpretation of the decay rate parameters and other constants in eqs (2.1) are as follows (Hassan and Ali 2001):

- (i) The rate γ_1^+ represents a total decay rate of the atomic inversion and polarization between levels $E_b \leftrightarrow E_j$ and $E_a \leftrightarrow E_j$.
- (ii) The rate γ_1^- represents an equilibrium value for the atomic inversion.
- (iii) The rates $\gamma_2^\pm$ are associated with the induced steady oscillations in the atomic inversion (w) due to the presence of the SV field ($M \neq 0$). Similarly, the rates $\lambda_{1,2}$ are associated with the induced steady oscillations in the atomic polarizations (r, s) due to the SV parameter M .
- (iv) The rate γ_3^- , due to quantum interference (QI) effect, is associated with the induced 2-photon polarizations (r, s) in the equation for ($\dot{w}$). As well, it is associated with the atomic inversion w in the equation for ($\dot{r}, \dot{s}$).
- (v) The rate γ_3^+ represents a steady oscillation source for the atomic polarization (r, s) due to the two-photon transition interference processes.
- (vi) The rates γ_4 and λ_3^- , due to QI effect, show that the SV field acts as an oscillating driving field for both atomic inversion (w) and polarization (r, s).
- (vii) The rate λ_3^+ , due to QI effect, is associated with the SV field parameter (M) that contributes to the steady state value of the atomic polarization components (r, s).

Derivation of the model equations (2.1) within a full quantized Hamiltonian approach is given in (Hassan and Ali 2001).

2.2 Floquet's Theorem

Consider the non-homogeneous linear system of ODEs of matrix form,

$$\dot{X}(t) = A(t)X(t) - I(t) \quad (2.4)$$

where $A(t)$ is a square periodic and complex matrix function of t and $I(t)$ is column vector which, in general, has time-dependent elements.

First, for the homogeneous part of the above system, namely,

$$\dot{X}(t) = A(t)X(t) \quad (2.5)$$

and according to Floquet's theorem (Hale 1996) every fundamental matrix solution for $X(t)$ in (2.5) has the form,

$$X(t) = B(t)e^{\beta t} \quad (2.6)$$

where the matrix $B(t)$ is of the same periodicity as $A(t)$ and β is a constant.

Second, the effect of the non-homogeneous part $I(t)$ in (2.4) is to modulate or persist the periodic behavior, particularly in the long time limit where $\dot{X}(t) = 0$ (e.g. Hassan and El-Deberky 2005).

Note that, the model eqs (2.1) are of the same matrix form (2.4) and according to theory (e.g. Hale 1996) there exists a unique solution of $X(t)$ on the whole interval $t \in [0, \infty)$ for arbitrary initial condition $X(t_0); t_0 \in [0, \infty)$ if $A(t)$ and $I(t)$ are continuous functions on that interval $[0, \infty)$. The continuity conditions on the matrix elements of $A(t)$ and the inhomogeneous vector $I(t)$ are physically met in our model under study.

2.3 Linear Independence of Functions

Consider the set of functions $F = \{f_1(t), f_2(t), \dots, f_n(t)\}$.

The elements $f_1(t), f_2(t), \dots, f_n(t)$ are said to be linearly independent over a certain interval I if the linear combination,

$$\alpha_1 f_1(t) + \alpha_2 f_2(t) + \dots + \alpha_n f_n(t) = 0, \quad t \in I \quad (2.7)$$

is valid with all constant coefficients $\alpha_i (i = 1 \rightarrow n) = 0$ (Goode and Annin 2007).

If, however, (2.7) is valid with some or all of the coefficients α_i are non-zero, then the elements $f_1(t), f_2(t), \dots, f_n(t)$ are said to be linearly dependent.

2.4 Fourier Series

The idea of waves is a common share in many branches of science: Mechanics, fluid mechanics, elasticity, sounds, optics etc.. Also, waves are part of our daily life practice: transmission of information via emails, fax machines, wireless communications, radio, television, music, seeing, hearing, talking, cooking and its use in many medical treatments, etc. (Markos and Soukoulis 2008).

The simple and basic modelling of waves is a vibrating string that vibrates at different frequencies $n\omega, n = 1, 2, \dots$. The least frequency ω is called the fundamental frequency and the next higher frequencies $n\omega, n \geq 2$ are called higher harmonics.

The vibration at different frequencies leads to the summation of a number of higher harmonic frequencies that gives rise to a complex periodic form.

Mathematically speaking, let $f(t)$ be a periodic function of period T that satisfies the Dirichlet's conditions, namely,

- (i) The integral $\int_{-T/2}^{T/2} |f(t)| dt$ is definite, and
- (ii) $f(t)$ is a piecewise continuous and piecewise monotonic function.

Then $f(t)$ can be represented in the general complex convergent FS form (Strauss 1992),

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \quad (2.8a)$$

where the coefficients c_n are given by

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega t} dt \quad (2.8b)$$

and the frequency $\omega = 2\pi/T$.

The FS form (2.8) is thus a decomposition (i.e. break down) of $f(t)$ into component waves of different frequencies and amplitudes that can be analysed separately.

In our case, the Dirichlet's conditions are practically met with all functions of interest in our study (namely, atomic variables). Further, positive (negative) values of n in the sum of eq (2.8a) are associated with

positive (negative) frequencies. For example, the complex function $e^{\pm i\omega t}$ of constant amplitude and phase ($\pm\omega t$) represents a counterclock- (clock-) wise rotation in the Argand diagram.

CHAPTER 3

Methodology: Fourier Series Decomposition

3.1 Truncated Fourier Series: The General Case

3.1.1 The Cascade Model

3.1.2 The Λ -Model

3.1.3 The V-Model

3.2 Adiabatic Fourier Series

3.2.1 The Cascade Model

3.2.2 The Λ - and V- Models

CHAPTER 3

Methodology: Fourier Series Decomposition

In this Chapter, the model system of linear DEs presented in Chapter 2, eqs (2.2), is treated via two approaches of FS decomposition, namely:

- (i) The truncated FS up to 1st harmonics, and
- (ii) The adiabatic FS, which depends on adiabatic elimination of some of the system variables to reach a single 1st order DE for one of the remaining variables that is treated by FS decomposition. The choice of the adiabatically eliminated variables depends on the system configuration hence its associated DEs, as shown below.

This is followed by specializing the discussion to the three cases of indirect atomic transition models via the set of intermediate levels $\{E_j\}$.

3.1 Truncated Fourier Series: The General Case

Due to the presence of the harmonic coefficients $e^{\pm 2i\omega_p t}$ in the linear system of DEs, (2.1) and in the light of Floquet's theorem (section 2.2) their solutions are expressed as FS of the harmonics $(2n\omega_p); n = 0, \pm 1, \dots$. Up to 1st harmonics ($n = \pm 1$), we use the truncated FS ansatz for the atomic inversion (w) and polarization (r, s) components,

$$w = w_0 + w_+ e^{2i\omega_p t} + w_- e^{-2i\omega_p t} \quad (3.1a)$$

$$r = r_0 + r_+ e^{2i\omega_p t} + r_- e^{-2i\omega_p t} \quad (3.1b)$$

$$s = s_0 + s_+ e^{2i\omega_p t} + s_- e^{-2i\omega_p t} \quad (3.1c)$$
$$= (r)^*.$$

Here $(w_0, r_0 = s_0^*)$ are the non-oscillatory (fundamental) envelope components, while, $(w_{\pm} = w_{\mp}^*, r_{\pm} = s_{\mp}^*)$ are the 1st harmonic envelope components of atomic inversion and polarization variables.

In an asymptotic stationary state where the envelope components in (3.1) are slowly varying functions in time $O(\frac{1}{\omega_p})$, i.e., (constants over an optical period),

$$\begin{aligned}\dot{w}_l &\ll \omega_p w_l, \\ \dot{r}_l &\ll \omega_p r_l, \\ \dot{s}_l &\ll \omega_p s_l \quad (l = 0, \pm)\end{aligned}\tag{3.2}$$

we set $\dot{w}_l = \dot{r}_l = \dot{s}_l = 0$. Substituting (3.1) into eq (2.1a) we then get,

$$\begin{aligned}&2i\omega_p(w_+e^{2i\omega_p t} - w_+^*e^{-2i\omega_p t}) \\&= \left[-\gamma_1^+ w_0 - \frac{\gamma_2^+ M}{2} w_+ - \frac{\gamma_2^+ M}{2} w_- - \gamma_1^- + \gamma_3^- r_+ + \gamma_3^- r_+^* \right. \\&\quad \left. + \frac{\gamma_4 M}{2} (r_0 + r_0^*) \right] \\&+ \left[-\gamma_1^+ w_+ - \frac{\gamma_2^+ M}{2} w_0 - \frac{\gamma_2^- M}{2} + \gamma_3^- r_0^* \right. \\&\quad \left. + \frac{\gamma_4 M}{2} (r_+ + r_-^*) \right] e^{2i\omega_p t} \\&+ \left[-\gamma_1^+ w_- - \frac{\gamma_2^+ M}{2} w_0 - \frac{\gamma_2^- M}{2} + \gamma_3^- r_0 \right. \\&\quad \left. + \frac{\gamma_4 M}{2} (r_- + r_+^*) \right] e^{-2i\omega_p t} + O(e^{\pm 4i\omega_p t}).\end{aligned}\tag{3.3a}$$

Similarly, the use of (3.1) into (2.1b) and (2.1c) gives,

$$\begin{aligned}
& 2i\omega_p(r_+e^{2i\omega_pt} - r_-e^{-2i\omega_pt}) \\
&= [-(i\Delta + \gamma_1^+)r_0 + M\lambda_1r_- + M\lambda_2r_+ - \gamma_3^-w_+^* - M\lambda_3^-w_0 - M\lambda_3^+] \\
&\quad + [-(i\Delta + \gamma_1^+)r_+ + M\lambda_1r_0 - \gamma_3^-w_0 - \gamma_3^+ - M\lambda_3^-w_+]e^{2i\omega_pt} \\
&\quad + [-(i\Delta + \gamma_1^+)r_- + M\lambda_2r_0 - M\lambda_3^-w_+^*]e^{-2i\omega_pt} \\
&\quad + O(e^{\pm 4i\omega_pt})
\end{aligned} \tag{3.3b}$$

and its complex conjugate, namely,

$$\begin{aligned}
& -2i\omega_p(r_+^*e^{-2i\omega_pt} - r_-^*e^{2i\omega_pt}) \\
&= [-(-i\Delta + \gamma_1^+)r_0 + M\lambda_1r_-^* + M\lambda_2r_+^* - \gamma_3^-w_+ - M\lambda_3^-w_0 \\
&\quad - M\lambda_3^+] \\
&\quad + [-(-i\Delta + \gamma_1^+)r_+^* + M\lambda_1r_0^* - \gamma_3^-w_0 - \gamma_3^+ \\
&\quad - M\lambda_3^-w_+^*]e^{-2i\omega_pt} \\
&\quad + [-(-i\Delta + \gamma_1^+)r_-^* + M\lambda_2r_0^* - M\lambda_3^-w_+]e^{2i\omega_pt} \\
&\quad + O(e^{\pm 4i\omega_pt}).
\end{aligned} \tag{3.3c}$$

Here, we discard the faster oscillatory terms in $(e^{\pm 4i\omega_pt})$ compared with those in $(e^{\pm 2i\omega_pt})$. This is a common approximation known as rotating wave approximation (RWA) (e.g. Agarwal 2013, Garrison and Chiao 2014). Within this context, it is worth to mention the following:

- (i) The dropping of the faster oscillatory terms is in conformity with the steady state (long time) regime.
- (ii) The original Bloch equations, eq (2.1), (Hassan and Ali 2001) have been derived within the RWA (i.e. by dropping terms in $e^{\pm 4i\omega_pt}$ and higher).
- (iii) In the FS truncated ansatz, (3.1), if the higher oscillatory terms $(e^{\pm 4i\omega_pt})$ are included, then the resulting expression in (3.3)

involves the higher oscillatory terms ($e^{\pm 6i\omega_p t}$). So, the FS truncation ansatz leads to a hierarchy of harmonics, due to the occurrence of the terms $w e^{\pm 2i\omega_p t}$, $r e^{\pm 2i\omega_p t}$ and $s e^{\pm 2i\omega_p t}$ in eqs (2.1).

Now, since the harmonic function coefficients $e^0, e^{\pm 2i\omega_p t}$ are linearly independent (see section 2.3), their net coefficients must be zero (or, equivalently, comparing the coefficients of equal harmonics). So we get the following system of linear algebraic equations,

$$-\gamma_1^+ w_0 - \frac{\gamma_2^+ M}{2} w_+ - \frac{\gamma_2^+ M}{2} w_+^* + \frac{\gamma_4 M}{2} r_0 + \frac{\gamma_4 M}{2} r_0^* + \gamma_3^- r_+ + \gamma_3^- r_+^* = \gamma_1^-, \quad (3.4a)$$

$$-\frac{\gamma_2^+ M}{2} w_0 - (\gamma_1^+ + 2i\omega_p) w_+ + \gamma_3^- r_0^* + \frac{\gamma_4 M}{2} r_+ + \frac{\gamma_4 M}{2} r_-^* = \frac{\gamma_2^- M}{2}, \quad (3.4b)$$

$$-\frac{\gamma_2^+ M}{2} w_0 - (\gamma_1^+ - 2i\omega_p) w_+^* + \gamma_3^- r_0 + \frac{\gamma_4 M}{2} r_+^* + \frac{\gamma_4 M}{2} r_- = \frac{\gamma_2^- M}{2}, \quad (3.4c)$$

$$-M\lambda_3^- w_0 - \gamma_3^- w_+^* - (i\Delta + \gamma_1^+) r_0 + M\lambda_2 r_+ + M\lambda_1 r_- = M\lambda_3^+, \quad (3.4d)$$

$$-\gamma_3^- w_0 - M\lambda_3^- w_+ + M\lambda_1 r_0 - (i\Delta + \gamma_1^+ + 2i\omega_p) r_+ = \gamma_3^+, \quad (3.4e)$$

$$-M\lambda_3^- w_+^* + M\lambda_2 r_0 - (i\Delta + \gamma_1^+ - 2i\omega_p) r_- = 0, \quad (3.4f)$$

$$-M\lambda_3^- w_0 - \gamma_3^- w_+ - (i\Delta + \gamma_1^+) r_0^* + M\lambda_2 r_+^* + M\lambda_1 r_-^* = M\lambda_3^+, \quad (3.4g)$$

$$-\gamma_3^- w_0 - M\lambda_3^- w_+^* + M\lambda_1 r_0^* - (i\Delta + \gamma_1^+ + 2i\omega_p) r_+^* = \gamma_3^+, \quad (3.4h)$$

$$-M\lambda_3^- w_+ + M\lambda_2 r_0^* - (-i\Delta + \gamma_1^+ + 2i\omega_p) r_-^* = 0. \quad (3.4i)$$

To reduce the number of equations in (3.4), we eliminate r_-, r_-^* from (3.4f) and (3.4i) to get,

$$r_- = \frac{M(\lambda_2 r_0 - \lambda_3^- w_+^*)}{\gamma_1^+ + i(\Delta - 2\omega_p)} \quad (3.5a)$$

$$r_-^* = \frac{M(\lambda_2 r_0^* - \lambda_3^- w_+)}{\gamma_1^+ - i(\Delta - 2\omega_p)}. \quad (3.5b)$$

Substituting (3.5a,b) into (3.4b,c) for $(r_-), (r_-^*)$, we get the reduced set of seven linear algebraic equations:

$$\begin{aligned}
-\gamma_1^+ w_0 - \frac{\gamma_2^+ M}{2} w_+ - \frac{\gamma_2^+ M}{2} w_+^* + \frac{\gamma_4 M}{2} r_0 + \gamma_3^- r_+ + \gamma_3^- r_+^* + \frac{\gamma_4 M}{2} r_0^* &= \gamma_1^-, \\
-\frac{\gamma_2^+ M}{2} w_0 - a_1 w_+ + \frac{\gamma_4 M}{2} r_+ + a_2 r_0^* &= \frac{\gamma_2^- M}{2}, \\
-\frac{\gamma_2^+ M}{2} w_0 - a_1^* w_+^* + a_2^* r_0 + \frac{\gamma_4 M}{2} r_+^* &= \frac{\gamma_2^- M}{2}, \\
-M\lambda_3^- w_0 - b_1 w_+^* - b_2 r_0 + M\lambda_2 r_+ &= M\lambda_3^+, \\
-M\lambda_3^- w_0 - b_1^* w_+^* + M\lambda_2 r_+^* - b_2^* r_0^* &= M\lambda_3^+, \\
-\gamma_3^- w_0 - M\lambda_3^- w_+ + M\lambda_1 r_0 - c_1 r_+ &= \gamma_3^+, \\
-\gamma_3^- w_0 - M\lambda_3^- w_+^* - c_1^* r_+^* + M\lambda_1 r_0^* &= \gamma_3^+
\end{aligned} \tag{3.6}$$

where, the coefficients $a_{1,2}, b_{1,2}, c_1$ are given by,

$$\begin{aligned}
a_1 &= \gamma_1^+ + 2i\omega_p + \frac{M\gamma_4\lambda_3^-}{2(\gamma_1^+ - i(\Delta - 2\omega_p))}, \\
a_2 &= \gamma_3^- + \frac{M^2\gamma_4\lambda_2}{2(\gamma_1^+ - i(\Delta - 2\omega_p))}, \\
b_1 &= \gamma_3^- + \frac{M^2\lambda_1\lambda_3^-}{\gamma_1^+ + i(\Delta - 2\omega_p)}, \\
b_2 &= \gamma_1^+ + i\Delta - \frac{M^2\lambda_1\lambda_2}{\gamma_1^+ + i(\Delta - 2\omega_p)}, \\
c_1 &= \gamma_1^+ + i(\Delta + 2\omega_p).
\end{aligned} \tag{3.7}$$

In matrix form, eqs (3.6) are:

$$AX = I \tag{3.8}$$

where

$$X = [w_0 \quad w_+ \quad w_+^* \quad r_0 \quad r_+ \quad r_+^* \quad r_0^*]^T, \tag{3.9a}$$

$$I = \left[\gamma_1^- \quad \frac{\gamma_2^- M}{2} \quad \frac{\gamma_2^- M}{2} \quad M\lambda_3^+ \quad M\lambda_3^+ \quad \gamma_3^+ \quad \gamma_3^+ \right]^T \tag{3.9b}$$

and

$$A = \begin{bmatrix} -\gamma_1^+ & -\frac{\gamma_2^+ M}{2} & -\frac{\gamma_2^+ M}{2} & \frac{\gamma_4 M}{2} & \gamma_3^- & \gamma_3^- & \frac{\gamma_4 M}{2} \\ -\frac{\gamma_2^+ M}{2} & -a_1 & 0 & 0 & \frac{\gamma_4 M}{2} & 0 & a_2 \\ -\frac{\gamma_2^+ M}{2} & 0 & -a_1^* & a_2^* & 0 & \frac{\gamma_4 M}{2} & 0 \\ -M\lambda_3^- & 0 & -b_1 & -b_2 & M\lambda_2 & 0 & 0 \\ -M\lambda_3^- & -b_1^* & 0 & 0 & 0 & M\lambda_2 & -b_2^* \\ -\gamma_3^- & -M\lambda_3^- & 0 & M\lambda_1 & -c_1 & 0 & 0 \\ -\gamma_3^- & 0 & -M\lambda_3^- & 0 & 0 & -c_1^* & M\lambda_1 \end{bmatrix} \quad (3.9c)$$

and the notation $[\quad]^T$ refers to the transpose matrix.

The formal matrix solution of (3.8) is,

$$X = A^{-1}I \quad (3.10)$$

where A is non-singular matrix (which is met in our model).

The *general* form of the elements of the vector I and the matrix A , eqs (3.9b, c), are given in (2.2). Note that, the model eqs (2.1), for the atomic variables (w, r, s) represent a 2-step (i.e. 2-photon processes) transition between the upper level (E_b) and lower level (E_a) via a set of intermediate levels $\{E_j\}$. There are three possibilities for this indirect transition examined in the following sub-sections.

3.1.1 The Cascade Model

In this case, $\omega_b > \omega_j > \omega_a$ for all j (Figure (1)), so the step functions appearing in the expressions for the decay constants given in (2.2) have the values:

$$\begin{aligned} \theta(\omega_{bj}) &= \theta(\omega_{ja}) = 1, \\ \theta(\omega_{jb}) &= \theta(\omega_{aj}) = 0. \end{aligned} \quad (3.11)$$

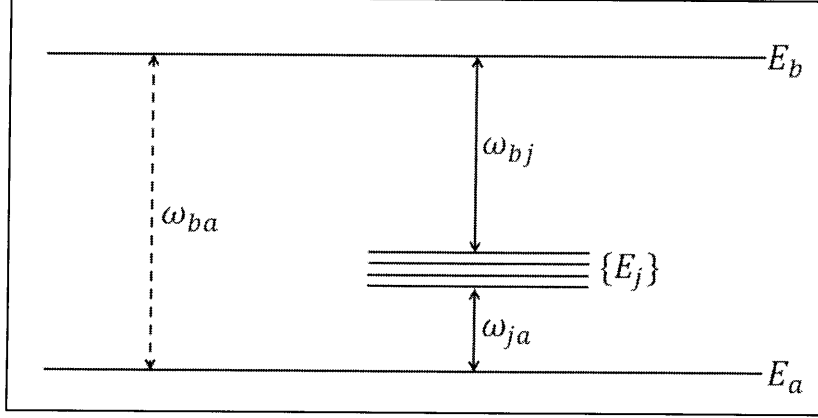


Figure (1): The cascade model ($E_b > E_j > E_a$)

For simplicity, we assume equal decay rates from levels $E_b \rightarrow E_j$ and $E_j \rightarrow E_a$, so (see (2.2)),

$$\gamma_{bj} = \gamma_{ja}, \quad \gamma_{bj}^{ab} = \gamma_{ja}^{ba},$$

$$\sum_j \gamma_{bj} = \sum_j \gamma_{ja} = \gamma, \quad \sum_j \gamma_{bj}^{ab} = \sum_j \gamma_{ja}^{ab} = \gamma^{ab}, \quad \gamma_1^- = \frac{1}{2}\gamma. \quad (3.12)$$

Hence, from (2.2), the elements of the vector I and the matrix A in (3.9b, c) normalized to γ_1^- (which amounts to put $\gamma_1^- = 1$) have the following forms,

$$\begin{aligned} \gamma_1^+ &= N_1, \quad N_1 = 1 + 2N, \quad \gamma_2^+ = 1, \quad \gamma_2^- = 0, \\ \gamma_3^+ &= N_1 f_1, \quad \gamma_3^- = f_1, \quad \gamma_4 = 0, \quad f_1 = \gamma^{ab}/2, \\ \lambda_1 &= 2, \quad \lambda_2 = 0, \quad \lambda_3^+ = 2f_1, \quad \lambda_3^- = 0, \\ a_1 &= N_1 + i\lambda, \quad a_2 = f_1, \quad \lambda = 2\omega_p, \\ b_1 &= f_1, \quad b_2 = N_1 + i\Delta, \\ c_1 &= N_1 + i(\Delta + \lambda). \end{aligned} \quad (3.13)$$

The parameter f_1 ($0 \leq f_1 \leq 1$) represents the interference measure between the two channels $E_b \rightarrow E_j$, $E_j \rightarrow E_a$ (see below eqs (2.2)).

With this simplification, the matrix A and the vector I in eqs (3.9b, c) have the following forms,

$$I = [1 \quad 0 \quad 0 \quad 2Mf_1 \quad 2Mf_1 \quad N_1f_1 \quad N_1f_1]^T \quad (3.14a)$$

and,

$$A = \begin{bmatrix} -N_1 & -\frac{M}{2} & -\frac{M}{2} & 0 & f_1 & f_1 & 0 \\ -\frac{M}{2} & -a_1 & 0 & 0 & 0 & 0 & f_1 \\ -\frac{M}{2} & 0 & -a_1^* & f_1 & 0 & 0 & 0 \\ 0 & 0 & -f_1 & -b_2 & 0 & 0 & 0 \\ 0 & -f_1 & 0 & 0 & 0 & 0 & -b_2^* \\ -f_1 & 0 & 0 & 2M & -c_1 & 0 & 0 \\ -f_1 & 0 & 0 & 0 & 0 & -c_1^* & 2M \end{bmatrix}. \quad (3.14b)$$

3.1.2 The Λ -Model

In this case $\omega_j > \omega_b > \omega_a$ for all j (Figure (2)), so the step functions in eqs (2.2) have the values:

$$\begin{aligned} \theta(\omega_{jb}) &= \theta(\omega_{ja}) = 1, \\ \theta(\omega_{bj}) &= \theta(\omega_{aj}) = 0. \end{aligned} \quad (3.15)$$

Unlike the previous case of cascade model, the assumption of equal decay rates from levels $E_j \rightarrow E_b, E_a$ lead to trivial and unphysical results.

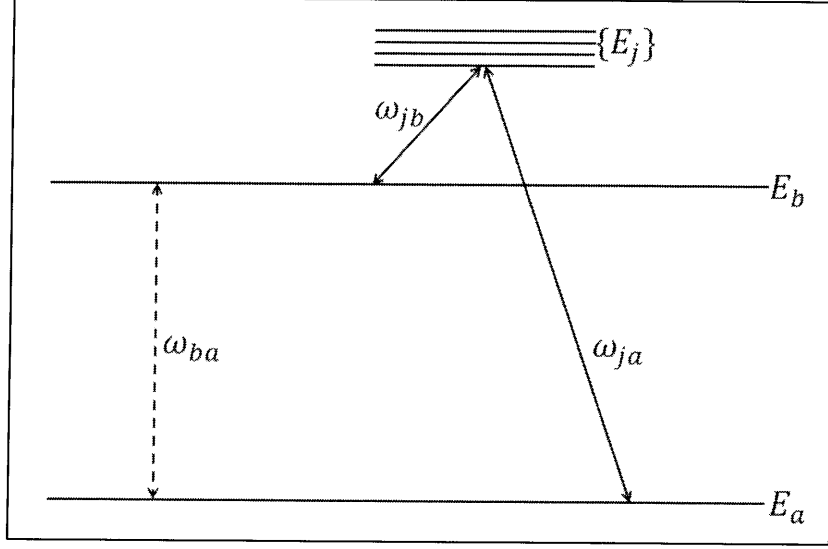


Figure (2): The Λ -model ($E_j > E_b > E_a$)

Instead, we assume that, the total decay rate $\gamma_b = \sum_j \gamma_{jb}$ is greater than the corresponding rate $\gamma_a = \sum_j \gamma_{ja}$. Further, the interference between the two channels $E_j \rightarrow E_b, E_a$ are assumed equal as in the previous cascade model. Hence, from (2.2) the elements of the vector I and matrix A in (3.9b, c) normalized to γ_b have the following forms:

$$\begin{aligned}
 \gamma_1^\pm &= N\mu_\pm, \quad \mu_\pm = \frac{1}{2}(1 \pm \mu), \quad \mu = \frac{\gamma_a}{\gamma_b} < 1, \\
 \gamma_2^\pm &= \mu_\pm, \gamma_3^+ = Nf_2, \quad \gamma_3^- = \gamma_4 = 0, \quad f_2 = \gamma_b^{ab}, \\
 \lambda_1 &= \frac{1}{2}\mu, \quad \lambda_2 = \frac{1}{2}, \lambda_3^\pm = \frac{1}{2}f_2, \\
 a_1 &= N\mu_+ + i\lambda, \quad a_2 = 0, \\
 b_1 &= \frac{1}{4} \frac{M^2 f_2 \mu}{(N\mu_+ + i(\Delta - \lambda))}, \\
 b_2 &= N\mu_+ + i\Delta - \frac{M^2 \mu}{4(N\mu_+ + i(\Delta - \lambda))}, \\
 c_1 &= N\mu_+ + i(\Delta + \lambda).
 \end{aligned} \tag{3.16}$$

The parameter f_2 ($0 \leq f_2 \leq 1$) represents the interference measure between the two channels $E_j \rightarrow E_b, E_a$ (see below eqs (2.2)).

So, the vector I and the matrix A , eqs (3.9b, c), have the forms,

$$I = \left[N\mu_- \quad -\frac{M\mu_-}{2} \quad -\frac{M\mu_-}{2} \quad \frac{1}{2}Mf_2 \quad \frac{1}{2}Mf_2 \quad Nf_2 \quad Nf_2 \right]^T \quad (3.17a)$$

and,

$$A = \begin{bmatrix} -N\mu_+ & -\frac{M\mu_+}{2} & -\frac{M\mu_+}{2} & 0 & 0 & 0 & 0 \\ -\frac{M\mu_+}{2} & -a_1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{M\mu_+}{2} & 0 & -a_1^* & 0 & 0 & 0 & 0 \\ -\frac{Mf_2}{2} & 0 & -b_1 & -b_2 & \frac{M}{2} & 0 & 0 \\ -\frac{Mf_2}{2} & -b_1^* & 0 & 0 & 0 & \frac{M}{2} & -b_2^* \\ 0 & -\frac{Mf_2}{2} & 0 & \frac{M\mu}{2} & -c_1 & 0 & 0 \\ 0 & 0 & -\frac{Mf_2}{2} & 0 & 0 & -c_1^* & \frac{M\mu}{2} \end{bmatrix}. \quad (3.17b)$$

3.1.3 The V-Model

In this case $\omega_b > \omega_a > \omega_j$ for all j (Figure (3)), so the step functions in eqs (2.2) have the values:

$$\begin{aligned} \theta(\omega_{bj}) &= \theta(\omega_{aj}) = 1, \\ \theta(\omega_{jb}) &= \theta(\omega_{ja}) = 0. \end{aligned} \quad (3.18)$$

Here, we assume that, the total decay rate $\gamma_a = \sum_j \gamma_{aj}$ is greater than the corresponding rate $\gamma_b = \sum_j \gamma_{bj}$. Further, the interference between the two channels $E_b, E_a \rightarrow E_j$ are assumed equal, as in the previous two models, sections 3.2, 3.3. Hence, from (2.2) and (3.7) the elements of the vector I and matrix A in (3.9b, c) normalized to γ_a have the following forms:

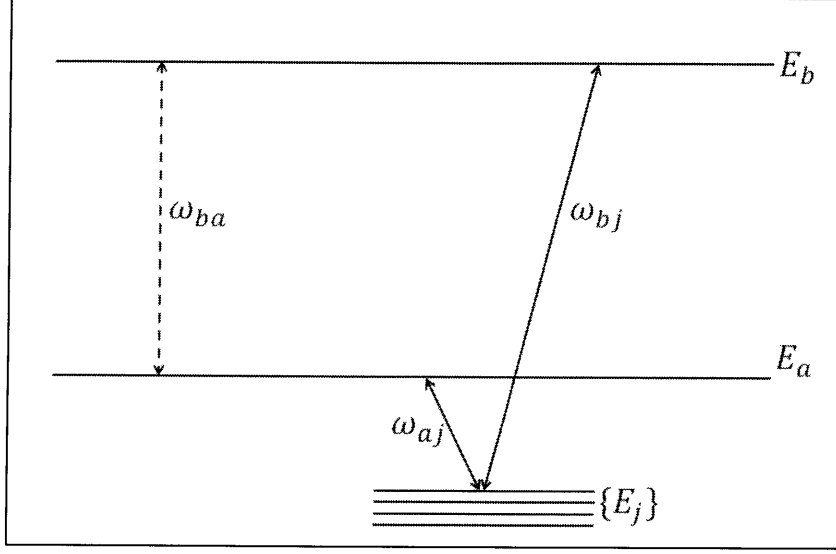


Figure (3): The V-model ($E_b > E_a > E_j$)

$$\begin{aligned}
\gamma_1^\pm &= (N+1)\kappa_\pm, \quad \kappa_\pm = \frac{1}{2}(1 \pm \kappa); \quad \kappa = \frac{\gamma_b}{\gamma_a} < 1, \\
\gamma_2^\pm &= \kappa_\pm, \quad \gamma_3^+ = (N+1)f_3, \quad \gamma_3^- = \gamma_4 = 0, \quad f_3 = \gamma_a^{ab}, \\
\lambda_1 &= \frac{1}{2}\kappa, \quad \lambda_2 = \frac{1}{2}, \quad \lambda_3^\pm = \pm \frac{1}{2}f_3, \\
a_1 &= (N+1)\kappa_+ + i\lambda, \quad a_2 = 0, \\
b_1 &= -\frac{M^2 f_3 \kappa}{4((N+1)\kappa_+ + i(\Delta - \lambda))}, \\
b_2 &= (N+1)\kappa_+ + i\Delta - \frac{M^2 \kappa}{4((N+1)\kappa_+ + i(\Delta - \lambda))}, \\
c_1 &= (N+1)\kappa_+ + i(\Delta + \lambda).
\end{aligned} \tag{3.19}$$

The parameter f_3 ($0 \leq f_3 \leq 1$) represents the interference measure between the two channels $E_b, E_a \rightarrow E_j$ (see below eqs (2.2)).

So, the vector I and the matrix A , eqs (3.9b, c), have the forms,

$$I = \left[(N+1)\kappa_-, \quad \frac{M\kappa_-}{2}, \quad \frac{M\kappa_-}{2}, \quad \frac{Mf_3}{2}, \quad \frac{Mf_3}{2}, \quad (N+1)f_3, \quad (N+1)f_3 \right]^T \tag{3.20a}$$

and,

$$A = \begin{bmatrix} -(N+1)\kappa_+ & -\frac{M\kappa_+}{2} & -\frac{M\kappa_+}{2} & 0 & 0 & 0 & 0 \\ -\frac{M\kappa_+}{2} & -a_1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{M\kappa_+}{2} & 0 & -a_1^* & 0 & 0 & 0 & 0 \\ \frac{Mf_3}{2} & 0 & -b_1 & -b_2 & \frac{M}{2} & 0 & 0 \\ \frac{Mf_3}{2} & -b_1^* & 0 & 0 & 0 & \frac{M}{2} & -b_2^* \\ 0 & \frac{Mf_3}{2} & 0 & \frac{M\kappa}{2} & -c_1 & 0 & 0 \\ 0 & 0 & \frac{Mf_3}{2} & 0 & 0 & -c_1^* & \frac{M\kappa}{2} \end{bmatrix}. \quad (3.20b)$$

The computational results for the fundamental and 1st harmonic envelope functions $(w_0, r_0 = s_0^*)$ and $(w_+ = w_-, r_+ = r_-^*)$, respectively, will be presented in the following Chapter 4. This is done by solving the algebraic system of linear equations, eqs (3.8) and (3.9) with I and A are given for the three different transition models by (3.14), (3.17) and (3.20).

3.2 Adiabatic Fourier Series

In this section, we resort to an alternative FS decomposition that avoids the earlier truncation in (3.1). The adopted FS decomposition procedure to the system (3.21), is of less labored approach (Goll and Haken 1980, Hassan et al 1982) and depends on adiabatic elimination of some of the system variables (r, s, w) to reach a single DE for one of the remaining variables where FS decomposition is used to solve this DE, as shown below.

The adiabatic approximation means that the eliminated variable reaches its steady state much faster than the rest of variables. Within the context of thermodynamics, it is a process during which there is no heat transfer (Daintith 2009).

3.2.1 The Cascade Model

The full model eqs (2.1) with decay constants given by (3.12) and (3.13) in the cascade model case and with interference parameter $f_1 = 1$ are of the normalized form (where we put $\gamma_1^- = 1$ – see below eq (3.12)),

$$\dot{w} = -\left(N_1 + \frac{M}{2}(e^{2i\omega_p t} + e^{-2i\omega_p t})\right)w - 1 + (re^{-2i\omega_p t} + se^{2i\omega_p t}), \quad (3.21a)$$

$$\dot{r} = -(N_1 + i\Delta - 2Me^{2i\omega_p t})r - (N_1 + w)e^{2i\omega_p t} - 2M, \quad (3.21b)$$

$$\dot{s} = (\dot{r})^*. \quad (3.21c)$$

The FS decomposition to the system (3.21), within this approach, is outlined as follows:

(1) *Adiabatic elimination of (r, s)*

In the model DEs ($\dot{w}$), eq (3.21a), depends on all variables (w, r, s) , while ($\dot{r}$), eq (3.21b), depends on (w, r) . Hence we choose to adiabatically eliminate $r = (s)^*$ in terms of (w) .

So, we set $\dot{r} = \dot{s} = 0$ in (3.21b, c) and solve for r, s in terms of w to get,

$$r = -[(N_1 + w)e^{2i\omega_p t} + 2M]G_1^{-1}, \quad (3.22a)$$

$$s = r^*, \quad (3.22b)$$

where,

$$G_1 = (N_1 + i\Delta - 2Me^{2i\omega_p t}). \quad (3.22c)$$

Hence, from (3.22a, b) we have,

$$\begin{aligned}
& re^{-2i\omega_p t} + se^{2i\omega_p t} \\
& = -|G_1|^{-2} \left[2N_1^2 + 2iM\Delta(e^{2i\omega_p t} - e^{-2i\omega_p t}) \right. \\
& \quad \left. - 4M^2(e^{4i\omega_p t} + e^{-4i\omega_p t}) \right. \\
& \quad \left. + w(2N_1 - 2M(e^{2i\omega_p t} + e^{-2i\omega_p t})) \right]
\end{aligned} \tag{3.23}$$

From (3.22c) the quantity $|G_1|^{-2}$ is given by,

$$\begin{aligned}
|G_1|^{-2} & = |N_1 + i\Delta - 2Me^{2i\omega_p t}|^{-2} \\
& = (N_1 + i\Delta)^{-1}(N_1 - i\Delta)^{-1}(1 - \beta_1 e^{2i\omega_p t})^{-1}(1 - \beta_1^* e^{-2i\omega_p t})^{-1} \\
& \simeq (N_1^2 + \Delta^2)^{-1}(1 + \beta_1 e^{2i\omega_p t})(1 + \beta_1^* e^{-2i\omega_p t}) \tag{3.24a}
\end{aligned}$$

$$= (N_1^2 + \Delta^2)^{-1}(1 + |\beta_1|^2 + \beta_1 e^{2i\omega_p t} + \beta_1^* e^{-2i\omega_p t}) \tag{3.24b}$$

where, the approximation in (3.24a) is valid for,

$$|\beta_1| = \left| \frac{2M}{N_1 + i\Delta} \right| < 1. \tag{3.24c}$$

Note, for the (real) parameter $M \leq \sqrt{N(N+1)}$, (e.g. Hassan et al 2005), $N_1 = 1 + 2N$ and $|\Delta| \geq 0$ the inequality (3.24c) is valid.

Substituting the expression (3.24b) for $|G_1|^{-2}$ into (3.23), we get

$$\begin{aligned}
& re^{-2i\omega_p t} + se^{2i\omega_p t} \\
& = -(N_1^2 + \Delta^2)^{-1}(1 + |\beta_1|^2 + \beta_1 e^{2i\omega_p t} + \beta_1^* e^{-2i\omega_p t}) \cdot \\
& \quad \cdot \left[2N_1^2 + 2iM\Delta(e^{2i\omega_p t} - e^{-2i\omega_p t}) \right. \\
& \quad \left. - 4M^2(e^{4i\omega_p t} + e^{-4i\omega_p t}) \right. \\
& \quad \left. + w(2N_1 - 2M(e^{2i\omega_p t} + e^{-2i\omega_p t})) \right].
\end{aligned} \tag{3.25}$$

(2) *Differential equation for (w)*

Now, substituting the expression (3.25) for $(re^{-2i\omega_p t} + se^{2i\omega_p t})$ into eq (3.21a) we get finally the following DE for w ,

$$\begin{aligned} \dot{w} = & [C_0 + (C_1 e^{2i\omega_p t} + C_2 e^{4i\omega_p t} + c.c.)]w + D_0 \\ & + (D_1 e^{2i\omega_p t} + D_2 e^{4i\omega_p t} + D_3 e^{6i\omega_p t} + c.c.). \end{aligned} \quad (3.26)$$

The coefficients C_i, D_i are given as follows:

$$\begin{aligned} C_0 &= -N_1 - \frac{2N_1}{(N_1^2 + \Delta^2)}, \\ C_1 &= -\frac{M}{2} - \frac{\alpha_1}{(N_1^2 + \Delta^2)}, \quad \alpha_1 = 2(N_1\beta_1 - M(1 + |\beta_1|^2)), \\ C_2 &= \frac{2M\beta_1}{(N_1^2 + \Delta^2)}, \\ D_0 &= -1 - \frac{2[N_1^2 + M(\beta_1(N_1 - i\Delta) + c.c.)]}{(N_1^2 + \Delta^2)}, \\ D_1 &= -\frac{\alpha_1 N_1}{(N_1^2 + \Delta^2)} - \frac{2M[(N_1 + i\Delta)(1 + |\beta_1|^2) - 2M\beta_1^*]}{(N_1^2 + \Delta^2)}, \\ D_2 &= \frac{4M^2(1 + |\beta_1|^2) - 2i\Delta M\beta_1}{(N_1^2 + \Delta^2)}, \\ D_3 &= \frac{4M^2\beta_1}{(N_1^2 + \Delta^2)}. \end{aligned} \quad (3.27)$$

(3) *FS ansatz for (w)*

In view of the harmonic coefficients in the DE (3.26), the steady state solution for w may take the FS ansatz,

$$w = w_0 + (w_1 e^{2i\omega_p t} + w_2 e^{4i\omega_p t} + w_3 e^{6i\omega_p t} + c.c.), \quad (3.28)$$

where the coefficients $w_{0,1,2,3}$ are time-independent.

Note that, in the FS, (3.28) we have included *all* possible harmonics (namely, up to $e^{\pm 6i\omega_p t}$) which affects the envelopes w_1, w_1^* of the 1st harmonic components $e^{\pm 2i\omega_p t}$.

Using (3.28) into (3.26) and by comparing coefficients of $e^{\pm 2in\omega_p t}$ ($n = 0,1,2,3$) we finally reach the following system of algebraic equations for the envelopes $w_{0,1,2,3}$ in matrix form:

$$A_1 X_1 = -I_1 \quad (3.29)$$

where,

$$X_1 = [w_0 \quad w_1 \quad w_1^* \quad w_2 \quad w_2^* \quad w_3 \quad w_3^*]^T, \quad (3.30a)$$

where,

$$I_1 = [D_0 \quad D_1 \quad D_1^* \quad D_2 \quad D_2^* \quad D_3 \quad D_3^*]^T, \quad (3.30b)$$

$$A_1 = \begin{bmatrix} C_0 & C_1^* & C_1 & C_2^* & C_2 & 0 & 0 \\ C_1 & C_0 - 2i\omega_p & C_2 & C_1^* & 0 & C_2^* & 0 \\ C_1^* & C_2^* & C_0 + 2i\omega_p & 0 & C_1 & 0 & C_2 \\ C_2 & C_1 & 0 & C_0 - 4i\omega_p & 0 & C_1^* & 0 \\ C_2^* & 0 & C_1^* & 0 & C_0 + 4i\omega_p & 0 & C_1 \\ 0 & C_2 & 0 & C_1 & 0 & C_0 - 6i\omega_p & 0 \\ 0 & 0 & C_2^* & 0 & C_1^* & 0 & C_0 + 6i\omega_p \end{bmatrix}. \quad (3.30c)$$

(4) Harmonic envelopes for (r)

Now, we derive the expressions for the fundamental and 1st harmonic envelopes for (r). First, the expression for G_1^{-1} using eq (3.22c) is approximated as follows,

$$G_1^{-1} = (N_1 + i\Delta)^{-1} (1 - \beta_1 e^{2i\omega_p t})^{-1}$$

$$G_1^{-1} \simeq (N_1 + i\Delta)^{-1} (1 + \beta_1 e^{2i\omega_p t}); \quad \beta_1 = \frac{2M}{N_1 + i\Delta} \quad (3.31)$$

valid for $|\beta_1| < 1$.

Using (3.31) and (3.28) into (3.22a) we have,

$$r = \frac{-(N_1 e^{2i\omega_p t} + 2M)(1 + \beta_1 e^{2i\omega_p t})}{(N_1 + i\Delta)} - \frac{(e^{2i\omega_p t} + \beta_1 e^{4i\omega_p t})}{(N_1 + i\Delta)} \cdot [w_0 + (w_1 e^{2i\omega_p t} + w_2 e^{4i\omega_p t} + w_3 e^{6i\omega_p t} + c.c.)]. \quad (3.32)$$

Up to 1st harmonics, (3.32) is reduced to,

$$r = r_0 + r_+ e^{2i\omega_p t} + r_- e^{-2i\omega_p t} \quad (3.33)$$

where the envelopes $r_{0,\pm}$ for the fundamental and 1st harmonic components, respectively, are given by,

$$r_0 = \frac{-(2M + w_1^* + \beta_1 w_2^*)}{(N_1 + i\Delta)}, \quad (3.34a)$$

$$r_- = -\frac{(w_2^* + \beta_1 w_3^*)}{(N_1 + i\Delta)}, \quad (3.34b)$$

$$r_+ = \frac{-(N_1 + w_0 + \beta_1(w_1^* + 2M))}{(N_1 + i\Delta)}, \quad (3.34c)$$

where the coefficients $w_{0,1,2,3}$ are determined numerically from the algebraic system (3.29) and (3.30).

3.2.2 The Λ - and V- Models

Both the Λ - and V- models can be treated together, since in this case the coupling constants $\gamma_3^- = \gamma_4 = 0$ (see eqs (3.16) and (3.19)). Hence, eq (2.1) reduce (for both models) to the following system of (non-normalized form) of DEs,

$$\dot{w} = - \left[\gamma_1^+ + \frac{\gamma_2^+ M}{2} (e^{2i\omega_p t} + e^{-2i\omega_p t}) \right] w - \left[\gamma_1^- + \frac{\gamma_2^- M}{2} (e^{2i\omega_p t} + e^{-2i\omega_p t}) \right], \quad (3.35a)$$

$$\begin{aligned}\dot{r} = & -[\gamma_1^+ + i\Delta - M(\lambda_1 e^{2i\omega_p t} + \lambda_2 e^{-2i\omega_p t})]r \\ & - \gamma_3^+ e^{2i\omega_p t} - M(\lambda_3^- w + \lambda_3^+),\end{aligned}\tag{3.35b}$$

$$\dot{s} = (\dot{r})^*.\tag{3.35c}$$

Unlike the system (3.21) for the cascade model configuration, eq (3.35a) for w is independent of the other variables (r, s) . In other words, the induced polarization field has no effect on the atomic population (w).

The FS decomposition to the system (3.35) proceeds as follows:

(1) *Adiabatic elimination of (w)*

Since the variable (w) in (3.35a) is independent of the variables (r, s) , it is convenient to eliminate (w) adiabatically. So we set $\dot{w} = 0$ in the steady state, to get,

$$w = -\left[\gamma_1^- + \frac{\gamma_2^- M}{2}(e^{2i\omega_p t} + e^{-2i\omega_p t})\right] \cdot G_2^{-1}\tag{3.36a}$$

where,

$$G_2 = \gamma_1^+ + \frac{\gamma_2^+ M}{2}(e^{2i\omega_p t} + e^{-2i\omega_p t}).\tag{3.36b}$$

Now,

$$\begin{aligned}G_2^{-1} &= (\gamma_1^+)^{-1} \left(1 + \frac{\gamma_2^+ M}{2\gamma_1^+}(e^{2i\omega_p t} + e^{-2i\omega_p t})\right)^{-1} \\ &= (\gamma_1^+)^{-1} \left(1 + \beta_2(e^{2i\omega_p t} + e^{-2i\omega_p t})\right)^{-1} \\ &\simeq (\gamma_1^+)^{-1} \left(1 - \beta_2(e^{2i\omega_p t} + e^{-2i\omega_p t})\right)\end{aligned}\tag{3.36c}$$

where $\beta_2 = \frac{\gamma_2^+ M}{2\gamma_1^+} < 1$.

Using (3.36c) into (3.36a) we have,

$$\begin{aligned}
w &= -(\gamma_1^+)^{-1} \left[\gamma_1^- + \frac{\gamma_2^- M}{2} (e^{2i\omega_p t} + e^{-2i\omega_p t}) \right] \\
&\quad \cdot (1 - \beta_2 (e^{2i\omega_p t} + e^{-2i\omega_p t})) \\
&= w_0 + w_1 (e^{2i\omega_p t} + e^{-2i\omega_p t}) + w_2 (e^{4i\omega_p t} + e^{-4i\omega_p t})
\end{aligned} \tag{3.37}$$

where the harmonic envelopes $w_{0,1,2}$ are given by,

$$\begin{aligned}
w_0 &= -\frac{(\gamma_1^- - \gamma_2^- M \beta_2)}{\gamma_1^+}, \\
w_1 &= -\frac{\left(\frac{\gamma_2^- M}{2} - \gamma_1^- \beta_2\right)}{\gamma_1^+}, \\
w_2 &= \frac{\gamma_2^- M}{2\gamma_1^+} \beta_2.
\end{aligned} \tag{3.38}$$

(2) Differential equation for (r)

Substituting the FS expansion (3.37) for w into (3.35b) we get the DE for r in the form,

$$\begin{aligned}
\dot{r} &= -[\gamma_1^+ + i\Delta - M(\lambda_1 e^{2i\omega_p t} + \lambda_2 e^{-2i\omega_p t})]r - \gamma_3^+ e^{2i\omega_p t} - M\lambda_3^+ \\
&\quad - M\lambda_3^- [w_0 + w_1 (e^{2i\omega_p t} + e^{-2i\omega_p t}) \\
&\quad + w_2 (e^{4i\omega_p t} + e^{-4i\omega_p t})] \\
&= [E_0 + E_1 e^{2i\omega_p t} + E_2 e^{-2i\omega_p t}]r + F_0 + F_1 e^{2i\omega_p t} + F_2 e^{-2i\omega_p t} \\
&\quad + F_3 (e^{4i\omega_p t} + e^{-4i\omega_p t})
\end{aligned} \tag{3.39}$$

where,

$$\begin{aligned}
E_0 &= -(\gamma_1^+ + i\Delta), \\
E_1 &= M\lambda_1, \\
E_2 &= M\lambda_2, \\
F_0 &= -M(\lambda_3^- w_0 + \lambda_3^+), \\
F_1 &= -(\gamma_3^+ + M\lambda_3^- w_1),
\end{aligned}$$

$$\begin{aligned}
F_2 &= -M\lambda_3^- w_1, \\
F_3 &= -M\lambda_3^- w_2.
\end{aligned} \tag{3.40}$$

(3) *FS ansatz for (r)*

In view of the harmonic coefficients in the DE (3.39), the steady state solution for (r) may take the form,

$$r = r_0 + r_{+,1}e^{2i\omega_p t} + r_{-,1}e^{-2i\omega_p t} + r_{+,2}e^{4i\omega_p t} + r_{-,2}e^{-4i\omega_p t} \tag{3.41}$$

where the coefficients $r_0, r_{\pm,1}, r_{\pm,2}$ are time-independent.

Using (3.41) into (3.39) and on comparing coefficients of equal harmonics $e^{\pm 2in\omega_p t}$, ($n = 0,1,2,$) we reach the following system of algebraic equations (in normalized form and with interference parameters $f_{2,3} = 1$) for the envelopes $r_0, r_{\pm,1}, r_{\pm,2}$ in matrix form,

$$A_2 X_2 = -I_2 \tag{3.42}$$

where,

$$X_2 = [r_0 \quad r_{+,1} \quad r_{-,1} \quad r_{+,2} \quad r_{-,2}]^T, \tag{3.43a}$$

$$I_2 = [F_0 \quad F_1 \quad F_2 \quad F_3 \quad F_3]^T, \tag{3.43b}$$

$$A_2 = \begin{bmatrix} E_0 & E_2 & E_1 & 0 & 0 \\ E_1 & E_0 - i\lambda & 0 & E_2 & 0 \\ E_2 & 0 & E_0 + i\lambda & 0 & E_1 \\ 0 & E_1 & 0 & E_0 - 2i\lambda & 0 \\ 0 & 0 & E_2 & 0 & E_0 + 2i\lambda \end{bmatrix} \tag{3.43c}$$

and the normalized parameter $\lambda = (2\omega_p/\gamma_a)$ or $(2\omega_p/\gamma_b)$ in the case of the Λ - or V- models, respectively.

The relevant coefficients in (3.38) and (3.40), namely, $\gamma_1^\pm, \gamma_2^\pm, \gamma_3^\pm, \lambda_{1,2}, \lambda_3^\pm$ for the Λ - and V- models are given in (3.16) and (3.19), respectively.

The computational results for the fundamental and 1st harmonic component envelopes within this adiabatic FS decomposition approach for the cascade model (sub-section 3.2.1), the Λ - and V- models (sub-section 3.2.2) are given in the next Chapter (4) and will be compared with the truncated FS decomposition of sub-sections (3.1.1 - 3.1.3).

CHAPTER 4

Results: Harmonic Envelopes

- 4.1 Physical Quantities
- 4.2 The Cascade Model
- 4.3 The Λ -Model
- 4.4 The V-Model
- 4.5 The One-Transition Case
- 4.6 Conclusion and Future work

CHAPTER 4

Results: Harmonic Envelopes

In this Chapter, the fundamental (zero harmonic) envelope components (w_0, r_0) and the 1st harmonic envelope components (w_+, r_-) are computationally investigated by solving the algebraic system of eqs (3.10), namely,

$$X = A^{-1}I \quad (4.1)$$

with the vector X , given by (3.9a), whose elements are the harmonic envelope components of atomic and polarization variables, while the vector I and the matrix A are given explicitly for the three transition models in sections (3.1) and (3.2).

4.1 Physical Quantities

Here, we investigate two specific physical quantities, namely:

- (i) *Atomic polarization field*, $r(t)$, which is the radiation field generated by the atomic system as a response to the external field. In our model (2.1), the external field is represented as the SV field, through the parameters (N, M) . The polarization field as a complex quantity is associated with absorption and dispersion lineshapes. If the current model is further probed by a weak pulse field, then the amount of energy absorbed or amplified (positive or negative absorption, respectively) by the pulse is proportional to $Im(r_-)$ (Hassan and Al-Harbey 2013). The real part, $Re(r_-)$, represents the dispersion effect that is related to the refractive index of the medium,

whose absolute value is the ratio of the pulse speed in vacua to that in the medium (Hassan et al 2003, Markes and Soukoulis 2008).

(ii) *The scattered intensity, $I(t)$, of the emitted light is proportional to the average atomic population of the upper state (E_b), i.e. the probability of atom being in its upper state. This is given by, (Hassan et al 1995),*

$$I(t) = 1 + w(t). \quad (4.2)$$

Using the FS form in (3.1a), we get

$$\begin{aligned} I(t) &= 1 + w_0 + (w_+ + w_-) \cos(2\omega_p t) + i(w_+ - w_-) \sin(2\omega_p t) \\ &= I_0 + I_c \cos(2\omega_p t) + I_s \sin(2\omega_p t) \end{aligned} \quad (4.3)$$

where,

$$I_0 = 1 + w_0 \quad (4.4a)$$

is the steady state fundamental intensity component, while,

$$I_c = 2\text{Re}(w_+), \quad I_s = -2\text{Im}(w_+) \quad (4.4b)$$

are the steady state in-phase and out-of-phase 1st harmonic intensity components.

Next, we present the computational results of the scattered intensity and the absorption-dispersion harmonic envelopes for the three transition models: the cascade, Λ - and V- models.

4.2 The Cascade Model

(A) Truncated FS results

Before presenting the computational result in the general case of SV field with interference effect $(M, N, f_1) \neq 0$, we look at the following special cases:

(i) *Thermal Field (TF) Case* ($N \neq 0, M = 0$)

For $f_1 = 0$ (no interference process) it is easy to get from the system of eqs (4.1) with I and A are given in (3.14 a, b),

$$r_{0,\pm} = 0, \quad (4.5a)$$

$$w_0 = -\frac{1}{1+2N}, \quad (4.5b)$$

$$w_{\pm} = 0. \quad (4.5c)$$

So in this case, no steady state polarization field, (4.5a), and the atomic inversion component w_0 , has the same result as the one-photon transition case (Hassan and Bullough 1998) with no 1st harmonic components ($w_{\pm} = 0$).

With interference effect ($f_1 = 1$) and in this case of no SV field $\Delta = \omega_{ba} - 2\omega_p \rightarrow \omega_{ba}$, the system (4.1), with (3.14a, b) reduces to,

$$-N_1 w_0 + r_+ + r_+^* = 1, \quad (4.6a)$$

$$-N_1 w_+ + r_0^* = 0, \quad (4.6b)$$

$$-N_1 w_+^* + r_0 = 0, \quad (4.6c)$$

$$-w_+^* - (N_1 + i\omega_{ba})r_0 = 0, \quad (4.6d)$$

$$-w_+ - (N_1 - i\omega_{ba})r_0^* = 0, \quad (4.6e)$$

$$-w_0 - (N_1 + i\omega_{ba})r_+ = N_1, \quad (4.6f)$$

$$-w_0 - (N_1 - i\omega_{ba})r_+^* = N_1. \quad (4.6g)$$

From eqs (4.6b) – (4.6e) we have,

$$r_0 = 0, \quad w_+ = 0, \quad (4.7a)$$

hence the polarization fundamental component (r_0) and the 1st harmonic components ($w_{\pm}$) are zero.

Solving eqs (4.6a, f, g) for w_0, r_+ we get the following,

$$w_0 = -\frac{1}{N_1} \left(\frac{3N_1^2 + \omega_{ba}^2}{N_1^2 + 2 + \omega_{ba}^2} \right) \quad (4.7b)$$

$$\simeq -\frac{1}{N_1} \quad (\text{since } \omega_{ba}^2 \gg 1)$$

$$= -\frac{1}{1+2N} \quad (4.7c)$$

and

$$r_+ = -\frac{(N_1^2 - 1)}{\omega_{ba}^2} + i \frac{(N_1^2 - 1)}{N_1 \omega_{ba}}. \quad (4.7d)$$

Eq (4.7c) means that the fundamental component of the inversion is essentially similar to the no interference result (4.5b) while (4.7d) means that the 1st harmonic component (r_+) of the polarization field is non-zero due to interference effect, with $Re(r_+) = O\left(\frac{1}{\omega_{ba}^2}\right), Im(r_+) = O\left(\frac{1}{\omega_{ba}}\right)$.

(ii) *SV Field Case* ($N, M \neq 0, f_1 = 0$)

In this case the system (4.1) with (3.14a, b) reduces to,

$$-N_1 w_0 - \frac{M}{2} (w_+ + w_+^*) + r_+ + r_+^* = 1, \quad (4.8a)$$

$$-\frac{M}{2} w_0 - a_1 w_+ = 0, \quad (4.8b)$$

$$-\frac{M}{2} w_0 - a_1^* w_+^* = 0, \quad (4.8c)$$

$$-b_2 r_0 = 0, \quad (4.8d)$$

$$-b_2^* r_0^* = 0, \quad (4.8e)$$

$$2Mr_0 - c_1 r_+ = 0, \quad (4.8f)$$

$$2Mr_0^* - c_1^* r_+^* = 0. \quad (4.8g)$$

Eqs (4.8d) – (4.8g) gives that

$$r_{0,+} = 0, \quad (4.9a)$$

so, no polarization field in the absence of interference effect.

Solving eqs (4.8a) – (4.8c) for the atomic inversion components $w_{0,\pm}$ we get,

$$w_0 = -\frac{1}{N_1} \frac{2(N_1^2 + \lambda^2)}{(7N^2 + 7N + 2 + 2\lambda^2)} \quad (4.9b)$$

$$\simeq -\frac{1}{N_1} \quad (\text{since } \lambda^2 = (2\omega_p)^2 \gg 1)$$

$$= -\frac{1}{1+2N} \quad (4.9c)$$

and

$$w_+ = -\frac{Mw_0}{2(N_1 + i\lambda)} \\ \simeq -\frac{M}{8\omega_p^2} + i\frac{M}{8N_1\omega_p}. \quad (4.9d)$$

So, the fundamental component (w_0) is essentially the same as the thermal field case (4.5b) while the 1st harmonic component (w_+) has real and imaginary parts of $O\left(\frac{1}{\omega_p^2}\right)$ and $O\left(\frac{1}{\omega_p}\right)$, respectively.

The results for the general case of SV field, $(N, M) \neq 0$, with interference effects ($f_1 = 1$) are obtained computationally by solving the system (4.1) with (3.14) using Maple software version 18 (Appendix A) and presented in Figure (4) where we notice the following:

(a) The fundamental absorption and dispersion components, $Im(r_0)$ and $Re(r_0)$, respectively, exhibit zero absorption- finite dispersion phenomenon (Hassan et al 1997) at exact resonance ($\Delta = 0$), Figure (4a), due to the quantum interference effect ($f_1 \neq 0$). For smaller $N = 0.1$, the same results are obtained but with narrowing widths. Physically, line narrowing effect means higher spectral resolution of separating closely

different frequencies (e.g. Garrison and Chiao 2014). For SV parameter $M \rightarrow -M$, the absorption shape, $Im(r_0)$, becomes the mirror image along the vertical axis at $\Delta = 0$ of the full lined curve in Figure (4a), while the dispersion shape (dotted line), $Re(r_0)$, becomes Lorentzian (See Appendix B for Lorentzian and dispersion curves).

There are no 1st harmonic polarization components, $r_{\pm} = 0$.

(b) Figure (4b) shows that the fundamental component of the intensity is constant, $I_0 = 0.66$ while the in- and out- of phase 1st harmonic components (I_c, I_s) have dispersive and inverted Lorentzian structure and both are small, $O(\lambda^{-1}) = O(10^{-7})$. For smaller $N = 0.1$ and negative M , we have the same effects as for the polarization field mentioned in (a) above.

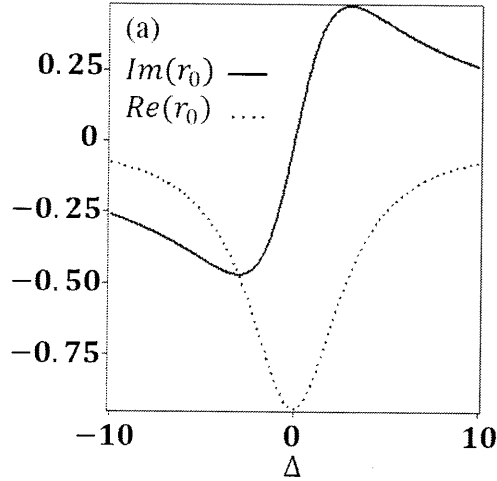


Figure (4a): The fundamental components $Im(r_0), Re(r_0)$ against the normalized detuning Δ for $N = 1, M = \sqrt{N(N+1)}, f_1 = 1$ and $\lambda = 10^7$ (in the optical frequency range)

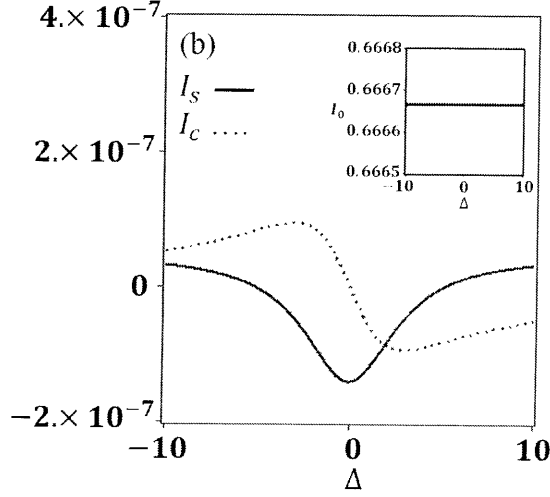


Figure (4b): As Figure (4a) but for intensity harmonics I_0, I_C, I_S .

(B) Adiabatic FS results

Within the adiabatic FS decomposition approach of sub-section (3.2.1) we notice the following:

(a) The corresponding results for the fundamental components $Im(r_0), Re(r_0)$ are shown in Figure (4c). The qualitative behavior is the same as in Figure (4a) using the truncated FS, eqs (3.1). In Figure (4d) the 1st harmonic components $Im(r_-), Re(r_-)$ are small $O(10^{-8})$ and show negative absorption (amplification) for all Δ , compared with vanishing components using the truncated FS, eqs (3.1).

(b) As for the intensity harmonics, the fundamental component $I_0 = 1 + w_0$ is shown in the inset of Figure (4e) where it exhibits variation with Δ , compared with constant value in Figure (4b). The 1st harmonic components $I_C = 2Re(w_1)$ and $I_S = -2Im(w_1)$ in Figure (4e) show essentially the same qualitative behavior as that in Figure (4b) for I_S, I_C , respectively.

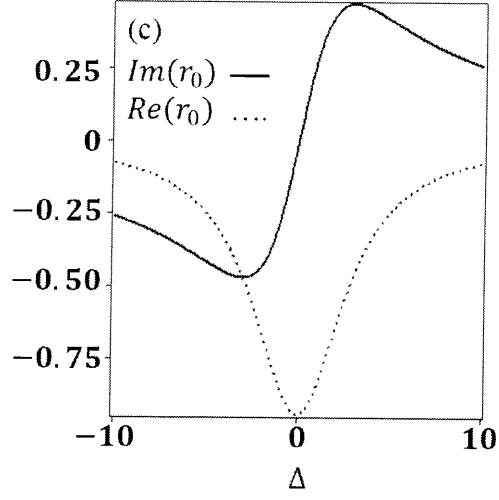


Figure (4c): As Figure (4a) but using the adiabatic FS decomposition, eq (3.34a)

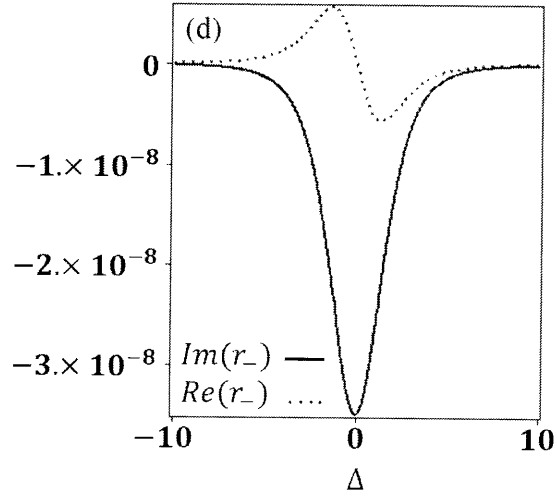


Figure (4d): $Im(r_-), Re(r_-)$ using the adiabatic FS decomposition, eq (3.34b)

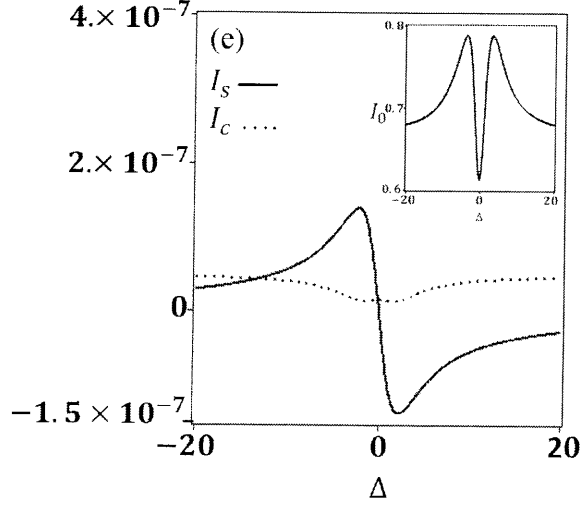


Figure (4e): As Figure (4b) but using the adiabatic FS decomposition, eqs (3.29) and (3.30)

4.3 The Λ -Model

(A) Truncated FS results

First, we look at the following special cases using eq (4.1) with I and A are given by (3.17a, b).

(i) Thermal Field Case ($N \neq 0, M = 0$)

For no interference effect ($f_2 = 0$), we have

$$w_0 = -\frac{(1-\mu)}{(1+\mu)}; \quad \mu < 1 \quad (4.10a)$$

$$w_{\pm} = r_{0,\pm} = 0. \quad (4.10b)$$

The only non-zero steady component is the fundamental atomic inversion w_0 which is only dependent on the damping ratio $\mu \equiv \frac{\gamma_a}{\gamma_b} < 1$ and all other higher harmonics $w_{\pm} = 0$. Also, no steady polarization field ($r_{0,\pm} = 0$) due to absence of interference.

With interference effect ($f_2 = 1$) and letting $\Delta = \omega_{ba} - 2\omega_p \rightarrow \omega_{ba}$ (where there is no SV field), we have from (4.1) and (3.17a, b) the same results (4.10) for the atomic inversion components and further,

$$r_0 = 0, \quad (4.11a)$$

$$r_+ = -\frac{N}{c_1} \simeq -\frac{N^2\mu_+}{4\omega_{ba}^2} + \frac{iN}{2\omega_{ba}}, \quad (4.11b)$$

since $\omega_{ba}^2 \gg 1$.

So, the polarization field has no fundamental component and the 1st harmonic component (r_+) has real and imaginary parts $O\left(\frac{1}{\omega_{ba}^2}\right)$ and $O\left(\frac{1}{\omega_{ba}}\right)$, respectively, similar to the cascade model case (eq 4.7d).

(ii) *SV Field Case* ($N, M \neq 0, f_2 = 0$)

In this case eq (4.1) with (3.17a, b) reduces to

$$-N\mu_+w_0 - \frac{M}{2}(w_+ + w_+^*) = N\mu_-, \quad (4.12a)$$

$$-\frac{M\mu_+}{2}w_0 - a_1w_+ = -\frac{M\mu_-}{2}, \quad (4.12b)$$

$$-\frac{M\mu_+}{2}w_0 - a_1^*w_+^* = -\frac{M\mu_-}{2}, \quad (4.12c)$$

$$-b_2r_0 + \frac{M}{2}r_+ = 0, \quad (4.12d)$$

$$-b_2^*r_0^* + \frac{M}{2}r_+^* = 0, \quad (4.12e)$$

$$\frac{M}{2}\mu r_0 - c_1r_+ = 0, \quad (4.12f)$$

$$\frac{M}{2}\mu r_0^* - c_1^*r_+^* = 0. \quad (4.12g)$$

Eqs (4.12d-g) gives

$$r_{0,\pm} = 0. \quad (4.13a)$$

So, no polarization field components due to absence of interference as in the cascade model case (eq 4.9a).

Solving eqs (4.13a-c) we get the following expressions (with $\omega_{ba}^2 \gg 1$) for the atomic inversion components,

$$w_0 \simeq -\frac{1-\mu}{1+\mu}, \quad (4.13b)$$

$$w_+ \simeq \frac{MN\mu_+\mu_-}{8\omega_{ba}^2} - i \frac{M\mu_-}{4\omega_{ba}^2}. \quad (4.13c)$$

The result (4.13b, c) are essentially of the same order as in the cascade model case (eqs 4.9c, d).

The general case of SV field (N, M) $\neq 0$ with interference effect ($f_2 = 1$) are obtained by solving the system (4.1) with (3.17) computationally and presented in Figure (5) where we notice the following:

(a) The fundamental absorption and dispersion components, $Im(r_0)$ and $Re(r_0)$, respectively, show the same qualitative behavior of zero-absorption-finite dispersion at exact resonance as in the cascade model (Figure 4a) but with narrowing effects.

The 1st harmonic components of the absorption and dispersion line shapes are non-zero; $O(\lambda^{-1})$, -Figure 5b- (unlike the cascade model) and show opposite behavior to that of Figure (5a), i.e. negative absorption (amplification) with zero-dispersion at exact resonance ($\Delta = 0$).

(b) All intensity components, the fundamental I_0 , and the 1st harmonic in- and out- of phase $I_{c,s}$ are constants with both $I_{c,s} = O(\lambda^{-1})$ (Figure 5c).

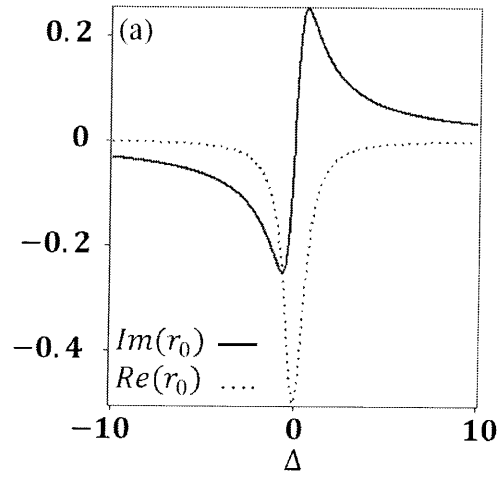


Figure (5a): As Figure (4a) but for the Λ -model with $\mu = 0.3$

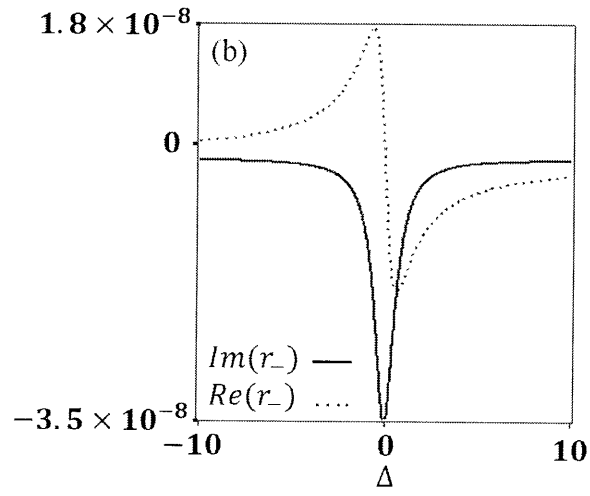


Figure (5b): The first harmonic components $Im(r_-), Re(r_-)$ against Δ for the Λ -model with $\mu = 0.3$

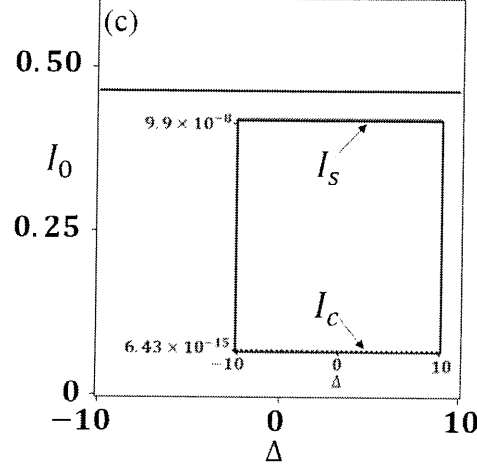


Figure (5c): As Figure (4b) but for the Λ -model with $\mu = 0.3$

(B) Adiabatic FS results

The computational results within the adiabatic FS decomposition for the fundamental components $Im(r_0), Re(r_0)$ and of the 1st harmonic components $Im(r_{-1}), Re(r_{-1})$ of the induced polarized field are shown in Figures (5d) and (5e), respectively. The results are qualitatively of the same behavior as for the truncated FS decomposition Figures (5a) and (5b). Note also that, the harmonic envelopes $w_{0,1,2}$ in eqs (3.38) are independent of the detuning parameter (Δ) (as in the truncated FS decomposition case, Figure (5c)) since w is independent of (r, s) .

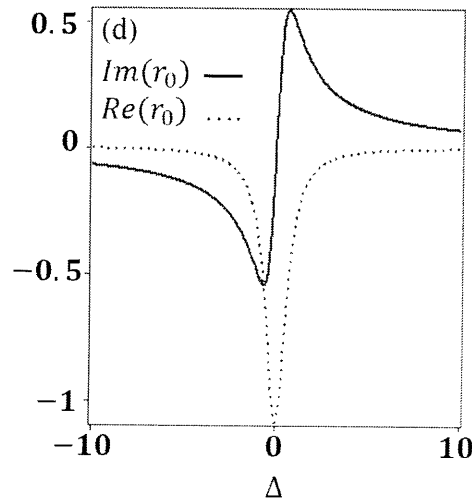


Figure (5d): As Figure (5a) but using the adiabatic FS decomposition, eqs (3.42) and (3.43)

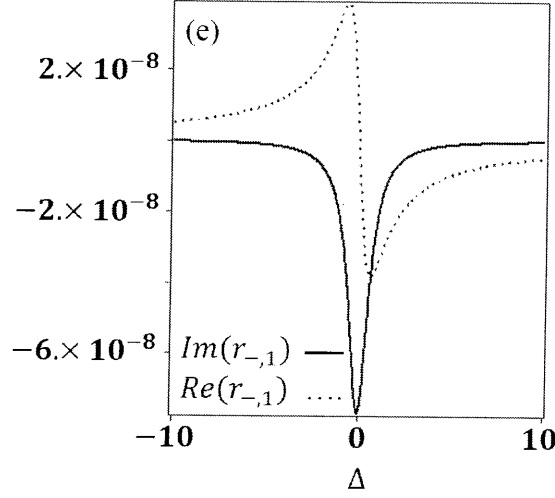


Figure (5e): As Figure (5d) but for $Im(r_{-,1}), Re(r_{-,1})$

4.4 The V-Model

(A) Truncated FS results

Here, the following special cases are very similar to the Λ -Model case.

(i) Thermal Field Case ($N \neq 0, M = 0$).

In the case of no interference ($f_3 = 0$), we have similar to (4.10),

$$w_0 \simeq -\left(\frac{1-k}{1+k}\right), \quad k < 1, \quad (4.14a)$$

$$w_{\pm} = 0, \quad (4.14b)$$

$$r_{0,\pm} = 0, \quad (4.14c)$$

and hence no higher harmonics for the atomics inversion and the polarization is zero due to absence of interference.

With interference effect ($f_3 = 1$) and by letting $\Delta \rightarrow \omega_{ba}$ (where no SV field), we have the same results for the inversion components (4.14a-c), while the polarization field has no fundamental component ($r_0 = 0$) but

its 1st harmonic components are of the form (compared with the Λ -model, eq (4.11b)),

$$r_+ \simeq -\frac{(N+1)k_+}{4\omega_{ba}^2} - i\frac{(N+1)}{2\omega_{ba}}. \quad (4.15)$$

(ii) *SV Field Case* $(N, M) \neq 0, f_3 = 0$

Similar to the Λ -model case, eqs (4.13), we have,

$$r_{0,\pm} = 0, \quad (4.16a)$$

$$w_0 \simeq -\left(\frac{1-k}{1+k}\right), \quad (4.16b)$$

$$w_+ \simeq \frac{(N+1)Mk_+k_-}{8\omega_{ba}^2} - i\frac{Mk_-}{4\omega_{ba}}. \quad (4.16c)$$

Hence, the atomic inversion components are all non-zero, while the polarization field is zero due to the absence of interference.

Effects of the interference ($f_3 = 1$) in the SV field case are examined by solving the system (4.1) with (3.20) computationally and are shown in Figure (6) where we notice the following:

(a) The fundamental and 1st harmonic components of the absorption and dispersion line shapes (Figure 6a, b) show the same behavior as in the Λ -model of Figure (5a, b), but with lesser narrowing effects.

(b) The fundamental I_0 and 1st harmonic components in- and out- of phase $I_{c,s}$ of the intensity are constants (Figure 6c), as in the Λ -model case (Figure 5c) with $I_s \simeq 0$.

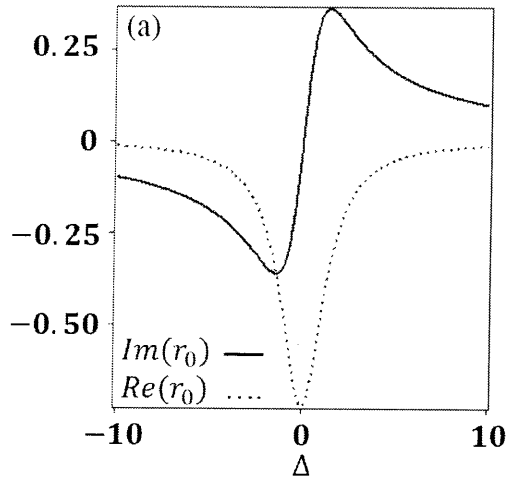


Figure (6a): As Figure (4a) but for the V-model with $k = 0.4$

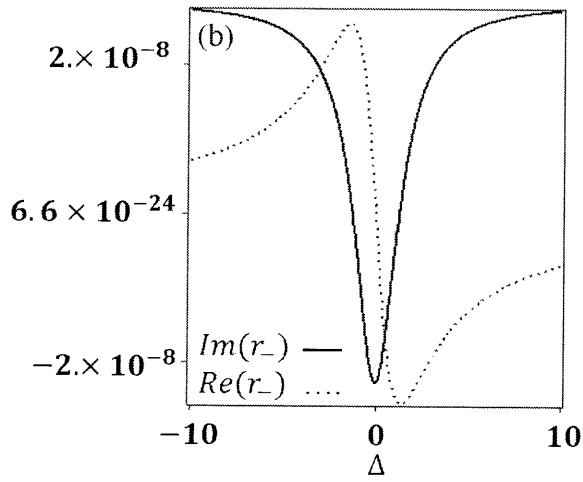


Figure (6b): The first harmonics $Im(r_-), Re(r_-)$ against Δ for the V-model with $k = 0.4$.

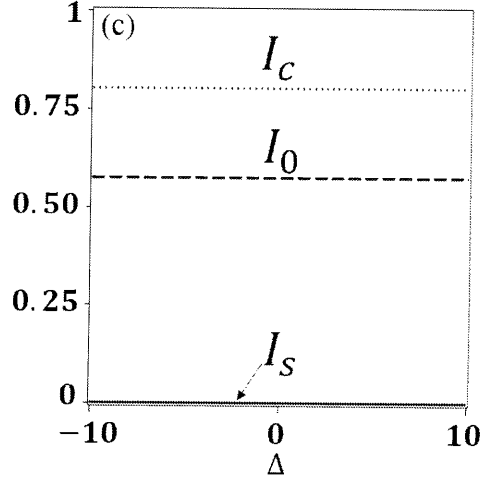


Figure (6c): As Figure (4b) but for the V-model with $k = 0.4$.

(B) Adiabatic FS results

Similar to the Λ - model, the computational results for the fundamental components $Im(r_0), Re(r_0)$ and of the 1st harmonic components $Im(r_{-1}), Re(r_{-1})$ – Figure (6d) and (6e) within the adiabatic decomposition are in close analogy with the results of the truncated FS decomposition, Figure (6a) and (6b), respectively. Further, the harmonic components $w_{0,1,2}$ are constants.

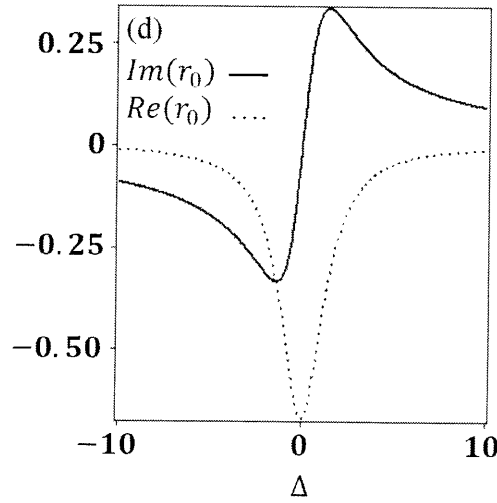


Figure (6d): As Figure (6a) but using the adiabatic FS decomposition, eqs (3.42) and (3.43)

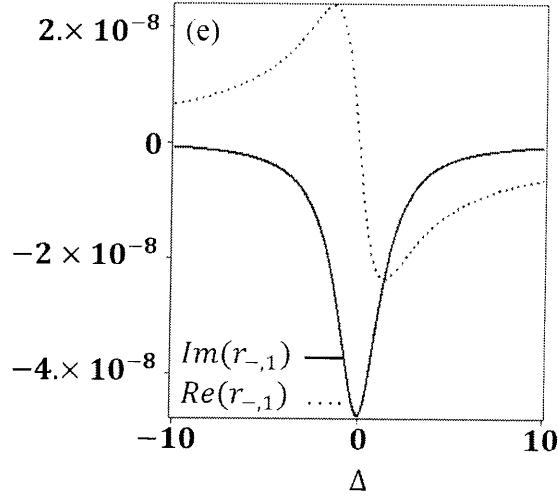


Figure (6e): As Figure (6d) but for $Im(r_{-,1}), Re(r_{-,1})$

4.5 The One-Transition Case

The corresponding Bloch equations modelling the interaction of a single 2-level atom with an off-resonant broadband SV field via a one-photon transition (so no intermediate states $\{E_j\}$) have the form (e.g. Hassan and Al-Harbey 2013),

$$\dot{w} = -\gamma - \gamma_1 w, \quad (4.17a)$$

$$\dot{r} = -\frac{\gamma_1}{2} r - \gamma M s e^{2i\Delta t} \quad (4.17b)$$

$$= (\dot{s})^*. \quad (4.17c)$$

Here, $w, r = s^*$ are the atomic inversion and polarization field, respectively, $\gamma_1 = \gamma(2N + 1)$, $\Delta = \omega_p - \omega_0$, with ω_p the central frequency of the SV field and ω_0 the atomic transition frequency, γ is the decay constant and N, M are the SV parameters.

Here, we notice from eqs (4.17) two main features:

(i) The atomic inversion w , eq (4.17a) is not coupled to the atomic polarization field ($r = s^*$), and it only depends on the SV parameter (N) as a broadening decay effect.

This is unlike the 2-photon transition case, eq (2.1a), where w is coupled to both polarization components (r, s) due to interference process (terms in γ_3^- and γ_4).

(ii) The two quadrature components of the polarization field (r, s), eqs (4.17b, c) are coupled through the SV parameter M (degree of squeezing) and independent of the atomic energy w . This is unlike the 2-photon transition case, eq (2.1b), as mentioned in (i) above.

In the steady state ($\dot{w} = \dot{r} = \dot{s} = 0$), eq (4.17a) simply gives, the usual thermal field result (Hassan and Bullough 1988),

$$w(\infty) = -\frac{1}{1+2N}, \quad (4.18a)$$

while, eqs (4.17b, c) simply shows that there is no steady polarization field,

$$r(\infty) = s(\infty) = 0. \quad (4.18b)$$

Note that, the exact time-dependent solutions of the homogenous DEs, eqs (4.17b, c), (see Appendix C) show that the polarization field is non-zero *due to* non-zero initial conditions, $(r(0), s(0)) \neq 0$.

4.6 Conclusion and Future work

The concerned mathematical model of non-autonomous Bloch equations represent the coupling of a single 2-level atom with a squeezed

light, where the atomic transition between the 2-energy levels occurs indirectly via a 2 way-transitions. Three shapes of this indirect transition is considered, namely, the cascade, Λ - and V- models. The non-autonomous differential equations (Bloch equations) with periodic coefficients are treated via two approaches of Fourier series (FS) decomposition:

(1) One approach is to use the truncated FS (up to 1st harmonics) for all variables (r, s, w) of the Bloch equations.

(2) The other approach, which does not resort to earlier truncation of FS, but uses adiabatic elimination of some of the variables to reach a single DE for only one of the rest of variables. In turn, FS decomposition ansatz is used to solve the concerned DE in the steady state. This depends on the form of the system of DEs (Bloch equations) and in turn on the atomic system configuration. Particularly:

(a) For the cascade model, the induced polarized field (r, s) affects the atomic inversion (w) and vice-versa, hence the evolution of any variable depends on the rest of variables in the model Bloch equations, (3.21).

In this case, we use adiabatic elimination of the two variables (r, s) of the polarization field to reach a DE for the third variable (w). The FS ansatz used to solve the resulting DE for w in the steady state takes account of all possible harmonics that affect the envelopes of the 1st harmonics of interest.

The results of this adiabatic FS decomposition for the cascade model case reveals two different aspects from the truncated FS approach in (1) above, namely: (i) the appearance of the 1st harmonics of the polarization field components (r, s), (ii) the variation of the fundamental intensity component (I_0) with the frequency detuning parameter (Δ).

(b) For both the Λ - and V- models, eqs (3.35), the induced polarized field (r, s) does not affect the atomic inversion (w) *but* the opposite is true. In this case, we adiabatically eliminate the variable (w) to reach a DE for the polarized field variable (r) , and in turn FS ansatz is used to solve in the steady state (up to 1st harmonics) for the harmonic envelopes of (r) . The results of this adiabatic FS decomposition approach for the Λ - and V- model cases are qualitatively similar to the truncated FS approach.

It is noted that within the adiabatic FS decomposition approach all possible higher harmonics, $e^{\pm 2in\omega_p t}; n \geq 2$, that affect the envelopes of the 1st harmonics $e^{\pm 2i\omega_p t}$ are taken into consideration (see eqs (3.28), (3.41)). This is unlike the truncated FS decomposition approach, eq (3.1).

Two characteristic quantities of the atomic medium were investigated analytically and computationally. These are the absorption-dispersion line shapes and scattered intensity (up to 1st harmonics). The results for the three atomic structures are essentially in qualitative similarity. A main result is that, the (quantum) interference process between the two-indirect transitions induces zero-absorption with finite dispersion at exact resonance.

The comparison is made with the direct one-transition case (so, no interference is present) where atomic polarization field (hence, absorption and dispersion) is absent for all time $t \geq 0$, unless the initial preparation of the atomic system has non-zero polarization.

Finally, we note that, for atomic system of 3-energy levels of various configuration (the cascade, Λ - and V- models) interacting with a SV

radiation reservoir, the model Bloch equations have constant coefficients at exact resonance conditions (e.g. Scully and Zubairy 1997, Sharaby et al 2006, Osman et al 2007). Otherwise, for non-resonance conditions the Bloch equations have time-dependent harmonic coefficients. The indirect transition takes place via a specific level, and not via a set of intermediate states as in the studied 2-level atomic structure (Hassan and Ali 2001).

Future work of the studied model of non-autonomous Bloch equations may include:

- (i) Iterative time-dependent solutions for the atomic variables (c.f. Batarfi et al 2010, Hassan and Al-Harbey 2013). Hence, calculations for the scattered and absorption spectra that depend on the two-time atomic auto-correlation functions, can be derived analytically and investigated computationally.
- (ii) Study of the present Bloch equations but with the replacement of the 1st order derivative by a fractional derivative of order α ($0 < \alpha < 1$) where non-local (time) effects are considered within this fractional approach (e.g. Bhalekar et al 2011, Petras 2011, Kumar et al 2014).
- (iii) Examination of the entropy and information transfer measures within the context of quantum information theory (Hassan et al 2007, Metwally and Hassan 2012).

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APPENDIX (A)

The Maple program to solve (4.1) and (3.14)

Solve a system of 7 eqs then plot the solution against Delta

```
> restart:
> with(plots):Digits:=75:
----- Defining the nature of the variables used -----
> assume(Delta,real):Digits:=20:
>
lambda:=10^(7);lambda31:=0:lambda2:=0:N:=1:M:=sqrt(N*(N+1)):
>
N1:=1+2*N;a1:=N1+I*lambda;b2:=N1+I*Delta;c1:=N1+I*(Delta+lambda);
>
> sol:=simplify(solve({-N1*wo-0.5*M*wplus-0.5*M*conjugate(wplus)+f1*rplus+f1*conjugate(rplus)=1,-0.5*M*wo-a1*wplus+f1*conjugate(ro)=0,-0.5*M*wo-conjugate(a1)*conjugate(wplus)+f1*ro=0,-f1*conjugate(wplus)-b2*ro=2*f1*M,-f1*(wplus)-conjugate(b2)*conjugate(ro)=2*f1*M,-f1*wo+2*M*ro-c1*rplus=f1*N1,-f1*wo+2*M*conjugate(ro)-conjugate(c1)*conjugate(rplus)=f1*N1},{wo,wplus,conjugate(wplus),ro,rplus,conjugate(rplus),conjugate(ro)})):
f:=eval(ro,sol):g:=eval(wo,sol):z:=eval(rplus,sol):k:=eval(wplus,sol):
> rminus:=M*(lambda2*f-lambda31*conjugate(k))/(1+2*N+I*(Delta-lambda)):
> Ia:=1+g:Ic:=2*Re(k):Is:=-2*Im(k):
> Im(f):Re(f):g1:=Im(g):g2:=Re(g):z1:=Im(z):z2:=Re(z):k1:=Im(k):k2:=Re(k):
> f1:=1;
> plot([Ia],Delta=-10..10,linestyle=[1,2,3],color=[black,black,black],axes=boxed,numpoints=700,tickmarks=[3,4],font=[1,1,14],thickness=2,view=[-10..10,0.6665..0.6668]);
> #display({P1,P2,P3});
> plot([Is,Ic],Delta=-10..10,linestyle=[1,2,3],color=[black,black,black],axes=boxed,numpoints=700,tickmarks=[3,4],font=[1,1,14],thickness=2);
> #
```

APPENDIX (B)

Lorentzian and Dispersion Lineshapes

(a) The *Lorentzian* function,

$$F_1(\Delta) = \frac{a^2}{a^2 + \Delta^2} \quad (\text{B.1})$$

is an even function of (Δ) with absolute maximum ($= 1$) at $\Delta = 0$ and of width (a) i.e. $F_1(\pm a) = \frac{1}{2}$. It approaches zero for large $|\Delta| \rightarrow \infty$ (Figure B.1).

(b) The *dispersive* lineshape function,

$$F_2(\Delta) = \frac{a\Delta}{a^2 + \Delta^2} \quad (\text{B.2})$$

is an odd function of (Δ) and vanishes at $\Delta = 0$ with absolute maximum and minimum values $(\pm \frac{1}{2})$ at $\Delta = \pm a$. It approaches zero for large $|\Delta| \rightarrow \infty$. (Figure B.1)

In Calculus books (e.g. Swokowski et al 1994) the functions $F_1(\Delta)$, $F_2(\Delta)$ are usually called the *Witch of Agnesi* and *Serpentine* curves, respectively.

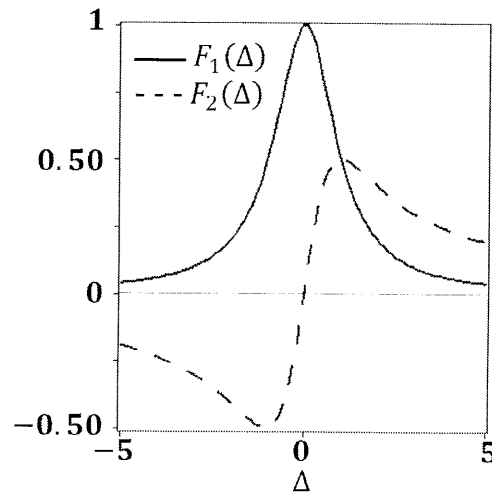


Figure (B.1): The functions $F_1(\Delta)$, $F_2(\Delta)$ against (Δ) for $a = 1$.

APPENDIX (C)

Solutions of Eqs (4.17)

Eq (4.17a) is a 1st order linear non-homogenous DE and its exact solution is given by,

$$w(t) = \left(w(0) + \frac{\gamma}{\gamma_1} \right) e^{-\gamma_1 t} - \frac{\gamma}{\gamma_1}. \quad (\text{C.1})$$

and its steady state value,

$$w(\infty) = -\frac{\gamma}{\gamma_1}.$$

To solve (4.17b, c), we introduce the transformations

$$\begin{aligned} r(t) &= R(t)e^{i\Delta t}, \\ s(t) &= S(t)e^{-i\Delta t}. \end{aligned} \quad (\text{C.2})$$

Substituting into (4.17b, c) we get the following DEs of constant coefficients (taking M as real for simplicity),

$$\begin{aligned} \dot{R} &= -\left(\frac{\gamma_1}{2} + i\Delta\right) R - \gamma MS, \\ \dot{S} &= -\left(\frac{\gamma_1}{2} - i\Delta\right) S - \gamma MR. \end{aligned} \quad (\text{C.3})$$

The solutions of (C.3) are obtained, e.g., using Laplace transform ($L.T.$).

Define,

$$\begin{aligned} L.T.(R(t)) &= \bar{R}(P) \\ &\equiv \int_0^\infty R(t)e^{-pt} dt, \end{aligned} \quad (\text{C.4})$$

etc., where $p > 0$ is the $L.T.$ parameter.

Hence the $L.T.$ of (C.3) are,

$$(p + \gamma_0)\bar{R}(p) = R(0) - \gamma M \bar{S}(p), \quad (C.5a)$$

$$(p + \gamma_0^*)\bar{S}(p) = S(0) - \gamma M \bar{R}(p), \quad (C.5b)$$

where $\gamma_0 = \frac{\gamma_1}{2} + i\Delta$ and $R(0), S(0)$ are arbitrary initial conditions.

Eq. (C.5b) gives,

$$\bar{S}(p) = \frac{S(0)}{(p + \gamma_0^*)} - \frac{\gamma M \bar{R}(p)}{(p + \gamma_0^*)}. \quad (C.6)$$

Substituting (C.6) into (C.5a), we get,

$$\bar{R}(p) = \frac{(p + \gamma_0^*)}{Q(p)} R(0) - \frac{\gamma M}{Q(p)} S(0), \quad (C.7)$$

where the quadratic $Q(p)$ is given by,

$$Q(p) = p^2 + \gamma_1 p + \frac{\gamma_1^2}{4} + \Delta^2 - \gamma^2 M^2. \quad (C.8)$$

The inverse $L.T.$, $(L.T.)^{-1}$, of (C.7) depends on the roots of $Q(p)$.

There are three cases:

$$Q(p) = \begin{cases} \left(p + \frac{\gamma_1}{2}\right)^2, & \Delta^2 - \gamma^2 M^2 = 0, \\ \left(p + \frac{\gamma_1}{2}\right)^2 - \frac{\xi_1^2}{4}, & \gamma^2 M^2 - \Delta^2 > 0, \\ \left(p + \frac{\gamma_1}{2}\right)^2 + \frac{\xi_2^2}{4}, & \Delta^2 - \gamma^2 M^2 > 0; \end{cases} \quad (C.9)$$

where: $\xi_1 = \sqrt{\gamma^2 M^2 - \Delta^2}$, $\xi_2 = \sqrt{\Delta^2 - \gamma^2 M^2}$.

Accordingly, the $(L.T.)^{-1}$ of (C.7) gives the solution for $r(t) = R(t)e^{i\Delta t} = s^*(t)$ in the above three cases as follows,

Case (1): $\Delta^2 - \gamma^2 M^2 = 0$

$$r(t) = [(1 - i\Delta t)r(0) - \gamma M t s(0)]e^{-\left(\frac{\gamma_1}{2} - i\Delta\right)t}. \quad (C.10a)$$

Case (2): $\gamma^2 M^2 - \Delta^2 > 0$

$$r(t) = \left\{ \left[\cosh\left(\frac{\xi_1}{2}t\right) - i(2\Delta/\xi_1) \sinh\left(\frac{\xi_1}{2}t\right) \right] r(0) - (2\gamma M/\xi_1) \sinh\left(\frac{\xi_1}{2}t\right) s(0) \right\} e^{-(\frac{\gamma_1}{2} - i\Delta)t}. \quad (\text{C.10b})$$

Case (3): $\Delta^2 - \gamma^2 M^2 > 0$

$$r(t) = \left\{ \left[\cos\left(\frac{\xi_2}{2}t\right) - i(2\Delta/\xi_2) \sin\left(\frac{\xi_2}{2}t\right) \right] r(0) - (2\gamma M/\xi_2) \sin\left(\frac{\xi_2}{2}t\right) s(0) \right\} e^{-(\frac{\gamma_1}{2} - i\Delta)t}. \quad (\text{C.10c})$$

The transient solutions (C.10) for the polarization field quadrature components have oscillatory (for $\Delta \neq 0$) decaying behavior provided that the initial conditions are non-zero, otherwise, the polarization field is zero for all $t \geq 0$. The case of exact resonance ($\Delta = 0$) is obtained from (C.10b) and shows that the solution is of purely decaying behavior. In all cases, the solutions (C.10) give vanishing steady state solutions,

$$r(\infty) = s(\infty) = 0.$$

ACCEPTED ARTICLE FOR PUBLICATION

Squeezing and Quantum Interference- Induced Zero and Amplified Absorption Harmonics with a 2-Photon 2-Level Atom

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The non-autonomous model of Bloch equations, modelling the coupling of a 2-photon 2-level single atom with a squeezed vacuum (SV) radiation reservoir, are treated via a Fourier Series (FS) decomposition approach. It is shown that both processes of quantum interference (QI), between the two indirect photon transition channels and the reservoir radiation squeezing, induce resonant zero absorption, accompanied by finite dispersion in the fundamental component and the 1st harmonic envelope component, which is associated with the positive frequency part of the atomic polarisation field. Negative absorption (amplification) can be seen with the corresponding envelope component of the negative frequency part.

Keywords: 2-photon transition, quantum interference, squeezed vacuum, absorption-dispersion harmonics.

1. INTRODUCTION

Within the context of non-linear optics and for a combined system of atom $\oplus$ field, the atomic absorption and dispersion response lineshape functions

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depend on the atomic configuration, as well as the state of the radiation field. One specific aim of such a study is the achievement of a transparent atomic medium with enhanced coherence (i.e. zero absorption with finite dispersion), which within the context of quantum information processing, has direct implications in transforming information (via a probe pulse) without loss or gain, and with controllable speed of propagation through the medium.

In the particular case of a dissipative 2-level atomic structure coupled to different states of radiation reservoirs, in the absence or presence of a driving field, the manipulation of the combined system parameters has led to various phenomena in the atomic absorption-dispersion lineshapes. There are two main cases of radiation reservoir states:

(a) Normal vacuum (NV) reservoir case

With the radiation reservoir in the NV case (i.e. zero average photon number and of zero temperature), and in the presence of a resonant or non-resonant strong driving field, the amplification of the absorption spectrum (i.e. negative absorption) of a weak probe field *around* the Rabi frequency side-bands, at the expense of the strong driving field, was first predicted theoretically [1] and later verified experimentally at optical frequencies [2]. The occurrence of these amplifications is accompanied with zero absorption, and occur *exactly* at the Rabi frequency side-bands [1,2]. As interpreted in [3,4], the positive dressed (combined) state population inversion is attributed to these amplifications. Zero absorption accompanied with finite dispersion (i.e finite refractive index), as demonstrated in the NV case [5, 6], is due to induced atomic coherence and hence a complex refractive index of the atomic medium (cf. [7]). Furthermore, for a 2-photon transition and bi-coherently driven 2-level atom in the NV case, the interplay of field-dependent damping rates and quantum interference processes (due to the 2-photon transition channels) were investigated in detail regarding the 2-photon induced atomic coherence [8].

(b) Squeezed vacuum (SV) reservoir case

For a weakly driven 2-level atomic system damped by a broadband SV reservoir, zero-absorption accompanied with enhanced coherence was predicted for single [9] and many cooperative atoms [10]. In the case of an intensely driven 2-level atom in the presence of a SV reservoir, where the damping coefficients become field-dependent [11], more enhanced coherence is predicted compared with the corresponding case of NV reservoir [12, 13]. In the SV case, this enhanced coherence with zero absorption is attributed to the coherent population oscillations, due to the

non-zero frequency detuning of the pump and probe field modes [9, 14]. In addition, the SV phase parameter related to the noise re-distribution of the SV radiation reservoir is one way to achieve zero absorption [15]. Analytical and computational investigations for the search of zero-absorption isoline regions, accompanied with finite or steep dispersion for a 2-level atomic system, coherently driven by mono- or bi-chromatic fields in the presence of a non-resonant SV reservoir, were recently given in [16].

In the present paper, we consider the model of a 2-photon 2-level atom, where only 2-photon transition processes are involved, with damping occurring in the presence of a broadband SV reservoir and in the absence of external driving fields [17]. The corresponding model of Bloch equations for the average atomic variables have time-dependent harmonic coefficients due to the SV reservoir, unlike the direct 1-photon transition case [18]. The non-autonomous model of Bloch equations [17] are treated via a FS harmonic decomposition approach. Amplitudes of the absorption-dispersion lineshapes, up to 1st harmonics, are investigated in detail. It is shown that quantum interference (QI), between the two indirect channel transitions in the presence of the SV parameters, induces zero-absorption accompanied with finite dispersion (at exact resonance) in the fundamental harmonic component, and amplified absorption in the 1st harmonic component.

The paper is presented as follows. In section II, the c-number non-autonomous Bloch equations for the averaged atomic variables are presented. The special case of no QI is discussed in detail. Otherwise, the full set of Bloch equations with QI included are treated in Section III via the FS decomposition approach. Absorption-dispersion lineshapes (up to 1st harmonic) are calculated and computationally investigated. Comparison with the 1-photon transition case is given in Section IV, followed by a summary in Section V.

2. THE MODEL BLOCH EQUATIONS

The energy level scheme of the 2-photon 2-level atom model is shown in Figure 1. The states $|a\rangle$ and $|b\rangle$ of energies $\hbar\omega_a$ and $\hbar\omega_b$, respectively, have the same parity, and the intermediate (dense) states $\{|j\rangle\}$ of energies $\{\hbar\omega_j\}$ are coupled to $|a\rangle$ or $|b\rangle$, or both by direct dipole transition. The zero dipole matrix element between the states $|a\rangle$ and $|b\rangle$ implies the absence of the direct 1-photon transition between them. It is assumed that the intermediate states $\{|j\rangle\}$ have no initial population, and are of the same parity, and therefore no transition and hence no interference effect occurs between them.

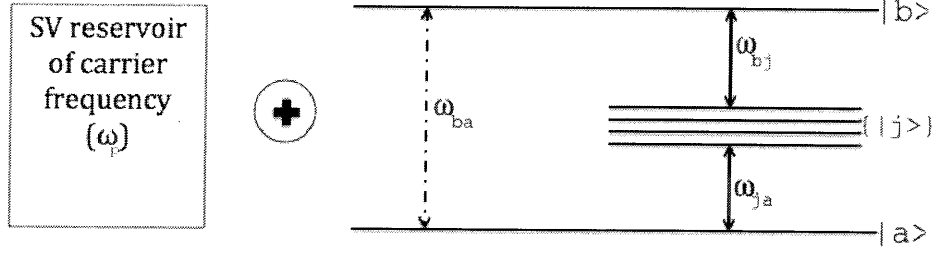


FIGURE 1

The 2-photon transition 2-level atom scheme coupled to a SV radiation reservoir.

The c-number model Bloch equation (after tracing over the SV radiation reservoir state [17]) for the averaged atomic inversion $w = \langle |b\rangle\langle b| - |a\rangle\langle a| \rangle$, and the (complex) induced polarisation $r = \langle |a\rangle\langle b| \rangle = s^*$ for the interacting system in Figure 1, have the following form, in a frame rotating at frequency $2\omega_p$:

$$\begin{aligned}
 w' = & - \left[\gamma_1^+ + \frac{\gamma_2^+ M}{2} (e^{2i\omega_p t} + e^{-2i\omega_p t}) \right] w \\
 & - \left[\gamma_1^- + \frac{\gamma_2^- M}{2} (e^{2i\omega_p t} + e^{-2i\omega_p t}) \right] \\
 & + \gamma_3^- (r e^{-2i\omega_p t} + s e^{2i\omega_p t}) + \frac{1}{2} \gamma_4 M (r + s), \quad (1a)
 \end{aligned}$$

$$\begin{aligned}
 r' = & - \left[\gamma_1^+ + i\Delta - M (\lambda_1 e^{2i\omega_p t} + \lambda_2 e^{-2i\omega_p t}) \right] r \\
 & - (\gamma_3^- w + \gamma_3^+) e^{2i\omega_p t} - M (\lambda_3^- w + \lambda_3^+) \quad (1b)
 \end{aligned}$$

$$s' = (r')^*, \quad (1c)$$

where $w' = \frac{dw}{dt}$, etc.

Derivation of the model equations (1) within a full quantised Hamiltonian approach is given in [17]. The notations in equations (1) are: The SV radiation reservoir parameters N and (real) $M = \sqrt{N(N+1)}$ (for maximum squeezing) are the averaged photon number and the degree of squeezing, respectively. The special case of a zero degree of squeezing ($M = 0$) represents the usual blackbody thermal reservoir case. The frequency detuning parameter is given as $\Delta = \omega_{ba} - 2\omega_p$, where $\omega_{ba} = \omega_b - \omega_a$ is the atomic frequency difference and ω_p is the central (carrier) frequency of the SV field. The various damping rates and constants in equations (1) are given explicitly [17] as

follows:

$$\begin{aligned}
\gamma_1^\pm &= \frac{1}{2} \sum_j (\Gamma_{bj} \pm \Gamma_{aj}), \\
\gamma_2^\pm &= \frac{1}{2} \sum_j \{(\gamma_{bj}\theta(\omega_{bj}) + \gamma_{jb}\theta(\omega_{jb})) \pm [\text{same with } b \rightarrow a]\}, \\
\gamma_3^\pm &= \frac{1}{2} \sum_j (\Gamma_{bj}^{ab} \pm \Gamma_{aj}^{ab}), \\
\gamma_4 &= \frac{1}{2} \sum_j \{(\gamma_{aj}^{ab}\theta(\omega_{aj}) + \gamma_{ja}^{ab}\theta(\omega_{ja})) \pm [\text{same with } a \rightarrow b]\}, \\
\Gamma_{ja} &= \gamma_{ja}\theta(\omega_{ja})(1 + N) + \gamma_{aj}\theta(\omega_{aj})N; \text{ etc.}, \\
\Gamma_{aj}^{ab} &= \gamma_{aj}^{ab}\theta(\omega_{aj})(1 + N) + \gamma_{ja}^{ab}\theta(\omega_{ja})N, \\
\lambda_1 &= \frac{1}{2} \sum_j [\gamma_{bj}\theta(\omega_{bj}) + \gamma_{ja}\theta(\omega_{ja})], \\
\lambda_2 &= \lambda_1(a \leftrightarrow b), \\
\lambda_3^\pm &= \frac{1}{2} \sum_j [\gamma_{ja}^{ab}\theta(\omega_{ja}) + \gamma_{bj}^{ab}\theta(\omega_{bj})]. \tag{2}
\end{aligned}$$

The parameters $\gamma_{bj} = \left(\frac{4}{3c^3\hbar}\right) |\underline{P}_{bj}| \omega_{bj}^3$ are the decay rates from the state $|b\rangle$ to one of the intermediate states $\{|j\rangle\}$, with $\underline{P}_{bj}$ is the dipole matrix element between states $|b\rangle$ and $|j\rangle$, while $\gamma_{aj}^{ab} = \left(\frac{4}{3c^3\hbar}\right) (\underline{P}_{aj} \cdot \underline{P}_{bj}) \omega_{aj}^3$ represents the quantum interference effect between the two transition channels $|a\rangle \leftrightarrow |j\rangle$ and $|b\rangle \leftrightarrow |j\rangle$; it is maximum if $\underline{P}_{aj}$ and $\underline{P}_{bj}$ are parallel, and vanishes if they are orthogonal [17] (similar expressions are for γ_{aj} , γ_{bj}^{ab} , etc.). In equations (1) and (2) we have assumed, for simplicity, that the SV parameters N and M are the same for all transition frequencies: $N(\omega_{aj}) = N(\omega_{bj}) = \dots = N$, $M(\omega_{aj}) = M(\omega_{bj}) = \dots = M$. The step function $\theta(x)$ is defined by:

$$\theta(x) = \begin{cases} 1 & , x > 0 \\ 0 & , x < 0 \end{cases} \tag{3}$$

and its role is determined by the configuration of the intermediate levels $\{|j\rangle\}$, as detailed below.

The physical interpretation of the decay rates in equation (2) are as follows [17]:

- (i) The rate γ_1^+ represents the total decay rate of the net atomic inversion and polarisation between levels $|b\rangle \leftrightarrow |j\rangle$ and $|a\rangle \leftrightarrow |j\rangle$.

- (ii) The rate γ_1^- represents an equilibrium value for the net atomic inversion.
- (iii) The rates $\gamma_2^\pm$ are associated with the induced steady oscillations in the atomic inversion (w) due to the presence of the SV field. Similarly, the rates $\lambda_{1,2}$ are associated with the induced steady oscillations in the atomic polarisation (r, s) due the SV parameter M .
- (iv) The rate γ_3^- (due to QI effect) is associated with the induced 2-photon polarisation (r, s) in the equation for w . It is also associated with the atom being in the state $|b\rangle$ or $|a\rangle$ in the equations for (r', s') .
- (v) The rate γ_3^+ represents a steady oscillation source for the atomic polarisation (r, s), due to the 2-photon transition interference processes.
- (vi) The rates γ_4 and λ_3^- (due to QI effect) show that the SV field acts as an oscillating driving field for both the atomic inversion w and polarisation (r, s).
- (vii) The rate λ_3^+ (due to QI effect) is associated with the SV field parameter (M) that contributes to the steady state value of the atomic polarisation component (r, s).

For the cascade configuration of the intermediate state $\{|j\rangle\}$, as shown in Figure 1, we have $\omega_b > \omega_j > \omega_a$ for all j , and therefore the step functions appearing in the expressions (2) have the values:

$$\begin{aligned}\theta(\omega_{bj}) &= \theta(\omega_{ja}) = 1, \\ \theta(\omega_{jb}) &= \theta(\omega_{aj}) = 0.\end{aligned}\tag{4}$$

To simplify the expressions in (2), we assume equal decay rates from levels $E_b \rightarrow E_j$ and $E_j \rightarrow E_a$ as follows:

$$\begin{aligned}\gamma_{bj} &= \gamma_{ja}, \quad \gamma_{bj}^{ab} = \gamma_{ja}^{ba}, \\ \sum_j \gamma_{bj} &= \sum_j \gamma_{ja} = \gamma, \quad \sum_j \gamma_{bj}^{ab} = \sum_j \gamma_{ja}^{ab} = \gamma^{ab}.\end{aligned}\tag{5}$$

Hence, the various constants in equation (2), appearing in equations (1), normalised to the damping constant $\gamma_1^- = \frac{1}{2}\gamma$, have the following forms:

$$\begin{aligned}\gamma_1^+ &= N_1, \gamma_2^+ = 1, \gamma_2^- = 0, \\ \gamma_3^+ &= N_1 f, \gamma_3^- = f, \gamma_4 = 0, \\ \lambda_1 &= 2, \lambda_2 = 0, \lambda_3^+ = 2f, \lambda_3^- = 0,\end{aligned}\tag{6}$$

where $N_1 = 1 + 2N$ and the normalised parameter $f = \frac{\gamma^{ab}}{\gamma}$ is the measure of QI ($0 \leq f \leq 1$). Hence, the model equations (1) in normalised form ($\tau = \gamma_1^{-1}t$) are as follows:

$$\frac{dw}{d\tau} = - \left[N_1 + \frac{1}{2}M \left(e^{2i\omega'_p\tau} + e^{-2i\omega'_p\tau} \right) \right] w - 1 + f \left(r e^{-2i\omega'_p\tau} + s e^{2i\omega'_p\tau} \right) \quad (7a)$$

$$\frac{dr}{d\tau} = - \left[N_1 + i\Delta - 2M e^{2i\omega'_p\tau} \right] r - f(N_1 + w) e^{2i\omega'_p\tau} - 2fM \quad (7b)$$

$$\frac{ds}{d\tau} = \left(\frac{dr}{d\tau} \right)^*, \quad (7c)$$

where $\omega'_p = \frac{\omega_p}{\gamma_1^{-1}}$.

In the absence of QI, $f = 0$, equations (7) reduce to the following form:

$$\frac{dw}{d\tau} = - \left[N_1 + \frac{1}{2}M \left(e^{2i\omega'_p\tau} + e^{-2i\omega'_p\tau} \right) \right] w - 1 \quad (8a)$$

$$\frac{dr}{d\tau} = - \left[N_1 + i\Delta - 2M e^{2i\omega'_p\tau} \right] r \quad (8b)$$

$$\frac{ds}{d\tau} = \left(\frac{dr}{d\tau} \right)^*. \quad (8c)$$

The rates of evolution of the atomic variables in equations (8) are independent of each other. In the steady state, a vanishing polarised field ($r(\infty) = s(\infty) = 0$) is observed. On the other hand, equation (8a) for the atomic inversion has the *same* form for the model of a single 2-level atom, with one direct photon transition, coupled to a SV reservoir outside the rotation wave approximation (RWA), where terms in $e^{\pm 2i\omega'_p\tau}$ are not discarded [19]. A formal analytical solution of equation (8a) [19] shows that the presence of the SV parameter M induces small steady oscillations in the atomic inversion, $O(\frac{\gamma}{\omega_p})^2$. To illustrate this within the FS approach, we assume the following form for w in the steady state:

$$w = w_o + \left(w_+ e^{2i\omega'_p\tau} + c.c. \right), \quad (9)$$

where $w_o, +$ are time-independent envelopes. Substituting equation (9) into equation (8a), and comparing the coefficients of $e^{2ni\omega'_p\tau}$ ($n = 0, \pm 1$) on both

sides, gives:

$$w_o = \frac{-1}{1+2N} + \frac{M^2}{2(1+2N)} \left(\frac{\gamma}{2\omega'_p} \right)^2 \quad (10a)$$

and

$$w_+ = \frac{-M}{2} \frac{w_o}{N_1 + 2i\omega'_p}. \quad (10b)$$

The expression w_o in equation (10a) agrees with that in [19] to $O\left(\frac{\gamma}{2\omega'_p}\right)^2$, and shows that the SV reservoir ($M \neq 0$), in the absence of QI effects, modifies the usual saturation formula, $-\frac{1}{1+2N}$, of the normal blackbody reservoir ($M = 0$) for the case of a 1-photon transition.

In the following section, we analyse the non-autonomous model of Bloch equations (7) in the case of maximum QI ($f = 1$) using a FS decomposition approach, and examine the steady state fundamental and 1st harmonic envelope components of the absorption and dispersion lineshapes.

3. ABSORPTION-DISPERSION HARMONICS

A convenient FS decomposition approach to treat the non-autonomous Bloch equations, equations (7), in the case of QI ($f = 1$) in the steady state $\left(\frac{dw}{d\tau} = \frac{dr}{d\tau} = \frac{ds}{d\tau} = 0\right)$, up to 1st harmonics in $e^{\pm 2i\omega'_p\tau}$, is outlined as follows (cf. [20, 21, 14]):

(1) Adiabatic elimination of (r, s)

We set $\frac{dr}{d\tau} = \frac{ds}{d\tau} = 0$ in equations (7) with $f = 1$ and solve for (r, s) in terms of w to get,

$$r = -\left[(N_1 + w)e^{2i\omega'_p\tau} + 2M\right] G_1^{-1} \quad (11a)$$

$$s = r^*, \quad (11b)$$

where

$$G_1 = \left(N_1 + i\Delta - 2Me^{2i\omega'_p\tau}\right). \quad (12)$$

Hence, from equations (11) we have:

$$\begin{aligned} r e^{-2i\omega'_p \tau} + s e^{2i\omega'_p \tau} = & -|G_1|^{-2} \left[2N_1^2 + 2iM\Delta \left(e^{2i\omega'_p \tau} - e^{-2i\omega'_p \tau} \right) \right. \\ & - 4M^2 \left(e^{4i\omega'_p \tau} + e^{-4i\omega'_p \tau} \right) \\ & \left. + w \left(2N_1 - 2M \left(e^{2i\omega'_p \tau} + e^{-2i\omega'_p \tau} \right) \right) \right]. \end{aligned} \quad (13)$$

From equation (12), the quantity $|G_1|^{-2}$ is given by:

$$\begin{aligned} |G_1|^{-2} = & |N_1 + i\Delta - 2Me^{2i\omega'_p \tau}|^{-2} \\ = & (N_1 + i\Delta)^{-1} (N_1 - i\Delta)^{-1} \left(1 - \beta_1 e^{2i\omega'_p \tau} \right)^{-1} \left(1 - \beta_1^* e^{-2i\omega'_p \tau} \right)^{-1} \\ \simeq & (N_1^2 + \Delta^2)^{-1} \left(1 + \beta_1 e^{2i\omega'_p \tau} \right) \left(1 + \beta_1^* e^{-2i\omega'_p \tau} \right) \end{aligned} \quad (14a)$$

$$= (N_1^2 + \Delta^2)^{-1} \left(1 + |\beta_1|^2 + \beta_1 e^{2i\omega'_p \tau} + \beta_1^* e^{-2i\omega'_p \tau} \right), \quad (14b)$$

where the approximation in equation (14a) is valid for:

$$|\beta_1| = \left| \frac{2M}{N_1 + i\Delta} \right| < 1. \quad (14c)$$

Note that, for the (real) parameters $M \leq \sqrt{N(N+1)}$, (e.g. [19]), $N_1 = 1 + 2N$ and $|\Delta| \geq 0$, the inequality (14c) is valid.

Substituting the expression (14b) for $|G_1|^{-2}$ into equation (13), gives:

$$\begin{aligned} r e^{-2i\omega'_p \tau} + s e^{2i\omega'_p \tau} = & - (N_1^2 + \Delta^2)^{-1} \left(1 + |\beta_1|^2 + \beta_1 e^{2i\omega'_p \tau} + \beta_1^* e^{-2i\omega'_p \tau} \right) \\ & \cdot \left[2N_1^2 + 2iM\Delta \left(e^{2i\omega'_p \tau} - e^{-2i\omega'_p \tau} \right) \right. \\ & - 4M^2 \left(e^{4i\omega'_p \tau} + e^{-4i\omega'_p \tau} \right) \\ & \left. + w \left(2N_1 - 2M \left(e^{2i\omega'_p \tau} + e^{-2i\omega'_p \tau} \right) \right) \right]. \end{aligned} \quad (15)$$

(2) Differential equation for w

Substituting expression (15) for $(r e^{-2i\omega'_p \tau} + s e^{2i\omega'_p \tau})$ into equation (7a) gives the following differential equation for w :

$$\begin{aligned} \frac{dw}{d\tau} = & \left[C_o + \left(C_1 e^{2i\omega'_p \tau} + C_2 e^{4i\omega'_p \tau} + c.c. \right) \right] w + D_o \\ & + \left(D_1 e^{2i\omega'_p \tau} + D_2 e^{4i\omega'_p \tau} + D_3 e^{6i\omega'_p \tau} + c.c. \right). \end{aligned} \quad (16)$$

The coefficients C_i and D_i are given as follows:

$$\begin{aligned}
C_o &= -N_1 - \frac{2N_1}{(N_1^2 + \Delta^2)}, \\
C_1 &= -\frac{M}{2} - \frac{\alpha_1}{(N_1^2 + \Delta^2)}, \quad \alpha_1 = 2(N_1\beta_1 - M(1 + |\beta_1|^2)), \\
C_2 &= \frac{2M\beta_1}{(N_1^2 + \Delta^2)}, \\
D_o &= -1 - \frac{2[N_1^2 + M(\beta_1(N_1 - i\Delta) + c.c.)]}{(N_1^2 + \Delta^2)}, \\
D_1 &= -\frac{\alpha_1 N_1}{(N_1^2 + \Delta^2)} - \frac{2M[(N_1 + i\Delta)(1 + |\beta_1|^2) - 2M\beta_1^*]}{(N_1^2 + \Delta^2)}, \\
D_2 &= \frac{4M^2(1 + |\beta_1|^2) - 2i\Delta M\beta_1}{(N_1^2 + \Delta^2)}, \\
D_3 &= \frac{4M^2\beta_1}{(N_1^2 + \Delta^2)}. \tag{17}
\end{aligned}$$

(3) FS ansatz for w

In view of the harmonic coefficients in the differential equation (16), the steady state solution for w may take the FS ansatz:

$$w = w_o + \left(w_1 e^{2i\omega'_p \tau} + w_2 e^{4i\omega'_p \tau} + w_3 e^{6i\omega'_p \tau} + c.c. \right), \tag{18}$$

where the coefficients $w_{o,1,2,3}$ are time-independent.

Note that, in the FS, equation (18), we have included all possible harmonics (namely, up to $e^{\pm 6i\omega'_p \tau}$) which affect the envelopes w_1, w_1^* of the 1st harmonic components $e^{\pm 2i\omega'_p \tau}$.

Substituting equation (18) into equation (16), and by comparing the coefficients of $e^{\pm 2ni\omega'_p \tau}$ ($n = 0, 1, 2, 3$), we obtain the following system of algebraic equations for the envelopes $w_{o,1,2,3}$ in matrix form:

$$AX = -I \tag{19}$$

where,

$$X = \begin{bmatrix} w_o & w_1 & w_1^* & w_2 & w_2^* & w_3 & w_3^* \end{bmatrix}^T, \tag{20a}$$

$$I = \begin{bmatrix} D_o & D_1 & D_1^* & D_2 & D_2^* & D_3 & D_3^* \end{bmatrix}^T, \tag{20b}$$

$$A = \begin{bmatrix} C_o & C_1^* & C_1 & C_2^* & C_2 & 0 & 0 \\ C_1 & C_o^* - 2i\omega'_p & C_2 & C_1^* & 0 & C_2^* & 0 \\ C_1^* & C_2^* & C_o + 2i\omega'_p & 0 & C_1 & 0 & C_2 \\ C_2 & C_1 & 0 & C_o - 4i\omega'_p & 0 & C_1^* & 0 \\ C_2^* & 0 & C_1^* & 0 & C_o + 4i\omega'_p & 0 & C_1 \\ 0 & C_2 & 0 & C_1 & 0 & C_o - 6i\omega'_p & 0 \\ 0 & 0 & C_2^* & 0 & C_1^* & 0 & C_o + 6i\omega'_p \end{bmatrix}. \quad (20c)$$

(4) Harmonic envelopes for r

Here, we derive the expression for the fundamental and the 1st harmonic envelopes for r . First, the expression for G_1^{-1} , equations (12), is approximated as follows:

$$G_1^{-1} = (N_1 + i\Delta)^{-1} \left(1 - \beta_1 e^{2i\omega'_p \tau}\right)^{-1}$$

$$G_1^{-1} \simeq (N_1 + i\Delta)^{-1} \left(1 + \beta_1 e^{2i\omega'_p \tau}\right)^{-1}, \quad \beta_1 = \frac{2M}{N_1 + i\Delta}, \quad (21)$$

valid for $|\beta_1| < 1$. Substituting equation (21) and equation (18) into equation (11a) gives:

$$r = \frac{-\left(N_1 e^{2i\omega'_p \tau} + 2M\right) \left(1 + \beta_1 e^{2i\omega'_p \tau}\right)}{N_1 + i\Delta} - \frac{e^{2i\omega'_p \tau} + \beta_1 e^{4i\omega'_p \tau}}{N_1 + i\Delta} \cdot \left[w_o + \left(w_1 e^{2i\omega'_p \tau} + w_2 e^{4i\omega'_p \tau} + w_3 e^{6i\omega'_p \tau} + c.c.\right)\right]. \quad (22)$$

Up to the 1st harmonics, equation (22) is reduced to:

$$r = r_o + r_+ e^{2i\omega'_p \tau} + r_- e^{-2i\omega'_p \tau}, \quad (23)$$

where the envelopes r_o and $r_{\pm}$ for the fundamental and 1st harmonic components, respectively, are given by:

$$r_o = \frac{-(2M + w_1^* + \beta_1 w_2^*)}{N_1 + i\Delta} \quad (24a)$$

$$r_- = \frac{-(w_2^* + \beta_1 w_3^*)}{N_1 + i\Delta} \quad (24b)$$

$$r_+ = \frac{-(N_1 + w_o + \beta_1(w_1^* + 2M))}{N_1 + i\Delta}, \quad (24c)$$

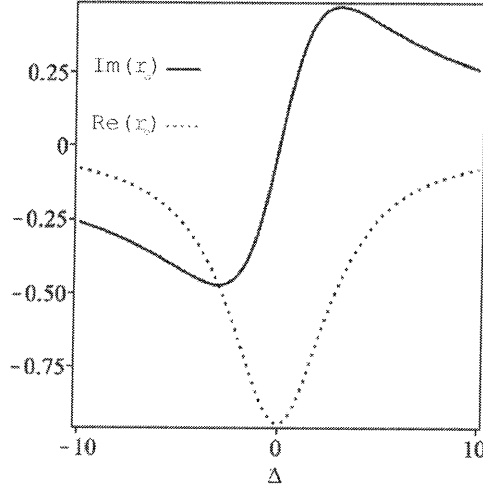


FIGURE 2

The fundamental absorption and dispersion components, $Im(r_o)$ and $Re(r_o)$, respectively, against the normalised Δ for $N = 1$, $M = \sqrt{N(N+1)}$, $\omega'_p = 10^7$.

with the coefficients $w_{o,1,2,3}$ determined numerically by solving the algebraic equations (19) and (20), using Maple software version 15.

For SV parameters, $N = 1$, $M = \sqrt{N(N+1)}$ and for $\omega'_p = 10^7$ (optical frequency range), the fundamental absorption and dispersion components, $Im(r_o)$ and $Re(r_o)$, respectively, exhibit zero absorption accompanied by finite dispersion [9] at *exact resonance* ($\Delta = 0$), Figure 2.

This is entirely due to the quantum interference effect between the two transition channels in the presence of the SV field. For $\Delta < 0$, the absorption is amplified (i.e. negative absorption). For smaller $N = 0.1$, the same results are obtained but with narrowing widths. For SV parameters $M \rightarrow -M$ (i.e. changing the SV phase parameter from $0 \rightarrow \pi$), the absorption shape $Im(r_o)$ becomes the mirror image along the vertical axis at $\Delta = 0$ of the full lined curve in Figure (2a); hence negative absorption occurs for $\Delta > 0$, while the dispersion shape (dotted line) $Re(r_o)$ becomes the usual Lorentzian profile. The 1st harmonic absorption and dispersion envelope components $Im(r_-)$ and $Re(r_-)$, respectively, associated with the positive frequency part of the atomic polarisation, equation (23), are small, $O\left(\frac{1}{\omega'_p}\right)$, and show negative absorp-

tion (amplification) for all Δ with the usual dispersion Lorentzian shape, Figure (3). On the other hand, the 1st harmonic envelope components, $Im(r_+)$ and $Re(r_+)$, associated with the negative frequency part of r , equation (23), have the same qualitative features as the fundamental components in Figure (2), however with larger extrema.

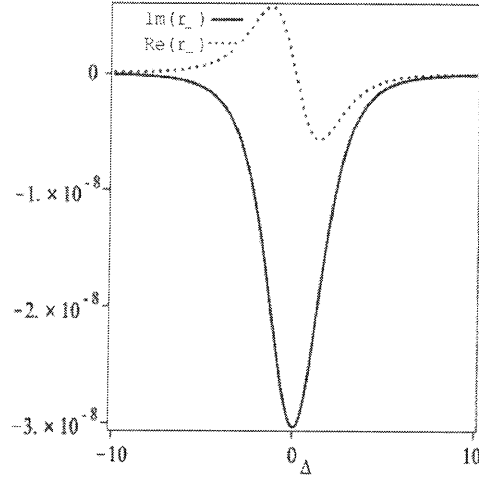


FIGURE 3

As Figure 2, but for the 1st harmonic absorption and dispersion components, $Im(r_-)$ and $Re(r_-)$, respectively

4. COMPARISON WITH THE ONE-TRANSITION CASE

The corresponding Bloch equations modelling the interaction of a single 2-level atom with an off-resonant broadband SV field via a 1-photon transition (therefore no intermediate states $\{|j\rangle\}$), within the RWA and in a rotating frame at ω_o , have the form (e.g. [19]):

$$\dot{w} = -\gamma - \gamma_1 w, \quad (25a)$$

$$\dot{r} = -\frac{\gamma_1}{2} r - \gamma M s e^{-iq\tau}, \quad (25b)$$

$$= (s \dot{)}^*. \quad (25c)$$

Here, w and $r = s^*$ are the atomic inversion and polarisation field, respectively, $\gamma_1 = \gamma(2N + 1)$, $q = 2\Delta$, $\Delta = \omega_p - \omega_o$, with ω_p the central frequency of the SV field and ω_o the atomic transition frequency, γ is the decay constant, and N, M are the SV parameters.

Two main features from equations (25) can be observed:

- (i) The atomic inversion $\dot{w}$ in equation (25a) is not coupled to the atomic polarisation field ($r = s^*$), and depends only on the SV parameter (N) as a broadening decay effect (c.f. [18]). This is unlike the present 2-photon transition case, equation (1a), where w is coupled to both polarisation components (r, s) due to the interference process (i.e. terms in γ_3^- and γ_4).

- (ii) The two quadrature components of the polarisation field (r, s) , equations (25b,c) are coupled through the SV parameter M (degree of squeezing) and independent of the atomic energy w . This is unlike the 2-photon transition case, equation (1b), as mentioned in (1) above. In the steady state $\dot{w} = \dot{r} = \dot{s} = 0$, equation (23a) gives the usual saturation formula for the thermal (blackbody) field [19]:

$$w(\infty) = -\frac{1}{1 + 2N} \quad (26a)$$

while equations (23b,c) shows that there is no steady polarisation field:

$$r(\infty) = s(\infty) = 0. \quad (26b)$$

Note that, from the exact time-dependent solutions of the homogenous differential equations (25b,c) for (r, s) , it is easy to show that the transient polarisation field is non-zero due to non-zero initial (coherence) conditions, $(r(0), s(0)) \neq 0$.

5. SUMMARY

We have investigated the non-autonomous model of Bloch equations representing the coupling of a single 2-level atom with a SV reservoir, where the atomic transition between the 2-energy levels occurs indirectly via a 2-photon-transition, through a set of intermediate states $\{|j\rangle\}$ [17]. For the cascade level structure, Figure 1, the equations for the atomic inversion w , equation (1a), and the induced polarisations (r, s) , equations (1b,c) are coupled due to both the QI processes between the 2-photon transition channels and the presence of the SV field. This is unlike the 1-photon transition case where no QI occurs, and the atomic inversion is not affected by the induced polarisation, and vice-versa. The non-autonomous Bloch equations with periodic coefficients are treated via the FS decomposition approach. The main results for the investigation of the absorption-dispersion envelop harmonics at exact atomic resonance ($\Delta = 0$) are:

- (i) The fundamental harmonic component shows zero-absorption accompanied with finite dispersion.
- (ii) The 1st harmonic envelope component associated with the positive frequency part is small, $O\left(\frac{1}{\omega'_p}\right)$, and shows maximum amplified (negative) absorption accompanied with zero dispersion. The corresponding

envelope associated with the negative frequency part resembles that of the fundamental harmonic component, but with larger extrema.

Both results are due to the combined presence of the QI process with the SV reservoir.

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الملخص

يتعلق موضوع الرسالة بدراسة نموذج رياضي لمجموعة معادلات تفاضلية عادية من الرتبة الأولى ذات معاملات تفاضلية متغيرة مع الزمن (Non-Autonomous Bloch Equations). نموذج المعادلات تحت الدراسة يمثل تفاعل ذرة واحدة ذات مستويين للطاقة مع مجال ضوئي منضغط غير كلاسيكي (Squeezed light)، حيث انتقال الذرة بين مستويي الطاقة يتم بطريقة غير مباشرة (2-photon transition).

تم تحليل نموذج المعادلات التفاضلية بطريقتين لمفكوك توافقيات فوريير (Fourier Series Decomposition):- الطريقة الأولى تعتمد على فرضية مفكوك فوريير ذات التوافقية الأولى فقط (Truncated Fourier Series).

أما الطريقة الثانية تعتمد على حذف بعض المتغيرات (Adiabatic Elimination) وبالتالي استنتاج معادلة تفاضلية منفصلة لأحد المتغيرات، ثم استخدام فرضية مفكوك فوريير ذات التوافقية المناسبة لحل المعادلة التفاضلية المعنية (Adiabatic Fourier Series Decomposition).

تم حساب بعض الكميات الفيزيائية مثل دوال الامتصاص والتشتت، وكذلك شدة الإشعاع الناتج (ذات التوافقيات الأساسية والأولى) وعرضها عددياً (Computational)، مع مقارنة نتائج طريقتي مفكوك فوريير السابق ذكرهما.

ومن أهم نتائج الدراسة هو انعدام دالة الامتصاص بسبب تأثير التداخل الكمي (Quantum interference) الناتج عن الانتقال الغير مباشر للذرة، ومن ثم وجود وسط مادي شفاف (Transparent Medium) يستخدم لانتقال المعلومات الضوئية دون أي تغيير.

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أطروحة مقدمة كجزء من متطلبات الحصول على درجة الماجستير في الرياضيات

إعداد

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