

UNIVERSITY OF BAHRAIN

**APPLICATIONS OF HADAMARD PRODUCT TECHNIQUES
TO
PROBLEMS IN UNIVALENT FUNCTION THEORY**

BY

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ABSTRACT

In this thesis we study applications of Hadamard product (convolution) techniques to problems in univalent function theory. Proofs of several conjectures made by different researchers are discussed. Using the results of Ruscheweyh and Sheil-Small developed in connection with their proof of Pólya-Schoenberg conjecture, we exploit the strengths and properties of various convolution operators. A unified approach to the study of certain properties of existing and new subclasses of univalent analytic functions is also given by introducing and studying the classes $E_T(\varphi, \psi; \alpha, \beta)$ and $E_T^p(\varphi, \psi; \alpha, \beta)$. The approach not only generalize corresponding known results but also give an analogous study for several other subclasses of univalent analytic functions.

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SYNOPSIS

The theory of univalent analytic functions, during the current century, has been one of the most fertile fields of investigation by eminent mathematicians who have enriched it by way of introduction of numerous new concepts, ingenious and powerful methods. It is primarily concerned with the interplay of analytic structure and geometric behavior of analytic functions of a complex variable. The settlement of sixty eight years old Bieberbach conjecture in 1984 by De Branges [12] not only created a land mark in the history of the subject but also aroused curiosity and interest of the non-specialist in this branch of study.

The hadamard product or convolution of two convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is defined as the power series $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. In 1958, Pólya and Schoenberg [41] conjectured that if $f(z)$ and $g(z)$ are convex in $U = \{z : |z| < 1\}$, then so is their convolution $(f * g)(z)$. The convolution is univalent (and in fact close-to-convex) was first proved by Suffridge [69]. It was also proposed by him that if $\phi(z)$ is convex and $f(z)$ is close-to-convex, then $(\phi * f)(z)$ is close-to-convex. Many researchers attacked these problems and obtained partial results. In 1973, Ruscheweyh and Sheil-Small [55] established the truth of both Pólya-Schoenberg conjecture and the proposition. Their methods enable to obtain a number of other similar results and settle an interesting subordination conjecture by Wilf [71] stronger than Pólya-Schoenberg conjecture.

The proof of these conjectures has increased interest in the use of Hadamard product techniques. Using the results and techniques of Ruscheweyh and Sheil-Small developed in [55], Barnard and Kellogg [4] exploited the strength and properties of convolution operators. The strength of these methods lie in their ability to unify a number of diverse

results. Several subclasses of univalent analytic functions have also been introduced and studied, using Hadamard product (see [25],[39],[54],[60] etc.).

The aim of this thesis is to study applications of Hadamard product techniques to problems in the theory of univalent analytic functions. In particular, we discuss the results of Ruscheweyh and Shiel-Small [55], Barnard and Kellogg [4], and extend the results of Juneja, Reddy and Mogra [25]. The thesis consists of four chapters.

The basic concepts and principal results concerning the subject, which will be used throughout the thesis, have been given in Chapter I.

In Chapter II, we shall discuss in detail the results obtained by Ruscheweyh and Sheil-Small [55]. In Section 2.2 we study certain geometric properties of convex and starlike functions and the expression of these in analytic terms. Section 2.3 of this chapter is devoted to discuss the proof of Pólya-Schoenberg conjecture and the Suffridge proposition. The structure of functions starlike of order $1/2$ is also preserved under convolutions. It is discussed in Section 2.4. The subordination conjecture of Wilf [71] is discussed in Section 2.5 of this chapter.

In 1947, R.M. Robinson [48] conjectured that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in S , the class of univalent analytic functions in U , then the function $1/2[f(z) + zf'(z)]$ is also belongs to S . This well-known Robinson's $1/2$ conjecture was verified by Livingston [33] in 1966 for convex, starlike and close-to-convex functions. Many researchers (see [29],[43] etc) have studied the appropriate minimum radii for different operators on various subclasses of S . Using the results and techniques of Ruscheweyh and Shiel-Small [55] discussed in Chapter II, Barnard and Kellogg [4] obtained these results and many of their generalizations.

Chapter III is devoted to study the results of Barnard and Kellogg [4]. In Section 3.2 we discuss appropriate minimum radii for the different operators on various subclasses of S , using convolution techniques. Section 3.3 of this chapter is devoted to discuss Robinson's $1/2$ conjecture in case of γ -spirallike functions. Robinson's $1/2$ conjecture for typically real functions will be given in Section 3.4. Several other results including a generalization of Bernardi's results [5] are discussed in Section 3.5 of this chapter.

The effect of second coefficient on various properties of the subclasses of univalent analytic functions has been studied by several researchers (see [23],[36],[65] etc.). However, the first attempt to investigate the properties of functions in the classes $S^*[\alpha]$ and $C[\alpha]$, of starlike and convex functions of order α ($0 \leq \alpha \leq 1$), with fixed second coefficient has been made by Silverman and Silvia [66], where they obtain extreme points, distortion theorems and radius of convexity bounds etc. A unified approach to the study of these properties for various subclasses of S , using Hadamard product, has been given by Juneja, Reddy and Mogra [25] which not only extend the results of Silverman and Silvia [66] but also leads to the corresponding results for various other subclasses of univalent analytic functions with negative coefficients.

In Chapter IV, we extend the results of Juneja, Reddy and Mogra [25] which gives an analogous study for several other subclasses of univalent analytic functions. In Section 4.2 we obtain a necessary and sufficient condition for a function to be in the class $E_T(\varphi, \psi; \alpha, \beta)$ which leads to coefficient estimates. The extreme points of $E_T^p(\varphi, \psi; \alpha, \beta)$ are obtained in Section 4.3 of this chapter. Finally, we obtain distortion and covering theorems for functions in $E_T(\varphi, \psi; \alpha, \beta)$ and $E_T^p(\varphi, \psi; \alpha, \beta)$ in the last section of this chapter.

CHAPTER I

PRELIMINARIES

In this chapter we shall give only those concepts and results of the theory of univalent functions and related topics that we shall need in the subsequent chapters of this thesis. These results have been given in the form of definitions, lemmas or theorems. Detailed proofs and further discussions on the subject may be found in the standard texts of Goluzin [17], Duren [13], Goodman [18] etc.

1.1. Introduction

Definition 1.1.1. A single-valued function f is said to be univalent (schlicht or simple) in a domain D in the complex plane if it takes every value at most once; that is, $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in D with $z_1 \neq z_2$.

Note that there exist functions which are univalent but not analytic in a given domain D . For example, the function

$$f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

is univalent in D , but it has a simple pole at $z = -d/c$ if it is one of the values taken by z in D . On the other hand, an analytic function need not be univalent. For example, $f(z) = e^z$ is analytic in any domain D but is not univalent in D since $f(z + 2n\pi) = f(z)$ for all $z \in D$ and n is an integer.

An analytic univalent function is called a conformal mapping because of its angle preserving property. In the study of univalent analytic functions, the Riemann mapping theorem plays a crucial role. The theorem states that if D is a simply connected domain properly contained in $\mathbb{C}$ and z_0 is a given point in D then there exists a unique univalent analytic function f which

maps D conformally onto the open unit disc $U = \{z: |z| < 1\}$ and has properties $f(z_0) = 0$, $f'(z_0) > 0$. Thus in view of Riemann mapping theorem, instead of studying the properties of univalent analytic functions in an arbitrary simply connected domain, we shall confine ourselves to the case in which D is the open unit disc U .

Let A denote the class of all those functions f which are analytic in U and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, let S denote the class of functions $f \in A$ which are univalent in U . Thus each $f \in S$ has a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \quad (1.1.2)$$

The leading example of a function of the class S is the Koebe function

$$k(z) = z(1 - z)^{-2} = z + \sum_{n=2}^{\infty} n z^n \quad (1.1.3)$$

which maps the disc U onto the entire plane minus the part of the negative real axis from $-1/4$ to $-\infty$. This is best seen by writing

$$k(z) = \frac{1}{4} \left(\frac{1+z}{1-z} \right)^2 - \frac{1}{4}$$

and observing that the function

$$w = \frac{1+z}{1-z}$$

maps U conformally onto the right half-plane $\Re\{w\} > 0$.

Other simple examples of functions in S are

- (i) $f(z) = z$, the identity mapping;
- (ii) $f(z) = z(1 - z)^{-1}$, which maps U conformally onto the half-plane $\Re\{f(z)\} > -1/2$;
- (iii) $f(z) = z(1 - z^2)^{-1}$, which maps U onto the entire plane minus the two half-lines $1/2 \leq u < \infty$ and $-\infty < u \leq -1/2$;
- (iv) $f(z) = 1/2 \log [(1 + z)/(1 - z)]$, which maps U onto the horizontal strip $-\pi/4 < \Im\{f(z)\} < \pi/4$;
- (v) $f(z) = z - 1/2 z^2 = 1/2 [1 - (1 - z)^2]$, which maps U onto the interior of a cardioid.

It is easy to check that the sum, difference, product and quotient of two functions in S need not be in S . The same is true for the derivative and integral of a function $f \in S$. However, the class S is preserved under a number of elementary transformations. Here is the partial list.

- (i) **Conjugation.** If $f \in S$ and $g(z) = \overline{f(\bar{z})} = z + \bar{a}_2 z^2 + \bar{a}_3 z^3 + \dots$, then $g \in S$.
- (ii) **Rotation.** If $f \in S$ and $g(z) = e^{-i\theta} f(ze^{i\theta})$, then $g \in S$.
- (iii) **Dilation.** If $f \in S$ and $g(z) = r^{-1} f(rz)$, where $0 < r < 1$, then $g \in S$.
- (iv) **Disc automorphism.** If $f \in S$ and

$$g(z) = \frac{f\left(\frac{z + \alpha}{1 + \bar{\alpha}z}\right) - f(\alpha)}{(1 - |\alpha|^2) f'(\alpha)}, \quad |\alpha| < 1,$$

then $g \in S$.

- (v) **Range transformation.** If $f \in S$ and ψ is a function analytic and univalent on the range of f , with $\psi(0) = 0$ and $\psi'(0) = 1$, then $g = \psi \circ f \in S$.
- (vi) **Omitted-value transformation.** If $f \in S$ and $f(z) \neq w$, then $g = wf/(w - f) \in S$.
- (vii) **Square-root transformation.** If $f \in S$ and $g(z) = \sqrt{f(z^2)}$, then $g \in S$.

In 1907, Koebe [27] proved that there exists a constant k (known as Koebe's constant) such that the boundary of the image of U under $w = f(z) \in S$ is always at a distance not less than k from $w = 0$. This was a curiosity provoking discovery and attracted many other mathematicians, including Bieberback [6], Gronwall [19] and Pick [40], to mention a few, to the field of univalent functions.

In 1916, Bieberback [6] proved that $k = 1/4$. In fact he proved the following results.

Theorem 1.1.4. (Koebe Covering Theorem). The range of every function of the class S contains the disc $\{w : |w| < 1/4\}$.

Theorem 1.1.5. (Distortion Theorem). For each $f \in S$,

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \quad (|z| = r < 1). \quad (1.1.6)$$

Theorem 1.1.7. (Growth Theorem). For each $f \in S$,

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \quad (|z| = r < 1). \quad (1.1.8)$$

Theorem 1.1.9. For each $f \in S$,

$$\frac{1-r}{1+r} \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \frac{1+r}{1-r}, \quad (|z| = r < 1). \quad (1.1.10)$$

The results in Theorems 1.1.5, 1.1.7, and 1.1.9 are sharp for the Koebe function. Bieberbach [6] also proved that $|a_2| \leq 2$ for all $f \in S$. The Koebe function was acting as an extremal function for most of the results he proved and since for this function $|a_n| = n$ for all $n \geq 2$, he made the following conjecture:

Bieberbach conjecture. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$, then $|a_n| \leq n$ for all $n \geq 2$.

This famous problem stood as a challenge until 1984 and was settled by De Branges [12] by giving a surprisingly short proof in view of the rich and continuing history of the problem. However the Bieberbach conjecture has inspired the development of many important and ingenious methods which now form the backbone of the subject. In fact, several subclasses of S have been introduced and studied. Most of these subclasses are determined by a fixed geometric property of the image domain. It is interesting to note that these subclasses play an important role in other branches of mathematics. For example, starlike functions play an important role in the solution of certain differential equation (see [16],[47] etc) whereas spirallike functions have been the source of useful counter examples in geometric function theory (see [14,28]). Some of these classes are as follows.

1.2. Convex and Starlike Functions

Definition 1.2.1. A function $f \in S$ is said to be convex in U if the image of U under f is a convex region; i.e, the line segment joining any two points of $f(U)$ lies entirely in $f(U)$. Let C denotes the class of all functions $f \in S$ which are convex and univalent in U .

A necessary and sufficient condition for a function $f \in C$ has been given by Robertson [45].

Theorem 1.2.2. A function $f \in C$ if, and only if,

$$\operatorname{Re} \left(1 + z \frac{f''(z)}{f'(z)} \right) > 0 \quad z \in U. \quad (1.2.3)$$

Robertson [45] also introduced the concept of order for convex functions as follows.

Definition 1.2.4. A function $f \in S$ is said to be convex of order α ($0 \leq \alpha < 1$) if

$$\operatorname{Re} \left(1 + z \frac{f''(z)}{f'(z)} \right) > \alpha \quad z \in U. \quad (1.2.5)$$

We shall denote this class by $C(\alpha)$. It is clear that $C(0) = C$.

Some of the results for convex univalent functions are as follows.

Theorem 1.2.6 [13]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C$, then $|a_n| \leq 1$ for $n = 2, 3, \dots$. This theorem is sharp with the extremal functions of the form

$$f(z) = z(1 - xz)^{-1} \quad |x| = 1.$$

Theorem 1.2.7 [13]. The range of every function $f \in C$ contains the disc $|w| < 1/2$.

Theorem 1.2.8 [13]. For every positive number $r \leq 2 - \sqrt{3}$, each function $f \in S$ maps the disc $|z| < r$ onto a convex region. This is false for every $r > 2 - \sqrt{3}$.

Theorem 1.2.9 [17]. If $f(z) \in C(\alpha)$, then

$$|\arg f'(z)| \leq 2(1 - \alpha) \sin^{-1} r \quad (|z| = r < 1), \quad (1.2.10)$$

for some α , $0 \leq \alpha < 1$.

A subclass of S which is wider than C is the class of starlike functions defined as follows.

Definition 1.2.11. A function $f \in S$ is said to be starlike with respect to the origin in U if the image of U under f containing the point $w = 0$ is a starlike region. $f(U)$ is a starlike region if the line segment joining $w = 0$ to any point of $f(U)$ lies completely in $f(U)$. We denote by S^* , the class of all functions in A which are univalent and starlike with respect to the origin.

The following theorem due to Robertson [45] gives a necessary and sufficient condition for a function $f \in S^*$.

Theorem 1.2.12. A function $f \in S^*$ if, and only if,

$$\operatorname{Re} \left(z \frac{f'(z)}{f(z)} \right) > 0, \quad z \in U. \quad (1.2.13)$$

As in the case of convex functions, the order α for starlike functions has also been defined by Robertson [45].

Definition 1.2.14. A function $f \in S$ is said to be starlike of order α ($0 \leq \alpha < 1$) if

$$\operatorname{Re} \left(z \frac{f'(z)}{f(z)} \right) > \alpha, \quad z \in U. \quad (1.2.15)$$

We shall denote by $S^*(\alpha)$, the class of starlike functions of order α ($0 \leq \alpha < 1$) in U . Clearly $S^*(0) \equiv S^*$.

It is clear from (1.2.5) and (1.2.15) that

$$f \in C(\alpha) \text{ if, and only if, } zf'(z) \in S^*(\alpha), \quad (1.2.16)$$

$$f \in S^*(\alpha) \text{ if, and only if, } \int_0^z \frac{f(t)}{t} dt = F(z) \in C(\alpha). \quad (1.2.17)$$

Some of the results for starlike functions are as follows.

Theorem 1.2.18 [13]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*$, then f is convex in $|z| < r$ where $r = 2 - \sqrt{3}$. The result is sharp.

The number r in the above theorem is called the radius of convexity for functions in S^* .

Theorem 1.2.19 [57]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(\alpha)$, then

$$|a_n| \leq \frac{\prod_{k=2}^n (k - 2\alpha)}{(n - 1)!} \quad (n = 2, 3, \dots). \quad (1.2.20)$$

Theorem 1.2.21 [57]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(\alpha)$ then

$$\frac{r}{(1+r)^{2(1-\alpha)}} \leq |f(z)| \leq \frac{r}{(1-r)^{2(1-\alpha)}}, \quad (|z| = r < 1). \quad (1.2.22)$$

The results in Theorems 1.2.19 and 1.2.21 are sharp with the extremal function $k_\alpha(z) = z(1-z)^{2\alpha-2}$.

Theorem 1.2.23 [57]. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A$. If for some $\alpha (0 \leq \alpha < 1)$

$$\sum_{n=2}^{\infty} (n - \alpha) a_n \leq (1 - \alpha), \quad (1.2.24)$$

then $f(z) \in S^*(\alpha)$.

The concept of "type" for starlike functions of order $\alpha (0 \leq \alpha < 1)$ was introduced by Juneja and Mogra [22] as follows.

Definition 1.2.25. A function $f \in A$ is said to be starlike of order α and type β if the condition

$$\left| \frac{z \frac{f'(z)}{f(z)} - 1}{z \frac{f'(z)}{f(z)} + 1 - 2\alpha} \right| < \beta \quad (1.2.26)$$

is satisfied for some $\alpha, \beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$ and for all $z \in U$. The class of starlike functions of order $\alpha (0 \leq \alpha < 1)$ and type $\beta (0 < \beta \leq 1)$ shall be denoted by $S^*(\alpha, \beta)$.

1.3. Functions with Positive Real Part

Closely related to the class C and S^* is the class P of functions

$$p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \quad (1.3.1)$$

which are analytic and satisfies $\Re \{p(z)\} > 0$ for $z \in U$. Considerable work has been done to study various aspects of this class. Some of the results for the class P are as follows.

Theorem 1.3.2 [13]. If $p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \in P$, Then $|b_n| \leq 2, n = 1, 2, \dots$. The equality holds for all n for the function

$$p(z) = \frac{1+z}{1-z} = 1 + 2 \sum_{n=1}^{\infty} z^n.$$

Theorem 1.3.3 [13]. $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*$ if, and only if, $zf'(z)/f(z) \in P$.

Theorem 1.3.4 [13]. $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C$ if, and only if, $(1 + zf''(z)/f'(z)) \in P$.

MacGregor [34] introduced the following class of functions $f \in A$ such that $f'(z) \in P$ and obtained several properties for this class.

Definition 1.3.5. A function $f \in A$ is said to be in the class R if

$$\Re \{f'(z)\} > 0, \quad z \in U. \quad (1.3.6)$$

Ezrohi [15] extended the results of MacGregor [34] by introducing the following class.

Definition 1.3.7. A function $f \in A$ is said to be in the class $R(\alpha)$ if

$$\Re \{f'(z)\} > \alpha \quad (1.3.8)$$

for some $\alpha(0 \leq \alpha < 1)$ and $z \in U$.

Several other subclasses of R have also been studied by Juneja and Mogra [24], Shaffer [58] and others.

1.4. Spirallike and Close-to-Convex Functions

A class wider than the class of starlike functions is the class of Spirallike functions introduced by Špac̆ek [68] in 1933.

Definition 1.4.1. A function $f \in A$ is said to be spirallike in U if

$$\Re \left\{ \xi \frac{z f'(z)}{f(z)} \right\} > 0, \quad z \in U \quad (1.4.2)$$

for some ξ such that $|\xi| = 1$.

It was shown by Špac̆ek [68] that Spirallike functions are univalent in U . Taking $\xi = e^{i\gamma}$, $|\gamma| < \pi/2$ Libera [30] introduced and studied the class S_p^γ of γ -Spirallike functions as

follows.

Definition 1.4.3. A function $f \in A$ is said to be a γ -Spirallike function; $f \in S_p^\gamma$, if

$$\Re \left\{ e^{i\gamma} \frac{z f'(z)}{f(z)} \right\} > 0 \quad (1.4.4)$$

for some real $\gamma(|\gamma| < \pi/2)$ and for all $z \in U$.

Another interesting subclass of S is the class of close-to-convex functions, introduced by Kaplan [26] in 1952.

Definition 1.4.5. A function $f \in A$ is said to be close-to-convex in U if there exists a function $g \in C$ such that

$$\operatorname{Re} \left\{ \frac{f'(z)}{g'(z)} \right\} > 0 \quad (1.4.6)$$

for all $z \in U$.

We shall denote by K the class of close-to-convex functions in U .

Remark 1.4.7. The condition (1.4.6) is equivalent to

$$\operatorname{Re} \left\{ z \frac{f'(z)}{g(z)} \right\} > 0 \quad (1.4.8)$$

for some $g \in S^*$ and for all $z \in U$.

In [26], Kaplan proved that every close-to-convex function is univalent; thus $K \subset S$. Every convex function is obviously close-to-convex. More generally, every starlike function is close-to-convex. Indeed, each $f \in S^*$ has the form $f(z) = zg'(z)$ for some $g \in C$, and

$$\operatorname{Re} \left\{ \frac{f'(z)}{g'(z)} \right\} = \operatorname{Re} \left\{ z \frac{f'(z)}{f(z)} \right\} > 0.$$

These remarks are summarized by the chain of proper inclusion

$$C \subset S^* \subset K \subset S.$$

1.5. Typically Real Functions

Definition 1.5.1. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in U is said to be typically real if it has real values on the real axis and nonreal values elsewhere.

Thus the sign of $\operatorname{Im} \{f(z)\}$ is constant in each of the semidisks above and below the real axis. It is clear that every typically real function has real coefficients because $a_n = f^{(n)}(0)/n!$. Let T_r denote the class of all typically real functions. This class was

introduced and studied by Rogosinski [49]. He has shown that the functions in T_r need not be univalent.

The following results are well-known for typically real functions.

Theorem 1.5.2 [13]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in T_r , then $|a_n| \leq n$, $n = 2, 3, \dots$.

Strict inequality occurs for all even n unless f is a Koebe function or its real rotation $-K(-z)$. Strict inequality occurs for all odd n unless f is a convex combination of these two functions.

Theorem 1.5.3 [13]. $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in T_r if, and only if,

$$f(z) = \frac{z}{1 - z^2} p(z) \quad (1.5.4)$$

where $p \in P$ and has real coefficients.

Let C_I denote the class of functions in T_r that have their images convex in the direction of the imaginary axis. This class was introduced by Robertson [45] and the following result is well-known.

Theorem 1.5.5. $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in C_I if, and only if, $zf' \in T_r$.

1.6. Extreme Points, Support Points, and Closed Convex Hull

Definition 1.6.1. Let X be a topological vector space over the field of complex numbers, and let E be a subset of X . A point $x \in E$ is called an extreme point of E if it has no representation of the form

$$x = ty + (1 - t)z, \quad 0 < t < 1,$$

as a proper convex combination of two distinct points y and z in E .

Definition 1.6.2. A point $x \in E$ is called a support point of E if there is a continuous linear functional L , not constant in E , such that $\Re e \{L(x)\} \geq \Re e \{L(y)\}$ for all $y \in E$.

Example 1.6.3. If $\mathcal{C}$ is the complex plane and E is the closed area bounded by the convex polygon, every boundary point is a support point but only the vertices are extreme points.

Definition 1.6.4. The convex hull of a set $E \subset X$ is the smallest convex set containing E . The closed convex hull of E is the smallest closed convex set containing E ; it is the closure of the convex hull of E .

Definition 1.6.5. A family $\mathcal{F}$ of functions analytic in U is called a normal family if every sequence of functions $f_n \in \mathcal{F}$ has a subsequence which converges uniformly on each compact subset of U . A family $\mathcal{F}$ is compact if whenever $f_n \in \mathcal{F}$ and $f_n(z) \rightarrow f(z)$ uniformly on compact subset of U , it follows that $f \in \mathcal{F}$.

The Krein-Milman theorem asserts that every compact subset of a locally convex topological vector space is contained in the closed convex hull of its extreme points. In particular, every compact set has extreme points. Furthermore, there is at least one extreme point among the support points associated with every continuous linear functional.

We observe that the family S of univalent analytic functions is normal and compact. In fact, from the right hand inequality (1.1.8), the family S is locally uniformly bounded at all points of the unit disc U and so by Montel's theorem it is a normal family. Further, it is well-known [17] that a converging sequence of univalent functions converges either to a univalent function or to a constant. Since all functions of S satisfy $f'(0) = 1$, the latter possibility is excluded. Hence S is compact. Thus in view of the Krein-Milman theorem, it is important to identify the extreme points of S , because the solution of any linear extremal problem over S can be reduced to its solution over the set of extreme points. For detail on extreme points of S and other classes of univalent functions see [8],[35] etc.

1.7. Univalent Functions with Negative Coefficient

Let T denote the subclass of S consisting of functions whose nonzero coefficients, from the second on, are negative. That is, a univalent analytic function f is in T if it can be expressed in the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \quad (1.7.1)$$

Further, let $S^*[\alpha] = S^*(\alpha) \cap T$ and $C[\alpha] = C(\alpha) \cap T$. Clearly the functions in the classes $S^*[\alpha]$ and $C[\alpha]$ are respectively, starlike of order α and convex of order α with negative coefficients.

In [61], Silverman obtained the following necessary and sufficient conditions for the classes $S^*[\alpha]$ and $C[\alpha]$ along with several other properties including extreme points.

Theorem 1.7.2. A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in S^*[\alpha]$ if, and only if,

$$\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha. \quad (1.7.3)$$

Theorem 1.7.4. A function $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in C[\alpha]$ if, and only if,

$$\sum_{n=2}^{\infty} n(n - \alpha) |a_n| \leq 1 - \alpha. \quad (1.7.5)$$

There have been numerous papers involving subclasses of univalent functions with a fixed second coefficient (see e.g., [23], [36], [65] etc.). Most of these papers deal with distortion theorems and give coefficient and radius of convexity bounds. Moreover, the first attempt to investigate the properties of functions in $S^*[\alpha]$ and $C[\alpha]$ with fixed second coefficient has been made by Silverman and Silvia [66], where they obtain, extreme points, distortion

theorems and radius of convexity bounds etc for these classes. One may refer [62] for certain open problems on univalent functions with negative coefficients.

1.8. Subordination

The concept of subordination can be traced to Lindelöf (1909), but Littlewood [32] and Rogosinski [50,51] introduced the term and discovered the basic relations. Over the years a substantial theory has been developed, and subordination now plays an important role in complex analysis.

Definition 1.8.1. Let $f(z)$ and $g(z)$ be analytic in the unit disc U with $f(0) = g(0)$. Then g is subordinate to f , denoted by $g \prec f$ if

$$g(z) = f(\omega(z)), \quad z \in U, \quad (1.8.2)$$

for some analytic function ω with $|\omega(z)| \leq |z|$. g is properly subordinated to f if $g(z) = f(\omega(z))$ where $|\omega(z)| < |z|$.

Theorem 1.8.3 (Littlewood Subordination Theorem). Let f and g be analytic in the unit disc U , and suppose $g \prec f$. Then for $0 < p < \infty$,

$$M_p(r, g) \leq M_p(r, f), \quad 0 \leq r < 1 \quad (1.8.4)$$

where

$$M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}.$$

Strict inequality holds for $0 < r < 1$ unless f is constant or $\omega(z) = \alpha z$, $|\alpha| = 1$.

Theorem 1.8.5. (Rogosinski's Theorem). Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belong to A and suppose that $g \prec f$. Then

$$\sum_{k=2}^n |b_k|^2 \leq \sum_{k=2}^n |a_k|^2, \quad n = 2, 3, \dots \quad (1.8.6)$$

If f is convex, starlike or typically real, we have the following result.

Theorem 1.8.7 [13]. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belong to A .

If $g \prec f$, and

- (i) If $f \in C$, then $|b_n| \leq 1$, $n = 2, 3, \dots$.
- (ii) If $f \in S^*$, then $|b_n| \leq n$, $n = 2, 3, \dots$.
- (iii) If $f \in T_r$, then $|b_n| \leq n$, $n = 2, 3, \dots$.

Rogosinski Conjecture. If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is analytic in U and $g \prec f$ for some $f \in S$, then $|b_n| \leq n$, $n = 2, 3, \dots$. This conjecture is also known as generalized Bieberbach conjecture and it still remains an open problem.

1.9. Hadamard Product (Convolution) of Univalent Analytic Functions

Definition 1.9.1. The Hadamard product or convolution of two convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n$$

in U is defined as the power series

$$h(z) = (f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \quad (1.9.2)$$

This product is associative, commutative and distributive over addition. The geometric series

$$I(z) = \sum_{n=1}^{\infty} z^n = \frac{z}{1-z} \quad (1.9.3)$$

acts as the identity element under convolution: $(f * I)(z) = f(z)$ for all f .

The integral convolution $H(z) = (f \odot g)(z)$ is defined by

$$H(z) = \sum_{n=1}^{\infty} \frac{a_n b_n}{n} z^n = \int_0^z \frac{h(\xi)}{\xi} d\xi. \quad (1.9.4)$$

An old conjecture attributed to Mandelbrojt and Schiffer asserted that univalence is preserved under integral convolution: $f \odot g \in S$ if $f \in S$ and $g \in S$. This implies the Bieberbach

Conjecture, because if any $f \in S$ had $|a_n| > n$ for some n , the integral convolution of f with itself sufficiently many times would produce a function with n th coefficient larger than en . However, the Mandelbrojt-Schiffer conjecture is false. Bshouty [9] showed that $f \circledast g$ need not be univalent even if f and g are univalent functions with real coefficients. The classes T_r and S^* of typically real and starlike functions are preserved under integral convolution. Robertson [46] showed that $f \in T_r$ and $g \in T_r$ imply $f \circledast g \in T_r$. Pólya and Schoenberg [41] in connection with their work on de la vallée poussin means, conjectured that $f \in S^*$ and $g \in S^*$ imply $f \circledast g \in S^*$. Since $zf'(z)$ is starlike if and only if f is convex, an equivalent statement is as follows.

Pólya-Schoenberg Conjecture. If $f(z)$ and $g(z)$ are convex, then so is $(f * g)(z)$.

Pólya and Schoenberg verified several special cases. Robertson [46] had shown that if f and g are convex and have real coefficients, then $(f * g)(z)$ is univalent and is convex in the direction of the imaginary axis, that is, each vertical line intersects its range in a segment. Suffridge [69] proved that the convolution of every pair of convex functions is univalent and close-to-convex.

In 1973, Ruscheweyh and Sheil-Small [55] finally succeeded in proving Pólya-Schoenberg conjecture. Their argument is ingenious and very intricate. It leads to a wealth of results, including both the Pólya-Schoenberg conjecture and its analogue for the class of closed-to-convex functions. Their methods also enable to settle two other conjectures. One of these is an interesting subordination conjecture of Wilf [71] stronger than the Pólya-Schoenberg conjecture.

Wilf's Subordination Conjecture. Let φ and ψ be convex in U and suppose that f is subordinate to ψ . Then $\varphi * f$ is subordinate to $\varphi * \psi$.

In [67], Silverman, Silvia and Telage gave characterization for convex, starlike and γ -spirallike functions in terms of convolutions. They proved the following results.

Theorem 1.9.5. $f \in C(\alpha)$, $0 \leq \alpha < 1$, if, and only if,

$$\frac{1}{z} \left[f(z) * \frac{z + \left(\frac{x + \alpha}{1 - \alpha} \right) z^2}{(1 - z)^3} \right] \neq 0 \quad (|z| = r < 1, |x| = 1). \quad (1.9.6)$$

Theorem 1.9.7. $f \in S^*(\alpha)$, $0 \leq \alpha < 1$, if, and only if,

$$\frac{1}{z} \left[f(z) * \frac{z + \left(\frac{x + 2\alpha - 1}{2 - 2\alpha} \right) z^2}{(1 - z)^2} \right] \neq 0 \quad (|z| = r < 1, |x| = 1). \quad (1.9.8)$$

Theorem 1.9.9. $f \in S_p^\gamma(\alpha)$, $-\pi/2 < \gamma < \pi/2$, if, and only if,

$$\frac{1}{z} \left[f(z) * \frac{z + \left(\frac{x - e^{-2i\gamma}}{1 + e^{-2i\gamma}} \right) z^2}{(1 - z)^2} \right] \neq 0 \quad (|z| = r < 1, |x| = 1). \quad (1.9.10)$$

Similar characterization for several others subclasses of analytic functions are given by Mogra and Juneja [38]. For further work involving Hadamard product, one may refer to [25,37,39,54,63]

1.10. Prestarlike Functions

Using the Hadamard product techniques, Ruscheweyh [54] introduced and studied an important subclass of A , the class of prestarlike functions of order α ($0 \leq \alpha < 1$) as follows.

Definition 1.10.1. A function $f \in A$ is said to be prestarlike of order α , $0 \leq \alpha < 1$, denoted by $D(\alpha)$, if $f(z) * z(1 - z)^{2\alpha-2} \in S^*(\alpha)$.

The function $k_\alpha(z) = z(1 - z)^{2\alpha-2}$ is the well-known extremal function for the class $S^*(\alpha)$.

If we let

$$c(\alpha, n) = \frac{\prod_{k=2}^n (k - 2\alpha)}{(n - 1)!} \quad (n = 2, 3, \dots), \quad (1.10.2)$$

then $k_\alpha(z)$ can be written as

$$k_\alpha(z) = z + \sum_{n=2}^{\infty} c(\alpha, n) z^n. \quad (1.10.3)$$

Note that $D(0) = C$ and $D(1/2) = S^*(1/2)$. In [54], Ruscheweyh proved several results for functions in $D(\alpha)$. Some of them are as follows.

Theorem 1.10.4 [54]. A function $f \in D(\alpha)$ if, and only if,

$$G(\alpha, z) = \frac{f(z) * \left(\frac{k_\alpha(z)}{1 - z} \right)}{f(z) * k_\alpha(z)} \quad (1.10.5)$$

satisfy

$$\Re \{ G(\alpha, z) \} > \frac{1}{2}, \quad z \in U. \quad (1.10.6)$$

Thus we say that $f(z)$ is prestarlike of order 1 if $\Re \{ f(z)/z \} > 1/2$.

Theorem 1.10.7 [54]. $D(\alpha) \subset D(\beta)$ for $0 \leq \alpha < \beta \leq 1$.

Theorem 1.10.8 [54]. If $f(z), g(z) \in D(\alpha)$, then $(f * g)(z) \in D(\alpha)$.

The functions of $D(\alpha)$ need not to be univalent. Silverman and Silvia proved the following result.

Theorem 1.10.9 [65]. $D(\alpha) \subset S$ if, and only if, $\alpha \leq 1/2$.

The following characterization for prestarlike functions of order α in terms of convolution is due to Ruscheweyh [54].

Theorem 1.10.10 [54]. A function $f \in D(\alpha)$ if, and only if,

$$f(z) * \frac{z - xz^2}{(1 - z)^{3-2\alpha}} \neq 0 \quad (0 < |z| < 1, |x| = 1). \quad (1.10.11)$$

If we let, $D[\alpha] = D(\alpha) \cap T$, then $D[\alpha]$ is the class of prestarlike functions of order α with negative coefficients introduced and studied by Silverman and Silvia [64].

We close this chapter with a remark that some other well-known results, which may be needed in the following chapters, will be presented along with the definitions and motivations in the chapters.

CHAPTER II

HADAMARD PRODUCTS OF SCHLICHT FUNCTIONS AND THE PÓLYA-SCHOENBERG CONJECTURE

2.1. Introduction

Recall that a function $\varphi(z)$ is said to be convex if it is univalent (schlicht) conformal mapping of the unit disc U onto a convex domain. $f(z)$ is said to be starlike if it is a univalent conformal mapping of the unit disc U onto a domain starlike with respect to the origin. Throughout this chapter we shall assume that our convex and starlike functions vanish at $z = 0$.

The Hadamard product or convolution of two power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is defined as the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

In 1958, G. Pólya and I.J. Schoenberg [41] made the following conjecture.

Pólya-Schoenberg Conjecture 2.1.1. If $f(z)$ and $g(z)$ are convex, then so is $(f * g)(z)$.

In [69], Suffridge proved that the convolution is univalent (and in fact close-to-convex). Pólya and Schoenberg themselves showed that the conjecture was true in several special cases, and after Suffridge other special cases of the following proposition were established.

Proposition 2.1.2. If $\varphi(z)$ is convex and $f(z)$ is close-to-convex, then $(\varphi * f)(z)$ is close-to-convex.

Many mathematicians attacked these problems and obtained partial results. In 1973, ST.Ruscheweyh and T.Sheil-Small established the truth of both Pólya-Schoenberg conjecture 2.1.1 and Proposition 2.1.2. Their methods enable to obtain a number of other similar results and settle two other conjectures. One of these is an interesting subordination conjecture of Wilf [71] stronger than the Pólya-Schoenberg conjecture.

In this chapter we shall discuss in detail the results obtained by Ruscheweyh and Sheil-Small [55]. In section 2.2 we study certain geometric properties of convex and starlike functions and the expression of these in analytic terms. Section 2.3 of this chapter is devoted to discuss the proof of Pólya-Schoenberg conjecture 2.1.1 and Proposition 2.1.2. The structure of functions starlike of order $1/2$ is also preserved under convolutions. It is discussed in Section 2.4 of this chapter. The subordination conjecture of Wilf [71] is discussed in Section 2.5.

Mandelbrojt-Schiffer conjecture appeared in Pólya and Schoenberg's paper, that if $\sum_{n=1}^{\infty} a_n z^n$

and $\sum_{n=1}^{\infty} b_n z^n$ are univalent in U , then $\sum_{n=1}^{\infty} (a_n b_n / n) z^n$ also is univalent in U . This conjecture

has been disproved and it is not even true that $\sum_{n=1}^{\infty} a_n z^n$ univalent implies $\sum_{n=1}^{\infty} (a_n / n) z^n$

univalent. In particular one cannot replace $f(z)$ in Proposition 2.1.2 by an arbitrary univalent function. The correct generalization of Proposition 2.1.2 is briefly discussed in the last section of this chapter.

2.2. Structural Inequalities for Starlike and Convex Functions

A function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ ($a_1 \neq 0$) is starlike univalent in the unit disc U if, and only if,

$$\operatorname{Re} \left(z \frac{f'(z)}{f(z)} \right) > 0, \quad z \in U, \quad (2.2.1)$$

and that $\varphi(z)$ is convex if, and only if,

$$\operatorname{Re} \left(1 + z \frac{\varphi''(z)}{\varphi'(z)} \right) > 0, \quad z \in U, \quad (2.2.2)$$

and $\varphi'(0) \neq 0$.

Geometrically the conditions (2.2.1) and (2.2.2) are local in nature and are analytic formulations of the fact that for a starlike function f , $w = f(z)$ turns monotonically about the origin as z traverses the circle $|z| = r$ ($0 < r < 1$), and that for a convex function φ the tangent vector increases monotonically. The global geometric structure of the image domains of the functions is therefore implicitly determined by these conditions. However it turns out that in the present problem it is essential to formulate analytic conditions of a more explicitly global nature.

The first condition of this type which we shall need has also been proved in [59,70] but we shall include the very short and simple proof of Suffridge [70] in the following theorem.

Theorem 2.2.3. If $\varphi(z)$ is convex in $|z| < 1$, then for each z_0 ($|z_0| < 1$) the function

$$z \left(\frac{\varphi(z) - \varphi(z_0)}{z - z_0} \right)^2 \quad (2.2.4)$$

is starlike.

Proof. The image D of $\varphi(z)$ being convex is in particular star-shaped relative to the point $\varphi(z_0)$, and hence if $|z| > |z_0|$

$$\operatorname{Re} \left(\frac{z \varphi'(z)}{\varphi(z) - \varphi(z_0)} \right) > 0. \quad (2.2.5)$$

Now if $h(z)$ denotes the function (2.2.4), we have

$$z \frac{h'(z)}{h(z)} = \frac{2 z \varphi'(z)}{\varphi(z) - \varphi(z_0)} - \frac{z + z_0}{z - z_0} \quad (2.2.6)$$

and this is analytic in z and z_0 . By (2.2.5) it has non-negative real part for $|z| = |z_0|$. The maximum principle immediately implies that (2.2.6) has positive real part for every z and z_0 , and the conclusion follows.

It can be shown that the condition (2.2.4) is an analytic formulation of the geometrical fact that the chord from $\varphi(z_0)$ to $\varphi(z)$ ($|z| = |z_0|$) turns monotonically as the circle is traversed (see [59]).

A function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ ($a_1 \neq 0$) is said to be starlike of order $1/2$ if

$$\operatorname{Re} \left(z \frac{f'(z)}{f(z)} \right) > \frac{1}{2} \quad (|z| < 1). \quad (2.2.7)$$

The condition has no immediate geometrical interpretation, but is convenient to handle analytically and has a close relation to the general class of starlike functions. It is easy to verify that $g(z)$ is starlike if, and only if, $f(z) = z \sqrt{g(z)/z}$ is starlike of order $1/2$. Also

$$h(z) = h_1 z + h_3 z^3 + h_5 z^5 + \dots$$

is an odd starlike function if, and only if,

$$f(z) = h_1 z + h_3 z^2 + h_5 z^3 + \dots$$

is starlike of order $1/2$. To see this, let

$$h(z) = z (h_1 + h_3 z^2 + h_5 z^4 + \dots) = z g(z^2).$$

Then for $|z| < 1$,

$$h(z) \text{ is odd starlike} \Leftrightarrow \operatorname{Re} \left(z \frac{h'(z)}{h(z)} \right) > 0$$

$$\Leftrightarrow \operatorname{Re} \left(1 + 2 z^2 \frac{g'(z^2)}{g(z^2)} \right) > 0$$

$$\Leftrightarrow \operatorname{Re} \left(1 + 2 z \frac{g'(z)}{g(z)} \right) > 0$$

$$\Leftrightarrow \operatorname{Re} \left(z \frac{g'(z)}{g(z)} \right) > -\frac{1}{2}$$

$$\Leftrightarrow \operatorname{Re} \left(z \frac{f'(z)}{f(z)} \right) > \frac{1}{2}$$

$$\Leftrightarrow f(z) \text{ is starlike of order } 1/2.$$

If we put $z_0 = 0$ in Theorem 2.2.3 we see that every convex function φ is starlike of order $1/2$, and in general for each z_0 ($|z_0| < 1$)

$$z \frac{\varphi(z) - \varphi(z_0)}{z - z_0}, \quad (|z| < 1) \quad (2.2.8)$$

is starlike of order $1/2$. The following theorem gives more general condition than (2.2.7) for starlike functions of order $1/2$.

Theorem 2.2.9. $f(z)$ is starlike of order $1/2$ if, and only if,

$$\operatorname{Re} \left(\frac{f(z_1) - f(z_2)}{z_1 - z_2} \frac{z_2}{f(z_2)} \right) > \frac{1}{2} \quad (2.2.10)$$

for $|z_1| < 1$, $|z_2| < 1$.

proof. Suppose the condition (2.2.10) holds for $|z_1| < 1$, $|z_2| < 1$. Then,

$$\lim_{z_2 \rightarrow z_1} \operatorname{Re} \left(\frac{f(z_1) - f(z_2)}{z_1 - z_2} \frac{z_2}{f(z_2)} \right) > \frac{1}{2}.$$

This implies that

$$\Re \left(z_1 \frac{f'(z_1)}{f(z_1)} \right) > \frac{1}{2}$$

and hence $f(z)$ is starlike of order $1/2$. For the necessity, let $g(z) = (f(z))^2/z$, so that $g(z)$ is starlike. Consider the principle branch of $\arg(1 - \sigma)$ for $|\sigma| < 1$, and observe that it extends continuously to $|\sigma| = 1$, $\sigma \neq 1$, to give

$$\arg(1 - e^{i\varphi}) = -\frac{\pi}{2} + \frac{\varphi}{2} \quad (0 < \varphi < 2\pi).$$

Writing $z = re^{i\theta}$ we have

$$\begin{aligned} \arg \left\{ (1 - e^{i\varphi}) \frac{f(ze^{i\varphi})}{ze^{i\varphi}} \frac{z}{f(z)} \right\} &= \arg(1 - e^{i\varphi}) + \arg \frac{f(re^{i(\theta+\varphi)})}{re^{i(\theta+\varphi)}} + \arg \frac{re^{i\theta}}{f(re^{i\theta})} \\ &= -\frac{\pi}{2} + \frac{\varphi}{2} + \frac{1}{2} \arg \frac{g(re^{i(\theta+\varphi)})}{re^{i(\theta+\varphi)}} - \frac{1}{2} \arg \frac{g(re^{i\theta})}{re^{i\theta}} \\ &= \frac{1}{2} \left[-\pi + \arg g(re^{i(\theta+\varphi)}) - \arg g(re^{i\theta}) \right] \end{aligned}$$

and since g is starlike this expression does not exceed $\pi/2$ in absolute value if $0 < \varphi < 2\pi$.

Thus

$$\Re \left\{ (1 - \sigma) \frac{f(\sigma z)}{\sigma z} \frac{z}{f(z)} \right\} > 0 \quad (|z| < 1) \quad (2.2.11)$$

for $|\sigma| = 1$, $\sigma \neq 1$. Consider now $|z_1| = |z_2| < 1$ and $z_1 \neq z_2$, so that for some σ satisfying $|\sigma| = 1$, $\sigma \neq 1$, we have $z_2 = z_1\sigma$. We then obtain

$$\begin{aligned} \Re \left(\frac{f(z_1) - f(z_2)}{z_1 - z_2} \frac{z_2}{f(z_2)} \right) &= \Re \left(\frac{f(z_1) - f(z_1\sigma)}{z_1 - z_1\sigma} \frac{z_1\sigma}{f(z_1\sigma)} \right) \\ &= \Re \left(\frac{z_1\sigma f(z_1)}{(1 - \sigma)z_1 f(z_1\sigma)} - \frac{\sigma}{1 - \sigma} \right) \end{aligned}$$

$$\begin{aligned}
&= - \Re \frac{\sigma}{1 - \sigma} + \Re \left(\frac{1}{1 - \sigma} \frac{z_1 \sigma}{f(z_1 \sigma)} \frac{f(z_1)}{z_1} \right) \\
&> - \Re \frac{\sigma}{1 - \sigma} = \frac{1}{2}
\end{aligned}$$

applying (2.2.11). The proof is now completed by application of the maximum principle.

If $\varphi(z)$ is convex, the function (2.2.8) is starlike of order $1/2$ and so by applying Theorem 2.2.9, we obtain the following three point condition for convexity.

Theorem 2.2.12. If $\varphi(z)$ is convex, then for all complex numbers σ_1, σ_2 and σ_3 satisfying $|\sigma_k| \leq 1$ ($k = 1, 2, 3$) we have

$$\Re \left(\frac{\varphi(z) * z (1 - \sigma_1 z)^{-1} (1 - \sigma_2 z)^{-1} (1 - \sigma_3 z)^{-1}}{\varphi(z) * z (1 - \sigma_1 z)^{-1} (1 - \sigma_2 z)^{-1}} \right) > \frac{1}{2} \quad (|z| < 1). \quad (2.2.13)$$

Proof. Suppose $\varphi(z)$ is convex in $|z| < 1$ and that

$$\varphi(z) = \sum_{n=1}^{\infty} a_n z^n$$

Then, $f(z) = z [\varphi(z) - \varphi(z_0)]/(z - z_0)$ is starlike of order $1/2$.

Simple calculation gives

$$\begin{aligned}
\varphi(z) * z (1 - \sigma_1 z)^{-1} (1 - \sigma_2 z)^{-1} &= \varphi(z) * \left[\frac{1}{\sigma_1 - \sigma_2} \left(\frac{1}{1 - \sigma_1 z} - \frac{1}{1 - \sigma_2 z} \right) \right] \\
&= \sum_{n=1}^{\infty} \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} a_n z^n
\end{aligned}$$

and

$$\begin{aligned}
& \varphi(z) * z (1 - \sigma_1 z)^{-1} (1 - \sigma_2 z)^{-1} (1 - \sigma_3 z)^{-1} \\
&= \varphi(z) * \left[\frac{\sigma_1}{(\sigma_2 - \sigma_1)(\sigma_3 - \sigma_1)} \frac{1}{1 - \sigma_1 z} + \frac{\sigma_2}{(\sigma_2 - \sigma_1)(\sigma_2 - \sigma_3)} \frac{1}{1 - \sigma_2 z} \right. \\
&\quad \left. + \frac{\sigma_3}{(\sigma_2 - \sigma_3)(\sigma_1 - \sigma_3)} \frac{1}{1 - \sigma_3 z} \right] \\
&= \sum_{n=1}^{\infty} \left(\frac{\sigma_1 \sigma_2 (\sigma_1^n - \sigma_2^n) + \sigma_2 \sigma_3 (\sigma_2^n - \sigma_3^n) + \sigma_3 \sigma_1 (\sigma_3^n - \sigma_1^n)}{(\sigma_2 - \sigma_1)(\sigma_2 - \sigma_3)(\sigma_3 - \sigma_1)} \right) a_n z^n
\end{aligned}$$

Using these relations and writing $z_0 = \sigma_1 z$, $z_1 = \sigma_2 z$, $z_2 = \sigma_3 z$, where σ_1 , σ_2 and σ_3 are complex numbers satisfying $|\sigma_k| \leq 1$, $k = 1, 2, 3$, we have

$$\begin{aligned}
& \frac{f(z_1) - f(z_2)}{z_1 - z_2} \frac{z_2}{f(z_2)} \\
&= \left[\frac{1}{z_1 - z_2} \left\{ z_1 \frac{\varphi(z_1) - \varphi(z_0)}{z_1 - z_0} - z_2 \frac{\varphi(z_2) - \varphi(z_0)}{z_2 - z_0} \right\} \frac{z_2 - z_0}{\varphi(z_2) - \varphi(z_0)} \right] \\
&= \frac{\sum_{n=1}^{\infty} [\sigma_1 \sigma_2 (\sigma_1^n - \sigma_2^n) + \sigma_2 \sigma_3 (\sigma_2^n - \sigma_3^n) + \sigma_3 \sigma_1 (\sigma_3^n - \sigma_1^n)] a_n z^n}{(\sigma_2 - \sigma_1)(\sigma_2 - \sigma_3) \sum_{n=1}^{\infty} (\sigma_1^n - \sigma_2^n) a_n z^n} \\
&= \frac{\varphi(z) * z (1 - \sigma_1 z)^{-1} (1 - \sigma_2 z)^{-1} (1 - \sigma_3 z)^{-1}}{\varphi(z) * z (1 - \sigma_1 z)^{-1} (1 - \sigma_2 z)^{-1}}.
\end{aligned}$$

Hence the result follows from Theorem 2.2.9.

The condition (2.2.10) can be similarly translated and we obtain: if $f(z)$ is starlike of order $1/2$, then for any pair σ_1, σ_2 satisfying $|\sigma_1| \leq 1, |\sigma_2| \leq 1$, we have

$$\operatorname{Re} \left(\frac{f(z) * z (1 - \sigma_1 z)^{-1} (1 - \sigma_2 z)^{-1}}{f(z) * z (1 - \sigma_1 z)^{-1}} \right) > \frac{1}{2} \quad (|z| < 1). \quad (2.2.14)$$

We shall also need a relation of different type for starlike functions $f(z)$. This is based on the geometrical observation that a domain D starlike with respect to the origin is characterized by the property that every straight line through the origin intersects D in a single segment (finite or infinite). In the terminology of Robertson [44] the domain is "starlike in every direction". In order to give a sharp characterization of the property in analytical terms a few boundary properties of starlike functions are required.

Let $f(z)$ be starlike and let

$$V(t) = \lim_{r \rightarrow 1} \arg f(re^{it}). \quad (2.2.15)$$

The limit $V(t)$ exists for all real t and has the following properties.

$$V(t) \text{ is monotonic increasing with } t. \quad (2.2.16)$$

$$V(t + 2\pi) = V(t) + 2\pi, \text{ for every } t. \quad (2.2.17)$$

$$V(t) = \frac{1}{2} [V(t + 0) + V(t - 0)], \text{ for every } t. \quad (2.2.18)$$

These results are established for example in [42]. We now set

$$h(t) = \inf \{ s: V(s) \geq V(t) + \pi \} \quad (2.2.19)$$

$$k(t) = \sup \{ s: V(s) \leq V(t) + \pi \}.$$

It is immediately verified that $h(t)$ and $k(t)$ are non-decreasing functions of t which satisfy

$$h(t + 2\pi) = h(t) + 2\pi, \quad k(t + 2\pi) = k(t) + 2\pi \quad (2.2.20)$$

$$t \leq h(t) \leq k(t) \leq t + 2\pi \quad (2.2.21)$$

for every real t . We then have

Theorem 2.2.22. For each real t let t^* denote any number satisfying $h(t) \leq t^* \leq k(t)$. Then

$$\operatorname{Im} \left\{ e^{-iV(t)} e^{1/2 i (t + t^*)} (1 - ze^{-it}) (1 - ze^{-it^*}) \frac{f(z)}{z} \right\} \geq 0 \quad (2.2.23)$$

in $|z| < 1$.

Proof. For fixed t we consider the function

$$W(\theta) = \lim_{r \rightarrow 1} \arg \left\{ e^{-iV(t)} e^{1/2 i (t + t^*)} (1 - re^{i(\theta - t)}) (1 - re^{i(\theta - t^*)}) \frac{f(re^{i\theta})}{re^{i\theta}} \right\}. \quad (2.2.24)$$

$W(\theta)$ is periodic in θ with period 2π and we aim to show that

$$0 \leq W(\theta) \leq \pi. \quad (2.2.25)$$

To do this it is sufficient to consider the range $t \leq \theta \leq t + 2\pi$. We consider four cases:

(i) $\theta = t$. Then for $t < t^* < t + 2\pi$,

$$\begin{aligned} \lim_{r \rightarrow 1} \arg \left\{ (1 - re^{i(\theta - t)}) (1 - re^{i(\theta - t^*)}) \right\} \\ &= \lim_{r \rightarrow 1} \arg \left\{ (1 - r) (1 - re^{i(t - t^*)}) \right\} \\ &= \arg (1 - e^{i(t - t^*)}) = \arg (1 - \cos(t - t^*) - i \sin(t - t^*)) \\ &= \arg \left(2 \sin^2 \left(\frac{t - t^*}{2} \right) - i 2 \sin \left(\frac{t - t^*}{2} \right) \cos \left(\frac{t - t^*}{2} \right) \right) \\ &= \tan^{-1} \left(-\cot \left(\frac{t - t^*}{2} \right) \right) = \frac{1}{2} (t - t^*) + \frac{\pi}{2} \end{aligned}$$

and so

$$\begin{aligned} \lim_{r \rightarrow 1} \arg \left\{ (1 - re^{i(\theta - t)}) (1 - re^{i(\theta - t^*)}) \right\} \\ &= \begin{cases} \frac{1}{2} (t - t^*) + \frac{\pi}{2} & \text{if } t < t^* < t + 2\pi \\ 0 & \text{if } t^* = t \text{ or } t^* = t + 2\pi \end{cases} \end{aligned}$$

Thus

$$W(\theta) = \begin{cases} \frac{\pi}{2} & \text{if } t < t^* < t + 2\pi \\ 0 & \text{if } t^* = t \\ \pi & \text{if } t^* = t + 2\pi \end{cases}$$

(ii) $t < \theta < t^*$. Then

$$\begin{aligned} \lim_{r \rightarrow 1} \arg \left\{ (1 - re^{i(\theta - t)}) (1 - re^{i(\theta - t^*)}) \right\} \\ = \arg (1 - e^{i(\theta - t)}) + \arg (1 - e^{i(\theta - t^*)}) \\ = \tan^{-1} \left\{ -\cot \left(\frac{\theta - t}{2} \right) \right\} + \tan^{-1} \left\{ -\cot \left(\frac{\theta - t^*}{2} \right) \right\} \\ = \theta - \frac{1}{2}(t + t^*) \end{aligned}$$

and hence $W(\theta) = V(\theta) - V(t)$. Since $V(\theta)$ is increasing, $W(\theta) \geq 0$. Since $\theta < t^* \leq k(t)$, then $V(\theta) \leq V(t) + \pi$. Thus (2.2.25) follows in this case.

(iii) $\theta = t^*$. If $t^* = t$ or $t^* = t + 2\pi$, then (2.2.25) follows from case (i). Thus we may assume that $t < t^* < t + 2\pi$. Then it is easy to verify that

$$\lim_{r \rightarrow 1} \arg \left\{ (1 - re^{i(\theta - t)}) (1 - re^{i(\theta - t^*)}) \right\} = \frac{1}{2}(t^* - t) - \frac{\pi}{2},$$

which gives $W(t^*) = -V(t) + V(t^*) - \pi/2$. Now if $h(t) < t^* < k(t)$, then $V(t^*) = V(t) + \pi$ and so $W(t^*) = \pi/2$. If $h(t) = t^* < k(t)$ then $V(t^*) \leq V(t) + \pi$ and so $W(t^*) \leq \pi/2$. Also $V(t^* + 0) \geq V(t) + \pi$ and $V(t^* - 0) \geq V(t)$ and therefore applying (2.2.18), $V(t^*) \geq V(t) + \pi/2$, which gives $W(t^*) \geq 0$. If $h(t) < t^* = k(t)$, we obtain in a similar manner $\pi/2 \leq W(t^*) \leq \pi$. Lastly, if $h(t) = k(t) = t^*$, then

$$V(t) + \pi \leq V(t^* + 0) \leq V(t) + 2\pi$$

$$V(t) \leq V(t^* - 0) \leq V(t) + \pi$$

and applying (2.2.18) we obtain $0 \leq W(t^*) \leq \pi$.

(iv) $t^* < \theta < t + 2\pi$. Then we have

$$\lim_{r \rightarrow 1} \arg \left\{ (1 - re^{i(\theta - t)}) (1 - re^{i(\theta - t^*)}) \right\} = \theta - \frac{1}{2}(t + t^*) - \pi$$

and so $W(\theta) = V(\theta) - V(t) - \pi$. Since $\theta > h(t)$, $V(\theta) \geq V(t) + \pi$ and so $W(\theta) \geq 0$. Also $V(\theta) \leq V(t + 2\pi) = V(t) + 2\pi$ and so $W(\theta) \leq \pi$.

We now have (2.2.25) in all cases. Let

$$v(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}e \left(\frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} \right) W(\theta) d\theta.$$

By Poisson's formula we have for $|z| < R < 1$

$$\begin{aligned} & \arg \left\{ e^{-iV(t)} e^{1/2 i(t + t^*)} (1 - ze^{-it}) (1 - ze^{-it^*}) \frac{f(z)}{z} \right\} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}e \left(\frac{Re^{i\theta} + z}{Re^{i\theta} - z} \right) \arg \left\{ e^{-iV(t)} e^{1/2 i(t + t^*)} (1 - Re^{i(\theta - t)}) \right. \\ & \quad \times \left. (1 - Re^{i(\theta - t^*)}) \frac{f(Re^{i\theta})}{Re^{i\theta}} \right\} d\theta \end{aligned}$$

$\rightarrow \nu(z)$ as $R \rightarrow 1$ by the Lebesgue bounded convergence theorem. From (2.2.25) we have $0 \leq \nu(z) \leq \pi$ and the theorem follows.

2.3. The Conjecture of Pólya and Schoenberg

Theorem 2.3.1. Let $\varphi(z)$ and $\psi(z)$ be convex univalent functions in the unit disc U . Then $(\varphi * \psi)(z)$ is convex univalent in this disc.

Theorem 2.3.2. Let $\varphi(z)$ be convex and $f(z)$ close-to-convex in the unit disc U . Then $(\varphi * f)(z)$ is close-to-convex in U .

The proof of these results go together and will occupy this section. The following lemma plays a central role in the discussion and is a modification of a lemma appeared in [52].

Lemma 2.3.3. Let $\varphi(z)$ and $g(z)$ be analytic in $|z| < 1$ and satisfying $\varphi(0) = g(0) = 0$, $\varphi'(0) \neq 0$, $g'(0) \neq 0$. Suppose that for each $\sigma(|\sigma| = 1)$ and $\alpha(|\alpha| = 1)$ we have

$$\varphi(z) * \frac{1 + \alpha\sigma z}{1 - \sigma z} g(z) \neq 0 \quad (0 < |z| < 1). \quad (2.3.4)$$

Then for each function $F(z)$ analytic in $|z| < 1$ and satisfying

$$\Re(F(z)) > 0 \quad (|z| < 1), \quad (2.3.5)$$

we have

$$\Re \left(\frac{(\varphi * Fg)(z)}{(\varphi * g)(z)} \right) > 0 \quad (|z| < 1). \quad (2.3.6)$$

Proof. We first show that

$$\mathcal{R}e \left(\frac{\varphi(z) * \frac{1 + \sigma z}{1 - \sigma z} g(z)}{\varphi(z) * g(z)} \right) > 0 \quad (|z| < 1). \quad (2.3.7)$$

The hypothesis (2.3.4) when $\alpha = -1$ implies that $\varphi * g \neq 0$ for $0 < |z| < 1$. Assume that $|\alpha| = 1$ and $\alpha \neq -1$. Then we have

$$\begin{aligned} \varphi(z) * \frac{1 + \alpha \sigma z}{1 - \sigma z} g(z) &= \varphi(z) * \left[\frac{1}{2} (1 + \alpha) \frac{1 + \sigma z}{1 - \sigma z} g(z) + \frac{1}{2} (1 - \alpha) g(z) \right] \\ &= \frac{1}{2} (1 + \alpha) \varphi(z) * \frac{(1 + \sigma z)}{(1 - \sigma z)} g(z) + \frac{1}{2} (1 - \alpha) \varphi(z) * g(z). \end{aligned}$$

Hence from (2.3.4) we have

$$\frac{1}{2} (1 + \alpha) \varphi(z) * \frac{1 + \sigma z}{1 - \sigma z} g(z) + \frac{1}{2} (1 - \alpha) \varphi(z) * g(z) \neq 0$$

or,

$$\frac{\varphi(z) * \frac{1 + \sigma z}{1 - \sigma z} g(z)}{\varphi(z) * g(z)} \neq -\frac{1 - \alpha}{1 + \alpha} \quad (|z| < 1).$$

The left-hand member of this relation therefore takes no value on the imaginary axis, but clearly has the value 1 at $z = 0$. Therefore (2.3.7) follows.

Consider then $F(z)$ satisfying (2.3.5). We may assume that $|F(0)| = 1$, and we then have by Herglotz formula

$$e^{-i\gamma} F(z) = \int_T \frac{1 + \beta \sigma z}{1 - \sigma z} d\mu(\sigma) \quad (2.3.8)$$

where μ is a probability mass on the unit circle T and β and $e^{-i\gamma}$ are uniquely determined constants such that $|\beta| = 1$, $\beta \neq -1$, and $\cos \gamma > 0$ ($w = e^{i\gamma} (1 + \beta z)/(1 - z)$ maps $|z| < 1$ onto $\Re w > 0$). Then

$$\begin{aligned} e^{-i\gamma}(\varphi * Fg)(z) &= \int_T \varphi(z) * g(z) \frac{1 + \beta \sigma z}{1 - \sigma z} d\mu(\sigma) \\ &= \frac{1}{2} (1 + \beta) \int_T \varphi(z) * g(z) \frac{1 + \sigma z}{1 - \sigma z} d\mu(\sigma) + \frac{1}{2} (1 - \beta) (\varphi * g)(z) \\ &= (\varphi(z) * g(z)) \left\{ \frac{1}{2} (1 + \beta) \int_T H_\sigma(z) d\mu(\sigma) + \frac{1}{2} (1 - \beta) \right\} \end{aligned}$$

where $H_\sigma(0) = 1$ and by (2.3.7), $\Re H_\sigma(z) > 0$. Thus

$$e^{-i\gamma} \frac{(\varphi * Fg)(z)}{(\varphi * g)(z)} = \frac{1}{2} (1 + \beta) M(z) + \frac{1}{2} (1 - \beta)$$

Where $M(0) = 1$ and $\Re M(z) > 0$, and the condition (2.3.6) follows immediately.

Remark 2.3.9. It is easily seen from this lemma that the condition (2.3.4) implies that $(\varphi * Fg)/(\varphi * g)$ takes only values in the convex hull of the range of F for every analytic F .

We require next what is essentially a special case of Theorem 2.3.2 (see [52]).

Lemma 2.3.10. Let $h(z)$ be analytic in $|z| < 1$ with $h(0) = 0$ and suppose that there exists constants α and β with $|\alpha| = |\beta| = 1$ such that

$$\Re \left\{ (1 - \alpha z) (1 - \beta z) \frac{h(z)}{z} \right\} > 0, \quad (|z| < 1). \quad (2.3.11)$$

Then for every convex function $\varphi(z)$

$$\varphi(z) * h(z) \neq 0, \quad (0 < |z| < 1). \quad (2.3.12)$$

Proof. By Herglotz formula there exists $\gamma (|\gamma| = 1)$ such that

$$(1 - \alpha z) (1 - \beta z) \frac{h(z)}{z} = h'(0) \int_T \frac{1 + \gamma \sigma z}{1 - \sigma z} d\mu(\sigma)$$

or,

$$h(z) = h'(0) \int_T \frac{z (1 + \gamma \sigma z)}{(1 - \alpha z) (1 - \beta z) (1 - \sigma z)} d\mu(\sigma) \quad (2.3.13)$$

for $|z| < 1$, where μ is a probability mass on the unit circle T . Thus

$$\frac{1}{h'(0)} (\varphi(z) * h(z)) = \int_T \varphi(z) * \frac{z (1 + \gamma \sigma z)}{(1 - \alpha z) (1 - \beta z) (1 - \sigma z)} d\mu(\sigma)$$

$$= \int_T \varphi(z) * \left[\frac{z (1 + \gamma) - \gamma z (1 - \sigma z)}{(1 - \alpha z) (1 - \beta z) (1 - \sigma z)} \right] d\mu(\sigma)$$

$$= \int_T \varphi(z) * \left[z(1 + \gamma) (1 - \alpha z)^{-1} (1 - \beta z)^{-1} (1 - \sigma z)^{-1} - \gamma z (1 - \alpha z)^{-1} (1 - \beta z)^{-1} \right] d\mu(\sigma)$$

$$= \left\{ \varphi(z) * z (1 - \alpha z)^{-1} (1 - \beta z)^{-1} \right\}$$

$$\times \left\{ (1 + \gamma) \int_T \frac{\varphi(z) * z (1 - \alpha z)^{-1} (1 - \beta z)^{-1} (1 - \sigma z)^{-1}}{\varphi(z) * z (1 - \alpha z)^{-1} (1 - \beta z)^{-1}} d\mu - \gamma \right\}.$$

The expression in the first parentheses cannot vanish for $0 < |z| < 1$ by (2.2.14) and the expression in the second parenthesis cannot vanish by (2.2.13). This proves the lemma.

Lemma 2.3.14. Let $\varphi(z)$ be convex and $g(z)$ starlike in $|z| < 1$. Then for each function $F(z)$ analytic in $|z| < 1$ and satisfying

$$\Re F(z) > 0 \quad (|z| < 1), \quad (2.3.15)$$

we have

$$\Re \left(\frac{(\varphi * Fg)(z)}{(\varphi * g)(z)} \right) > 0 \quad (|z| < 1). \quad (2.3.16)$$

Proof. By Lemma 2.3.3 it will be sufficient to show that

$$\varphi(z) * \frac{1 + \alpha\sigma z}{1 - \sigma z} g(z) \neq 0 \quad (0 < |z| < 1) \quad (2.3.17)$$

for every α and σ satisfying $|\alpha| = |\sigma| = 1$. According to Lemma 2.3.10 a sufficient condition for this is that for each α and σ we find constants a, b and c with $|a| = |b| = |c| = 1$ such that

$$\Re \left\{ a (1 - bz) (1 - cz) \frac{1 + \alpha\sigma z}{1 - \sigma z} \frac{g(z)}{z} \right\} > 0 \quad (2.3.18)$$

in $|z| < 1$. However it is clear from Theorem 2.2.22 that such a relation holds, for if we set $b = \sigma$, then c can be chosen $c = e^{-it^*}$ where $-\alpha\sigma = e^{-it}$. By the maximum principle the possibility of equality occurring in the theorem can be ignored by appropriate choice of a .

Proof of Theorem 2.3.1 and Theorem 2.3.2. If f is close-to-convex, then for a suitable starlike function g , we have

$$zf'(z) = g(z) F(z)$$

where $\Re F(z) > 0$. From (2.3.16) we obtain

$$\Re \frac{z (\varphi * f)'}{\varphi * g} = \Re \frac{\varphi * zf'}{\varphi * g} = \Re \frac{\varphi * Fg}{\varphi * g} > 0. \quad (2.3.19)$$

Thus $\varphi * f$ is close-to-convex if $\varphi * g$ is starlike, and hence Theorem 2.3.2 follows if we can show that $\varphi * g$ is starlike, which is equivalent to theorem 2.3.1. To show that $\varphi * g$ is starlike we must establish the inequality

$$\operatorname{Re} \left(z \frac{(\varphi * g)'}{\varphi * g} \right) > 0 \quad (|z| < 1). \quad (2.3.20)$$

Since g is starlike, $\operatorname{Re}[zg'(z)/g(z)] > 0$, and hence (2.3.20) follows from (2.3.16) on putting $F(z) = zg'(z)/g(z)$. The two theorems are thus established.

Remark 2.3.21. If we say " f is close to φ " when

$$\operatorname{Re} \left(\frac{f'(z)}{\varphi'(z)} \right) > 0 \quad (|z| < 1),$$

then we have shown that $\varphi * f$ is close to $\varphi * \psi$ for every φ and ψ convex with f close to ψ .

2.4. Convolution of Functions Starlike of Order 1/2

In this section we shall prove that the structure of functions starlike of order 1/2 is also preserved under convolution. We need the following lemmas.

Lemma 2.4.1. Let $h(z)$ be analytic in $|z| < 1$ with $h(0) = 0$, and suppose that there exists β satisfying $|\beta| = 1$ such that

$$\operatorname{Re} \left\{ (1 - \beta z) \frac{h(z)}{z} \right\} > 0 \quad (|z| < 1). \quad (2.4.2)$$

Then if $\varphi(z)$ is starlike of order 1/2

$$\varphi(z) * h(z) \neq 0 \quad (0 < |z| < 1). \quad (2.4.3)$$

Proof. By Herglotz formula we can write

$$\frac{h(z)}{h'(0)} = \int_T \frac{z(1 + \gamma\sigma z)}{(1 - \sigma z)(1 - \beta z)} d\mu(\sigma)$$

where $|\gamma| = 1, \gamma \neq -1$. Thus

$$\begin{aligned} \frac{1}{h'(0)} \varphi(z) * h(z) &= \int_T \varphi(z) * \frac{z(1 + \gamma\sigma z)}{(1 - \sigma z)(1 - \beta z)} d\mu(\sigma) \\ &= \int_T \varphi(z) * \frac{z(1 + \gamma) - \gamma z(1 - \sigma z)}{(1 - \sigma z)(1 - \beta z)} d\mu(\sigma) \\ &= \int_T \left[(1 + \gamma) \varphi(z) * z(1 - \sigma z)^{-1} (1 - \beta z)^{-1} - \gamma \varphi(z) * z(1 - \beta z)^{-1} \right] d\mu(\sigma) \\ &= \left\{ \varphi(z) * z(1 - \beta z)^{-1} \right\} \int_T \left\{ (1 + \gamma) \frac{\varphi(z) * z(1 - \sigma z)^{-1} (1 - \beta z)^{-1}}{\varphi(z) * z(1 - \beta z)^{-1}} - \gamma \right\} d\mu(\sigma). \end{aligned}$$

But $\varphi(z) * z(1 - \beta z)^{-1} = \varphi(\beta z)/\beta$, it follows that

$$\frac{1}{h'(0)} \varphi(z) * h(z) = \frac{\varphi(\beta z)}{\beta} \int_T \left\{ (1 + \gamma) \frac{\varphi(z) * z(1 - \sigma z)^{-1} (1 - \beta z)^{-1}}{\varphi(z) * z(1 - \beta z)^{-1}} - \gamma \right\} d\mu(\sigma).$$

The integrand lies in a half-plane not containing 0 by (2.2.14) and $\varphi(\beta z) \neq 0$ ($0 < |z| < 1$) so the lemma follows.

Lemma 2.4.4. Let φ and ψ be starlike of order $1/2$. Then for each function $F(z)$ satisfying

$$\operatorname{Re} F(z) > 0 \quad (|z| < 1) \quad (2.4.5)$$

we have

$$\operatorname{Re} \left(\frac{\varphi(z) * F(z) \psi(z)}{\varphi(z) * \psi(z)} \right) > 0 \quad (|z| < 1). \quad (2.4.6)$$

Proof. By Lemma 2.3.3 it will be sufficient to show that for every α and σ satisfying $|\alpha| = |\sigma| = 1$, we have

$$\varphi(z) * \frac{1 + \alpha\sigma z}{1 - \sigma z} \psi(z) \neq 0 \quad (0 < |z| < 1).$$

By Lemma 2.4.1 this relation will be proved if for each such α and σ we can find β and a ($|a| = |\beta| = 1$) such that

$$\operatorname{Re} \left\{ a (1 - \beta z) \frac{1 + \alpha\sigma z}{1 - \sigma z} \frac{\psi(z)}{z} \right\} > 0 \quad (|z| < 1). \quad (2.4.7)$$

Since $\psi(z)$ is starlike of order $1/2$, it is immediately verified that for each σ ($|\sigma| = 1$) the function $[\psi(z)/(1 - \sigma z)]$ is starlike, and hence if we apply Theorem 2.2.22 to this function with $\beta = e^{-it}$, $\alpha\sigma = -e^{-it*}$ we obtain the required relation (2.4.7).

Theorem 2.4.8. If φ and ψ are starlike of order $1/2$, then so is $\varphi * \psi$.

Proof. The result will follow if we prove that

$$\operatorname{Re} \frac{z (\varphi * \psi)'}{\varphi * \psi} = \operatorname{Re} \frac{\varphi * z \psi'}{\varphi * \psi} > \frac{1}{2}. \quad (2.4.9)$$

Since $\psi(z)$ is starlike of order $1/2$, we have

$$\operatorname{Re} \left(z \frac{\psi'(z)}{\psi(z)} \right) > \frac{1}{2}.$$

Thus applying Lemma 2.4.4 with $F(z) = [z \psi'(z)/\psi(z)]$ we obtain

$$\operatorname{Re} \frac{\varphi(z) * F(z) \psi(z)}{\varphi(z) * \psi(z)} = \operatorname{Re} \frac{\varphi(z) * z \psi'(z)}{\varphi(z) * \psi(z)} > 0 \quad (|z| < 1).$$

Now (2.4.9) follows from remark 2.3.9.

Corollary 2.4.10. The Hadamard product of two odd starlike functions is starlike.

Theorem 2.4.11. Let φ and ψ be starlike of order $1/2$, and suppose that f satisfies

$$\operatorname{Re} \left(z \frac{f'(z)}{\psi(z)} \right) > 0 \quad (|z| < 1). \quad (2.4.12)$$

Then

$$\operatorname{Re} \left(\frac{\varphi(z) * z f'(z)}{(\varphi * \psi)(z)} \right) > 0 \quad (|z| < 1) \quad (2.4.13)$$

and in particular $\varphi * f$ is close-to-convex.

Proof. Applying Lemma 2.4.4 with $F(z) = [z f'(z)/\psi(z)]$, we have

$$\operatorname{Re} \left(\frac{\varphi(z) * z f'(z)}{(\varphi * \psi)(z)} \right) > 0, \quad (|z| < 1)$$

or,

$$\operatorname{Re} \left(\frac{z(\varphi * f)'}{\varphi * \psi} \right) > 0, \quad (|z| < 1).$$

But by Theorem 2.4.8, $\varphi * \psi$ is starlike of order $1/2$ and hence $\varphi * f$ is close-to-convex in $|z| < 1$.

2.5. A Subordination Theorem

To prove the subordination conjecture of Wilf [71], we need the following lemma.

Lemma 2.5.1. If $k(z)$ is convex and $h(z)$ is analytic, then $h(z)$ is properly subordinate to $k(z)$ if, and only if, for each σ ($|\sigma| = 1$)

$$\operatorname{Re} \left(\frac{zk'(z)}{k(z) - h(\sigma z)} \right) > 0 \quad (|z| < 1). \quad (2.5.2)$$

Proof. Suppose $h(z)$ is properly subordinate to $k(z)$. Then $h(z) = k(\omega(z))$ for some analytic function ω with $|\omega(z)| < |z|$. The image D of $k(z)$ being convex is in particular starlike relative to the point $k(z_0)$ and hence if $|z_0| < |z| < 1$,

$$\operatorname{Re} \left(\frac{zk'(z)}{k(z) - k(z_0)} \right) > 0.$$

Now (2.5.2) immediately follows from the maximum principle by choosing $z_0 = \omega(\sigma z)$.

Conversely, if the condition (2.5.2) holds, then it holds for every σ such that $|\sigma| \leq 1$.

Suppose that for some w , $k(z) \neq w$, where $w = h(z_1)$ and $|z_1| < 1$. Then if $|z_1| < |z| < 1$,

$$\operatorname{Re} \left(\frac{zk'(z)}{k(z) - w} \right) = \operatorname{Re} \left(\frac{zk'(z)}{k(z) - h(\sigma z)} \right) > 0$$

where $\sigma = z_1/z$, so that $|\sigma| < 1$. Since $k(z) \neq w$, the maximum principle gives

$$\operatorname{Re} \left(\frac{zk'(z)}{k(z) - w} \right) > 0 \quad (|z| < 1),$$

which is clearly false for $z = 0$. Since also (2.5.2) implies that $h(0) = k(0)$, we deduce that h is subordinate to k .

Theorem 2.5.3 (Wilf's Subordination Conjecture). Let φ and ψ be convex in $|z| < 1$ and suppose that f is subordinate to ψ . Then $\varphi * f$ is subordinate to $\varphi * \psi$.

Proof. We may assume that f is properly subordinate to ψ . We then have

$$\operatorname{Re} \left(\frac{z \psi'(z)}{\psi(z) - f(\sigma z)} \right) > 0 \quad (|z| < 1, |\sigma| = 1). \quad (2.5.4)$$

In view of Lemma 2.5.1, it is sufficient to show that

$$\operatorname{Re} \left(\frac{z(\varphi * \psi)'(z)}{(\varphi * \psi)(z) - (\varphi * f)(\sigma z)} \right) > 0$$

or,

$$\operatorname{Re} \left(\frac{\varphi(z) * z \psi'(z)}{(\varphi * \psi)(z) - (\varphi * f)(\sigma z)} \right) > 0 \quad (|z| < 1, |\sigma| < 1).$$

This is equivalent to

$$\operatorname{Re} \left(\frac{\varphi(z) * \frac{\psi(z) - f(\sigma z)}{z \psi'(z)} z \psi'(z)}{\varphi(z) * z \psi'(z)} \right) > 0$$

which follows from (2.5.4) and Lemma 2.3.14.

2.6. Applications

(a) The de la Vallée Poussin Means

The de la Vallée Poussin means of an analytic function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ are given by

$$V_n(z, f) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^n \binom{2n}{n+k} a_k z^k \quad (2.6.1)$$

for $n = 1, 2, \dots$ Pólya and Schoenberg [41] showed that these means are convex (starlike) whenever f is convex (starlike) (and conversely). In particular the polynomials

$$V_n(z) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^n \binom{2n}{n+k} z^k \quad (2.6.2)$$

are convex on putting $f(z) = z(1-z)^{-1}$. This apparently weaker statement now implies the full result, and furthermore the de la Vallée Poussin means of a close-to-convex function are close-to-convex. In addition Pólya and Schoenberg showed that $V_n(z, f)$ is subordinate to f for every convex f , and conjectured that in this case the stronger subordination condition

$$V_n(z, f) \prec V_{n+1}(z, f) \quad (n = 1, 2, \dots)$$

held. This follows from Theorem 2.5.3 as soon as it is shown for the polynomials (2.6.2).

Pólya and Schoenberg verified this case, although the proof was not given.

(b) Univalence of Partial Sums

Let $P_n(z, f)$ denote the n th partial sum of the analytic function f . If f is convex then $P_n(z, f)$ is convex in $|z| < r_1$, where r_1 is the radius of convexity of the polynomial

$$P_n(z) = z + z^2 + \cdots + z^n,$$

and if f is close-to-convex, then $P_n(z, f)$ is close-to-convex for $|z| < r_1$. Again if r_2 is the radius of close-to-convexity of $P_n(z)$, then $P_n(z, f)$ is close-to-convex in $|z| < r_2$ for every convex f . These results, which are sharp, have been obtained and the values of r_1 and r_2 have been computed [53,56].

It is also easy to show from our results that if $f(z)$ is starlike of order $1/2$ then

$$\operatorname{Re} \left(\frac{f(z)}{P_n(z, f)} \right) > \frac{1}{2} \quad (|z| < 1) \quad (2.6.3)$$

In fact, we can write

$$\begin{aligned} P_n(z, f) &= P_n(z) * f(z) \\ &= \frac{z}{1-z} (1-z^n) * f(z) \\ &= g(z)F(z) * f(z) \end{aligned}$$

where $g(z) = z/(1-z)$ is starlike of order $1/2$ and $F(z) = 1-z^n$ lies in a circular disc with center at 1 and radius 1. Thus by Lemma 2.4.4 and remark 2.3.9 it follows that

$$\frac{P_n(z, f)}{f(z)} = \frac{f(z) * g(z)F(z)}{f(z) * g(z)}$$

lies in a circular disc with center 1 and radius 1, i.e.,

$$\left| \frac{P_n(z, f)}{f(z)} - 1 \right| < 1. \quad (2.6.4)$$

Now, if we let $P_n(z, f)/f(z) = u + iv$ then (2.6.3) follows from (2.6.4)

Writing $zg'(z) = f(z)$ and using (2.6.3) we have

$$\begin{aligned} \Re \left(z \frac{(P_n(z, g))'}{f(z)} \right) &= \Re \left(\frac{P_n(z) * zg'(z)}{f(z)} \right) \\ &= \Re \left(\frac{P_n(z) * f(z)}{f(z)} \right) \\ &= \Re \left(\frac{P_n(z, f)}{f(z)} \right) > 0. \end{aligned}$$

Hence the polynomials $P_n(z, g)$ are close-to-convex in $|z| < 1$.

We conclude this chapter by pointing out the following extension of Theorems 2.3.1 and 2.3.2.

Theorem 2.6.5. Let ψ be convex and suppose that for a function f , $f'\psi'$ takes all its values in a convex domain D . Then for every convex φ

$$\frac{\varphi * zf'}{\varphi * z\psi'}$$

takes all its values in D .

Proof. This follows immediately from Lemma 2.3.14 and Remark 2.3.9.

CHAPTER III

APPLICATIONS OF CONVOLUTION OPERATORS TO PROBLEMS IN UNIVALENT FUNCTION THEORY

3.1. Introduction

In this chapter we shall exploit the strengths and properties of convolution operators. The strength of these methods lies in their ability to unify a number of diverse results.

In 1947, R.M. Robinson [48] conjectured that if $f \in S$, then the function $1/2[f(z) + zf'(z)]$ also belongs to S for $|z| < 1/2$. This well-known Robinson's $1/2$ conjecture was verified by Livingston [33] in 1966 for convex, starlike and close-to-convex functions. In fact, he proved that if $f \in C$, S^* or K , then $1/2[f(z) + zf'(z)]$ is convex, starlike, or close-to-convex for $|z| < 1/2$. Many researchers (see [29], [43], etc) have studied the appropriate minimum radii for the different operators on various subclasses of S . Using the convolution results and techniques of Ruscheweyh and Sheil-Small [55] discussed in Chapter II, Bernard and Kellogg [4] obtained these results and many of their generalizations. They also obtained a number of new results including a verification of Robinson's $1/2$ conjecture in the case of γ -spirallike functions. It is interesting to note that the proofs using convolution operators are clearer and more concise and point out how the unifying linear structure that is common to so many problems can be used to solve them via convolution operator techniques.

The aim of this chapter is to study in detail the results of Bernard and Kellogg [4]. In Section 3.2 we discuss appropriate minimum radii for the different operators on various subclasses of S , using convolution techniques. Section 3.3 of this chapter is devoted to

study Robinson's 1/2 conjecture in the case of γ -spirallike functions. Robinson's 1/2 conjecture for typically real functions will be discussed in Section 3.4. Finally, several other results including a generalization of Bernardi's results [5] are discussed in the last section of this chapter.

3.2. Minimum Radii for the Different Operators on Various Subclasses of S

In this section we shall use Theorems 2.3.1 and 2.3.2 of Ruscheweyh and Sheil-Small [55] discussed in chapter II. In order to apply these results we shall need the following notation. Let Γ_i , $0 \leq i \leq 4$ be the linear operators defined on A by the equations below.

$$\Gamma_0 f(z) = z f'(z) \qquad \Gamma_1 f(z) = \frac{1}{2} [f(z) + z f'(z)]$$

$$\Gamma_2 f(z) = \int_0^z \frac{f(\xi) - f(0)}{\xi} d\xi \qquad \Gamma_3 f(z) = \frac{2}{z} \int_0^z f(\xi) d\xi$$

$$\Gamma_4 f(z) = \int_0^z \frac{f(\xi) - f(x\xi)}{\xi - x\xi} d\xi, \quad |x| \leq 1, x \neq 1.$$

We note that Γ_4 , generalizing Γ_2 , was first used by Pommerenke in [43]. It is easy to verify that each of these operators can be written as a convolution operator given by

$$\Gamma_i f = h_i * f, \quad 0 \leq i \leq 4$$

where

$$\begin{aligned} h_0(z) &= \sum_{n=1}^{\infty} n z^n = \frac{z}{(1-z)^2} & h_1(z) &= \sum_{n=1}^{\infty} \frac{n+1}{2} z^n = \frac{z - z^2/2}{(1-z)^2} \\ h_2(z) &= \sum_{n=1}^{\infty} \frac{1}{n} z^n = -\log(1-z) \\ h_3(z) &= \sum_{n=1}^{\infty} \frac{2}{n+1} z^n = \frac{-2[z + \log(1-z)]}{z} \end{aligned}$$

$$h_4(z) = \sum_{n=1}^{\infty} \frac{1 - x^n}{(1 - x)n} z^n = \frac{1}{1 - x} \log \left[\frac{1 - xz}{1 - z} \right], \quad |x| \leq 1, x \neq 1.$$

For example, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\Gamma_0 f = h_0 * f = \sum_{n=1}^{\infty} n z^n * \sum_{n=0}^{\infty} a_n z^n = \sum_{n=1}^{\infty} n a_n z^n = z f'(z),$$

$$\Gamma_1 f = h_1 * f = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n * \sum_{n=0}^{\infty} a_n z^n = \frac{1}{2} \left[\sum_{n=1}^{\infty} n a_n z^n + \sum_{n=0}^{\infty} a_n z^n \right]$$

$$= \frac{1}{2} [z f'(z) + f(z)]$$

$$\Gamma_2 f = h_2 * f = \sum_{n=1}^{\infty} \frac{1}{n} z^n * \sum_{n=0}^{\infty} a_n z^n = \sum_{n=1}^{\infty} \frac{1}{n} a_n z^n = \int_0^z \frac{f(\xi) - f(0)}{\xi} d\xi$$

etc.

For a given compact subclass X of A , let $r_S(X)$ denote the minimum radius of univalence over all functions f in X . We shall use the corresponding notation for the other subclasses of S . For example $r_{S^*}(X)$ denote the minimum radius of starlikeness over all functions f in X . Robinson [48] observed that for any f in S , the derivative of $\Gamma_1 f(z)$ does not vanish for $|z| < 1/2$. He also noted that for the koebe function $k(z) = z(1 - z)^{-2}$, $r_S[\Gamma_1(k)] = 1/2$. He conjectured that $r_X[\Gamma_1(X)] = 1/2$ for $X = S$. Although $1/2$ has been verified to be the correct radius when X is replaced by many of the subclasses of S , Bernard has proved in [3] that $.49 \leq r_S[\Gamma_1(S)] \leq .50$.

Many authors have studied the appropriate minimum radii for the different operators Γ_i , $0 \leq i \leq 4$ on various subclasses of S . We list below a few of these results. Each of these and many of their generalizations can be obtained by the techniques presented in this chapter.

The classical results of Alexander in [1] shows that

$$r_S * [\Gamma_0(S)] = r_S * [\Gamma_0(S^*)] = 2 - \sqrt{3} .$$

In [33], A. Livingston proved that

$$r_C[\Gamma_1(C)] = r_S * [\Gamma_1(S^*)] = r_K[\Gamma_1(K)] = 1/2.$$

and that

$$r_R[\Gamma_1(R)] = \frac{\sqrt{5} - 1}{2} .$$

Generalizations of these results have been given by Libera and Livingston in [31] and by Bernardi in [5]. It has been shown by Causey in [11] and others that $r_K[\Gamma_2(K)] = 1$. The example of Krzyz and Lewandoski in [28] shows that $r_S[\Gamma_2(S)] < 1$. In [29] Libera showed that

$$r_C[\Gamma_3(C)] = r_S * [\Gamma_3(S^*)] = r_K[\Gamma_3(K)] = 1$$

and these results have been generalized by Bernardi in [5]. Pommerenke in [43] has shown that $r_K[\Gamma_4(K)] = 1$.

It is easy to verify that the radius of convexity of each of the functions h_i , $0 \leq i \leq 4$, previously defined, that is, $r_C(h_0) = 2 - \sqrt{3}$, $r_C(h_1) = 1/2$ and

$$r_C(h_2) = r_C(h_3) = r_C(h_4) = 1 .$$

These facts together with Theorems 2.3.1 and 2.3.2 yield the following theorem and its consequences.

Theorem 3.2.1. If f is in C , S^* , or K then $\Gamma_i f = h_i * f$ is convex, starlike or close-to-convex, respectively, up to $r_C(h_i)$ for each i , $0 \leq i \leq 4$.

It is clear that Theorem 3.2.1 encompasses most of the major results previously mentioned concerning the various radii with the sharpness following by using the standard extremal functions for the specific subclasses. Livingston's results in [33] follow from Theorem

3.2.1. That is , Robinson's $1/2$ conjecture is valid when X is replaced by C , S^* or K simply because $r_C(h_1) = 1/2$.

3.3. Robinson's $1/2$ Conjecture in the case of γ -Spirallike Functions

In this section we shall prove that X can also be replaced by S_p^γ the class of γ -spirallike functions. However, the result does not follow directly from the convexity of h_1 up to $1/2$ because, unlike C , S^* , and K , S_p^γ is not preserved under convolution with convex functions (see [28]) . We shall still be able to obtain the result using convolution techniques by applying the lemmas proved in this section.

In the proofs of Theorems 2.3.1 and 2.3.2, Ruscheweyh and Sheil-Small proved a most interesting key lemma. We give below the following slightly more general version of their lemma stated without proof in their paper. For completeness we include the proof suggested by D. Styer and D. Wright which is more concise than the authors original.

Lemma 3.3.1. Let φ and g be analytic in $|z| < 1$ with $\varphi(0)=g(0)=0$ and $\varphi'(0), g'(0) \neq 0$. Suppose that for each $\alpha(|\alpha| = 1)$ and $\sigma(|\sigma| = 1)$ we have

$$\left[\varphi * \left(\frac{1 + \alpha \sigma z}{1 - \sigma z} \right) g \right](z) \neq 0 \quad \text{on} \quad 0 < |z| < r \leq 1 . \quad (3.3.2)$$

Then for each F in A the image of $|z| < r$ under $(\varphi * Fg)/(\varphi * g)$ is a subset of the convex hull of $F(U)$.

Remark 3.3.3. We note that in Lemma 2.3.14 we proved that if $\varphi \in C$ and $g \in S^*$ then (3.3.2) is satisfied for all $z \in U$.

Proof. Since $\varphi(z) * [(1 + \alpha\sigma z) / (1 - \sigma z)] g(z) \neq 0$ for $0 < |z| < r \leq 1$ is equivalent to $\varphi(rz) * [(1 + \alpha\sigma z) / (1 - \sigma z)] g(z) \neq 0$ for $0 < |z| < 1$ we can assume $r = 1$. In Lemma 2.3.3 we proved that if F has positive real part and (3.3.2) is satisfied, then $\Re\{[(\varphi * gF) / (\varphi * g)](z)\} > 0$ for z in U . For arbitrary F in A the convex hull of $F(U)$ is defined to be the total intersection of all half planes containing $F(U)$. If we denote by $\overline{F(U)}$ the closure of $F(U)$ then a line of support ℓ of $F(U)$ is the boundary of a half plane containing $F(U)$ such that $B_\ell = \ell \cap \overline{F(U)} \neq \emptyset$. For a given support line ℓ let b be a point in B_ℓ . Then there exists an α such that the half plane defined by the set $\{e^{-i\alpha}[(1+z)/(1-z)] + b; z \in U\}$ contains $F(U)$. For this α and b , if F_1 is defined by $F_1(z) = e^{i\alpha}[F(z) - b]$ for z in U we have that $\Re F_1(z) > 0$ for z in U . Thus we can apply the Ruscheweyh - Sheil- Small result to F_1 to obtain

$$\Re\left\{\frac{\varphi * gF_1}{\varphi * g}(z)\right\} = \Re\left\{e^{i\alpha}\frac{\varphi * gF}{\varphi * g}(z) - b\right\} > 0, \quad z \in U.$$

Therefore, for each z in U we have that $[(\varphi * gF)/(\varphi * g)](z)$ lies in the appropriate half plane for each support line ℓ of $F(U)$. Hence, it follows that $[(\varphi * gF)/(\varphi * g)](U)$ lies in the convex hull of $F(U)$ as claimed.

Lemma 3.3.4 . Let f be in S and $F(z) = 1 + a_1z + \dots$ be regular in U . Then, the image of $|z| < 1/2$ under $(h_1 * fF)/(h_1 * f)$ is a subset of the convex hull of $F(U)$.

Proof. This result follows from Lemma 3.3.1 upon showing that for all α and $\sigma(|\alpha| = |\sigma| = 1)$,

$$h_1(z) * \frac{1 + \alpha\sigma z}{1 - \sigma z} f(z) = H(z) \neq 0 \quad \text{for } 0 < |z| < 1/2.$$

From the definition of h_1 , we see that

$$H(z) = \frac{z - z^2/2}{(1 - z)^2} * \frac{1 + \alpha\sigma z}{1 - \sigma z} f(z)$$

$$= \frac{1}{2} \left[\frac{1 + \alpha \sigma z}{1 - \sigma z} f(z) + z \left(\frac{1 + \alpha \sigma z}{1 - \sigma z} f(z) \right)' \right]$$

or,

$$2H(z) = \left(\frac{1 + \alpha \sigma z}{1 - \sigma z} \right) f(z) \left[1 + \frac{zf'(z)}{f(z)} + \frac{(1 + \alpha) \sigma z}{(1 - \sigma z)(1 + \alpha \sigma z)} \right].$$

Since f is in S , if we put $\xi = \sigma z$, it is sufficient to show that for all $\alpha(|\alpha| = 1)$

$$1 + z \frac{f'(z)}{f(z)} + \frac{(1 + \alpha) \xi}{(1 - \xi)(1 + \alpha \xi)} \neq 0$$

for $|z|, |\xi| < 1/2$. We note that since f is in S we have that

$$\left| \log \frac{zf'(z)}{f(z)} \right| \leq \log \frac{1+r}{1-r} \text{ for } |z| \leq r.$$

Let $r = 1/2$, we then have that

$$\left| \log \frac{zf'(z)}{f(z)} \right| \leq \log 3 \text{ for } |z| \leq \frac{1}{2}.$$

Now, we claim that $|1 + e^\omega| \geq 4/3$ if $|\omega| \leq \log 3$. To verify this, it clearly suffices to consider $\omega = -\rho e^{i\theta}$ for θ in $(-\pi, \pi]$ and $\rho > 0$, and note that

$$\left| 1 + e^\omega \right|^2 = \left| 1 + e^{-\rho \cos \theta - i\rho \sin \theta} \right|^2 = 1 + e^{-2\rho \cos \theta} + 2 [\cos(\rho \sin \theta)] e^{-\rho \cos \theta}.$$

It is clear that replacement of θ by $-\theta$ does not change the above expression and thus it suffices to consider $0 \leq \theta \leq \pi$. Letting $h(\rho, \theta) = \left| 1 + e^\omega \right|^2$, we see that

$$\frac{\partial h}{\partial \theta} = 2\rho e^{-2\rho \cos \theta} [\sin \theta + \sin(\theta - \rho \sin \theta) e^{\rho \cos \theta}].$$

It now follows that $\partial h / \partial \theta \geq 0$ and therefore $h(\rho, \theta)$ is an increasing function of θ , and hence

$$|1 + e^{\omega}|^2 \geq h(\rho, 0) = |1 + e^{-\rho}|^2 \geq (4/3)^2$$

and the claim follows. We now have that $|1 + zf'(z)/f(z)| > 4/3$ for $|z| < 1/2$.

Lemma 3.3.4 will follow upon showing that for all $\alpha (|\alpha| = 1)$,

$$\frac{(1 + \alpha)\xi}{(1 - \xi)(1 + \alpha\xi)} < \frac{4}{3} \quad (3.3.5)$$

Provided $|\xi| < 1/2$. Inequality (3.3.5) will follow from next lemma which we will use a number of times in this chapter.

Lemma 3.3.6. For $|\alpha| = 1$, Let

$$f_{\alpha}(z) = \frac{(1 + \alpha)z}{(1 - z)(1 + \alpha z)}.$$

If $|z| \leq r < 1$, then $|f_{\alpha}(z)| \leq 2r/(1 - r^2)$.

Proof. Write $\alpha = e^{2i\varphi}$, $-\pi/2 < \varphi \leq \pi/2$ and $z = re^{i\theta}$. If we put $t = \theta + \varphi$ we see that

$$\begin{aligned} f_{\alpha}(z) &= \frac{r(e^{-i\varphi} + e^{i\varphi})e^{it}}{1 + re^{it}(e^{i\varphi} - e^{-i\varphi}) - r^2 e^{2it}} \\ &= \frac{2r \cos \varphi}{(1 - r^2) \cos t + i[2r \sin \varphi - (1 + r^2) \sin t]}. \end{aligned}$$

Now,

$$\begin{aligned} &\left| (1 - r^2) \cos t + i[2r \sin \varphi - (1 + r^2) \sin t] \right|^2 \\ &= (1 - r^2)^2 + 4r^2 \sin^2 t - 4r(1 + r^2) \sin \varphi \sin t + 4r^2 \sin^2 \varphi \end{aligned}$$

$$\begin{aligned}
&= 4r^2 \left[\sin t - \frac{(1+r^2)}{2r} \sin \varphi \right]^2 + (1-r^2)^2 \cos^2 \varphi \\
&\geq (1-r^2)^2 \cos^2 \varphi
\end{aligned}$$

and hence

$$|f_\alpha(z)| = \frac{|2r \cos \varphi|}{|(1-r^2) \cos t + i[2r \cos \varphi - (1+r^2) \sin t]|} \leq \frac{2r}{1-r^2} .$$

With these results we now prove below that Robinson's 1/2 conjecture is valid when S is replaced by S_p^γ .

Theorem 3.3.7. $r_{S_p^\gamma}[\Gamma_1(S_p^\gamma)] = \frac{1}{2}$.

Proof. Since f is in S_p^γ there exists a real γ such that $H(z) = e^{i\gamma} z f'(z) / f(z)$ has positive real part in U . To show that $\Gamma_1 f = h_1 * f$ is γ -spirallike in $|z| < 1/2$, we define H_1 , by

$$H_1(z) = \frac{e^{i\gamma} h_1 * z f'(z)}{h_1 * f(z)} = \frac{h_1 * f(z) H(z)}{h_1 * f(z)} .$$

Then Lemma 3.3.4 assures that $H_1(|z| < 1/2)$ is contained in the convex hull of $H(U)$. The result follows by noting that $k(z) = z(1-z)^{-2}$ is spirallike for $\gamma = 0$, and the radius of γ -spirallikeness of $\Gamma_1(k)$ is $1/2$.

The following result of Livingston [33] can be proved directly by using Lemma 3.3.1.

Theorem 3.3.8. $r_R[\Gamma_1(R)] = (\sqrt{5} - 1) / 2$.

Proof. Let $f \in R$. Then $f' \in P$. Thus

$$\operatorname{Re} f'(z) > 0, \quad z \in U .$$

Since

$$(h_1 * f)'(z) = \frac{h_1(z) * z f'(z)}{z} = \frac{h_1(z) * z f'(z)}{h_1(z) * z}$$

we need only to show that $\operatorname{Re}\{[h_1(z) * z f'(z)]/[h_1(z) * z]\} > 0$ for $|z| < r_0$

where $r_0 = (\sqrt{5} - 1)/2$. By Lemma 3.3.1 it suffices to show that

$$H(z) = h_1(z) * z \frac{1 + \alpha \sigma z}{1 - \sigma z} \neq 0 \text{ for } 0 < |z| < r_0, |\alpha| = |\sigma| = 1.$$

However from the definition of h_1 we have that

$$\begin{aligned} H(z) &= \frac{z - z^2/2}{(1 - z)^2} * \frac{z(1 + \alpha \sigma z)}{1 - \sigma z} \\ &= \frac{1}{2} \left[\frac{z(1 + \alpha \sigma z)}{1 - \sigma z} + z \left(\frac{z + \alpha \sigma z^2}{1 - \sigma z} \right)' \right]. \end{aligned}$$

On simplification we have

$$2H(z) = z \left(\frac{1 + \alpha \sigma z}{1 - \sigma z} \right) \left[2 + \frac{(1 + \alpha) \sigma z}{(1 - \sigma z)(1 + \alpha \sigma z)} \right].$$

By using Lemma 3.3.6 we obtain

$$\left| 2 + \frac{(1 + \alpha) \sigma z}{(1 - \sigma z)(1 + \alpha \sigma z)} \right| \geq 2 - \left| \frac{(1 + \alpha) \sigma z}{(1 - \alpha z)(1 + \alpha \sigma z)} \right| \geq 2 - \frac{2r}{(1 - r^2)} > 0,$$

for $0 \leq r < r_0$. The result then follows by considering the function f_1 defined by

$$f_1(z) = -2 \log(1 - z) - z.$$

3.4. Robinson's 1/2 Conjecture for Typically Real Functions with Functions in C_I

Let T_r be Rogosinski's class of typically real functions on U , noting that functions in T_r need not be univalent. Let C_I be Robertson's (see [44]) class of functions in T_r that have their images convex in the direction of the imaginary axis. Recall Fejer's observation that h is in C_I if and only if zh' is in T_r . We include a new proof of Robertson's result in [46] showing that T_r is invariant under convolution with functions in C_I . We then give a corollary showing its application to Robinson's 1/2 conjecture.

Theorem 3.4.1. If h is in C_I and f is in T_r then $h * f$ is in T_r .

Proof. It is well-known that f is in T_r if and only if $f(z) = z(1 - z^2)^{-1} p(z)$, where p is in P and has real coefficients. Also, for any function g in T_r there exists a non-decreasing function μ_g on $[0, \pi]$ with $\mu_g(\pi) - \mu_g(0) = \pi$ and such that

$$g(z) = \frac{1}{\pi} \int_0^\pi \frac{z}{1 - 2z \cos t + z^2} d\mu_g(t).$$

For each t in $[0, \pi]$ the function g_t defined by

$$zg'_t(z) = z(1 - 2z \cos t + z^2)^{-1}, g(0) = 0,$$

is convex because zg'_t is starlike. Thus, for t and s in $[0, \pi]$, $g(z, s, t) = zg'_s(z) * g_t(z)$ is starlike by Theorem 2.3.1 and has real coefficients so that $g(z, t, s)$ is in T_r . Hence $g(z, s, t) = z(1 - z^2)^{-1} p(z, s, t)$ where p is in P . By using these facts and the properties of convolution, we obtain for h in C_I that

$$(h * f)(z) = \frac{1}{\pi} \int_0^\pi h(z) * \frac{z}{1 - 2z \cos t + z^2} d\mu_f(t)$$

$$\begin{aligned}
&= \frac{1}{\pi} \int_0^\pi z h'(z) * \int_0^\pi \frac{d\xi}{1 - 2\xi \cos t + \xi^2} d\mu_f(t) \\
&= \frac{1}{\pi} \int_0^\pi \frac{1}{\pi} \left[\int_0^\pi \frac{z}{1 - 2z \cos s + z^2} d\mu_{zh'}(s) \right] * g_t(z) d\mu_f(t) \\
&= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi z g'_s(z) * g_t(z) d\mu_{zh'}(s) d\mu_f(t) \\
&= \frac{z}{1 - z^2} \frac{1}{\pi^2} \int_0^\pi \int_0^\pi P(z, s, t) d\mu_{zh'}(s) d\mu_f(t) \\
&= \frac{z}{1 - z^2} p_1(z)
\end{aligned}$$

where p_1 has real coefficients and is in P from the properties of $\mu_{zh'}$ and μ_f . Thus, from the characterization of T_r the theorem is proved.

Corollary 3.4.2. If f is in S and has real coefficients then $h_1 * f$ is typically real for $|z| < 1/2$.

Proof. This follows from Theorem 3.4.1 since $r_C(h_1) = r_{C_I}(h_1) = 1/2$ and any function in S with real coefficients is in T_r .

3.5. Generalization of Bernardi's Results

In this section we discuss several results including a generalization of Bernardi's results proved by Bernard and Kellogg [4]. Throughout Bernardi's paper he considered c a positive integer. We shall consider c a complex number with $c \neq -1$. In each case when c is considered a positive integer we obtain Bernardi's results. We define d_c by

$$d_c(z) = \sum_{n=1}^{\infty} \frac{n+c}{1+c} z^n = \frac{z - [c/(1+c)] z^2}{(1-z)^2}.$$

For f in A let the operator $\Gamma_c: A \rightarrow A$ be defined by $\Gamma_c(f) = d_c * f$.

Theorem 3.5.1. (i) If $\Re\{c\} > 0$ then $r_K[\Gamma_c(C)] = 1$.

(ii) If

$$\Re\{c\} > \frac{4r - 1 - r^2}{1 - r^2} \quad (3.5.2)$$

and f is in C , S^* or K then $d_c * f$ is convex, starlike, or close-to-convex, respectively, for $|z| < r$. If c is real and greater than -1 then

$$r_C[\Gamma_c(C)] = r_{S^*}[\Gamma_c(S^*)] = r_K[\Gamma_c(K)] = r_0 \quad (3.5.3)$$

where $r_0 = \{2 - (3 + c^2)^{1/2}\}/(1 - c)$ for $c \neq 1$ and for $c = 1$, $r_0 = 1/2$.

(iii) If

$$|c + 1| > \frac{2r}{(1 - r^2)} \quad (3.5.4)$$

and f is in R , then $\Re\{(d_c * f)'(z)\} > 0$ for $|z| < r$. If c is real and greater than -1 , then $r_R[\Gamma_c(R)] = r_1$ where $r_1 = [-1 + (2 + 2c + c^2)^{1/2}]/(1 + c)$.

Proof. Part (i) follows from Theorem 3.2.1 by the known fact that d_c is in K if, and only if,

$$\left| \frac{c}{(1+c)} - \frac{1}{2} \right| \leq \frac{1}{2}$$

which is equivalent to $\Re\{c\} > 0$.

The first part of (ii) will follow from Theorem 3.2.1 upon showing that whenever inequality (3.5.2) holds then $r \leq r_C(d_c)$. Consider, for $k(z) = z/(1 - z)^2$, that

$$1 + \frac{z d_c''(z)}{d_c'(z)} = \frac{d_c * k * k}{d_c * k}(z) = \frac{d_c * k [(k * k)/k]}{d_c * k}(z). \quad (3.5.5)$$

Since $[(k * k)/k](z) = (1 + z)/(1 - z)$, Lemma 3.3.1 assures that the term on the right hand side of (3.5.5) has positive real part whenever

$$d_c * \left[\frac{k(1 + \alpha \sigma z)}{(1 - \sigma z)} \right] \neq 0 \text{ for all } \alpha, \sigma, |\alpha| = |\sigma| = 1 \quad (3.5.6)$$

and $0 < |z| < r$. Thus we need only to show that this r is determined by the condition (3.5.2). From the definition of d_c and the comparison of their corresponding Taylor series we have that

$$\begin{aligned} d_c * k \frac{1 + \alpha \sigma z}{1 - \sigma z} &= \frac{1}{1 + c} \left[c k \frac{1 + \alpha \sigma z}{1 - \sigma z} + z \left(k \frac{1 + \alpha \sigma z}{1 - \sigma z} \right)' \right] \\ &= \frac{1}{1 + c} k \frac{1 + \alpha \sigma z}{1 - \sigma z} \left[c + z \frac{k'(z)}{k(z)} + \frac{(1 + \alpha) \sigma z}{(1 - \sigma z)(1 + \alpha \sigma z)} \right] \end{aligned} \quad (3.5.7)$$

Thus, it suffices to show that the bracketed term in (3.5.7) is nonzero for $|z| < r$ determined by (3.5.2). From the definition of $k(z)$ and Lemma 3.3.6 we have

$$\begin{aligned} \left| c + z \frac{k'(z)}{k(z)} + \frac{(1 + \alpha) \sigma z}{(1 - \sigma z)(1 + \alpha \sigma z)} \right| &= \left| c + \frac{1 + z}{1 - z} + \frac{(1 + \alpha) \sigma z}{(1 - \sigma z)(1 + \alpha \sigma z)} \right| \\ &\geq \left| c + \frac{1 + z}{1 - z} \right| - \left| \frac{(1 + \alpha) \sigma z}{(1 - \sigma z)(1 + \alpha \sigma z)} \right| \\ &\geq \Re \left(c + \frac{1 + z}{1 - z} \right) - \frac{2r}{1 - r^2} > 0 \end{aligned}$$

whenever $\Re \{c + (1+z)/(1-z)\} > 2r/(1-r^2)$. This holds when

$$\Re \{c\} > \frac{2r}{1-r^2} - \frac{1-r}{1+r} = \frac{4r-1-r^2}{1-r^2}. \quad (3.5.8)$$

Hence the bracketed term in (3.5.7) is nonzero for $|z| < r$ if $\Re \{c\} > (4r-1-r^2)/(1-r^2)$, as claimed. For real $c > -1$, (3.5.8) is equivalent to $c+1-4r+(1-c)r^2 > 0$, which holds whenever $0 \leq r < r_0$. In order to complete the verification of (3.5.3) we consider the cases of sharpness. For the convex case we convolute $d_c(z)$ with $z/(1-z)$ to obtain

$$1+z \frac{d_c''(z)}{d_c'(z)} = \frac{(1+c)+4z+(1-c)z^2}{(1-z)[(1+c)+(1-c)z]} = J(z).$$

It easily follows that $J(-r_0) = 0$. For the starlike and close-to-convex case we convolute d_c with $k(z)$ to obtain

$$\frac{z(d_c * k)'(z)}{d_c * k(z)} = 1+z \frac{d_c''(z)}{d_c'(z)} = J(z).$$

So that again $J(-r_0) = 0$. Since for $c = 1$, $d_c * f = h_1 * f = \Gamma_1 f$ the case $c = 1$ follows from our previous results.

We prove part (iii) by using Theorem 3.2.1 and noting as before that

$$(d_c * f)'(z) = \frac{d_c(z) * zf'(z)}{d_c(z) * z}.$$

Since f is in R , Lemma 3.3.1 assures that $\Re \{(d_c * f)'\} > 0$ whenever

$$d_c * \left(z \frac{1 + \alpha \sigma z}{1 - \sigma z} \right) \neq 0, \quad |\alpha| = |\sigma| = 1 \quad (3.5.9)$$

and $0 < |z| < r$. We need only to show that this r is determined by condition (3.5.4). We have

$$\begin{aligned}
 d_c * z \frac{1 + \alpha \sigma z}{1 - \sigma z} &= \frac{1}{1 + c} \left[\frac{cz(1 + \alpha \sigma z)}{1 - \sigma z} + z \left(\frac{z + \alpha \sigma z^2}{1 - \sigma z} \right)' \right] \\
 &= \frac{z}{1 + c} \left[\frac{c(1 + \alpha \sigma z)}{1 - \sigma z} + \frac{1 + 2\alpha \sigma z - \alpha \sigma^2 z^2}{(1 - \sigma z)^2} \right] \\
 &= \frac{z}{1 + c} \left[\frac{c(1 + \alpha \sigma z)}{1 - \sigma z} + \frac{(1 + \alpha \sigma z)(1 - \sigma z) + (1 + \alpha) \sigma z}{(1 - \sigma z)^2} \right] \\
 &= \frac{1 + \alpha \sigma z}{1 - \sigma z} \left[1 + c + \frac{(1 + \alpha) \sigma z}{(1 - \sigma z)(1 + \alpha \sigma z)} \right] \frac{z}{1 + c} .
 \end{aligned}$$

Thus applying Lemma 3.3.6 we have that inequality (3.5.9) holds whenever inequality (3.5.4) holds as claimed. For $c > -1$, inequality (3.5.4) is equivalent to

$$|z| < r_1 = \frac{-1 + (2 + 2c + c^2)^{1/2}}{1 + c} .$$

That $r_1 = r_R [\Gamma_c(R)]$ follows by convoluting $d_c(z)$ with $f_1(z) = -z - 2 \log(1 - z)$ to obtain

$$(d_c * f_1)'(z) = \frac{(1 + c) + 2z - (1 + c)z^2}{(1 - z)^2} = g(z)$$

where $g(-r_1) = 0$. Hence the theorem.

Another type of problem to which we can apply these techniques is the following. Let $G(z) = -f'(-z)/f(z)$ and $F(z) = -f'(-z)/f'(z)$. In [10], Burdick, Keogh, and Merkes determined

the smallest α and β such that $\Re \{ G(z) \}^\alpha > 0$ and $\Re \{ F(z) \}^\beta > 0$, z in U , for f in C, S^* and K . Noting that

$$G(z) = \frac{f(z) * \frac{z}{1+z}}{f(z) * \frac{z}{1-z}}$$

and that

$$F(z) = \frac{k(z) * f(z) * \frac{z}{1+z}}{k(z) * f(z) * \frac{z}{1-z}}$$

we obtain the following generalizations of their work. Given β , $0 \leq \beta < 1$, let

$$S^*(\beta) = \left\{ f \in S : \Re \left(z \frac{f'(z)}{f(z)} \right) > \beta, z \in U \right\}.$$

Lemma 3.5.10. If f is in $S^*(\beta)$ then

$$\Re \left\{ \frac{f(z) * \frac{z}{1+z}}{f(z)} \right\}^{1/2(1-\beta)} > 0, \quad z \in U. \quad (3.5.11)$$

The result is sharp.

Proof. This result follows readily from Herglotz representation for functions in $S^*(\beta)$.

Since f is in $S^*(\beta)$ there exists a probability measure μ such that

$$\log \left(\frac{f(z)}{z} \right) = -2(1-\beta) \int_{-\pi}^{\pi} \log (1 - ze^{it}) d\mu(t).$$

Thus we obtain

$$-\frac{f(-z)}{f(z)} = \exp \left[2(1-\beta) \int_{-\pi}^{\pi} \log \frac{1 - ze^{it}}{1 + ze^{it}} d\mu(t) \right],$$

and the result follows. The sharpness follows by considering the function

$$f(z) = k_\beta(z) = z(1-z)^{2(\beta-1)}.$$

Theorem 3.5.12. If f is in K then

$$\mathcal{Re} \left\{ \left[\frac{k_{\beta}(z) * f(z) * \frac{z}{1+z}}{k_{\beta}(z) * f(z)} \right]^{1/(2(2-\beta))} \right\} > 0, \quad z \in U. \quad (3.5.13)$$

The result is sharp.

Proof. Let $K_{\beta}(z) = \int_0^z (k_{\beta}(\xi)/\xi) d\xi$. Since f is in K , $zf'(z) = g(z)p(z)$ for g starlike and $\mathcal{Re} \{p(z)\} > 0$, z in U . Let p_i designate a function with $\mathcal{Re} \{p_i(z)\} > 0$, z in U , for $i = 1, 2, 3$ and 4. Using these notations, Lemma 3.3.1 and Remark 3.3.3 a number of times we have

$$\begin{aligned} \frac{k_{\beta}(z) * f(z) * \frac{z}{1+z}}{k_{\beta}(z) * f(z)} &= \frac{K_{\beta}(z) * zf'(z) * \frac{z}{1+z}}{K_{\beta}(z) * zf'(z)} \\ &= \frac{K_{\beta}(z) * g(z) p_1(z) * \frac{z}{1+z}}{K_{\beta}(z) * g(z) p_1(z)} \\ &= \frac{[K_{\beta}(z) * g(z)] p_2(z) * \frac{z}{1+z}}{[K_{\beta}(z) * g(z)] p_2(z)} \\ &= \frac{\{[K_{\beta}(z) * g(z)] * \frac{z}{1+z}\} p_3(z)}{[K_{\beta}(z) * g(z)] p_2(z)} \\ &= \frac{\left\{ \left[k_{\beta}(z) * \int_0^z \frac{g(\xi)}{\xi} d\xi \right] * \frac{z}{1+z} \right\} p_3(z)}{\left[k_{\beta}(z) * \int_0^z \frac{g(\xi)}{\xi} d\xi \right] p_2(z)} \end{aligned} \quad (3.5.14)$$

$$= \left\{ \frac{[k_{\beta}(z) * \int_0^z \frac{g(\xi)}{\xi} d\xi] * \frac{z}{1+z}}{k_{\beta}(z) * \int_0^z \frac{g(\xi)}{\xi} d\xi} \right\} p_3(z) p_2^{-1}(z).$$

It is also easy to prove that if h is in C then $h * k_{\beta}$ is in $S^*(\beta)$. So from the convexity of $\int_0^z [g(\xi)/\xi] d\xi$, Lemma 3.5.10, and Remark 3.3.3, we have that (3.5.14) equals $p_4^{2(1-\beta)} p_3(z) p_2^{-1}(z)$. Inequality (3.5.13) follows. Sharpness is proved by considering the function

$$f(z) = d_c(z) = \frac{z - \left(\frac{c}{1+c} \right) z^2}{(1-z)^2}.$$

A straightforward calculation gives that

$$\begin{aligned} & \left| \arg \left\{ \frac{k_{\beta}(z) * d_c(z) * \frac{z}{1+z}}{k_{\beta}(z) * d_c(z)} \right\} \right| \\ &= \left| \arg \left\{ \left(\frac{1-z}{1+z} \right)^{3-2\beta} \left(\frac{1 + \left(\frac{1-2\beta}{1+c} \right) z}{1 - \left(\frac{1-2\beta}{1+c} \right) z} \right) \right\} \right| \quad (3.5.15) \end{aligned}$$

If we let $(1-2\beta)/(1+c) = Re^{i\varphi}$ and $z = e^{i\theta}$, then, for $\theta = -\varphi + \pi/2$, (3.5.15) becomes

$$\begin{aligned} & \left| \arg \left\{ \frac{k_{\beta}(z) * d_c(z) * \frac{z}{1+z}}{k_{\beta}(z) * d_c(z)} \right\} \right| \\ &= \left| (3-2\beta) \arg \left(\frac{1-e^{i\theta}}{1+e^{i\theta}} \right) + \arg \left(\frac{1+Re^{i(\theta+\varphi)}}{1-Re^{i(\theta+\varphi)}} \right) \right| \end{aligned}$$

$$= \left| (3 - 2\beta) \frac{\pi}{2} + \operatorname{Arcsin} \left(\frac{2R}{1 + R^2} \right) \right|. \quad (3.5.16)$$

Since R approaches 1 as $|c|$ approaches ∞ we have that (3.5.16) approaches $(2 - \beta)\pi$ and the result follows.

As another application we answer problem 6.45 posed by Krzyż in Research problems in Complex Analysis in [2]. For $0 < \alpha \leq 1$ let S_α^* be the class of strongly-starlike function defined by Brannan and Kirwan in [7]; that is,

$$S_\alpha^* = \{f \in S : |\arg [zf'(z)/f(z)]| < \alpha\pi/2, |z| < 1\}.$$

Let " $\odot$ " be defined on $A \times A$ as follows: for $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ define $f \odot g$ by

$$(f \odot g)(z) = \sum_{n=1}^{\infty} \frac{a_n b_n}{n} z^n$$

Krzyż posed the following problems:

- (a) Prove (or disprove) that S_α^* is closed under $\odot$.
- (b) Prove (or disprove) that if f is in S_α^* and g is in S_β^* then $(f \odot g)$ is in S_γ^* for some $\gamma = \gamma(\alpha, \beta)$.
- (c) One could ask the same question for different convolutions in place of $\odot$.

We first make the observation that $(f \odot g)(z) = \int_0^z [f(\xi)/\xi] d\xi * g(z)$. Since f is in S_α^* ,

$\int_0^z [f(\xi)/\xi] d\xi$ is convex, thus the Remark 3.3.3 applies.

If we let $h(z) = (f \circledast g)(z)$, then

$$\frac{zh'(z)}{h(z)} = \frac{\int_0^z \frac{f(\xi)}{\xi} d\xi \circledast [g(z)] \left[\frac{zg'(z)}{g(z)} \right]}{\int_0^z \frac{f(\xi)}{\xi} d\xi \circledast g(z)}$$

lies in the convex hull of the image of U under zg'/g , that is ,

$$\left| \arg \left(z \frac{h'(z)}{h(z)} \right) \right| < \alpha\pi/2$$

since g is in S_α^* . Therefore S_α^* is closed under $\circledast$ and part (a) is resolved. for part (b),

since $f(z) \circledast \int_0^z [g(\xi)/\xi] d\xi = \int_0^z [f(\xi)/\xi] d\xi \circledast g(z)$ it follows that $\gamma = \min [\alpha, \beta]$ will

work and if $\beta = 1$ and $0 < \alpha \leq 1$ sharpness follows using $f(z) = k(z)$.

For part (c) if we denote a general type of weighted convolution by $\circledast_\omega$, there are still many interesting open problems. It is interesting to note that if one defines $\circledast_\omega$ for f and g in S_α^* by

$$(f \circledast_\omega g)(z) = (f * g * \omega)(z),$$

then an argument similar to that showing that S_α^* is closed under $\circledast$ shows that a sufficient condition for S_α^* to be closed under this $\circledast_\omega$ is that $z\omega'(z)$ be convex.

CHAPTER IV

A UNIFIED APPROACH FOR UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS USING HADAMARD PRODUCT

4.1. Introduction

Let A denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic in the unit disc U . Denote by $S^*(\alpha, \beta)$, the class of function $f \in A$ which satisfy the condition

$$\left| z \frac{f'(z)}{f(z)} - 1 \right| < \beta \left| z \frac{f'(z)}{f(z)} + 1 - 2\alpha \right|$$

for some $\alpha, \beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$ and for all $z \in U$; and let $C(\alpha, \beta)$ be the class of functions $f(z)$ for which $zf'(z) \in S^*(\alpha, \beta)$. The functions in $S^*(\alpha, \beta)$ and $C(\alpha, \beta)$ are respectively called starlike of order α and type β and convex of order α and type β in U .

Let $R(\alpha, \beta)$ be the class of functions $f(z)$ which satisfy the condition

$$\left| f'(z) - 1 \right| < \beta \left| f'(z) + 1 - 2\alpha \right|$$

for some $\alpha, \beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$ and all $z \in U$.

The classes $S^*(\alpha, \beta)$, $C(\alpha, \beta)$ and $R(\alpha, \beta)$ have been introduced and studied in [22,24].

Note that $S^*(\alpha, 1) = S^*(\alpha)$, $C(\alpha, 1) = C(\alpha)$ and $R(\alpha, 1) = R(\alpha)$. A function $f \in A$ is said to be prestarlike of order $\alpha (0 \leq \alpha < 1)$ and type $\beta (0 < \beta \leq 1)$, denoted by $D(\alpha, \beta)$, if

$f(z) * z(1-z)^{2\alpha-2} \in S^*(\alpha, \beta)$. The class $D(\alpha, 1) = D(\alpha)$ has been studied by Ruscheweyh [54] and others.

Let T denote the class of functions of the form $z - \sum_{n=2}^{\infty} |a_n| z^n$ that are analytic and univalent in U , and let $S_T^*(\alpha, \beta) = S^*(\alpha, \beta) \cap T$, $C_T(\alpha, \beta) = C(\alpha, \beta) \cap T$, $R_T(\alpha, \beta) = R(\alpha, \beta) \cap T$ and $D_T(\alpha, \beta) = D(\alpha, \beta) \cap T$. Several properties for the classes $S_T^*(\alpha, \beta)$, $C_T(\alpha, \beta)$ and $R_T(\alpha, \beta)$ including the characterization of extreme points of their closed convex hull have been obtained by Gupta and Jain [20,21].

The effect of second coefficients on various properties of the subclasses of univalent analytic functions has been studied by several researchers (see [23],[36],[65] etc.). However, the first attempt to investigate the effect of second coefficient of functions in the classes $S_T^*(\alpha, 1) \equiv S^*[\alpha]$ and $C_T(\alpha, 1) \equiv C[\alpha]$, of starlike and convex functions of order α with negative coefficients has been made by Silverman and Silvia [66], where they obtain extreme points, distortion theorems and radius of convexity bounds etc. A unified approach to the study of these properties for various subclasses of S , using Hadamard product, has been given by Juneja, Reddy and Mogra [25]. To determine properties analogous to Silverman and Silvia [66] and those obtained in [25] for the above mentioned subclasses of T , we introduce the following new classes.

Definition 4.1.1. Given the functions $\varphi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$, $\psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$,

analytic in U , such that $\lambda_n \geq 0$, $\mu_n \geq 0$ and $\lambda_n \geq \mu_n$ for $n = 2, 3, \dots$, we say that $f \in A$ is in $E(\varphi, \psi; \alpha, \beta)$ if $f(z) * \psi(z) \neq 0$ and the inequality

$$\left| \frac{f(z) * \varphi(z)}{f(z) * \psi(z)} - 1 \right| < \beta \left| \frac{f(z) * \varphi(z)}{f(z) * \psi(z)} + 1 - 2\alpha \right| \quad (4.1.2)$$

holds for some $\alpha, \beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$ and for all $z \in U$.

We shall denote by $E_T(\varphi, \psi; \alpha, \beta)$ the subclass of functions in $E(\varphi, \psi; \alpha, \beta)$ that have their nonzero coefficients, from second onward, all negative. Thus $E_T(\varphi, \psi; \alpha, \beta) = E(\varphi, \psi; \alpha, \beta) \cap T$.

It is easy to check that various subclasses of T referred to above can be represented as $E_T(\varphi, \psi; \alpha, \beta)$ for suitable choices of φ and ψ . For example,

$$E_T\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha, \beta\right) = S_T^*(\alpha, \beta),$$

$$E_T\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, \beta\right) = C_T(\alpha, \beta),$$

$$E_T\left(\frac{z}{(1-z)^2}, z; \alpha, \beta\right) = R_T(\alpha, \beta),$$

$$E_T\left(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}}; \alpha, \beta\right) = D_T(\alpha, \beta),$$

etc. In fact many new subclasses of T can be defined by suitably choosing $\varphi(z)$ and $\psi(z)$. It may be noted that $f \in E_T(\varphi, \psi; \alpha, \beta)$ if, and only if,

$$zf'(z) \in E_T\left(\int_0^z \frac{\varphi(t)}{t} dt, \int_0^z \frac{\psi(t)}{t} dt; \alpha, \beta\right)$$

or,

$$\int_0^z \frac{f(t)}{t} dt \in E_T(z\varphi', z\psi'; \alpha, \beta).$$

In Section 4.2 we obtain a necessary and sufficient condition for a function to be in the class $E_T(\varphi, \psi; \alpha, \beta)$ which leads to coefficient estimates. The extreme points of the class $E_T^P(\varphi, \psi; \alpha, \beta)$, defined later, are obtained in Section 4.3 of this chapter. Finally, we obtain distortion and covering theorems for functions in $E_T(\varphi, \psi; \alpha, \beta)$ and $E_T^P(\varphi, \psi; \alpha, \beta)$ in the last section of this chapter. The results thus obtained not only generalize the corresponding results of Silverman and Silvia [66] and those obtained in [25] but also give an analogous study for the classes $S_T(\alpha, \beta)$, $C_T(\alpha, \beta)$, $R_T(\alpha, \beta)$ and $D_T(\alpha, \beta)$.

It will be assumed throughout that φ and ψ satisfy the conditions stated in Definition 4.1.1 and that $f(z) * \psi(z) \neq 0$ for $z \in U$.

4.2. Characterization of the Class $E_T(\varphi, \psi; \alpha, \beta)$

In this section we find a necessary and sufficient condition for a function to be in $E_T(\varphi, \psi; \alpha, \beta)$.

Theorem 4.2.1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in A . If

$$\sum_{n=2}^{\infty} [(1 + \beta)\lambda_n - (1 - \beta + 2\alpha\beta)\mu_n] |a_n| \leq 2\beta(1 - \alpha), \quad (4.2.2)$$

$0 \leq \alpha < 1$, $0 < \beta \leq 1$, then $f \in E(\varphi, \psi; \alpha, \beta)$.

Proof. Suppose the condition (4.2.2) holds for all admissible values of α and β . Then for $z \in U$,

$$\begin{aligned} & \left| f(z) * \varphi(z) - f(z) * \psi(z) \right| - \beta \left| f(z) * \varphi(z) + (1 - 2\alpha)f(z) * \psi(z) \right| \\ &= \left| \sum_{n=2}^{\infty} a_n (\lambda_n - \mu_n) z^n \right| - \left| 2\beta(1 - \alpha)z + \sum_{n=2}^{\infty} \{ \beta\lambda_n + (1 - 2\alpha)\beta\mu_n \} a_n z^n \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left[\sum_{n=2}^{\infty} (\lambda_n - \mu_n) |a_n| - 2\beta(1 - \alpha) + \sum_{n=2}^{\infty} \{ \beta \lambda_n + (1 - 2\alpha) \beta \mu_n \} |a_n| \right] |z| \\
&< \left[\sum_{n=2}^{\infty} \{ (1 + \beta) \lambda_n - (1 - \beta + 2\alpha\beta) \mu_n \} |a_n| - 2\beta(1 - \alpha) \right] \\
&\leq 0,
\end{aligned}$$

in view of (4.2.2). Therefore it follows that

$$\left| \frac{f(z) * \varphi(z)}{f(z) * \psi(z)} - 1 \right| < \beta \left| \frac{f(z) * \varphi(z)}{f(z) * \psi(z)} + 1 - 2\alpha \right|$$

and hence $f \in E(\varphi, \psi; \alpha, \beta)$.

Theorem 4.2.3. If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in A , then $f \in E_T(\varphi, \psi; \alpha, \beta)$ if, and only if, (4.2.2) is satisfied.

Proof. In view of Theorem 4.2.1 it is sufficient to show the "only if" part. Thus, let

$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ be in $E_T(\varphi, \psi; \alpha, \beta)$. Then

$$\begin{aligned}
&\left| \frac{\frac{f(z) * \varphi(z)}{f(z) * \psi(z)} - 1}{\frac{f(z) * \varphi(z)}{f(z) * \psi(z)} + 1 - 2\alpha} \right| \\
&= \left| \frac{\sum_{n=2}^{\infty} (\lambda_n - \mu_n) |a_n| z^n}{2(1 - \alpha)z - \sum_{n=2}^{\infty} \{ \lambda_n + (1 - 2\alpha) \mu_n \} |a_n| z^n} \right| < \beta
\end{aligned}$$

theorem leads to necessary and sufficient conditions for a function f to be in $S_T^*(\alpha, \beta)$, $C_T(\alpha, \beta)$, $R_T(\alpha, \beta)$, $D_T(\alpha, \beta)$ etc.

4.3. Extreme Points of the Class $E_T^P(\varphi, \psi; \alpha, \beta)$

Since by Theorem 4.2.3 for $f \in E_T(\varphi, \psi; \alpha, \beta)$,

$$|a_n| \leq 2\beta(1 - \alpha) / \{(1 + \beta)\lambda_n - (1 - \beta + 2\alpha\beta)\mu_n\}, \quad n = 2, 3, \dots,$$

we may write

$$|a_2| = \{2\beta(1 - \alpha)p\} / \{(1 + \beta)\lambda_2 - (1 - \beta + 2\alpha\beta)\mu_2\}, \text{ so that } 0 \leq p \leq 1.$$

We denote by $E_T^P(\varphi, \psi; \alpha, \beta)$, the functions in $E_T(\varphi, \psi; \alpha, \beta)$ that are of the form

$$z - \frac{2\beta(1 - \alpha)p}{(1 + \beta)\lambda_2 - (1 - \beta + 2\alpha\beta)\mu_2} z^2 - \sum_{n=3}^{\infty} |a_n| z^n,$$

where p satisfying $0 \leq p \leq 1$ is kept fixed. From Theorem 4.2.3 it follows that the family $E_T(\varphi, \psi; \alpha, \beta)$ is closed under convex linear combinations so that the closed convex hull of $E_T^P(\varphi, \psi; \alpha, \beta)$ is same as $E_T^P(\varphi, \psi; \alpha, \beta)$. We now determine the extreme points of $E_T^P(\varphi, \psi; \alpha, \beta)$.

Theorem 4.3.1. Let $f_2(z) = z - [2\beta(1 - \alpha)p / \{(1 + \beta)\lambda_2 - (1 - \beta + 2\alpha\beta)\mu_2\}] z^2$

and

$$f_n(z) = z - \frac{2\beta(1 - \alpha)p}{(1 + \beta)\lambda_2 - (1 - \beta + 2\alpha\beta)\mu_2} z^2 - \frac{2\beta(1 - \alpha)(1 - p)}{(1 + \beta)\lambda_n - (1 - \beta + 2\alpha\beta)\mu_n} z^n \quad (4.3.2)$$

$n = 3, 4, \dots$, then $f \in E_T^P(\varphi, \psi; \alpha, \beta)$ if, and only if, it can be expressed in the form

$$f(z) = \sum_{n=2}^{\infty} \rho_n f_n(z) \text{ where } \rho_n \geq 0 \text{ and } \sum_{n=2}^{\infty} \rho_n = 1.$$

Proof. Suppose $f(z) = \sum_{n=2}^{\infty} \rho_n f_n(z)$ where $\rho_n \geq 0$ and $\sum_{n=2}^{\infty} \rho_n = 1$, and $f_2(z), f_n(z), n \geq 3$ are as given in the statement of the theorem. It then follows that

$$f(z) = z - \frac{2\beta(1-\alpha)p}{(1+\beta)\lambda_2 - (1-\beta+2\alpha\beta)\mu_2} z^2 - \sum_{n=3}^{\infty} \frac{2\beta(1-\alpha)(1-p)\rho_n}{(1+\beta)\lambda_n - (1-\beta+2\alpha\beta)\mu_n} z^n.$$

Since

$$\begin{aligned} & \frac{(1+\beta)\lambda_2 - (1-\beta+2\alpha\beta)\mu_2}{2\beta(1-\alpha)} \frac{2\beta(1-\alpha)p}{(1+\beta)\lambda_2 - (1-\beta+2\alpha\beta)\mu_2} \\ & + \sum_{n=3}^{\infty} \frac{(1+\beta)\lambda_n - (1-\beta+2\alpha\beta)\mu_n}{2\beta(1-\alpha)} \frac{2\beta(1-\alpha)(1-p)\rho_n}{(1+\beta)\lambda_n - (1-\beta+2\alpha\beta)\mu_n} \\ & = p + \sum_{n=3}^{\infty} \rho_n(1-p) = p + (1-p)(1-\rho_2) \leq 1, \end{aligned}$$

it follows from (4.2.2) that $f \in E_T^P(\varphi, \psi; \alpha, \beta)$. Conversely suppose

$$f(z) = z - \frac{2\beta(1-\alpha)p}{(1+\beta)\lambda_2 - (1-\beta+2\alpha\beta)\mu_2} z^2 - \sum_{n=3}^{\infty} |a_n| z^n \in E_T^P(\varphi, \psi; \alpha, \beta).$$

Since by (4.2.2) we have

$$|a_n| \leq \frac{2\beta(1-\alpha)(1-p)}{(1+\beta)\lambda_n - (1-\beta+2\alpha\beta)\mu_n}, \quad n = 3, 4, \dots$$

by setting $\rho_n = \left[\frac{\{(1+\beta)\lambda_n - (1-\beta+2\alpha\beta)\mu_n\} |a_n|}{\{2\beta(1-\alpha)(1-p)\}} \right]$, $n = 3, 4, \dots$

and $\rho_2 = 1 - \sum_{n=3}^{\infty} \rho_n$, we have $f(z) = \sum_{n=2}^{\infty} \rho_n f_n(z)$. Hence the theorem.

Corollary 4.3.3. The extreme points of $E_T^P(\varphi, \psi; \alpha, \beta)$ are the functions $f_n(z)$, $n = 2, 3, \dots$

The same procedure as in the last theorem leads to the following characterization for extreme points of the class $E_T(\varphi, \psi; \alpha, \beta)$.

Theorem 4.3.4. Let $g_1(z) = z$, $g_n(z) = z - \left[2\beta(1 - \alpha) / \{(1 + \beta)\lambda_n - (1 - \beta + 2\alpha\beta)\mu_n\} \right] z^n$, $n = 2, 3, \dots$, then $f \in E_T(\varphi, \psi; \alpha, \beta)$ if, and only if, it can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \rho_n g_n(z)$, where $\rho_n \geq 0$, and $\sum_{n=1}^{\infty} \rho_n = 1$.

It may be noted that whereas the function $g_1(z) = z$ is an extreme point of $E_T(\varphi, \psi; \alpha, \beta)$, it is not even a member of $E_T^P(\varphi, \psi; \alpha, \beta)$, unless $p = 0$.

4.4. Distortion and Covering Theorems for Functions in $E_T^P(\varphi, \psi; \alpha, \beta)$

The extreme points found in Theorem 4.3.4 immediately yield the following distortion theorems for functions in $E_T(\varphi, \psi; \alpha, \beta)$.

Theorem 4.4.1. Let $f \in E_T(\varphi, \psi; \alpha, \beta)$ and $\sigma_n = (1 + \beta)\lambda_n - (1 - \beta + 2\alpha\beta)\mu_n$, $n = 2, 3, \dots$. If $\{\sigma_n\}_{n=2}^{\infty}$ is a non-decreasing sequence, then

$$r - \frac{2\beta(1 - \alpha)}{\sigma_2} r^2 \leq |f(re^{i\theta})| \leq r + \frac{2\beta(1 - \alpha)}{\sigma_2} r^2. \quad (4.4.2)$$

Further, if $\{n/\sigma_n\}_{n=2}^{\infty}$ forms a non-decreasing sequence, then

$$1 - \frac{4\beta(1 - \alpha)}{\sigma_2} r \leq |f'(re^{i\theta})| \leq 1 + \frac{4\beta(1 - \alpha)}{\sigma_2} r. \quad (4.4.3)$$

Equality holds at both extremes in (4.4.2) and (4.4.3) for $g_2(z)$.

The sharp upper and lower bounds for $|f|$ as well as $|f'|$ for a function $f \in E_T^P(\varphi, \psi; \alpha, \beta)$ will occur at one of the extreme points of $E_T^P(\varphi, \psi; \alpha, \beta)$. However the bound are not straightforward as noted by Silverman and Silvia [66]. We need the following lemmas which generalize the corresponding lemmas of Silverman and Silvia [66]. For notational convenience we shall henceforth denote $\sigma_n = (1 + \beta)\lambda_n - (1 - \beta + 2\alpha\beta)\mu_n$, $n = 2, 3, \dots$.

Lemma 4.4.4. Let

$$p_0 = \frac{-\{\sigma_3 + 4\sigma_2 - 2\beta(1-\alpha)\} + [\{\sigma_3 + 4\sigma_2 - 2\beta(1-\alpha)\}^2 + 32\beta(1-\alpha)\sigma_2]^{1/2}}{4\beta(1-\alpha)}$$

$$r_0 = \frac{-2(1-p)\sigma_2 + [4(1-p)^2\sigma_2^2 + 2p^2\beta(1-\alpha)(1-p)\sigma_3]^{1/2}}{2\beta(1-\alpha)(1-p)p}$$

and

$$f_3(z) = z - \frac{2\beta(1-\alpha)p}{\sigma_2} z^2 - \frac{2\beta(1-\alpha)(1-p)}{\sigma_3} z^3,$$

then for $0 \leq r < 1$, $0 \leq p \leq 1$,

$$|f_3(re^{i\theta})| \geq r - \frac{2\beta(1-\alpha)p}{\sigma_2} r^2 - \frac{2\beta(1-\alpha)(1-p)}{\sigma_3} r^3,$$

with equality for $\theta = 0$. For either $0 \leq p < p_0$ and $0 \leq r \leq r_0$ or $p_0 \leq p \leq 1$,

$$|f_3(re^{i\theta})| \leq r + \frac{2\beta(1-\alpha)p}{\sigma_2} r^2 - \frac{2\beta(1-\alpha)(1-p)}{\sigma_3} r^3,$$

with equality for $\theta = \pi$. For $0 \leq p < p_0$ and $r_0 \leq r \leq 1$,

$$\begin{aligned} |f_3(re^{i\theta})| \leq r & \left[\frac{4(1-p)\sigma_2^2 + 2\beta(1-\alpha)p^2\sigma_3}{4(1-p)\sigma_2^2} + \frac{2\beta^2(1-\alpha)^2p^2\sigma_3 + 4\beta(1-\alpha)(1-p)\sigma_2^2}{\sigma_2^2\sigma_3} r^2 \right. \\ & \left. + \frac{(1-p)\beta^2(1-\alpha)^2 \{4(1-p)\sigma_2^2 + 2\beta(1-\alpha)p^2\sigma_3\}}{\sigma_2^2\sigma_3^2} r^4 \right]^{1/2} \end{aligned}$$

with equality for

$$\theta = \cos^{-1} \left[\frac{2\beta(1-\alpha)(1-p)pr^2 - p\sigma_3}{4(1-p)\sigma_2r} \right] .$$

Proof. Following the same lines as those of Silverman and Silvia [66] we may easily show that $\partial|f_3|^2/\partial\theta = 0$ for $\theta_1 = 0$, $\theta_2 = \pi$ and

$$\theta_3 = \cos^{-1} \left[\frac{2\beta(1-\alpha)(1-p)pr^2 - p\sigma_3}{4(1-p)\sigma_2r} \right] .$$

Since θ_3 exists only when

$$-1 \leq \frac{2\beta(1-\alpha)(1-p)pr^2 - p\sigma_3}{4(1-p)\sigma_2r} \leq 1 ,$$

We have a third root if, and only if, $r_0 \leq r < 1$ and $0 \leq p < p_0$. Thus comparing the extremal values $|f(re^{i\theta_k})|$, $k = 1, 2, 3$ on the different intervals, we get the result stated in the lemma.

Lemma 4.4.5. If $\sigma_n = (1 + \beta)\lambda_n - (1 - \beta + 2\alpha\beta)\mu_n$ is a non-decreasing function of n . Then for $n \geq 4$, $|f_n(re^{i\theta})| \leq |f_4(-r)|$.

Proof. The assumption made in the lemma implies that r^n/σ_n is a decreasing function of n . Thus, we have

$$\begin{aligned} \left| f_n(re^{i\theta}) \right| &\leq r + \frac{2\beta(1-\alpha)p}{\sigma_2} r^2 + \frac{2\beta(1-\alpha)(1-p)}{\sigma_n} r^n \\ &\leq r + \frac{2\beta(1-\alpha)p}{\sigma_2} r^2 + \frac{2\beta(1-\alpha)(1-p)}{\sigma_4} r^4 \\ &= -f_4(-r). \end{aligned}$$

Hence the lemma follows.

Lemma 4.4.4 and Lemma 4.4.5 lead to,

Theorem 4.4.6. If $f \in E_T^P(\varphi, \psi; \alpha, \beta)$ and $\sigma_n = (1 + \beta)\lambda_n - (1 - \beta + 2\alpha\beta)\mu_n$ is a non-decreasing function of n , then

$$\left| f(re^{i\theta}) \right| \geq r - \frac{2\beta(1-\alpha)p}{\sigma_2} r^2 - \frac{2\beta(1-\alpha)(1-\rho)}{\sigma_3} r^3 \quad (0 \leq r < 1)$$

with equality for $f_3(z)$ at $z = r$ and

$$\left| f(re^{i\theta}) \right| \leq \max \left\{ \max_{\theta} \left| f_3(re^{i\theta}) \right|, -f_4(-r) \right\},$$

where $\max_{\theta} \left| f_3(re^{i\theta}) \right|$ is given by Lemma 4.4.4

In view of the fact that $f \in E_T(\varphi, \psi; \alpha, \beta)$ if, and only if,

$$zf'(z) \in E_T \left(\int_0^z \frac{\varphi(t)}{t} dt, \int_0^z \frac{\psi(t)}{t} dt ; \alpha, \beta \right),$$

the following Lemma 4.4.7 and Theorem 4.4.8 for $f(z)$ follow easily from Lemma 4.4.4 and Theorem 4.4.6 respectively.

Lemma 4.4.7. Let

$$p_1 = \frac{-\{\sigma_3 + 6\sigma_2 - 6\beta(1-\alpha)\} + [\{\sigma_3 + 6\sigma_2 - 6\beta(1-\alpha)\}^2 + 144\beta(1-\alpha)\sigma_2]^{1/2}}{12\beta(1-\alpha)}$$

and

$$r_1 = \frac{-3(1-p)\sigma_2 + [\{9(1-p)^2\sigma_2^2 + 6\beta(1-\alpha)(1-p)p^2\sigma_3\}]^{1/2}}{6\beta(1-\alpha)(1-p)p}$$

and

$$f_3(z) = z - \frac{2\beta(1-\alpha)p}{\sigma_2} z^2 - \frac{2\beta(1-\alpha)(1-p)}{\sigma_3} z^3,$$

then for $0 \leq r < 1$, $0 \leq p < 1$,

$$\left| f_3'(re^{i\theta}) \right| \geq 1 - \frac{4\beta(1-\alpha)p}{\sigma_2} r - \frac{6\beta(1-\alpha)(1-p)}{\sigma_3} r^2,$$

with equality for $\theta = 0$. For either $0 \leq p < p_1$ and $0 \leq r \leq r_1$ or $p_1 \leq p < 1$,

$$\left| f_3'(re^{i\theta}) \right| \leq 1 + \frac{4\beta(1-\alpha)p}{\sigma_2} r - \frac{6\beta(1-\alpha)(1-p)}{\sigma_3} r^2,$$

with equality for $\theta = \pi$. For $0 \leq p \leq p_1$ and $r_1 \leq r < 1$,

$$|f_3'(re^{i\theta})| \leq \left[\frac{3(1-p)\sigma_2^2 + 2\beta(1-\alpha)p^2\sigma_3}{3(1-p)\sigma_2^2} + \frac{4\beta(1-\alpha)\{2\beta(1-\alpha)p^2\sigma_3 + 3(1-p)\sigma_2^2\}}{\sigma_2^2\sigma_3} r^2 \right. \\ \left. + \frac{12\beta^2(1-\alpha)^2(1-p)\{3(1-p)\sigma_2^2 + 2\beta(1-\alpha)p^2\sigma_2\}}{\sigma_2^2\sigma_3^2} r^4 \right]^{1/2}$$

with equality for

$$\theta = \cos^{-1} \left[\frac{6\beta(1-\alpha)(1-p)pr^2 - p\sigma_3}{6(1-p)\sigma_2 r} \right].$$

Theorem 4.4.8. If $f \in E_T^P(\varphi, \psi; \alpha, \beta)$, then

$$|f'(re^{i\theta})| \geq 1 - \frac{4\beta(1-\alpha)p}{\sigma_2} r - \frac{6\beta(1-\alpha)(1-p)}{\sigma_3} r^2 \quad (0 \leq r < 1),$$

with equality for $f_3'(z)$ at $z = r$ and

$$|f'(re^{i\theta})| \leq \max \left\{ \max_{\theta} |f_3'(re^{i\theta})|, -f_4'(-r) \right\},$$

where $\max |f_3'(re^{i\theta})|$ is given by Lemma 4.4.7.

Corollary 4.4.9. If $f \in E_T^P(\varphi, \psi; \alpha, \beta)$, then $f(U)$ contains a disc of radius

$$1 - \frac{2\beta(1-\alpha)p}{\sigma_2} - \frac{2\beta(1-\alpha)(1-p)}{\sigma_3}, \quad 0 \leq p \leq 1, \quad 0 \leq \alpha < 1, \quad 0 < \beta \leq 1.$$

Remark 4.4.10. For suitable choices of $\varphi(z)$ and $\psi(z)$ as mentioned in section 4.1, the results of this section lead to distortion and covering theorems for function in $S_T^*(\alpha, \beta)$, $C_T(\alpha, \beta)$, $R_T(\alpha, \beta)$, $D_T(\alpha, \beta)$ etc, when their second coefficient is kept fixed.

Theorem 4.4.11. Let $f \in E_T^P(\varphi, \psi; \alpha, \beta)$ and $g(z) = z + \sum_{n=2}^{\infty} \lambda_n' z^n$, $h(z) = z + \sum_{n=2}^{\infty} \mu_n' z^n$, such that $\lambda_n' \geq 0$ and $\mu_n' \geq 0$ and $\lambda_n' \geq \mu_n'$. Let

$$\delta = \frac{\sigma_2 \sigma_3 - (1-\alpha) [(1+\beta) \{p \lambda_2' \sigma_3 + (1-p) \lambda_3' \sigma_2\} - (1-\beta) \{p \mu_2' \sigma_3 + (1-p) \mu_3' \sigma_2\}]}{\sigma_2 \sigma_3 - 2\beta(1-\alpha) [p \mu_2' \sigma_3 + (1-p) \mu_3' \sigma_2]}.$$

where $\sigma_n' = (1 + \beta) \lambda_n' - (1 - \beta + 2\alpha\beta) \mu_n'$, $n = 2, 3, \dots, n$. If $\sigma_n' = (1 + \beta) \lambda_n' - (1 - \beta + 2\delta\beta) \mu_n'$ and $w(n) = \sigma_n' / \sigma_n$ for $n \geq 3$, is a decreasing function

of n , then $f \in E_T(g, h; \delta, \beta)$. The result is sharp with

$$f^*(z) = z - \frac{2\beta(1-\alpha)P}{\sigma_2} z^2 - \frac{2\beta(1-\alpha)(1-p)}{\sigma_3} z^3.$$

Proof. Since

$$\begin{aligned} & \frac{\sigma_2'}{2\beta(1-\delta)} \frac{2\beta(1-\alpha)p}{\sigma_2} + \frac{\sigma_3'}{2\beta(1-\delta)} \frac{2\beta(1-\alpha)(1-p)}{\sigma_3} \\ &= p \frac{\sigma_2'}{\sigma_2} \frac{1-\alpha}{(1-\delta)} + \frac{(1-\alpha)(1-p)}{1-\delta} \frac{\sigma_3'}{\sigma_3} = 1, \end{aligned} \quad (4.4.12)$$

we have $f^* \in E_T(g, h; \delta, \beta)$. To show that arbitrary

$$f(z) = z - \frac{2\beta(1-\alpha)p}{\sigma_2} z^2 - \sum_{n=2}^{\infty} |a_n| z^n ,$$

is in $E_T^P(g, h; \delta, \beta)$, we must show that

$$\frac{\sigma_2'}{\sigma_2} \frac{(1-\alpha)p}{1-\delta} + \sum_{n=3}^{\infty} \frac{\sigma_n'}{2\beta(1-\delta)} |a_n| \leq 1 . \quad (4.4.13)$$

By Theorem 4.3.1, we may set, for $n \geq 3$, $|a_n| = \{2\beta(1-\alpha)(1-p)\rho_n\}/\sigma_n$, where $\sum_{n=3}^{\infty} \rho_n \leq 1$. Thus

$$\begin{aligned} & \frac{\sigma_2'}{\sigma_2} \frac{(1-\alpha)p}{1-\delta} + \sum_{n=3}^{\infty} \frac{\sigma_n'}{2\beta(1-\alpha)} |a_n| \\ &= \frac{\sigma_2'}{\sigma_2} \frac{(1-\alpha)p}{1-\delta} + \sum_{n=3}^{\infty} \frac{\sigma_n'}{2\beta(1-\delta)} \frac{2\beta(1-\alpha)(1-p)\rho_n}{\sigma_n} \\ &= \frac{\sigma_2'}{\sigma_2} \frac{(1-\alpha)p}{1-\delta} + \frac{(1-\alpha)(1-p)}{1-\delta} \sum_{n=3}^{\infty} \rho_n \frac{\sigma_n'}{\sigma_n} \\ &\leq \frac{\sigma_2'}{\sigma_2} \frac{(1-\alpha)p}{1-\delta} + \frac{(1-\alpha)(1-p)}{1-\delta} \frac{\sigma_3'}{\sigma_3} \sum_{n=3}^{\infty} \rho_n \\ &\leq \frac{\sigma_2'}{\sigma_2} \frac{(1-\alpha)p}{1-\delta} + \frac{(1-\alpha)(1-p)}{1-\delta} \frac{\sigma_3'}{\sigma_3} = 1 , \end{aligned}$$

since σ_n'/σ_n is a decreasing function of n and by (4.4.12) the result follows.

Remark 4.4.14. Choosing $\varphi(z) = (z + z^2)/(1 - z)^3$, $\psi(z) = z/(1 - z)^2$ and $g(z) = z/(1 - z)^2$, $h(z) = z/(1 - z)$, so that $\lambda_n = n^2$, $\mu_n = n$, $\lambda'_n = n$, $\mu'_n = 1$ with $\beta = 1$, we get the corresponding result due to Silverman and Silvia [66]. The corresponding result for the classes $S_T^*(\alpha, \beta)$, $C_T(\alpha, \beta)$, $R_T(\alpha, \beta)$, $D_T(\alpha, \beta)$ etc, when there second coefficient is kept fixed, can be obtained for suitable choices of the function $\varphi(z)$, $\psi(z)$, $g(z)$ and $h(z)$ as mentioned in Section 4.1.

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