

University of Bahrain  
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6



**SINGLE CHANNEL QUEUES  
WITH BREAKDOWNS AND DELAYED  
OR TWO-PHASE REPAIRS**

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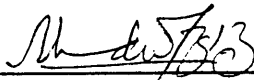
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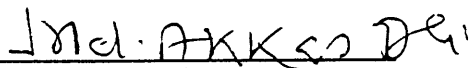
**"Single Channel Queues with Breakdowns  
and Delayed or Two-Phase Repairs"**

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satisfactory and acceptable for the Master of Science Degree in  
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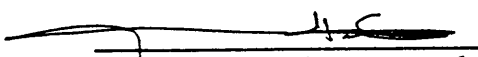
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O ye who believe! shall I lead you to a bargain that will save you from a grievous penalty?. That ye believe in God and his a postle, and that ye strive (your utmost) in the cause property and your persons. That will be best for you if ye but knew! He will forgive your sins, and admit you to garden beneath which rivers flow, and to beautiful mansions in gardens of eternity: that is indeed the supreme achievement.

[The Holy Quran : Sura LXI, Saff].

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## ABSTRACT

We study a queueing system wherein the service channel is subject to breakdowns occurring randomly in time. Immediately on a failure the service of units is suspended and the service channel waits for repairs to start. This waiting time is termed as delay time which has been assumed to have exponential distribution whereas the repair times are also exponential. A time-dependent solution in terms of the probability generating functions has been obtained. Finally, the dependence of the average number in the system on the delay parameter has been shown graphically.

The above problem has been generalised in the sense that instead of exponential delay times, we have considered them to have a general (arbitrary) distribution.

Next, we study a queueing system with bulk arrivals wherein the service channel is subject to random failures and the repairs are performed in two phases. Units wait until both phases of repairs of the service channel are complete. First, the service times have been considered to be exponentially distributed. A time-dependent solution in terms of the probability generating functions has been obtained and steady state results have been derived.

Later we have generalised this model by considering the service times to have general (arbitrary) distribution.

# SECTION 1

## 1.1 INTRODUCTION

All of us have experienced the annoyance of having to wait in line. Unfortunately, this phenomenon is becoming more and more prevalent in our increasingly congested and urbanized society. We wait in a line in our cars in traffic jams or at a doctor's office, we wait in line at supermarkets to check out, we wait in line at post offices, and so on. Not only human beings, jobs waiting to be processed at computer systems, messages waiting to be transmitted in communication systems, air planes waiting to land at an airport, machines waiting to be repaired, assembly lines, conveyor belts and production lines etc. are some other examples of queues.

Queuing theory is a branch of applied probability theory utilising concepts from the field of stochastic processes. The subject grew out of pioneering work done in this field by a Danish telephone engineer, A.K. ERLANG [5], who, in 1909, published a paper on the theory of probabilities and telephone conversations. The development in the field followed by the work done by POLLACZEK [19], KHINCHINE [10] and PALM [18]. And later some other significant contributions in the field were made by KENDALL [11], COX [3], JACKSON [7], and LINDLEY[12].

A queueing system may be described as on having a service facility at which units of some kind, generally called 'customers' arrive for service and a queue or a waiting line develops if the service is not available immediately. The units are served according to some preassigned rule and they leave after being served.

### CHARACTERISTICS OF QUEUEING SYSTEM

- Arrival pattern
- Service pattern
- Queue discipline
- System capacity
- Number of service channels
- Number of service stages

## 1 THE ARRIVAL PATTERN

Suppose the customers arrive at the system at time marks  $t_0 < t_1 < t_2 < \dots < t_{k-1} < t_k < \dots$ . The random variables  $X_k = t_k - t_{k-1}$  ( $k = 1, 2, 3, \dots$ ) are called interarrival times. Quite generally it is assumed that  $X_1, X_2, \dots$  form a sequence of independent random variables with the distribution

$$A(x) = P[X_k \leq x]$$

Here  $A(x)$  is called the arrival time distribution. A common form of  $A(x)$  is the negative exponential distribution defined by

$$A(x) = 1 - e^{-\lambda x} \quad \text{where } \frac{1}{\lambda} \text{ is called the mean interarrival time.}$$

## 2. SERVICE PATTERN

The time taken by a service channel to serve a unit or a batch of units is called the service time. Let the service times  $V_1, V_2, \dots$  form a sequence of independent random variables with the distribution

$B(x) = P[V_k \leq x]$ . Then  $B(x)$  is called the service time distribution. If the service times are exponentially distributed, then

$$B(x) = 1 - e^{-\mu x}, \quad \text{where } \frac{1}{\mu} \text{ is called the mean service time.}$$

## 3. QUEUE DISCIPLINE

Queue discipline is referred to the manner in which units are chosen for service. The most generally followed queue discipline is First in, First out (FIFO). However, there are several other queue disciplines e.g. Last in, First out (LIFO), priorities, random selection etc.

#### 4. SYSTEM CAPACITY

This means the waiting space for customers. A queueing system may have a finite or infinite capacity.

#### 5. NUMBER OF SERVICE CHANNELS

A queueing system may have one or more service channels which could be arranged in parallel or series or arranged in a network.

#### 6. NUMBER OF SERVICE STAGES

A unit may be served in one or more phases / stages.

### QUEUE NOTATION

A classification of queueing systems given by KENDALL[11] has the form  $A/B/C/k/m/z$  where A describes the arrival pattern, B the service pattern, C the number of servers, k the system capacity, m the number in the source and z the queue discipline. However queue notation with only first three letters are quite often used. For example the simple single server queueing system with Poisson arrivals (exponential interarrival times) and exponential service is denoted as M/M/ 1.

### THE PRESENT WORK

In many real-life situations, particularly where the service facility consists of a mechanical or an electronic device, there are breakdowns in the system which result in the stopping of service for some time. Many authors have studied such systems. Among them, GAVAR [6], SHANTHIKUMAR [22], DOSHI[4], MADAN [14], CRAMER [2] and KEILSON and SERVI [9] are a few to be mentioned.

In most of such studies, it is assumed that as soon as the service facility breakdowns, it instantly goes into repairs. Refer to MADAN [14], [15] and [16]. However, in the present work, we have assumed that there is a time-lag (delay time) for the repairs to start on the failed channel and later we have assumed that instead of a time-lag the repairs are performed in two phases.

We give below brief chapter-wise summary

## SUMMARY

In section 2, we have considered a single channel queueing system with random failures and exponential delay time. The probability generating functions of the number in the system have been found and the corresponding steady state solutions have been derived. Mean number in the system has been obtained and the dependence of mean number in the system and the idle state probability on the delay parameter  $\beta$  has been shown graphically.

Section 3 deals with a queueing model which is generalisation of the model discussed in section 2 in the sense that instead of single arrivals, there are batch arrivals and in addition the delay times have a general distribution.

In section 4, we study queueing system with random failures and two phases repairs. The probability generating functions of the queue length have been found and the corresponding steady state solution have been derived. The dependence of the mean queue length on failure and repair parameters has been shown graphically.

In section 5, we considered arrivals in batches of variable size and two-phases repairs. The results of section 4 have been derived.

Finally, in section 6, we study a system with two phases of repairs and instead of one by one exponential service, we have assumed that the service times have general (arbitrary) distribution. Again the results of section 4 have been derived.

## 1.2 SOME DEFINITIONS AND NOTATIONS

### UNITS

The customers arriving at a system for a service.

### STOCHASTIC PROCESS

A continuous time stochastic process  $\{X(t), t \geq 0\}$  is generally defined as a family of random variables, where  $X(t)$  describe the law of probability at time  $t$ .

The set of all possible values of  $X(t)$  is called a state space and  $T$ , the set of all values of  $t$  (generally  $t \geq 0$ ) is called the index set.

### MARKOV PROCESS

A stochastic process is said to be Markovian if the future development of the system is completely determined by the present state and is independent of the way in which the present state has developed.

A Markovian process whose state space is discrete is called a Markovian Chain.

### OPERATIVE TIME

The continuous stretch of the time period from the instant the service channel is put into operation until the instant it fails again is defined as the 'operative time' of the service channel.

### DELAY TIME

The continuous stretch of the time period from the instant it fails until the instant it actually goes for repairs is defined as the 'delay time'.

## REPAIR TIME

The continuous stretch of the time period from the instant the repairs actually start to the instant the service channel again becomes operative.

## TIME - INDEPENDENT OR STEADY STATE SOLUTION

In this case the probabilities are independent of time but yet the system is not deterministic. The queue continues to fluctuate following a distribution independent of time. In some cases, the steady state is never reached.

## TIME - DEPENDENT OR TRANSIENT SOLUTION

In some cases, the steady state solutions either do not exist or are inadequate. It is then desirable to go in for transient solutions from which the steady state (if it exists) could be determined either by letting  $t \rightarrow \infty$  or by setting the derivative with respect to  $t$  equal to zero and solving the resulting steady state solution.

## LAPLACE TRANSFORM

Define the Laplace transform of a function  $f(t)$  as follows

$$f(s) = L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt, \text{Re}(s) > 0 \quad (1.1)$$

The Laplace transform of  $\frac{d}{dt} f(t)$  is given by

$$L\left[\frac{d}{dt} f(t)\right] = sf(s) - f(0) \quad (1.2)$$

## GENERATING FUNCTION

Consider a function  $G(z)$  which has a power series expansion

$$G(z) = \sum_{n=0}^{\infty} g_n z^n = g_0 + g_1 z + g_2 z^2 + \dots \quad (1.3)$$

If the series converges for some range of  $z$ ,  $G(z)$  is called the generating function of the sequence  $g_0, g_1, g_2, \dots$

If this sequence is replaced by the sequence of probabilities  $p_0, p_1, p_2, \dots$  ( $0 \leq p_n \leq 1$ ) then the generating function

$$P(z) = \sum_{n=0}^{\infty} p_n z^n \quad (|z| \leq 1) \quad (1.4)$$

is defined as the probability generating function, where  $n$  is the number in the system.

The probability generating function,  $P(z) = \sum_{n=0}^{\infty} p_n z^n$  ( $z$  complex with  $|z| \leq 1$ )

Then the mean number in the system is given by

$$L = E(n) = \sum_{n=0}^{\infty} n p_n = \frac{d}{dz} P(z) \text{ at } z = 1 \quad (1.5)$$

## TAUBERIAN PROPERTY

$$\lim_{s \rightarrow 0} s f(s) = \lim_{t \rightarrow \infty} f(t) \quad (1.6)$$

If the limit on the right exists.

## ROUCHE'S THEOREM

If  $f(z)$  and  $g(z)$  are functions, analytic inside and on a closed contour  $C$  and if  $|g(z)| < |f(z)|$  on  $C$ , then  $f(z)$  and  $f(z) + g(z)$  have the same number of zeros inside  $C$ .

## UTILIZATION FACTOR

The ratio  $\rho = \frac{\lambda}{\mu}$ , where  $\lambda$  is the rate of arrival and  $\mu$  is the rate of service per unit time, is often called the utilization factor, since it is by definition, a measure of the average use of the service facility. It is, in fact, the expected number of arrivals per mean service time in the limit, and it is therefore often also called the traffic intensity.

## LITTLE'S FORMULAS

$$L = \lambda W, \quad L_q = \lambda W_q, \quad W = W_q + \frac{1}{\mu} \quad (1.7)$$

where

$L$  is mean number of units in the system

$L_q$  is mean number of units in the queue

$\lambda$  is arrival rate

$W$  is average waiting time in the system

$W_q$  is average waiting time in the queue

$\mu$  is service rate

## L' HÔPITAL'S RULE

Suppose that  $f$  and  $g$  are differentiable on an open interval  $(a,b)$  containing  $c$ , except possibly at  $c$  itself. If  $f(x) / g(x)$  has the indeterminate form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$  at  $x=c$  and if  $g'(x) \neq 0$  for  $x \neq c$ , then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} ,$$

provided either

$$\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} \text{ exists} \quad \text{or} \quad \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = \infty$$

## SECTION 2

### A SINGLE CHANNEL QUEUEING SYSTEM WITH RANDOM FAILURES AND EXPONENTIAL DELAY TIME

#### 2.1 INTRODUCTION

In this section we study a single channel queueing system in which the service channel is subject to occasional failures. We have assumed that the operative times are exponentially distributed. In most of the earlier work, it was assumed that as soon as the service channel fails it immediately goes to repairs. However, in this study, we have assumed that the failed service channel has to wait for the repairs to start. This waiting time of the service channel has been termed as 'delay time' which has been assumed to be exponentially distributed. As soon as the delay time is over, the failed channel undergoes repairs which have again been assumed to be exponentially distributed. Immediately after the repairs are complete, the service channel becomes operational and starts providing service to the waiting units, if any, in the system.

#### 2.2 THE MATHEMATICAL MODEL

The following assumptions describe the mathematical model.

- Units arrive one by one in a Poisson stream with mean arrival rate  $\lambda$  ( $\lambda > 0$ ) and queue up before the service channel if it is busy.
- The units are served one by one in order of their arrival exponentially with mean service time  $\frac{1}{\mu}$  ( $\mu > 0$ ).
- Let  $\alpha dt$  be the first order probability that the service channel will fail during the short interval of time  $dt$ .
- As soon as the service channel fails, it stops providing service and waits for the repairs to start. We assume that the waiting time of the failed channel (i.e, the time from the instant it fails to the instant its repairs start) follows the exponential distribution with mean waiting time  $\frac{1}{\beta}$  ( $\beta > 0$ ).
- Let  $\gamma dt$  be the first order probability that the repairs of the service channel will be completed during the short interval of time  $dt$ .
- As soon as the repairs are complete the service channel instantaneously starts operating.

It has been assumed that the various stochastic processes are independent of each other.

### 2.3 DEFINITIONS AND EQUATIONS GOVERNING THE SYSTEM

We define

$W_n(t)$  : as the probability that at time  $t$  there are  $n(\geq 0)$  units in the system (including one in service, if any) and the service channel is operative.

$D_n(t)$  : as the probability that at time  $t$  there are  $n(\geq 0)$  units in the system and the service channel is down and waiting for repairs to start.

$R_n(t)$  : as the probability that at time  $t$  the service channel is being repaired.

and

$P_n(t)$  : as the probability that at time  $t$  there are  $n(\geq 0)$  units in the system irrespective of the state of the service channel.

The system is governed by the following set of differential – difference equations:

$$\frac{d}{dt} W_n(t) + (\lambda + \mu + \alpha) W_n(t) = \lambda W_{n-1}(t) + \mu W_{n+1}(t) + \gamma R_n(t) \quad (n > 0) \quad (2.1)$$

$$\frac{d}{dt} W_0(t) + (\lambda + \alpha) W_0(t) = \mu W_1(t) + \gamma R_0(t) \quad (2.2)$$

$$\frac{d}{dt} D_n(t) + (\lambda + \beta) D_n(t) = \lambda D_{n-1}(t) + \alpha W_n(t) \quad (n > 0) \quad (2.3)$$

$$\frac{d}{dt} D_0(t) + (\lambda + \beta) D_0(t) = \alpha W_0(t) \quad (2.4)$$

$$\frac{d}{dt} R_n(t) + (\lambda + \gamma) R_n(t) = \lambda R_{n-1}(t) + \beta D_n(t) \quad (n > 0) \quad (2.5)$$

$$\frac{d}{dt} R_0(t) + (\lambda + \gamma) R_0(t) = \beta D_0(t) \quad (2.6)$$

We assume that initially the system starts when there are  $i$  units in the system and the service channel is in working order so that the initial condition is :

$$W_n(0) = \delta_{n,i} \quad (2.7)$$

where  $\delta_{n,i}$  is the Kronecker delta

i.e.  $\delta_{n,i} = 1 \quad \text{if } n = i$

$$= 0 \quad \text{if } n \neq i$$

## 2.4 THE TIME - DEPENDENT SOLUTION

We take Laplace transform of equations (2.1) to (2.6) and use equation (2.7). We have

$$(s + \lambda + \mu + \alpha) W_n(s) = \delta_{n,i} + \lambda W_{n-1}(s) + \mu W_{n+1}(s) + \gamma R_n(s) \quad (2.8)$$

$$(s + \lambda + \alpha) W_0(s) = \delta_{0,i} + \mu W_1(s) + \gamma R_0(s) \quad (2.9)$$

$$(s + \lambda + \beta) D_n(s) = \lambda D_{n-1}(s) + \alpha W_n(s) \quad (2.10)$$

$$(s + \lambda + \beta) D_0(s) = \alpha W_0(s) \quad (2.11)$$

$$(s + \lambda + \gamma) R_n(s) = \lambda R_{n-1}(s) + \beta D_n(s) \quad (2.12)$$

$$(s + \lambda + \gamma) R_0(s) = \beta D_0(s) \quad (2.13)$$

We define the following probability generating functions in terms of their Laplace transforms

$$W(s,z) = \sum_{n=0}^{\infty} z^n W_n(s)$$

$$D(s,z) = \sum_{n=0}^{\infty} z^n D_n(s)$$

$$R(s,z) = \sum_{n=0}^{\infty} z^n R_n(s)$$

$$P(s,z) = \sum_{n=0}^{\infty} z^n P_n(s) \quad (2.14)$$

We perform  $\sum_{n=1}^{\infty} z^{n+1}$  times equation (2.8) +  $z$  times equation (2.9) and obtain

$$[(s + \lambda - \lambda z + \mu + \alpha) z - \mu] W(s,z) = z^{i+1} + \mu (z-1) W_0(s) + \gamma z R(s,z) \quad (2.15)$$

We perform  $\sum_{n=1}^{\infty} z^n$  times equation (2.10) + equation (2.11)

$$(s + \lambda + \beta - \lambda z) D(s,z) - \alpha W(s,z) = 0 \quad (2.16)$$

and we perform  $\sum_{n=1}^{\infty} z^n$  times equation (2.12) + equation (2.13), make use of (2.14) and simplify

$$(s + \lambda + \gamma - \lambda z) R(s,z) = \beta D(s,z) \quad (2.17)$$

From (2.16), We have

$$D(s,z) = \frac{\alpha W(s,z)}{(s + \lambda + \beta - \lambda z)} \quad (2.18)$$

From (2.17) and (2.18) We have

$$R(s,z) = \frac{\beta D(s,z)}{(s + \lambda + \gamma - \lambda z)} = \frac{\beta \alpha W(s,z)}{(s + \lambda + \beta - \lambda z) (s + \lambda + \gamma - \lambda z)} \quad (2.19)$$

Using equation (2.19) in equation (2.15) and simplifying we get

$$W(s,z) = \frac{(s + \lambda + \beta - \lambda z) (s + \lambda + \gamma - \lambda z) [z^{i+1} + \mu (z-1) W_0(s)]}{(s + \lambda + \beta - \lambda z) (s + \lambda + \gamma - \lambda z) [(s + \lambda - \lambda z + \mu + \alpha) z - \mu] - \gamma \beta \alpha z} \quad (2.20)$$

Then using equation (2.20), equations (2.18) and (2.19) can be re-written as :

$$D(s,z) = \frac{\alpha (s + \lambda + \gamma - \lambda z) [z^{i+1} + \mu (z-1) W_0(s)]}{[(s + \lambda - \lambda z + \mu + \alpha) z - \mu] (s + \lambda - \lambda z + \beta) [s + \lambda - \lambda z + \gamma] - \alpha \beta \gamma z} \quad (2.21)$$

and (2.19) gives, on using (2.20)

$$R(s,z) = \frac{\beta \alpha (z^{i+1} + \mu (z-1) W_0(s))}{[(s + \lambda - \lambda z + \mu + \alpha) z - \mu] (s + \lambda - \lambda z + \beta) [s + \lambda - \lambda z + \gamma] - \alpha \beta \gamma z} \quad (2.22)$$

We will apply Rouché's Theorem to show that the common denominator of the right hand side of equations (2.20), (2.21) and (2.22) has one zero inside and on the unit circle  $|z| = 1$

let

$$f(z) = [(s + \lambda - \lambda z + \mu + \alpha) z - \mu] [s + \lambda - \lambda z + \beta] [s + \lambda - \lambda z + \gamma]$$

and

$g(z) = \alpha\beta\gamma z$ , then on  $|z| = 1$ , we have

$$|g(z)| = |\alpha\beta\gamma z| = \alpha\beta\gamma$$

and

$$\begin{aligned} |f(z)| &= |(s + \lambda - \lambda z + \mu + \alpha) z - \mu| |s + \lambda - \lambda z + \beta| |s + \lambda - \lambda z + \gamma| \\ &= |(s + \lambda - \lambda z + \mu + \alpha) z - \mu| |s + \lambda - \lambda z + \beta| |s + \lambda - \lambda z + \gamma| \\ &\geq (|s + \lambda - \lambda z + \mu + \alpha| - \mu) (s + \lambda + \beta - \lambda) (s + \lambda + \gamma - \lambda) \\ &\geq ((s + \lambda + \mu + \alpha - \lambda) - \mu) (s + \beta) (s + \gamma) \\ &= (s + \alpha) (s + \beta) (s + \gamma) \\ &\geq \alpha\beta\gamma = |g(z)| \end{aligned}$$

Thus on  $|z| = 1$ ,  $|f(z)| > |g(z)|$ . Hence by Rouché's Theorem  $f(z) - g(z)$  has the same number of zeroes as that of  $f(z)$  inside the unit circle  $|z| = 1$ . Clearly  $f(z)$  has one zero inside the unit circle and, therefore,  $f(z) - g(z)$  i.e. the denominator in the right hand side of equations (2.20), (2.21) and (2.22) has one zero inside the unit circle  $|z| = 1$ .

So we can determine the only unknown  $W_0(s)$  appearing in the numerator of equation (2.20), (2.21) and (2.22). Hence  $W(s, z)$ ,  $D(s, z)$ ,  $R(s, z)$  and, for that matter  $P(s, z)$  can be completely determined.

## 2.5 THE STEADY STATE SOLUTION

Let  $W(t)$ ,  $D(t)$ ,  $R(t)$  and  $P(t)$  respectively denote the steady state probabilities corresponding to  $W_n(t)$ ,  $D_n(t)$ ,  $R_n(t)$  and  $P_n(t)$  and for that matter,  $W(z)$ ,  $D(z)$ ,  $R(z)$  and  $P(z)$  are the corresponding steady state probability generating functions. Then the steady state solution can be obtained by using the Tauberian property (1.6) in equations (2.20), (2.21), (2.22) and have :

$$W(z) = \frac{(\lambda - \lambda z + \beta) (\lambda - \lambda z + \gamma) \mu (z - 1) W_0}{[(\lambda - \lambda z + \mu + \alpha) z - \mu] (\lambda - \lambda z + \beta) (\lambda - \lambda z + \gamma) - \alpha \beta \gamma z} \quad (2.23)$$

$$D(z) = \frac{\mu \alpha (\lambda - \lambda z + \gamma) (z - 1) W_0}{[(\lambda - \lambda z + \mu + \alpha) z - \mu] (\lambda - \lambda z + \beta) (\lambda - \lambda z + \gamma) - \alpha \beta \gamma z} \quad (2.24)$$

$$R(z) = \frac{\mu \alpha \beta (z - 1) W_0}{[(\lambda - \lambda z + \mu + \alpha) z - \mu] (\lambda - \lambda z + \beta) (\lambda - \lambda z + \gamma) - \alpha \beta \gamma z} \quad (2.25)$$

Let us denote the common denominator of the right hand side of equations (2.23) to (2.25) by  $K(z)$ . We have:

$$\begin{aligned} K(1) &= [(\lambda - \lambda + \mu + \alpha) - \mu] (\lambda - \lambda + \beta) (\lambda - \lambda + \gamma) - \alpha \beta \gamma \\ &= \alpha \beta \gamma - \alpha \beta \gamma = 0 \end{aligned}$$

which implies that  $(z-1)$  is a factor of  $K(z)$ . By synthetic division  $K(z)$  can be factored as

$$\begin{aligned} K(z) &= (z-1) [-\lambda^3 z^3 + \lambda^2 (2\lambda + \mu + \alpha + \beta + \gamma) z^2 - \left\{ \lambda^3 + \lambda^2 (2\mu + \alpha + \beta + \gamma) + \right. \\ &\quad \left. \lambda (\mu \beta + \alpha \beta + \mu \gamma + \alpha \gamma + \beta \gamma) \right\} z + [\lambda^2 \mu + \lambda \mu \beta + \lambda \mu \gamma + \mu \beta \gamma]] \end{aligned} \quad (2.26)$$

Cancelling out the factor  $(z-1)$  from the numerator and the denominator of each of the equations (2.23) to (2.25), we can write these equations respectively as:

$$W(z) = \frac{\mu (\lambda - \lambda z + \beta) (\lambda - \lambda z + \gamma) W_0}{H(z)} \quad (2.27)$$

$$D(z) = \frac{\mu \alpha (\lambda - \lambda z + \gamma) W_0}{H(z)} \quad (2.28)$$

$$R(z) = \frac{\mu \alpha \beta W_0}{H(z)} \quad (2.29)$$

where

$$H(z) = -\lambda^3 z^3 + \lambda^2 (2\lambda + \mu + \alpha + \beta + \gamma) z^2 - \left\{ \lambda^3 + \lambda^2 (2\mu + \alpha + \beta + \gamma) + \lambda(\mu\beta + \alpha\beta + \mu\gamma + \alpha\gamma + \beta\gamma) \right\} z + (\lambda^2 \mu + \lambda\mu\beta + \lambda\mu\gamma + \lambda\beta\gamma) \quad (2.30)$$

Since  $P(z) = W(z) + D(z) + R(z)$ , we have on adding equations (2.27), (2.28) and (2.29)

$$P(z) = \frac{\mu [(\lambda - \lambda z + \beta) (\lambda - \lambda z + \gamma) + \alpha (\lambda - \lambda z + \gamma) + \alpha\beta] W_0}{H(z)} \quad (2.31)$$

where  $H(z)$  is given by equation (2.30). Since  $P(1)$  must equal 1, we have on setting  $z=1$  in equation (2.31)

$$P(1) = \frac{\mu [(\lambda - \lambda + \beta) (\lambda - \lambda + \gamma) + \alpha(\lambda - \lambda + \gamma) + \alpha\beta] W_0}{H(1)}$$

$$\begin{aligned} \text{where } H(1) &= -\lambda^3 + \lambda^2(2\lambda + \mu + \alpha + \beta + \gamma) - \{\lambda^3 + \lambda^2(2\mu + \alpha + \beta + \gamma) + \\ &\quad \lambda(\mu\beta + \alpha\beta + \mu\gamma + \alpha\gamma + \beta\gamma)\} + [\lambda^2\mu + \lambda\mu\beta + \lambda\mu\gamma + \mu\beta\gamma] \\ &= \mu\beta\gamma - \lambda(\alpha\beta + \alpha\gamma + \beta\gamma) \end{aligned}$$

Finally

$$P(1) = \frac{\mu(\beta\gamma + \alpha\gamma + \alpha\beta) W_0}{\mu\beta\gamma - \lambda(\alpha\beta + \alpha\gamma + \beta\gamma)} = 1$$

$$\text{which yields } W_0 = \frac{\mu\beta\gamma - \lambda(\alpha\beta + \beta\gamma + \gamma\alpha)}{\mu(\alpha\beta + \beta\gamma + \gamma\alpha)} \quad (2.32)$$

Equation (2.32) gives the steady state probability that there is no unit in the system and the service channel is in working order but idle.

From (2.32), We obtain the steady state condition

$$\frac{\lambda}{\mu} < \frac{\beta\gamma}{\alpha\beta + \beta\gamma + \gamma\alpha} \quad (2.33)$$

In the special case when there is no delay for the repairs to start i.e. as soon as the service channel breaks down, the repair process starts instantly, this implies that the mean delay time for repairs to start is zero i.e.  $\frac{1}{\beta} \rightarrow 0$  and hence from equation (2.32) we have

$$W_0 = \frac{\mu \gamma - \lambda (\alpha + \gamma)}{\mu (\alpha + \gamma)} \quad (2.34)$$

and the steady state condition (2.33) reduces to :

$$\frac{\lambda}{\mu} < \frac{\gamma}{\alpha + \gamma} \quad (2.35)$$

Further, if the service channel does not fail at all, then setting  $\alpha = 0$  in equation (2.32) or (2.34) yields :

$$W_0 = \frac{\mu \gamma - \lambda \gamma}{\mu \gamma} = 1 - \frac{\lambda}{\mu} \quad (2.36)$$

and the steady state (2.33) or (2.35) reduces to :

$$\frac{\lambda}{\mu} < 1 \quad (2.37)$$

The results (2.36) and (2.37) are the well-known results of the system M/M/1.

## 2.6 MEAN NUMBER IN THE SYSTEM

Let  $L$  denote the steady state average number in the system.

Then,

$$L = \sum_{n=0}^{\infty} nP_n = \frac{d}{dz} P(z) \text{ at } z = 1 \quad (2.38)$$

which, on using (2.31) and simplifying, yields

$$L = \frac{\lambda\mu\alpha (\beta^2 + \beta\gamma + \gamma^2) + \lambda (\alpha\beta + \beta\gamma + \gamma\alpha)^2}{[\mu\beta\gamma - \lambda(\alpha\beta + \beta\gamma + \gamma\alpha)] [\alpha\beta + \beta\gamma + \gamma\alpha]} \quad (2.39)$$

If the delay time is zero, then  $\frac{1}{\beta} \rightarrow 0$  and hence equation (2.39) gives

$$L = \frac{\lambda\mu\alpha + \lambda (\alpha + \gamma)^2}{[\mu\gamma - \lambda(\alpha + \gamma)] (\alpha + \gamma)} \quad (2.40)$$

If there are no failures at all, then setting  $\alpha = 0$  in (2.40). We have

$$L = \frac{\lambda}{\mu - \lambda} \quad (2.41)$$

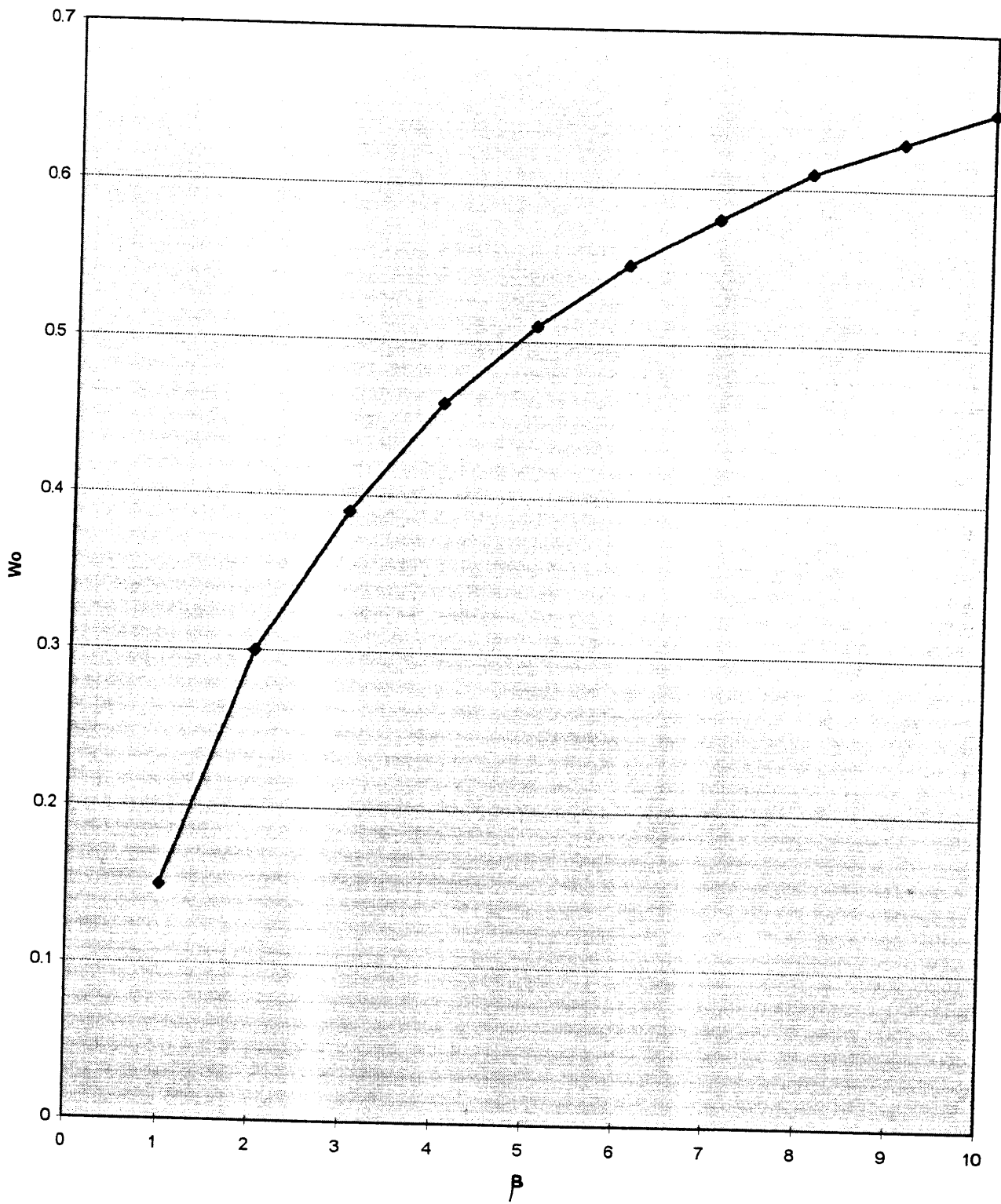
The result (2.41) is a well-known result of the system M/M/1.

## 2.7 DEPENDENCE OF $W_0$ AND L ON DELAY PARAMETER ( $\beta$ )

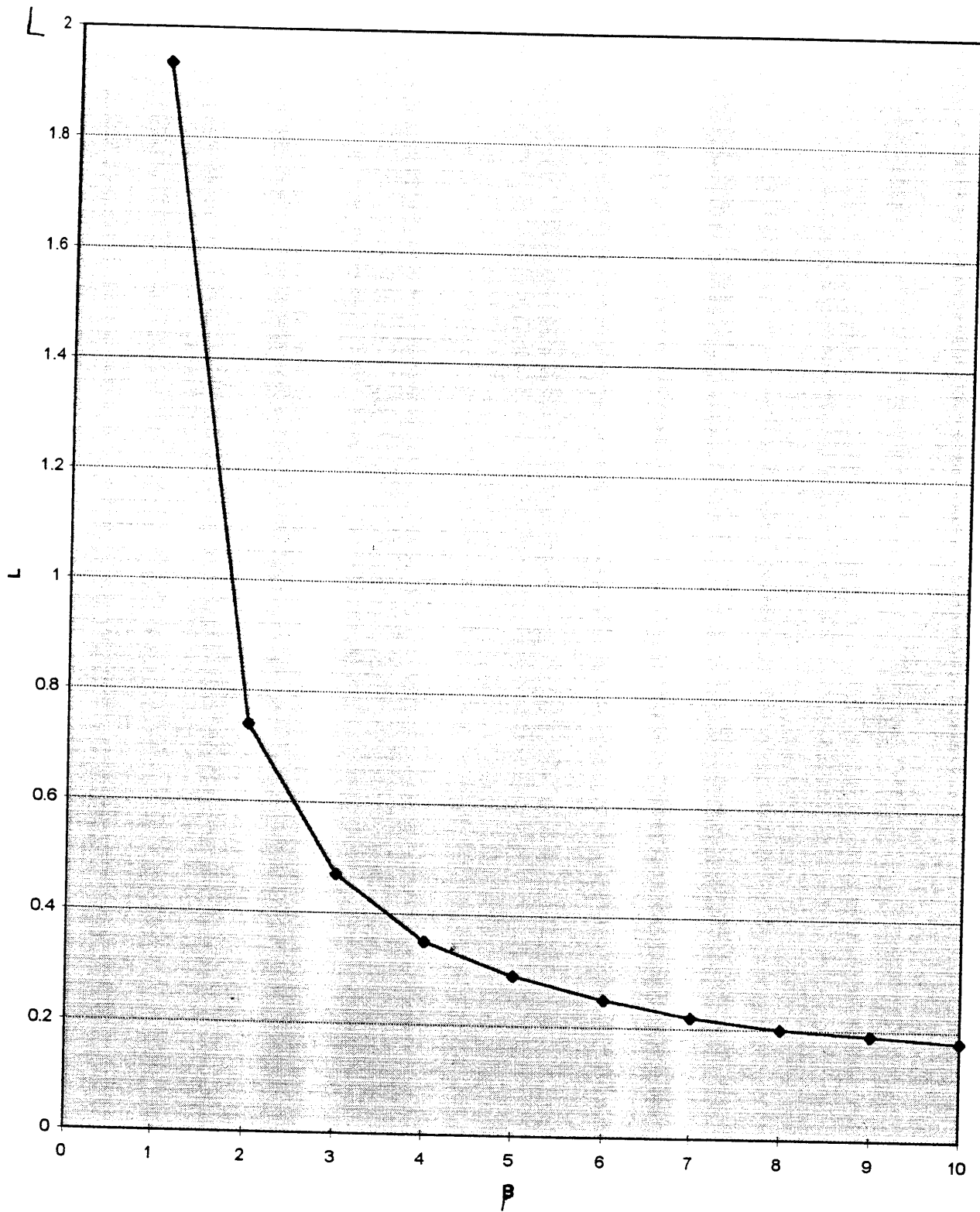
We have chosen some arbitrary values of  $\lambda, \mu, \alpha, \gamma$  and  $\beta$  such that this set of values satisfy the steady state condition given by (2.33) and evaluated the steady state probability  $W_0$  (Idle state) and L (the mean number in the system) using equation (2.32)

| $\lambda$ | $\mu$ | $\alpha$ | $\gamma$ | $\beta$ | $W_0$ | L     |
|-----------|-------|----------|----------|---------|-------|-------|
| 1         | 10    | 3        | 100      | 1       | 0.148 | 1.93  |
|           |       |          |          | 2       | 0.295 | 0.744 |
|           |       |          |          | 3       | 0.393 | 0.467 |
|           |       |          |          | 4       | 0.462 | 0.350 |
|           |       |          |          | 5       | 0.513 | 0.287 |
|           |       |          |          | 6       | 0.554 | 0.249 |
|           |       |          |          | 7       | 0.586 | 0.224 |
|           |       |          |          | 8       | 0.612 | 0.206 |
|           |       |          |          | 9       | 0.633 | 0.192 |
|           |       |          |          | 10      | 0.652 | 0.182 |

Graph of  $W_o$



Graph of L



## SECTION 3

### BATCH ARRIVALS AND GENERAL DELAY TIMES

#### 3.1 INTRODUCTION

In this section we generalise the mathematical model discussed in section 2 in the sense that instead of single arrivals we have now assumed that units arrive at the system in batches of variable size. Let  $\lambda c_i dt$  ( $i = 1, 2, 3, \dots$ ) denote the first order probability that a batch of  $i$  units arrives in the short interval of time  $dt$ , where  $0 \leq c_i \leq 1$  and  $\sum_{i=1}^{\infty} c_i = 1$  and  $\lambda > 0$  is the mean arrival rate of batches. In addition, instead of exponential delay times, we have assumed that the delay times of the failed channel follow a general (arbitrary) distribution with the distribution function  $B(x)$ .

#### 3.2 DEFINITIONS AND EQUATIONS GOVERNING THE SYSTEM

Let  $D_n(x, t)$  be the probability that at time  $t$  there are  $n$  ( $\geq 0$ ) units in the system and the service channel is down and waiting for repairs to start with elapsed waiting time lying between  $x$  and  $x + dx$ .

The system is governed by the following set of difference – differential equations:

$$\frac{d}{dt} W_n(t) + (\lambda + \mu + \alpha) W_n(t) = \sum_{i=1}^n \lambda c_i W_{n-i}(t) + \mu W_{n+1}(t) + \gamma R_n(t) \quad (n > 0) \quad (3.1)$$

$$\frac{d}{dt} W_0(t) + (\lambda + \alpha) W_0(t) = \mu W_1(t) + \gamma R_0(t) \quad (3.2)$$

$$\frac{\partial}{\partial x} D_n(x, t) + \frac{\partial}{\partial t} D_n(x, t) + (\lambda + \beta(x)) D_n(x, t) = \sum_{i=1}^n \lambda c_i D_{n-i}(x, t) \quad (n > 0) \quad (3.3)$$

$$\frac{\partial}{\partial x} D_0(x,t) + \frac{\partial}{\partial t} D_0(x,t) + (\lambda + \beta(x))D_0(x,t) = 0 \quad (3.4)$$

$$\frac{d}{dt} R_n(t) + (\lambda + \gamma) R_n(t) = \sum_{i=1}^n \lambda c_i R_{n-i}(t) + \int_0^\infty D_n(x,t) \beta(x) dx \quad (n > 0) \quad (3.5)$$

$$\frac{d}{dt} R_0(t) + (\lambda + \gamma) R_0(t) = \int_0^\infty D_0(x,t) \beta(x) dx \quad (n > 0) \quad (3.6)$$

where  $\beta(x)$  is the first order probability that the repairs will start in time  $x$  and  $x + dx$  conditioned that the same had not started till time  $x$  and is related to  $B(x)$  by the relation

$$B(x) = \beta(x) e^{-\int_0^x \beta(x) dx} \quad (3.7)$$

The above equations are to be solved subject to the boundary condition

$$D_n(0,t) = \alpha W_n(t) \quad (n \geq 0) \quad (3.8)$$

### 3.3 TIME - DEPENDENT SOLUTION

We take Laplace transform of equations (3.1) to (3.6) and (3.8) and using (2.7) to (2.9) we have :

$$(s + \lambda + \mu + \alpha) W_n(s) = \sum_{i=1}^n \lambda c_i W_{n-i}(s) + \mu W_{n+1}(s) + \gamma R_n(s) + \delta_{n,i} \quad (n > 0) \quad (3.9)$$

$$(s + \lambda + \alpha) W_0(s) = \mu W_1(s) + \gamma R_0(s) + \delta_{0,i} \quad (3.10)$$

$$\frac{\partial}{\partial x} D_n(x,s) + (s + \lambda + \beta(x)) D_n(x,s) = \sum_{i=1}^n \lambda c_i D_{n-i}(x,s) \quad (n > 0) \quad (3.11)$$

$$\frac{\partial}{\partial x} D_0(x,s) + (s + \lambda + \beta(x)) D_0(x,s) = 0 \quad (3.12)$$

$$(s + \lambda + \gamma) R_n(s) = \sum_{i=1}^n \lambda c_i R_{n-i}(s) + \int_0^\infty D_n(x,s) \beta(x) dx \quad (n > 0) \quad (3.13)$$

$$(s + \lambda + \gamma) R_0(s) = \int_0^\infty D_0(x,s) \beta(x) dx \quad (3.14)$$

$$D_n(0,s) = \alpha W_n(s) \quad (n \geq 0) \quad (3.15)$$

In addition to the definitions in equation (2.14), we define the following :

$$D(x,s,z) = \sum_{n=0}^{\infty} z^n D_n(x,s) \quad , \quad C(z) = \sum_{i=1}^{\infty} z^i c_i \quad (3.16)$$

Where  $D_n(s)$  is the Laplace transform of the probability that at time  $t$  there are  $n$  units in the system and the service channel is down and waiting for repairs to start irrespective of the values of  $x$  so that

$$D_n(s) = \int_0^{\infty} D_n(x,s) dx \quad (3.17)$$

and therefore

$$D(s,z) = \int_0^{\infty} D(x,s,z) dx \quad (3.18)$$

We perform  $\sum_{n=1}^{\infty} z^{n+1}$  times equation (3.9) +  $z$  times equation (3.10), use equations (2.14) and (3.16). Thus we have

$$((s + \lambda + \mu + \alpha - \lambda C(z))z - \mu) W(s,z) = \gamma z R(s,z) + \mu(z - 1) W_0(s) + z^{i+1} \quad (3.19)$$

Performing the operation  $\sum_{n=1}^{\infty} z^n$  times equation (3.11) + equation (3.12), make use of (2.14), (3.16) and simplify

$$\frac{\partial}{\partial x} D(x,s,z) + (s + \lambda + \beta(x) - \lambda C(z)) D(x,s,z) = 0 \quad (3.20)$$

We perform  $\sum_{n=1}^{\infty} z^n$  times equation (3.13) + equation (3.14) make use of (2.14), (3.16) and simplify

$$(s + \lambda + \gamma - \lambda C(z)) R(s,z) = \int_0^{\infty} D(x,s,z) \beta(x) dx \quad (3.21)$$

Performing  $\sum_{n=0}^{\infty} z^n$  times equation (3.15), make use of (2.14), (3.16) and simplify, Thus we have

$$D(0,s,z) = \alpha W(s,z) \quad (3.22)$$

We integrate equation (3.20) between the limits 0 and x, so we have

$$D(x,s,z) = D(0,s,z) e^{[-(s + \lambda - \lambda C(z))x - \int_0^x \beta(x) dx]} \quad (3.23)$$

On using equation (3.22), equation (3.23) becomes :

$$D(x,s,z) = \alpha W(s,z) e^{[-(s + \lambda - \lambda C(z))x - \int_0^x \beta(x) dx]} \quad (3.24)$$

We integrate equation (3.24) by parts with respect to x, use equation (3.7) and (3.18). Thus we have :

$$D(s,z) = \alpha W(s,z) \left[ \frac{1 - B(s + \lambda - \lambda C(z))}{(s + \lambda - \lambda C(z))} \right] \quad (3.25)$$

Where

$$B(s + \lambda - \lambda C(z)) = \int_0^{\infty} e^{-(s + \lambda - \lambda C(z))x} B(x) dx \quad (3.26)$$

Let us substitute the value of  $D(x,s,z)$  from equation (3.24) in equation (3.21). Thus we have, on simplifying,

$$(s + \lambda - \lambda C(z) + \gamma) R(s,z) = \alpha W(s,z) \int_0^\infty e^{[-(s + \lambda - \lambda C(z))x - \int_0^x \beta(x) dx]} \beta(x) dx \quad (3.27)$$

Note that on using (3.7), we can write:

$$\int_0^\infty e^{[-(s + \lambda - \lambda C(z))x - \int_0^x \beta(x) dx]} \beta(x) dx = \int_0^\infty e^{-(s + \lambda - \lambda C(z))x} B(x) dx = B(s + \lambda - \lambda C(z)) \quad (3.28)$$

and therefore, equation (3.27) becomes :

$$R(s,z) = \left[ \frac{\alpha B(s + \lambda - \lambda C(z))}{(s + \lambda - \lambda C(z) + \gamma)} \right] W(s,z) \quad (3.29)$$

On using equation (3.29), equation (3.19) gives, on simplifying

$$W(s,z) = \frac{(s + \lambda - \lambda C(z) + \gamma) [z^{i+1} + \mu(z-1)W_0(s)]}{[(s + \lambda - \lambda C(z) + \mu + \alpha)z - \mu] [(s + \lambda - \lambda C(z) + \gamma) - \alpha\gamma z B(s + \lambda - \lambda C(z))]} \quad (3.30)$$

Then using equation (3.30), equation (3.25) can be re-written as :

$$D(s,z) = \frac{\alpha \left[ \frac{1 - B(s + \lambda - \lambda C(z))}{(s + \lambda - \lambda C(z))} \right] (s + \lambda - \lambda C(z) + \gamma) (z^{i+1} + \mu (z - 1) W_0(s))}{[(s + \lambda - \lambda C(z) + \mu + \alpha)z - \mu] [(s + \lambda - \lambda C(z) + \gamma) - \alpha \gamma z B(s + \lambda - \lambda C(z))]} \quad (3.31)$$

and (3.29) gives, on using (3.30)

$$R(s,z) = \frac{\alpha B(s + \lambda - \lambda C(z)) [z^{i+1} + \mu(z - 1) W_0(s)]}{[(s + \lambda - \lambda C(z) + \mu + \alpha)z - \mu] [(s + \lambda - \lambda C(z) + \gamma) - \alpha \gamma z B(s + \lambda - \lambda C(z))]} \quad (3.32)$$

It can be shown that the common denominator of the right hand side of equations (3.30), (3.31) and (3.32) has one zero inside the unit circle  $|z| = 1$  by applying Rouché's Theorem as follow

$$\text{Let } f(z) = [(s + \lambda - \lambda C(z) + \mu + \alpha)z - \mu] [s + \lambda - \lambda C(z) + \gamma]$$

$$g(z) = \alpha \gamma B(s + \lambda - \lambda C(z)) z$$

Then on  $|z| = 1$ , We have

$$\begin{aligned} |g(z)| &= |\alpha \gamma B(s + \lambda - \lambda C(z)) z| \\ &= |\alpha \gamma| |B(s + \lambda - \lambda C(z))| |z| \\ &= \alpha \gamma \end{aligned}$$

and

$$|f(z)| = |(s + \lambda - \lambda C(z) + \mu + \alpha)z - \mu| |s + \lambda - \lambda C(z) + \gamma|$$

$$\geq ((s + \lambda + \mu + \alpha) - \lambda |C(z)| - \mu) (s + \lambda + \gamma - \lambda |C(z)|)$$

$$\geq (s + \lambda + \alpha - \lambda) (s + \lambda + \gamma - \lambda)$$

$$= (s + \alpha) (s + \gamma)$$

$$> \alpha\gamma = |g(z)|$$

Thus on  $|z| = 1$ ,  $|f(z)| > |g(z)|$ . Hence by Rouché's Theorem  $f(z) - g(z)$  has one zero inside the unit circle  $|z| = 1$ , same as  $f(z)$ , which should enable us to determine the only unknown  $W_0(s)$  appearing in the numerator. Hence  $W(s,z)$ ,  $D(s,z)$ ,  $R(s,z)$  and, for that matter,  $P(s,z)$  can be completely determined.

Let the delay times for repairs to start be exponential with mean delay time  $1/\beta$  and  $c_1 = 1$ ,  $c_i = 0$  for  $i \neq 1$ . Then we have

$$B(s + \lambda - \lambda C(z)) = \frac{\beta}{s + \lambda - \lambda z + \beta} \quad (3.33)$$

With this substitution, equations (3.30), (3.31) and (3.32) would become:

$$W(s,z) = \frac{(s + \lambda - \lambda z + \beta) (s + \lambda - \lambda z + \gamma) [z^{i+1} + \mu(z-1)W_0(s)]}{[(s + \lambda - \lambda z + \mu + \alpha) z - \mu] [s + \lambda - \lambda z + \beta] [s + \lambda - \lambda z + \gamma] - \alpha\beta\gamma z} \quad (3.34)$$

$$D(s,z) = \frac{\alpha (s + \lambda - \lambda z + \gamma) [z^{i+1} + \mu(z-1)W_0(s)]}{[(s + \lambda - \lambda z + \mu + \alpha) z - \mu] [s + \lambda - \lambda z + \beta] [s + \lambda - \lambda z + \gamma] - \alpha\beta\gamma z} \quad (3.35)$$

$$R(s,z) = \frac{\alpha\beta [z^{i+1} + \mu(z-1)W_0(s)]}{[(s + \lambda - \lambda z + \mu + \alpha) z - \mu] [s + \lambda - \lambda z + \beta] [s + \lambda - \lambda z + \gamma] - \alpha\beta\gamma z} \quad (3.36)$$

The unknown  $W_0(s)$  can be determined as before.

The results in equations (3.34), (3.35) and (3.36) agree with the results of section 2.

## SECTION 4

### AN M/M/1 QUEUEING SYSTEM WITH RANDOM FAILURES AND TWO PHASE REPAIRS

#### 4.1 INTRODUCTION

This section deals with a single server queueing system in which units arrive at the system one by one according to Poisson stream. The service is rendered one by one with exponential service times. However, as soon as the service channel fails it has to go to two phases of repairs, phase 1 and phase 2, with different repair rates.

#### 4.2 THE UNDERLYING ASSUMPTIONS

The following assumptions briefly describe the mathematical model :

- Units arrive at the system one by one. There are single arrivals. Let  $\lambda dt$  denote the first order probability of an arrival in the short interval of time  $dt$ , where  $\lambda > 0$  is the mean arrival rate.
- Let the service times of successive arrivals be distributed exponentially with mean service time  $\frac{1}{\mu}$  ( $\mu > 0$ ).
- Let  $\zeta dt$  be the first order probability that the service channel will fail during the short interval of time  $dt$ . We have further assumed that failures occur only while the service channel is providing service.
- As soon as the service channel fails, it stops providing service and enters phase 1 of repairs after the completion of which it goes in phase 2 of repairs. As soon as the repairs are complete the service channel instantaneously starts operating.
- Let  $\eta_j dt$  be the first order probability that  $j$  th phase of repairs of the service channel will be complete during the short interval of time  $dt$  ( $j = 1, 2$ ).
- Various stochastic processes involved in the system are independent of each other.

### 4.3 EQUATIONS GOVERNING THE SYSTEM

We define:

$W_n(t)$  : as the probability that at time  $t$  there are  $n$  ( $n \geq 0$ ) units in the system (excluding the unit in service, if any).

$R_n^{(j)}(t)$  : as the probability that at time  $t$  there are  $n$  ( $n \geq 0$ ) units in the system excluding the unit with suspended service, if any, and the service channel is in  $j$  th phase of repairs ( $j = 1, 2$ ).

$Q(t)$  : as the probability that at time  $t$  there is no unit either in the queue or in service i.e. the service channel, though operative, is idle.

$P_n(t)$  : as the probability that there are  $n$  ( $n \geq 0$ ) units in the queue irrespective of the state of the service channel so that

$$P_n(t) = W_n(t) + R_n^{(1)}(t) + R_n^{(2)}(t) \quad (n \geq 0)$$

The system is governed by the following set of difference- differential equations :

$$\frac{d}{dt} W_n(t) + (\lambda + \mu + \zeta) W_n(t) = \lambda W_{n-1}(t) + \mu W_{n+1}(t) + \eta_2 R_n^{(2)}(t) \quad (n > 0) \quad (4.1)$$

$$\frac{d}{dt} W_0(t) + (\lambda + \mu + \zeta) W_0(t) = \lambda Q(t) + \mu W_1(t) + \eta_2 R_0^{(2)}(t) \quad (4.2)$$

$$\frac{d}{dt} R_n^{(1)}(t) + (\lambda + \eta_1) R_n^{(1)}(t) = \lambda R_{n-1}^{(1)}(t) + \zeta W_n(t) \quad (n > 0) \quad (4.3)$$

$$\frac{d}{dt} R_0^{(1)}(t) + (\lambda + \eta_1) R_0^{(1)}(t) = \zeta W_0(t) \quad (4.4)$$

$$\frac{d}{dt} R_n^{(2)}(t) + (\lambda + \eta_2) R_n^{(2)}(t) = \lambda R_{n-1}^{(2)}(t) + \eta_1 R_n^{(1)}(t) \quad (n > 0) \quad (4.5)$$

$$\frac{d}{dt} R_0^{(2)}(t) + (\lambda + \eta_2) R_0^{(2)}(t) = \eta_1 R_0^{(1)}(t) \quad (4.6)$$

$$\frac{d}{dt} Q(t) + (\lambda + \zeta) Q(t) = \mu W_0(t) \quad (4.7)$$

We assume that initially the system starts when there are  $m$  units in the system and the service channel is in working order so that the initial condition is :

$$W_n(0) = \delta_{n,m}$$

Where  $\delta_{n,m}$  is the Kronecker delta (4.8)

$$\begin{aligned} \text{i.e} \quad \delta_{n,m} &= 1 & \text{if } n &= m \\ &= 0 & \text{if } n &\neq m \end{aligned}$$

We take Laplace transform of equations (4.1) to (4.7). We thus have :

$$(s + \lambda + \mu + \zeta) W_n(s) = \delta_{n,m} + \lambda W_{n-1}(s) + \mu W_{n+1}(s) + \eta_2 R_n^{(2)}(s) \quad (n > 0) \quad (4.9)$$

$$(s + \lambda + \mu + \zeta) W_0(s) = \delta_{0,m} + \lambda Q(s) + \mu W_1(s) + \eta_2 R_0^{(2)}(s) \quad (4.10)$$

$$(s + \lambda + \eta_1) R_n^{(1)}(s) = \lambda R_{n-1}^{(1)}(s) + \zeta W_n(s) \quad (n > 0) \quad (4.11)$$

$$(s + \lambda + \eta_1) R_0^{(1)}(s) = \zeta W_0(s) \quad (4.12)$$

$$(s + \lambda + \eta_2) R_n^{(2)}(s) = \lambda R_{n-1}^{(2)}(s) + \eta_1 R_n^{(1)}(s) \quad (n > 0) \quad (4.13)$$

$$(s + \lambda + \eta_2) R_0^{(2)}(s) = \eta_1 R_0^{(1)}(s) \quad (4.14)$$

$$(s + \lambda + \zeta) Q(s) = \mu W_0(s) \quad (4.15)$$

Equations (4.18) and (4.19) yield :

$$R^{(1)}(s,z) = \frac{\zeta W(s,z)}{(s + \lambda + \eta_1 - \lambda z)} \quad (4.20)$$

and

$$R^{(2)}(s,z) = \frac{\zeta \eta_1 W(s,z)}{(s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z)} \quad (4.21)$$

Substituting for  $R^{(2)}(s,z)$  from equation (4.21) in equation (4.17), We have :

$$W(s,z) = \frac{z^{m+1} - \mu W_0(s) + \lambda z Q(s) + \eta_2 z R^{(2)}(s,z)}{[(s + \lambda + \mu + \zeta - \lambda z) z - \mu]} \quad (4.21)$$

$$W(s,z) = \frac{N}{D} \quad (4.22)$$

Where

$$N = z^{m+1} - \mu W_0(s) + \lambda z Q(s) \quad (4.23)$$

$$D = [(s + \lambda + \mu + \zeta - \lambda z) z - \mu] - \frac{\eta_1 \eta_2 \zeta z}{(s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z)} \quad (4.24)$$

$$W(s,z) = \frac{[z^{m+1} - \mu W_0(s) + \lambda z Q(s)] (s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z)}{[(s + \lambda + \mu + \zeta - \lambda z) z - \mu] (s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z) - \eta_1 \eta_2 \zeta z} \quad (4.25)$$

In order to determine the unknowns  $Q(s)$  and  $W_0(s)$  appearing in the numerator of the right hand side of equation (4.25) we shall apply Rouché's Theorem

$$\text{Let } f(z) = [(s + \lambda + \mu + \zeta - \lambda z)z - \mu] (s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z)$$

$$g(z) = \eta_1 \eta_2 \zeta z$$

Both  $f(z)$  and  $g(z)$  are differentiable inside and continuous on the contour  $|z| = 1$ . Also, on  $|z| = 1$ ,

$$|f(z)| = |(s + \lambda + \mu + \zeta - \lambda z)z - \mu| |s + \lambda + \eta_1 - \lambda z| |s + \lambda + \eta_2 - \lambda z|$$

$$\geq (|s + \lambda + \mu + \zeta - \lambda z| - \mu) (s + \lambda + \eta_1 - \lambda) (s + \lambda + \eta_2 - \lambda)$$

$$\geq ((s + \lambda + \mu + \zeta - \lambda) - \mu) (s + \eta_1) (s + \eta_2)$$

$$= (s + \zeta) (s + \eta_1) (s + \eta_2)$$

$$> \eta_1 \eta_2 \zeta = |g(z)|$$

Thus on  $|z| = 1$ ,  $|f(z)| > |g(z)|$ . Hence by Rouché's Theorem  $f(z) - g(z)$  has the same number of zeros as that of  $f(z)$  inside the unit circle  $|z| = 1$ . Clearly  $f(z)$  has one zero inside the unit circle and, therefore,  $f(z) - g(z)$ , i.e. the denominator in the right hand side of equation (4.26) has one zero inside the unit circle  $|z| = 1$ .

Since  $W(s, z)$  is regular inside the unit circle  $|z| = 1$ , the numerator  $N$  must vanish for the zero of  $D$ , along with equation (4.15) are sufficient to determine the unknowns  $Q(s)$  and  $W_0(s)$ . Hence  $W(s, z)$  and, for that matter  $R^{(1)}(s, z)$  and  $R^{(2)}(s, z)$  can be completely determined.

## 4.5 THE STEADY STATE SOLUTION

let  $W_n, R_n^{(1)}, R_n^{(2)}$  and  $Q$  be the respective steady state probabilities corresponding to  $W_n(t), R_n^{(1)}(t), R_n^{(2)}(t)$  and  $Q(t)$  and for that matter  $W(z), R^{(1)}(z)$  and  $R^{(2)}(z)$  are the probability generating functions corresponding to  $W(s,z), R^{(1)}(s,z)$  and  $R^{(2)}(s,z)$ . Then the steady state solution can be obtained by using the Tauberian property (1.6).

Thus equation (4.22) yields

$$W(z) = \frac{N}{D} \quad (4.26)$$

Where

$$N = \lambda z Q - \mu W_0 \quad (4.27)$$

$$D = [(\lambda + \mu + \zeta - \lambda z) z - \mu] - \frac{\eta_1 \eta_2 \zeta z}{(\lambda + \eta_1 - \lambda z)(\lambda + \eta_2 - \lambda z)} \quad (4.28)$$

Similarly equations (4.20) and (4.21) respectively yield

$$R^{(1)}(z) = \frac{\zeta W(z)}{(\lambda + \eta_1 - \lambda z)} \quad (4.29)$$

$$R^{(2)}(z) = \frac{\zeta \eta_1 W(z)}{(\lambda + \eta_1 - \lambda z)(\lambda + \eta_2 - \lambda z)} \quad (4.30)$$

From equation (4.26)

$$W(z) = \frac{(\lambda z Q - \mu W_0)(\lambda + \eta_1 - \lambda z)(\lambda + \eta_2 - \lambda z)}{[(\lambda + \mu + \zeta - \lambda z) z - \mu](\lambda - \lambda z + \eta_1)(\lambda + \eta_2 - \lambda z) - \zeta \eta_1 \eta_2 z} \quad (4.31)$$

Using Tauberian property (1.6), equation (4.15), in the steady state, yields  $\mu W_0 = \lambda Q$  with this relation equation (4.31) can be re-written as :

$$W(z) = \frac{\lambda(z-1) (\lambda + \eta_1 - \lambda z) (\lambda + \eta_2 - \lambda z) Q}{[(\lambda + \mu + \zeta - \lambda z) z - \mu] (\lambda - \lambda z + \eta_1) (\lambda + \eta_2 - \lambda z) - \zeta \eta_1 \eta_2 z} \quad (4.32)$$

With this value of  $W(z)$ , We have from equations (4.29) and (4.30)

$$R^{(1)}(z) = \frac{\zeta \lambda(z-1) (\lambda + \eta_2 - \lambda z) Q}{[(\lambda + \mu + \zeta - \lambda z) z - \mu] (\lambda + \eta_1 - \lambda z) (\lambda + \eta_2 - \lambda z) - \zeta \eta_1 \eta_2 z} \quad (4.33)$$

$$R^{(2)}(z) = \frac{\zeta \eta_1 \lambda(z-1) Q}{[(\lambda + \mu + \zeta - \lambda z) z - \mu] (\lambda - \lambda z + \eta_1) (\lambda + \eta_2 - \lambda z) - \eta_1 \eta_2 \zeta z} \quad (4.34)$$

To determine the only unknown  $Q$  we would use the normalising condition :

$$W(1) + R^{(1)}(1) + R^{(2)}(1) + Q = 1 \quad (4.35)$$

We observe from equations (4.32), (4.33) and (4.34) that each of  $W(z)$ ,  $R^{(1)}(z)$  and  $R^{(2)}(z)$  are indeterminate (of the  $\frac{0}{0}$  form) for  $z = 1$  and hence, on using L' Hopital's rule we have:

$$\begin{aligned} W(1) &= \lim_{z \rightarrow 1} W(z) \\ &= \frac{\lambda \eta_1 \eta_2 Q}{\mu \eta_1 \eta_2 - \lambda (\zeta \eta_1 + \eta_1 \eta_2 + \zeta \eta_2)} \end{aligned} \quad (4.36)$$

$$\begin{aligned}
R^{(1)}(1) &= \lim_{z \rightarrow 1} R^{(1)}(z) \\
&= \frac{\lambda \zeta \eta_2 Q}{\mu \eta_1 \eta_2 - \lambda (\zeta \eta_1 + \eta_1 \eta_2 + \zeta \eta_2)}
\end{aligned} \tag{4.37}$$

$$\begin{aligned}
R^{(2)}(1) &= \lim_{z \rightarrow 1} R^{(2)}(z) \\
&= \frac{\lambda \zeta \eta_1 Q}{\mu \eta_1 \eta_2 - \lambda (\zeta \eta_1 + \eta_1 \eta_2 + \zeta \eta_2)}
\end{aligned} \tag{4.38}$$

Using equations (4.36), (4.37) and (4.38) in equation (4.35) and simplifying, we have

$$Q = 1 - \frac{\lambda (\zeta \eta_1 + \eta_1 \eta_2 + \zeta \eta_2)}{\mu \eta_1 \eta_2} \tag{4.39}$$

This is the steady state probability that the service channel is in working state but idle. We note that the utilisation factor is

$$\rho = \frac{\lambda (\zeta \eta_1 + \eta_1 \eta_2 + \zeta \eta_2)}{\mu \eta_1 \eta_2} \tag{4.40}$$

and that the steady state condition is given by

$$\frac{\lambda}{\mu} < \frac{\eta_1 \eta_2}{\zeta \eta_1 + \eta_1 \eta_2 + \eta_2 \zeta} \tag{4.41}$$

If there are no failures in the service channel then setting  $\zeta = 0$  in equations (4.39), (4.40) and (4.41), we have

$$Q = 1 - \frac{\lambda}{\mu} \quad (4.42)$$

$$\rho = \frac{\lambda}{\mu} \quad (4.43)$$

$$\frac{\lambda}{\mu} < 1 \quad (4.44)$$

The results (4.42), (4.43) and (4.44) are the well - known results of the M/M/1 system.

Now, in addition to the previous assumptions, if the repair rates in the two phases are equal, then setting  $\eta_1 = \eta_2 = \eta$  in equations (4.32), (4.33), (4.34) and (4.39) we have the corresponding steady state results as follows

$$W(z) = \frac{\lambda(z-1) (\lambda - \lambda z + \eta)^2 Q}{[(\lambda - \lambda z + \mu + \zeta) z - \mu](\lambda - \lambda z + \eta)^2 - \zeta \eta^2 z} \quad (4.45)$$

$$R^{(1)}(z) = \frac{\lambda(z-1) \zeta (\lambda - \lambda z + \eta) Q}{[(\lambda - \lambda z + \mu + \zeta) z - \mu](\lambda - \lambda z + \eta)^2 - \zeta \eta^2 z} \quad (4.46)$$

$$R^{(2)}(z) = \frac{\lambda(z-1) \zeta \eta Q}{[(\lambda - \lambda z + \mu + \zeta) z - \mu](\lambda - \lambda z + \eta)^2 - \zeta \eta^2 z} \quad (4.47)$$

where

$$Q = 1 - \frac{\lambda(2\zeta + \eta)}{\mu\eta} \quad (4.48)$$

And from equation (4.41) we have the steady state condition as :

$$\frac{\lambda}{\mu} < \frac{\eta}{2\zeta + \eta} \quad (4.49)$$

Now we proceed to find the mean queue length.

## 4.6 THE MEAN QUEUE LENGTH

On adding equation (4.45), (4.46) and (4.47) and using the value of Q from (4.48) we have

$$P(z) = \frac{\lambda(z-1) [(\lambda - \lambda z + \eta)^2 + \zeta (\lambda - \lambda z + \eta) + \zeta \eta] \left( 1 - \frac{\lambda(2\zeta + \eta)}{\mu\eta} \right)}{[(\lambda - \lambda z + \mu + \zeta) z - \mu] (\lambda - \lambda z + \eta)^2 - \zeta \eta^2 z} \quad (4.50)$$

Let us denote the denominator of the right hand side of equation (4.50) by D(z). We can see that since D(1) = 0, it follows that (z - 1) is a factor of D(z) and hence by synthetic division we can write

$$\begin{aligned} D(z) = (z-1) & [-\lambda^3 z^3 + \lambda^2 (2\lambda + \mu + \zeta + 2\eta) z^2 - \{ \lambda^3 + \lambda^2 (2\mu + \zeta + 2\eta) \\ & + \lambda \eta (2\mu + 2\zeta + \eta) \} z + (\lambda + \eta)^2 \mu] \end{aligned} \quad (4.51)$$

Cancelling out the factor (z-1) with that of the numerator we can finally write equation (4.50) as :

$$P(z) = \frac{\lambda [(\lambda - \lambda z + \eta)^2 + \zeta (\lambda - \lambda z + \eta) + \zeta \eta] \left( 1 - \frac{\lambda(2\zeta + \eta)}{\mu\eta} \right)}{[-\lambda^3 z^3 + \lambda^2 (2\lambda + \mu + \zeta + 2\eta) z^2 - \{ \lambda^3 + \lambda^2 (2\mu + \zeta + 2\eta) + \lambda \eta (2\mu + 2\zeta + \eta) \} z + (\lambda + \eta)^2 \mu]} \quad (4.52)$$

Let  $L$  denote the mean queue length (excluding those being served or in suspended service), then

$$L = \sum_{n=0}^{\infty} n P_n = \frac{d}{dz} P(z) \text{ at } z=1 \quad (4.53)$$

Evaluating the derivative at  $z = 1$  and simplifying we have, finally:

$$L = \frac{\lambda^2 [\zeta(3\mu + 4\eta + 4\zeta) + \eta^2]}{\mu\eta (\mu\eta - \lambda(2\zeta + \eta))} \quad (4.54)$$

If there are no failures in the service channel then setting  $\zeta = 0$  in equations (4.52) and (4.54) we have

$$P(z) = \frac{\lambda(z-1) \left(1 - \frac{\lambda}{\mu}\right)}{(\lambda - \lambda z + \mu) z - \mu} \quad (4.55)$$

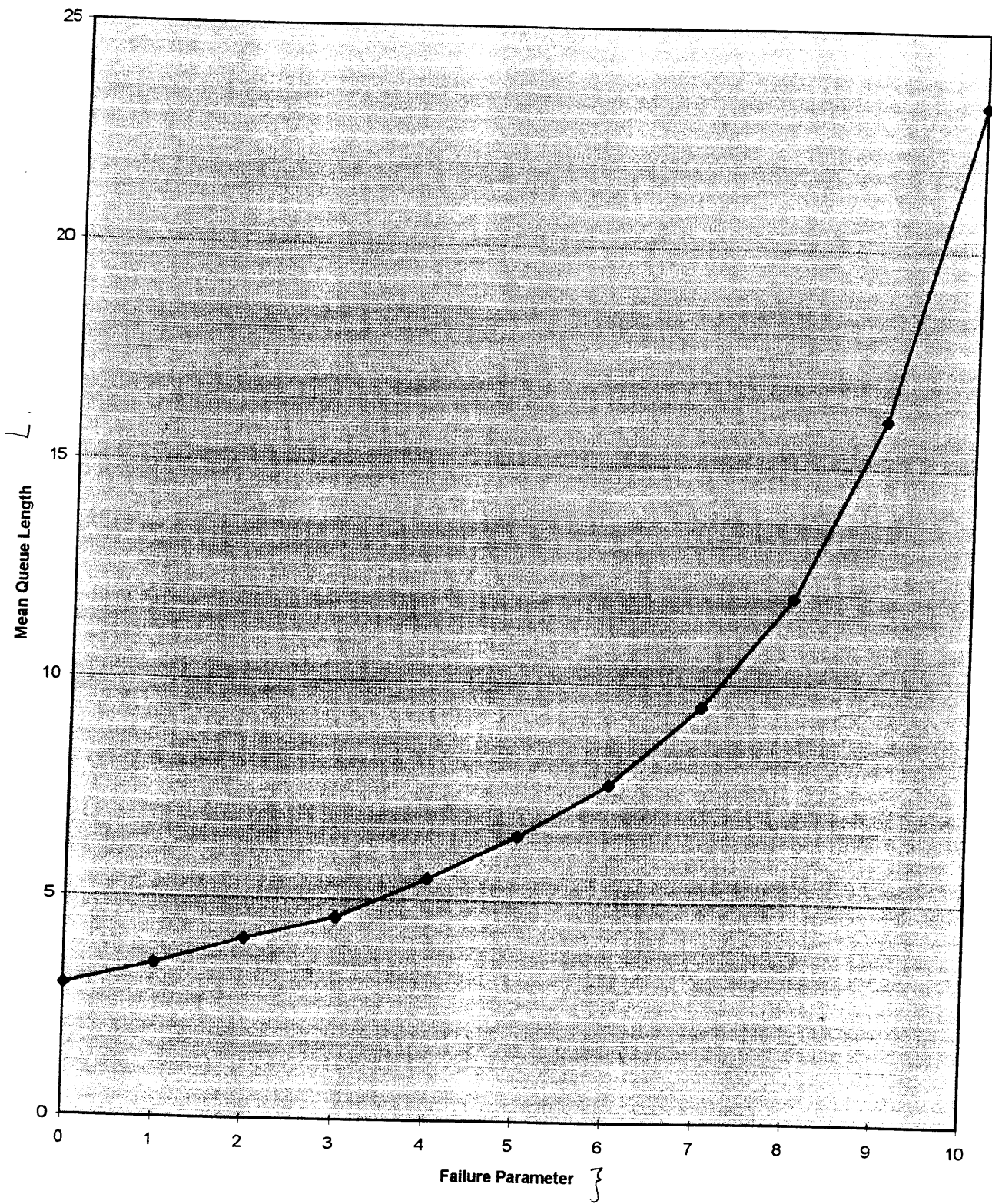
$$L = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\lambda^2/\mu^2}{1 - \frac{\lambda}{\mu}} = \frac{\rho^2}{1 - \rho} \quad (4.56)$$

#### 4.7 DEPENDENCE OF THE MEAN QUEUE LENGTH ON FAILURE AND REPAIR RATES

We have chosen some arbitrary values of  $\lambda$ ,  $\mu$ ,  $\zeta$  and  $\eta$  such that this set of values satisfy the steady state condition given by (4.49) and evaluated the mean queue length  $L$  using equation (4.54). To see the effect of failure and repair parameters, the following calculations for the mean queue length were obtained once by varying the failure parameter  $\zeta$ , keeping  $\lambda$ ,  $\mu$  and  $\eta$  fixed and then by varying the repair parameter  $\eta$ , keeping  $\lambda$ ,  $\mu$  and  $\zeta$  fixed.

| $\lambda$ | $\mu$ | $\eta$ | $\zeta$ | $L$      |
|-----------|-------|--------|---------|----------|
| 4         | 5     | 100    | 10      | 23.28    |
|           |       |        | 9       | 16.06743 |
|           |       |        | 8       | 12.06756 |
|           |       |        | 7       | 9.528    |
|           |       |        | 6       | 7.77477  |
|           |       |        | 5       | 6.493334 |
|           |       |        | 4       | 5.517177 |
|           |       |        | 3       | 4.749895 |
|           |       |        | 2       | 4.13181  |
|           |       |        | 1       | 3.624    |
|           |       |        | 0       | 3.2      |

Figure (1)



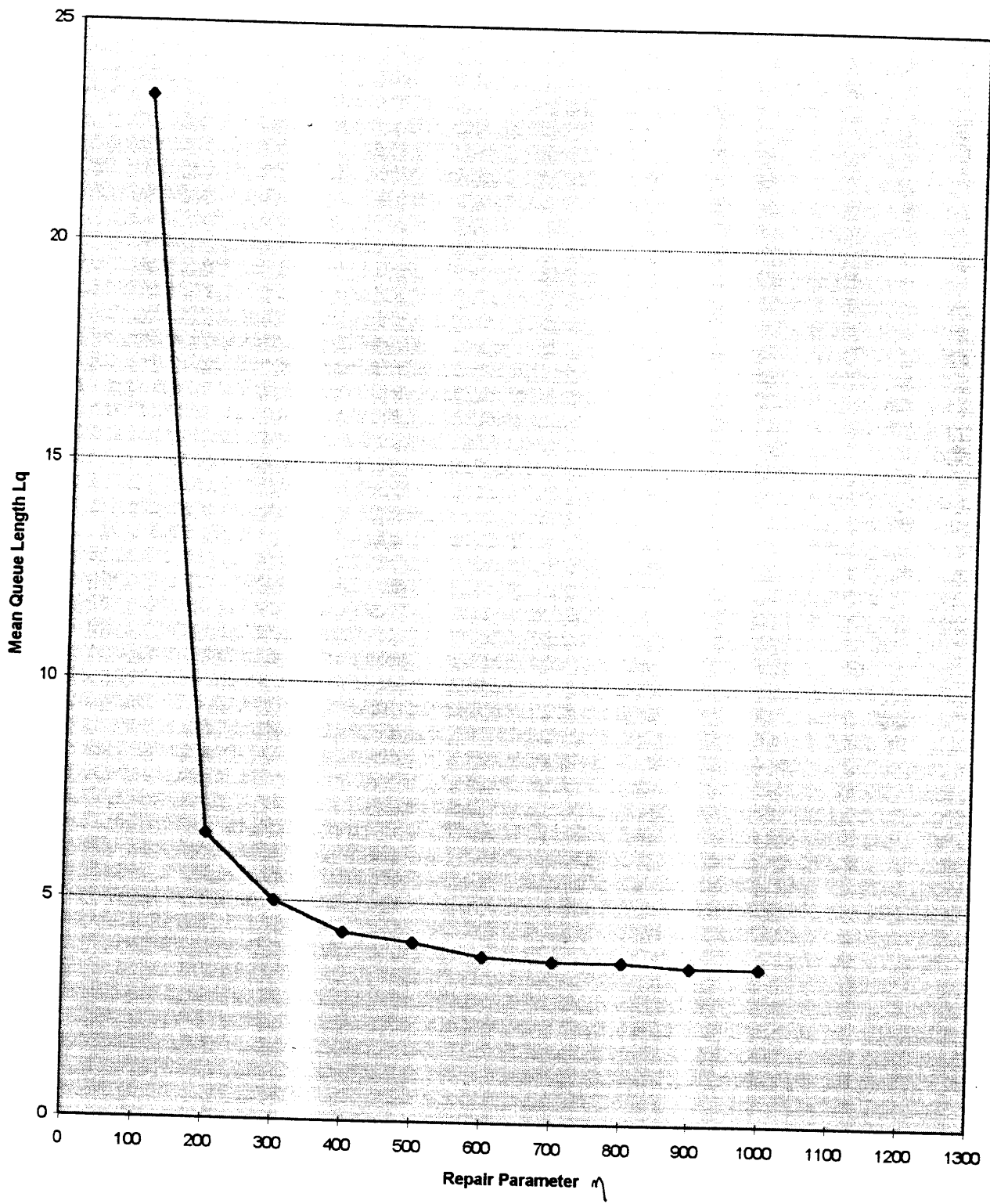
| $\lambda$ | $\mu$ | $\zeta$ | $\eta$  | L        |
|-----------|-------|---------|---------|----------|
| 4         | 5     | 10      | 100     | 23.28    |
|           |       |         | 200     | 6.473334 |
|           |       |         | 300     | 4.972121 |
|           |       |         | 400     | 4.41375  |
|           |       |         | 500     | 4.122667 |
|           |       |         | 600     | 3.944103 |
|           |       |         | 700     | 3.82341  |
|           |       |         | 800     | 3.736389 |
|           |       |         | 900     | 3.670678 |
|           |       |         | 1000    | 3.619305 |
|           |       |         | 10000   | 3.238728 |
|           |       |         | 100000  | 3.203843 |
|           |       |         | 1000000 | 3.200384 |

Finally, the dependence of the mean queue length on  $\zeta$  and  $\eta$  is shown graphically in figures 1 and 2. We can see from figure 1 that as the failure rate  $\zeta$  decreases the mean queue length goes on reducing finally approaching to 3.2, the steady state value which can be directly obtained by equation (4.56) as follows

$$L = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{4^2}{5(5-4)} = \frac{16}{5} = 3.2$$

Similarly figure 2 depicts that as the value of  $\eta$  increases (which means the repair times are shorter and shorter), the mean queue length goes on decreasing, also finally reducing to 3.2

Figure 2



## SECTION 5

### ARRIVALS IN BATCHES OF VARIABLE SIZE AND TWO-PHASE REPAIRS

#### 5.1 INTRODUCTION

In this section, we have considered a bulk input model in which arrivals occurs in batches of variable size with the first order probability  $\lambda c_i dt$  that during short interval of time  $dt$ , a batch of  $i$  units will arrive at the system, where  $0 \leq c_i \leq 1$ , and  $\sum_{i=1}^{\infty} c_i = 1$ .

#### 5.2 EQUATIONS GOVERNING THE SYSTEM

The system is governed by the following set of difference - differential equations:

$$\frac{d}{dt} W_n(t) + (\lambda + \mu + \zeta) W_n(t) = \sum_{i=1}^n \lambda c_i W_{n-i}(t) + \mu W_{n+1}(t) + \eta_2 R_n^{(2)}(t) \quad (n > 0) \quad (5.1)$$

$$\frac{d}{dt} W_0(t) + (\lambda + \mu + \zeta) W_0(t) = \lambda Q(t) + \mu W_1(t) + \eta_2 R_0^{(2)}(t) \quad (5.2)$$

$$\frac{d}{dt} R_n^{(1)}(t) + (\lambda + \eta_1) R_n^{(1)}(t) = \sum_{i=1}^n \lambda c_i R_{n-i}^{(1)}(t) + \zeta W_n(t) \quad (n > 0) \quad (5.3)$$

$$\frac{d}{dt} R_0^{(1)}(t) + (\lambda + \eta_1) R_0^{(1)}(t) = \zeta W_0(t) \quad (5.4)$$

$$\frac{d}{dt} R_n^{(2)}(t) + (\lambda + \eta_2) R_n^{(2)}(t) = \sum_{i=1}^n \lambda c_i R_{n-i}^{(2)}(t) + \eta_1 R_n^{(1)}(t) \quad (n > 0) \quad (5.5)$$

$$\frac{d}{dt} R_0^{(2)}(t) + (\lambda + \eta_2) R_0^{(2)}(t) = \eta_1 R_0^{(1)}(t) \quad (5.6)$$

$$\frac{d}{dt} Q(t) + (\lambda + \zeta) Q(t) = \mu W_0(t) \quad (5.7)$$

We assume that the initial condition (4.8) hold

We take Laplace transform of equations (5.1) to (5.7)

$$(s + \lambda + \mu + \zeta) W_n(s) = \delta_{n,m} + \sum_{i=1}^{\infty} \lambda c_i W_{n-i}(s) + \mu W_{n+1}(s) + \eta_2 R_n^{(2)}(s) \quad (n > 0) \quad (5.9)$$

$$(s + \lambda + \mu + \zeta) W_0(s) = \delta_{0,m} + \lambda Q(s) + \mu W_1(s) + \eta_2 R_0^{(2)}(s) \quad (5.10)$$

$$(s + \lambda + \eta_1) R_n^{(1)}(s) = \sum_{i=1}^{\infty} \lambda c_i R_{n-i}^{(1)}(s) + \zeta W_n(s) \quad (n > 0) \quad (5.11)$$

$$(s + \lambda + \eta_1) R_0^{(1)}(s) = \zeta W_0(s) \quad (5.12)$$

$$(s + \lambda + \eta_2) R_n^{(2)}(s) = \sum_{i=1}^n \lambda c_i R_{n-i}^{(2)}(s) + \eta_1 R_n^{(1)}(s) \quad (n > 0) \quad (5.13)$$

$$(s + \lambda + \eta_2) R_0^{(2)}(s) = \eta_1 R_0^{(1)}(s) \quad (5.14)$$

$$(s + \lambda + \zeta) Q(s) = \mu W_0(s) \quad (5.15)$$

### 5.3 THE PROBABILITY GENERATING FUNCTIONS OF THE QUEUE LENGTH

We define the probability generating functions same as in the last section (section 4). In

addition, we define  $C(z)$  as :  $C(z) = \sum_{i=1}^{\infty} c_i z^i$

We perform the operations  $z$  times equation (5.10) +  $\sum_{i=1}^n z^{n+1}$  times equation (5.9) and simplify:

$$(s + \lambda + \mu + \zeta) [zW_0(s) + \sum_{n=1}^{\infty} z^{n+1} W_n(s)] = z\delta_{0,m} + \sum_{n=1}^{\infty} z^{n+1} \delta_{n,m} + \lambda zQ(s) + \mu zW_1(s) + \eta_2 zR_0^{(2)}(s) \\ + \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \lambda c_i z^{n+1} W_{n-i}(s) + \mu \sum_{n=1}^{\infty} z^{n+1} W_{n+1}(s) + \eta_2 \sum_{n=1}^{\infty} z^{n+1} R_n^{(2)}(s)$$

$$(s + \lambda + \mu + \zeta) zW(s, z) = z^{m+1} + \lambda zQ(s) + \mu W(s, z) - \mu W_0(s) + \eta_2 zR^{(2)}(s, z) \\ + \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \lambda c_i W_{n-i}(s) z^i z^{n-i+1}$$

$$(s + \lambda + \mu + \zeta) zW(s, z) = z^{m+1} + \lambda zQ(s) + \mu W(s, z) - \mu W_0(s) + \eta_2 zR^{(2)}(s, z) + \\ + \lambda \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} c_i z^i W_{n-i}(s) z^{n-i+1}$$

$$[(s + \lambda + \mu + \zeta) z - \mu] W(s, z) = z^{m+1} + \lambda zQ(s) - \mu W_0(s) + \eta_2 zR^{(2)}(s, z) + \lambda \sum_{n=1}^{\infty} C(z) W_{n-i}(s) z^{n-i+1}$$

We claim that  $C(z) W(s, z) = \sum_{n=1}^{\infty} C(z) W_{n-i}(s) z^{n-i}$

$$[(s + \lambda + \mu + \zeta) z - \mu] W(s, z) = z^{m+1} + \lambda zQ(s) - \mu W_0(s) + \eta_2 zR^{(2)}(s, z) + \lambda zC(z)W(s, z)$$

$$[(s + \lambda + \mu + \zeta - \lambda C(z)) z - \mu] W(s, z) = z^{m+1} + \lambda zQ(s) - \mu W_0(s) + \eta_2 zR^{(2)}(s, z) \quad (5.16)$$

Performing the operation  $\sum_{n=1}^{\infty} z^n$  times equation (5.11) + equation (5.12), and simplify

$$(s + \lambda + \eta_1) \sum_{n=1}^{\infty} z^n R_n^{(1)}(s) + (s + \lambda + \eta_1) R_0^{(1)}(s) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} z^n \lambda c_i R_{n-i}^{(1)}(s) + \zeta \sum_{n=1}^{\infty} z^n W_n(s)$$

$$(s + \lambda + \eta_1) R^{(1)}(s, z) = \lambda \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} z^i c_i z^{n-i} R_{n-i}^{(1)}(s) + \zeta W(s, z)$$

$$(s + \lambda + \eta_1) R^{(1)}(s, z) = \lambda \sum_{n=1}^{\infty} C(z) z^{n-i} R_{n-i}^{(1)}(s) + \zeta W(s, z)$$

$$\text{We claim that } \sum_{n=1}^{\infty} C(z) z^{n-i} R_{n-i}^{(1)}(s) = C(z) R(s, z)$$

$$[s + \lambda + \eta_1 - \lambda C(z)] R^{(1)}(s, z) = \zeta W(s, z) \quad (5.17)$$

And similar operation on equation (5.13) and (5.14) yields :

$$(s + \lambda - \lambda C(z) + \eta_2) R^{(2)}(s, z) = \eta_1 R^{(1)}(s, z) \quad (5.18)$$

Equations (5.17) and (5.18) yield:

$$R^{(1)}(s, z) = \frac{\zeta W(s, z)}{(s + \lambda + \eta_1 - \lambda C(z))} \quad (5.19)$$

$$R^{(2)}(s, z) = \frac{\eta_1 R^{(1)}(s, z)}{(s + \lambda - \lambda C(z) + \eta_2)}$$

$$R^{(2)}(s, z) = \frac{\eta_1 \zeta W(s, z)}{(s + \lambda + \eta_1 - \lambda C(z)) (s + \lambda - \lambda C(z) + \eta_2)} \quad (5.20)$$

substituting for  $R^{(2)}(s, z)$  from equation (5.20) in equation (5.16), we have:

$$\left[ [(s + \lambda + \mu + \zeta - \lambda C(z))z - \mu] \frac{\eta_2 z \zeta \eta_1}{[s + \lambda + \eta_1 - \lambda C(z)] [s + \lambda - \lambda C(z) + \eta_2]} \right] W(s, z)$$

$$= z^{m+1} + \lambda z Q(s) - \mu W_0(s)$$

So, we have

$$W(s, z) = \frac{N}{D} \quad (5.21)$$

where

$$N = z^{m+1} + \lambda z Q(s) - \mu W_0(s) \quad (5.22)$$

and

$$D = [(s + \lambda - \lambda C(z) + \mu + \zeta)z - \mu] - \frac{\zeta \eta_1 \eta_2 z}{[s + \lambda - \lambda C(z) + \eta_1][s + \lambda - \lambda C(z) + \eta_2]} \quad (5.23)$$

In order to determine the unknowns  $Q(s)$  and  $W_0(s)$  appearing in the numerator  $N$  of the right hand side of equation (5.21), we will apply Rouché's Theorem to show that  $D$  has only one zero as follow

Let

$$f(z) = [(s + \lambda - \lambda C(z) + \mu + \zeta)z - \mu][s + \lambda - \lambda C(z) + \eta_1][s + \lambda - \lambda C(z) + \eta_2] \text{ and} \\ g(z) = \eta_1 \eta_2 \zeta z$$

so,  $f(z)$  and  $g(z)$  are continuous and differentiable inside and on, the contour  $|z| = 1$ ,  
Also on  $|z| = 1$ , we have :

$$|f(z)| \geq ((s + \lambda + \mu + \zeta) - \lambda |C(z)| - \mu)(s + \lambda + \eta_1 - \lambda |C(z)|)(s + \lambda - \lambda |C(z)| + \eta_2)$$

$$= (s + \lambda + \zeta) - \lambda |C(\zeta)|)(s + \lambda + \eta_1 - \lambda |C(z)|)(s + \lambda - \lambda |C(z)| + \eta_2)$$

$$\geq (s + \lambda + \zeta - \lambda)(s + \lambda + \eta_1 - \lambda)(s + \lambda - \lambda + \eta_2)$$

$$= (s + \zeta)(s + \eta_1)(s + \eta_2) > \eta_1 \eta_2 \zeta = |g(z)|$$

so, by Rouché's theorem  $f(z) - g(z)$  has the same number of zeros as  $f(z)$ . Clearly  $f(z)$  has only one zero inside the unit circle  $|z|=1$ , so  $D$  has only one zero.

Since  $W(s, z)$  is regular inside the unit circle  $|z|=1$ , the numerator  $N$  must vanish for this zero, along with equation (5.15), are sufficient to determine the unknowns  $Q(s)$  and  $W_0(s)$ . Hence  $W(s, z)$  and for that matter  $R^{(1)}(s, z)$  and  $R^{(2)}(s, z)$  can be completely determined.

## 5.4 THE STEADY STATE SOLUTION

The steady state solution can be obtained by using the Tauberian property (1.6). Thus equation (5.21) yields.

$$W(z) = \frac{N}{D} \quad (5.24) \text{ where}$$

$$N = \lambda z Q - \mu W_0 \quad (5.25) \text{ and}$$

$$D = [(\lambda - \lambda C(z) + \mu + \zeta)z - \mu] - \frac{\zeta \eta_1 \eta_2 z}{[\lambda - \lambda C(z) + \eta_1][\lambda - \lambda C(z) + \eta_2]} \quad (5.26)$$

Similarly equations (5.19) and (5.20) respectively yield:

$$R^{(1)}(z) = \frac{\zeta W(z)}{\lambda - \lambda C(z) + \eta_1} \quad \text{and (5.27)}$$

$$R^{(2)}(z) = \frac{\zeta \eta_1 W(z)}{(\lambda - \lambda C(z) + \eta_1)[\lambda - \lambda C(z) + \eta_2]} \quad (5.28)$$

In order to determine the unknowns  $Q$  and  $W_0$  appearing in the numerator  $N$  of the right hand side of equation (5.24) we will apply Rouché's Theorem By letting

$$f(z) = [(\lambda - \lambda C(z)\mu + \zeta)z - \mu] (\lambda - \lambda C(z) + \eta_1) (\lambda - \lambda C(z) + \eta_2)$$

$$g(z) = \zeta \eta_1 \eta_2 z$$

and show that  $D$  has one zero inside the unit circle  $|z| = 1$ , along with equation (5.15), are sufficient to determine the unknowns  $Q(s)$  and  $W_0(s)$ .

## SECTION 6

### TWO PHASES OF REPAIRS AND GENERAL (ARBITRARY) SERVICE TIMES

#### 6.1 INTRODUCTION

In this section we generalise the queueing model discussed in section 4 in the sense that instead of a single exponential service, we have assumed that the service times of units follow a general (arbitrary) distribution with the distribution function  $B(x)$ . The probability generating functions of the queue length have been obtained for the time-dependent as well as the steady state cases. The result of section 4 have been derived as a particular case.

#### 6.2 DEFINITIONS AND EQUATIONS GOVERNING THE SYSTEM

We define

$W_n(x, t)$  : as the probability that at time  $t$  there are  $n$  units ( $n \geq 0$ ) in the queue and a unit is in service with elapsed service time lying between  $x$  and  $x + dx$ , and  $\mu(x)$  is the first order probability that the service will be completed in time  $x$  and  $x + dx$  conditioned that the same were not complete till time  $x$  and is related to  $B(x)$  by the relation

$$B(x) = \mu(x) e^{-\int_0^x \mu(x) dx} \quad (6.1)$$

Assumptions needed to obtain the difference – differential equations are similar to those of section 4. We shall not reproduce them.

The system is governed by the following set of difference – differential equations

$$\frac{\partial}{\partial x} W_n(x, t) + \frac{\partial}{\partial t} W_n(x, t) + (\lambda + \mu(x) + \zeta) W_n(x, t) = \lambda W_{n-1}(x, t) \quad (n > 0) \quad (6.2)$$

$$\frac{\partial}{\partial x} W_0(x, t) + \frac{\partial}{\partial t} W_0(x, t) + (\lambda + \mu(x) + \zeta) W_0(x, t) = 0 \quad (6.3)$$

$$\frac{d}{dt} R_n^{(1)}(t) + (\lambda + \eta_1) R_n^{(1)}(t) = \lambda R_{n-1}^{(1)}(t) + \zeta \int_0^\infty W_n(x, t) dt \quad (n > 0) \quad (6.4)$$

$$\frac{d}{dt} R_0^{(1)}(t) + (\lambda + \eta_1) R_0^{(1)}(t) = \zeta \int_0^\infty W_0(x, t) dt \quad (6.5)$$

$$\frac{d}{dt} R_n^{(2)}(t) + (\lambda + \eta_2) R_n^{(2)}(t) = \lambda R_{n-1}^{(2)}(t) + \eta_1 R_n^{(1)}(t) \quad (n > 0) \quad (6.6)$$

$$\frac{d}{dt} R_0^{(2)}(t) + (\lambda + \eta_2) R_0^{(2)}(t) = \eta_1 R_0^{(1)}(t) \quad (6.7)$$

$$\frac{d}{dt} Q(t) + (\lambda + \zeta) Q(t) = \int_0^\infty \mu(x) W_0(x, t) dt \quad (6.8)$$

We assume that initially the system starts when there are  $m$  units in the system and the service channel is under phase one of repairs so that the initial condition is

$$R_n^{(1)}(0) = \delta_{n,m} ,$$

Where  $\delta_{n,m}$  is the Kronecker delta (6.9)

i.e.

$$\delta_{n,m} = 1 \quad \text{if} \quad n = m$$

$$= 0 \quad \text{if} \quad n \neq m$$

The above equations are to be solved subject to the boundary conditions

$$W_n(0,t) = \int_0^\infty W_{n+1}(x,t) \mu(x) dx + \eta_2 R_n^{(2)}(t) \quad (n > 0) \quad (6.10)$$

$$W_0(0,t) = \int_0^\infty W_1(x,t) \mu(x) dx + \eta_2 R_0^{(2)}(t) + \lambda Q(t) \quad (6.11)$$

We take Laplace transform of equations (6.2) to (6.8) with (6.10) and (6.11), we thus have

$$\frac{\partial}{\partial x} W_n(x,s) + (s + \lambda + \mu(x) + \zeta) W_n(x,s) = \lambda W_{n-1}(x,s) \quad (n > 0) \quad (6.12)$$

$$\frac{\partial}{\partial x} W_0(x,s) + (s + \lambda + \mu(x) + \zeta) W_0(x,s) = 0 \quad (6.13)$$

$$(s + \lambda + \eta_1) R_n^{(1)}(s) = \lambda R_{n-1}^{(1)}(s) + \zeta \int_0^\infty W_n(x,s) dx + \delta_{n,m} \quad (n > 0) \quad (6.14)$$

$$(s + \lambda + \eta_1) R_0^{(1)}(s) = \zeta \int_0^\infty W_0(x,s) dx + \delta_{0,m} \quad (6.15)$$

$$(s + \lambda + \eta_2) R_n^{(2)}(s) = \lambda R_{n-1}^{(2)}(s) + \eta_1 R_n^{(1)}(s) \quad (n > 0) \quad (6.16)$$

$$(s + \lambda + \eta_2) R_0^{(2)}(s) = \eta_1 R_n^{(1)}(s) \quad (6.17)$$

$$(s + \lambda + \zeta) Q(s) = \int_0^\infty W_0(x, s) \mu(x) dx \quad (6.18)$$

$$W_n(0, s) = \int_0^\infty W_{n+1}(x, s) \mu(x) dx + \eta_2 R_n^{(2)}(s) \quad (n > 0) \quad (6.19)$$

$$W_n(0, s) = \int_0^\infty W_1(x, s) \mu(x) dx + \eta_2 R_0^{(2)}(s) + \lambda Q(s) \quad (6.20)$$

### 6.3 THE PROBABILITY GENERATING FUNCTIONS OF THE QUEUE LENGTH

Define the following probability generating functions in terms of their Laplace transforms

$$W(x,s,z) = \sum_{n=0}^{\infty} W_n(x,s) z^n$$

$$R^{(1)}(s,z) = \sum_{n=0}^{\infty} R_n^{(1)}(s) z^n$$

$$R^{(2)}(s,z) = \sum_{n=0}^{\infty} R_n^{(2)}(s) z^n$$

$$P(s,z) = \sum_{n=0}^{\infty} P_n(s) z^n \quad (6.21)$$

Where  $W_n(s)$  is the Laplace transform of the probability that at time  $t$  there are  $n$  units in the queue and the service channel is in working order so that

$$W_n(s) = \int_0^{\infty} W_n(x,s) dx \quad (6.22)$$

and, therefore

$$W(s,z) = \int_0^{\infty} W(x,s,z) dx \quad (6.23)$$

We perform the operations  $\sum_{n=1}^{\infty} z^n$  times equation (6.12) + equation (6.13), use equation (6.21) and simplify, we have

$$\frac{\partial}{\partial x} W(x,s,z) + (s + \lambda + \mu(x) + \zeta - \lambda z) W(x,s,z) = 0 \quad (6.24)$$

Performing the operation  $\sum_{n=1}^{\infty} z^n$  times equation (6.14) + equation (6.15), using equation (6.21)

$$(s + \lambda + \eta_1 - \lambda z) R^{(1)}(s,z) = \zeta \int_0^{\infty} W(x,s,z) dx + z^m \quad (6.25)$$

And similar operation on equations (6.16) and (6.17) yields

$$(s + \lambda + \eta_2 - \lambda z) R^{(2)}(s,z) = \eta_1 R^{(1)}(s,z) \quad (6.26)$$

We perform the operation  $\sum_{n=1}^{\infty} z^{n+1}$  times equation (6.19) +  $z$  times equation (6.20), using (6.21), we have

$$z W(0,s,z) = \int_0^\infty W(x,s,z) \mu(x) dx - \int_0^\infty W_0(x,s) \mu(x) dx + \eta_2 z R^{(2)}(s,z) + \lambda z Q(s) \quad (6.27)$$

$$\text{Let } H = z^{-1} \int_0^\infty W(x,s,z) \mu(x) dx - z^{-1} \int_0^\infty W_0(x,s) \mu(x) dx + \lambda Q(s) \quad (6.28)$$

$$\text{so, } W(0,s,z) = H + \eta_2 R^{(2)}(s,z) \quad (6.29)$$

Equations (6.25) and (6.26) yields

$$R^{(1)}(s,z) = \frac{\zeta W(s,z) + z^m}{(s + \lambda + \eta_1 - \lambda z)} \quad (6.30)$$

$$R^{(2)}(s,z) = \frac{\eta_1 (\zeta W(s,z) + z^m)}{(s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z)} \quad (6.31)$$

We integrate equation (6.24) between the limits 0 and x. Thus we have

$$W(x,s,z) = W(0,s,z) e^{[-(s + \lambda + \zeta - \lambda z) x - \int_0^x \mu(x) dx]} \quad (6.32)$$

On using equation(6.29), equation (6.32) becomes

$$W(x,s,z) = (H + \eta_2 R^{(2)}(s,z)) e^{[-(s + \lambda + \zeta - \lambda z)x - \int_0^x \mu(x) dx]} \quad (6.33)$$

We integrate equation (6.33) by parts with respect to x and use equation (6.1) and (6.23) thus we have

$$W(s,z) = (H + \eta_2 R^{(2)}(s,z)) \left[ \frac{1 - B(s + \lambda + \zeta - \lambda z)}{(s + \lambda + \zeta - \lambda z)} \right] \quad (6.34)$$

where

$$B_*(s + \lambda + \zeta - \lambda z) = \int_0^\infty e^{-(s + \lambda + \zeta - \lambda z)x} B(x) dx$$

$$\text{Let } M = \frac{1 - B(s + \lambda + \zeta - \lambda z)}{(s + \lambda + \zeta - \lambda z)} \quad \text{and} \quad K = H \cdot M, \quad \text{then}$$

$$W(s,z) = K + \eta_2 M R^{(2)}(s,z) \quad (6.35)$$

Solving (6.31) with (6.35) we have

$$W(s,z) = \frac{K(s + \lambda + \eta_1 - \lambda z)(s + \lambda + \eta_2 - \lambda z) + \eta_1 \eta_2 M z^m}{(s + \lambda + \eta_1 - \lambda z)(s + \lambda + \eta_2 - \lambda z) - \eta_1 \eta_2 \zeta M} \quad (6.36)$$

In order to determine the unknowns appearing in the numerator of the right hand side of equation (6.36). We can apply Rouché's Theorem as before.

To get the result of section 4, we put  $\mu(x) = \mu$  in equation (6.1) then

$$B(x) = \mu e^{-\int_0^x \mu dx}$$

Which implies

$$B(x) = \mu e^{-\mu x}$$

$$\begin{aligned} \text{So, } B_*(s + \lambda + \zeta - \lambda z) &= \int_0^\infty e^{-(s + \lambda + \zeta - \lambda z)x} \mu e^{-\mu x} dx \\ &= \int_0^\infty \mu e^{-(s + \lambda + \zeta - \lambda z + \mu)x} dx \\ &= \frac{\mu}{s + \lambda + \zeta - \lambda z + \mu} \end{aligned}$$

Then we have

$$\begin{aligned} M &= \frac{1 - B_*(s + \lambda + \zeta - \lambda z)}{s + \lambda + \zeta - \lambda z} \\ &= 1 - \frac{\mu}{s + \lambda + \zeta - \lambda z + \mu} \\ &= \frac{s + \lambda + \zeta - \lambda z + \mu - \mu}{(s + \lambda + \zeta - \lambda z + \mu)(s + \lambda + \zeta - \lambda z)} \end{aligned}$$

$$= \frac{1}{s + \lambda + \zeta - \lambda z + \mu}$$

and

$$H = z^{-1} \int_0^{\infty} W(x,s,z) \mu(x) dx - z^{-1} \int_0^{\infty} W_0(x,s) \mu(x) dx + \lambda Q(s)$$

$$= z^{-1} \int_0^{\infty} W(x,s,z) \mu dx - z^{-1} \int_0^{\infty} W_0(x,s) \mu dx + \lambda Q(s)$$

$$\text{But } \int_0^{\infty} W(x,s,z) dx = W(s,z) \text{ and}$$

$$\int_0^{\infty} W_0(x,s) dx = W_0(s)$$

$$\text{So, } H = z^{-1} \mu W(s,z) - z^{-1} \mu W_0(s) + \lambda Q(s)$$

Which implies  $K = H.M$

$$K = \frac{z^{-1} \mu W(s,z) - z^{-1} \mu W_0(s) + \lambda Q(s)}{(s + \lambda + \zeta - \lambda z + \mu)}$$

$$K = \frac{\mu W(s,z) - \mu W_0(s) + \lambda z Q(s)}{z (s + \lambda + \zeta - \lambda z + \mu)}$$

Therefore, equation (6.36) give :-

$$W(s,z) = \frac{\left[ \frac{\mu W(s,z) - \mu W_0(s) + \lambda z Q(s)}{z(s + \lambda + \zeta - \lambda z + \mu)} \right] (s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z) + \eta_1 \eta_2 z^m}{(s + \lambda + \zeta - \lambda z + \mu) (s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z) - \frac{\eta_1 \eta_2 \zeta}{(s + \lambda + \zeta - \lambda z + \mu)}}$$

$$= \frac{[\mu W(s,z) - \mu W_0(s) + \lambda z Q(s)] (s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z) + \eta_1 \eta_2 z^{m+1}}{z [(s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z) (s + \lambda + \zeta - \lambda z + \mu) - \eta_1 \eta_2 \zeta]}$$

$$\text{so, } z [(s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z) (s + \lambda + \zeta - \lambda z + \mu) - \eta_1 \eta_2 \zeta] W(s,z)$$

$$= \mu (s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z) W(s,z) - \mu (s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z) W_0(s) + \lambda z (s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z) Q(s) + \eta_1 \eta_2 z^{m+1}$$

$$\left\{ (s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z) [(s + \lambda + \zeta - \lambda z + \mu) z - \mu] - \eta_1 \eta_2 \zeta z \right\} W(s,z)$$

$$= (s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z) (\lambda z Q(s) - \mu W_0(s)) + \eta_1 \eta_2 z^{m+1}$$

$$W(s,z) = \frac{(s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z) (\lambda z Q(s) - \mu W_0(s)) + \eta_1 \eta_2 z^{m+1}}{[(s + \lambda + \zeta - \lambda z + \mu) z - \mu] (s + \lambda + \eta_1 - \lambda z) (s + \lambda + \eta_2 - \lambda z) - \eta_1 \eta_2 \zeta z}$$

(6.37)

## 6.4 THE STEADY STATE SOLUTION

The steady state solution can be obtained by using the Tauberian property (1.6).

Thus equation (6.37) yields

$$\lim_{s \rightarrow 0} sW(s, z) = \lim_{s \rightarrow 0} \frac{(s + \lambda + \eta_1 - \lambda z)(s + \lambda + \eta_2 - \lambda z)(\lambda z sQ(s) - \mu sW_0(s)) + \eta_1 \eta_2 s z^{m+1}}{[(s + \lambda + \zeta - \lambda z + \mu)z - \mu](s + \lambda + \eta_1 - \lambda z)(s + \lambda + \eta_2 - \lambda z) - \eta_1 \eta_2 \zeta z}$$

$$W(z) = \frac{(\lambda + \eta_1 - \lambda z)(\lambda + \eta_2 - \lambda z)(\lambda z Q - \mu W_0)}{[(\lambda + \zeta - \lambda z + \mu)z - \mu](\lambda + \eta_1 - \lambda z)(\lambda + \eta_2 - \lambda z) - \eta_1 \eta_2 \zeta z} \quad (6.38)$$

Since  $\lim_{s \rightarrow 0} sQ(s) = Q$  and  $\lim_{s \rightarrow 0} sW_0(s) = W_0$  by defination

Similarly equation (6.30) and (6.31) respectively yield

$$R^{(1)}(z) = \lim_{s \rightarrow 0} sR^{(1)}(s, z) = \lim_{s \rightarrow 0} \frac{\zeta sW(s, z) + s z^m}{(s + \lambda + \eta_1 - \lambda z)} = \frac{\zeta W(z)}{(\lambda + \eta_1 - \lambda z)} \quad (6.39)$$

and

$$\begin{aligned} R^{(2)}(z) &= \lim_{s \rightarrow 0} sR^{(2)}(s, z) = \lim_{s \rightarrow 0} \frac{\eta_1 (\zeta sW(s, z) + s z^m)}{(s + \lambda + \eta_1 - \lambda z)(s + \lambda + \eta_2 - \lambda z)} \\ &= \frac{\eta_1 \zeta W(z)}{(\lambda + \eta_1 - \lambda z)(\lambda + \eta_2 - \lambda z)} \end{aligned} \quad (6.40)$$

Now, when we substitute in equation (6.18) for the value of  $\mu(x) = \mu$  equation (6.18) will be similar to equation (4.15).

Using Tauberian property (1.6), equation (6.18), in the steady state, yields  $\mu W_0 = \lambda Q$ , with this relation equation (6.38) can be re-written as

$$W(z) = \frac{\lambda (z-1) (\lambda + \eta_1 - \lambda z) (\lambda + \eta_2 - \lambda z) Q}{[(\lambda + \mu + \zeta - \lambda z) z - \mu] (\lambda + \eta_1 - \lambda z) (\lambda - \lambda z + \eta_2) - \eta_1 \eta_2 \zeta z} \quad (6.41)$$

Equations (6.39), (6.40) and (6.41) agree with the results of section 4.

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