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College of Science

Department of Mathematics



Generating Relations of Generalized Hermite Polynomials by Weisner's Group Theoretic Method

A Thesis Submitted in Partial Fulfillment of the Requirements for the
Master of Science Degree in Mathematics

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The Discussion Committee met on 10-Dec-18 In S41-2000. Upon discussing the student whose details are stated above in public, the discussion committee recommends the following:

The student is awarded a grade A in the thesis subject.

- ☐ Without making any changes to the thesis.
- ☒ After making the modifications attached with this decision.

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We would like to report that the student has carried out all the required modifications made by the discussion committee. Upon this, we recommend awarding the student a grade A in the thesis course number MATH599 based on the Discussion Committee recommendation.

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Abstract

In this thesis we use Weisner's group theoretic method to find different generating relations for generalized Hermite polynomials and functions.

The cases that we discuss in this work include, Hermite two variables polynomial, Hermite matrix polynomial and function, and Hermite polynomial associated with parabolic cylinder.

All the steps are shown in details starting from finding recurrence relations, commutative operator relations, the extended form of the groups generated by the operators, and ending with some new generalizations of well-known generating relations.

Dedication

To my mother, the best teacher ever

To my father, who always inspires

Acknowledgment

“Who never thank people, never thank God”

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Chapter 1: Introduction

1.1 Statement of the Problem

1.2 Justification of the Problem

1.3 Literature Review

1.4 Theoretical Framework

1.5 Methods and Procedures

1.6 Proposed Chapter Outlines

1.7 Background and History

1.7.1 Hermite Functions

1.7.2 Lie Algebra

1.7.3 Weisner's Group Theoretic Method

Introduction

1.1 Statement of the Problem

The purpose of this work is to show how to derive generating relations of some generalized Hermite polynomials by Weisner's method.

1.2 Justification of the Problem

(Wesnier 1955) has derived a method for obtaining generating functions for sets of functions which satisfy certain conditions. Among the functions which do satisfy these conditions are the Hermite functions.

Recently (Shahwan 2003, 2006, 2009), (Pathan 2003) and (Khan 2009, 2010) derived many generating functions for generalized Hermite polynomials. Here, we study some of these works in details and use the same technique in finding the generating relations for generalized Hermite polynomials associated with functions of parabolic cylinder introduced by (Wunsche 2003).

1.3 Literature Review

Many special functions including generalized Hermite polynomials have their applications in many branches of science, e.g. (Andrews 1998) and (Lebedev 1972).

In physics and electronics, the solution of Laplace equation in terms of Legendre polynomials arise when we try to find the electric potential due to a hollow sphere. Bessel function also plays a big role in lengthening pendulum problems.

If we try to determine the temperature in homogenous solid when the distribution of temperature throughout the solid is known at time $t=0$ and temperature at every boundary is known, or when we try to describe the wave motion of natural or mechanical system such as vibrating membrane, subject to initial and boundary conditions, special functions arise when solving the relevant model differential equations since they form many of the solutions of theses equations.

Hermite function also have many applications in physics. For instance, single- variable Hermite function $H_n(x)$ is just the wave-function of quantum harmonic oscillator. Their importance has been recognized only recently (Dattoli 1993) and has been exploited to deal with quantum mechanical and optical beam transport problems. Within this last framework, generalized forms of harmonic oscillator eigenfunctions have been proposed and the relevant annihilation-creation operator algebra has been developed.

Hermite polynomials are orthogonal and orthogonal polynomials in general and the classical orthogonal polynomials in particular are connected with numerous problems of applied mathematics, theoretical physics, chemistry, approximation theory and several other mathematical branches. In particular, their applications are being widely used in theories as Pade approximants, continued fractions, spectral study of Schrodinger discrete operators, polynomial solutions of second-order differential equations and others.

Important applications of the orthogonal polynomials are to the development of quadrature rules of maximum algebraic degree of exactness, most notably Gauss- type quadrature rules, Gauss- kronrod and Gauss-Tur an quadratures. Modification algorithms and discretization methods find applications in constructing quadrature rules exactly not only for polynomials but also for rational functions with prescribed poles. Gauss-type quadrature rules are applicable also for constructing Cauchy principal value integrals.

Due to the noticeable importance of generating functions of Hermite polynomials and its generalization, further generating functions of these functions have been obtained. The details of methods and ideas leading to generating relations are based on the approach of (Weisner's 1955).

1.4 Theoretical Framework

In chapter 1, we study the most important definitions and notification of the Lie group and Lie algebra, special functions and generating relations.

Chapters 2 and 3 explain in details the Weisner's group theoretic method in getting generating relations of generalized Hermite polynomials, Hermite matrix polynomials and functions, respectively. Some new generating relations of the generalized Hermite polynomials associated with functions of parabolic cylinder are derived in chapter 4 by using the same technique as in (Wunsche 2003).

1.5 Methods and Procedures

Using the recurrence relations of the generalized Hermite polynomials, we introduce first order differential operators, called raising, lowering and stationary operators, satisfying certain commutation relations, where these operators generate a Lie group (Chon 1961, Miller 1968). We find the extended form of the group generated by the given operators. Then, we use a method suggested by (Weisner 1955) which is based on Taylor's series formula to get some new generating relations (McBride 1971).

1.6 Proposed Chapter Outlines

Chapter 1: Introduction:

Definitions and notations, Lie groups and Lie algebra, special functions, generating relations.

Chapter 2: Generating relations of two variable Hermite polynomials by Weisner's group theoretic method:

Generating function of generalized Hermite polynomial $H_n(x, y)$ are obtained by giving suitable interpretations to the indices (n) with the help of (Weisner 1955) groups theoretic method.

Chapter 3: Generating relations of Hermite matrix polynomials and Hermite matrix functions by Weisner's group theoretic method:

Generating relations for Hermite matrix polynomials $H_n(x, A)$ are obtained by giving suitable interpretations to the index (n) with the help of (Weisner 1955) group theoretic method.

Chapter 4: Generating relations of generalized Hermite polynomials associated with functions of parabolic cylinder:

Generating relations of generalized Hermite polynomials $D_n(z)$ associated with functions of parabolic cylinder are derived.

This is followed by a summary of thesis work.

1.7 Background and History

1.7.1 Hermite Function

There are many special functions in mathematics which play a very important role in many physical and statistical applications. These special functions are the solutions of certain differential equations and integrals.

Special functions are analytic and most of them are considered as functions of a complex variables.

Hermite polynomials and functions are one of the main and most important special functions, which are used in many applications such as probability, numerical analysis and physics.

The first Hermite polynomial $H_n(x)$ was introduced by the French mathematician (Charlize Hermite 1864) which contains only one variable. It is defined by means of the relation

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \quad (1.1)$$

which is valid for all finite x and t .

It follows from that

$$H_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n! (2x)^{n-2k}}{k! (n-2k)!}$$

This function has been developed and modified by many mathematicians including (Dattoli 1992, 1993, 1995, 1999, 2000, 2005, 2007) and (Wunsche 2003).

Through this thesis, we consider three of these generalizations which are: Hermite two variables polynomial, Hermite matrix polynomial and matrix function, and Hermite polynomial associated with functions of parabolic cylinder.

1.7.2 Lie Algebra

The theory of special functions is closely related to the theory of Lie groups and Lie algebras.

A Lie algebra is an algebra that is anti-commutative and satisfies Jacobi identity. That is: If g is an algebra over a field of real numbers F , with product $[x,y]$, then g is a Lie algebra if:

1. $[x, x] = 0$, $\forall x \in g$
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ $\forall x, y, z \in g$

Anti-commutative may be written as $[x,y] = -[y,x]$

An abelian Lie algebra is a Lie algebra with commutative Lie bracket. That is: $[x,y] = [y,x]$. Hence, the Lie bracket in abelian Lie algebra is the identity which equals to 0.

The commutator product of x and y is a Lie bracket defined by:

$[x,y] = xy - yx$ on associative algebra. Lie brackets are defined similarly throughout of this thesis.

1.7.3 Weisner's Group Theoretic Method

There are many methods to derive generating relations of functions, including Rainsville's method, Truesdell method, Miller's method, series manipulation method and Weisner's method.

Each of these methods suits some type of functions that satisfy certain conditions.

Among the functions which do satisfy Wiesner's conditions are the Hermite functions which will be discussed through our work in this thesis.

Finding generating relations for Bessel polynomials, generalized Laguerre polynomials, Gegenbauer polynomials can also be done by using Weisner's method, as they satisfied Weisner's conditions too.

Weisner's method is a good method for obtaining generating functions for special functions starting from their polynomials using representation theory of Lie groups and Lie algebras. This method has been introduced by the Canadian mathematician Louis Weisner in 1955.

By this method we use some partial differential equations beside the use of Lie algebra.

To illustrate this method, we will discuss its main steps in the next page and hence use it in more details in coming chapters.

Weisner's Group Theoretic Method Steps:

1. Recurrence Relations:

Finding differential recurrence relations where the function satisfies.

2. First Order (Linear) Differential Operators:

Finding three types of operators which are called:

1. Raising operator (J^+)
2. Lowering operator (J^-)
3. Stationary operator (J^3)

Every linear differential operator of the first order generates a one parameter Lie group. If we know the linear differential operator we will be able to determine the extended form of the group generated by it. In other words we will be able to determine the effect of the operator e^{cC} on $f(x, y)$ where,

$$e^{cC} \equiv 1 + cC + \frac{(cC)^2}{2!} + \cdots + \frac{(cC)^k}{k!} + \cdots \equiv \sum_{k=0}^{\infty} \frac{(cC)^k}{k!}$$

and C is assumed to be an operator.

3. Commutative Relations of the Operators (Group of Operators)

Using the commutator notation, $[x, y] = xy - yx$ to find the three relations:

1. $[J^3, J^-]$
2. $[J^3, J^+]$
3. $[J^+, J^-]$

These commutator relations show that the operators J^3 , J^+ , J^- generate a Lie group.

4. The Extended Form of The Group Generated by Operators

Here we are seeking to find $e^{bJ^-} f(x, y)$ and $e^{cJ^+} f(x, y)$, where $f(x, y)$ is an arbitrary function, with b and c are arbitrary constants,

$$e^{cJ^+} f(x, y) = \varphi(x, y) g(x, y)$$

which is the extended form of the transformation group generated by C .

5. Generating Functions

Finding generating relations using Weisner's method which is based on Taylor's series in the form of

$$\exp \left[c \frac{d}{dX} F(X) \right] = F(X + C)$$

We will consider the following three cases:

1. The effect of both J^+ and J^- together (i.e. $b \neq 0$ & $c \neq 0$)
2. The effect of J^- only ($c = 0$)
3. The effect of J^+ only ($b = 0$)

We do that with the help of power series, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\text{as } e^{cJ^+} = \sum_{k=0}^{\infty} \frac{(cJ^+)^k}{k!} \quad \text{and} \quad e^{bJ^-} = \sum_{L=0}^{\infty} \frac{(bJ^-)^L}{L!} .$$

Chapter 2: Generating Relations of Two Variable Hermite Polynomials by Weisner's Group Theoretic Method

2.1 Introduction

2.2 Generating Relations using Weisner's Method

2.2.1 Recurrence Relations

2.2.2 First Order Differential Operators

2.2.3 Commutative Relations of the Operators

2.2.4 Extended Form of the Group Generated by Operators

2.2.5 Some Generating Functions

2.3 Some Applications

Generating Relations of Two Variable Hermite Polynomials by Weisner's Group Theoretic Method

2.1 Introduction

Hermite Function has been modified by many mathematicians. One of the main generalizations is the Hermite polynomial of two variables which has the following form (Lebedev 1972):

$$\sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} = \exp(2xt - yt^2)$$

Which gives:

$$H_n(x, y) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k! (n-2k)!} (2x)^{(n-2k)} y^k \quad (2.1)$$

where $[\frac{n}{2}]$ is the greatest integer less than or equal to $\frac{n}{2}$.

The generalized Hermite polynomial $H_n(x, y)$ is a solution of the following partial differential equation:

$$-y \frac{\partial^2 v}{\partial x^2} + 2x \frac{\partial v}{\partial x} - 2nv = 0$$

This leads to

$$H_n(x, y) = y^{\frac{n}{2}} H_n\left(\frac{x}{\sqrt{y}}\right) \quad (2.2)$$

2.2 Generating Relations using Weisner's Method

2.2.1 Recurrence Relations

Here, we derive some pure and differential recurrence relations. We differentiate Hermite generating function of two variables which considered in (2.1) partially with respect to each of the variables x, y and t .

Case (i) Differentiating (2.1) with respect to x , gives:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\partial}{\partial x} H_n(x, y) \frac{t^n}{n!} &= 2t \exp(2xt - yt^2) \\ \sum_{n=0}^{\infty} \frac{\partial}{\partial x} H_n(x, y) \frac{t^n}{n!} &= 2t \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} \\ \sum_{n=0}^{\infty} \frac{\partial}{\partial x} H_n(x, y) \frac{t^n}{n!} &= 2 \sum_{n=0}^{\infty} H_n(x, y) \frac{t^{n+1}}{n!} \\ &\quad \uparrow \\ &\quad \text{Replace } n \text{ by } n-1 \\ \sum_{n=0}^{\infty} \frac{\partial}{\partial x} H_n(x, y) \frac{t^n}{n!} &= 2 \sum_{n=1}^{\infty} H_{n-1}(x, y) \frac{t^n}{(n-1)!} \\ \sum_{n=0}^{\infty} \frac{\partial}{\partial x} H_n(x, y) \frac{t^n}{n!} &= 2 \sum_{n=1}^{\infty} H_{n-1}(x, y) \frac{n t^n}{n!}\end{aligned}$$

By comparing t^n coefficients, we get the following first recurrence relation:

$$\frac{\partial}{\partial x} H_n(x, y) = 2n H_{n-1}(x, y), \quad n \geq 1 \quad (2.3)$$

Case (ii) Differentiating (2.1) with respect to y, gives:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\partial}{\partial y} H_n(x, y) \frac{t^n}{n!} &= -t^2 \exp(2xt - yt^2) \\
\sum_{n=0}^{\infty} \frac{\partial}{\partial y} H_n(x, y) \frac{t^n}{n!} &= -t^2 \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} \\
\sum_{n=0}^{\infty} \frac{\partial}{\partial y} H_n(x, y) \frac{t^n}{n!} &= - \sum_{n=0}^{\infty} H_n(x, y) \frac{t^{n+2}}{n!} \\
&\quad \uparrow \\
&\quad \text{Replace n by n-2} \\
\sum_{n=0}^{\infty} \frac{\partial}{\partial y} H_n(x, y) \frac{t^n}{n!} &= - \sum_{n=2}^{\infty} H_{n-2}(x, y) \frac{t^n}{(n-2)!} \\
\sum_{n=0}^{\infty} \frac{\partial}{\partial y} H_n(x, y) \frac{t^n}{n!} &= - \sum_{n=2}^{\infty} H_{n-2}(x, y) \frac{n(n-1)t^n}{n!}
\end{aligned}$$

By comparing t^n coefficients, we get the following second recurrence relation:

$$\frac{\partial}{\partial y} H_n(x, y) = -n(n-1)H_{n-2}(x, y), \quad n \geq 2 \quad (2.4)$$

Case (iii) Differentiating (2.1) with respect to t, gives:

$$\begin{aligned}
\sum_{n=1}^{\infty} H_n(x, y) \frac{n t^{n-1}}{n!} &= (2x - 2yt) \exp(2xt - yt^2) \\
\sum_{n=1}^{\infty} H_n(x, y) \frac{n t^{n-1}}{n!} &= (2x - 2yt) \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}
\end{aligned}$$

$$\sum_{n=1}^{\infty} H_n(x, y) \frac{n t^{n-1}}{n!} = 2x \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} - 2yt \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}$$

$$\sum_{n=1}^{\infty} H_n(x, y) \frac{t^{n-1}}{(n-1)!} = 2x \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} - 2y \sum_{n=0}^{\infty} H_n(x, y) \frac{t^{n+1}}{n!}$$

$\uparrow$ Replace n by n+1 $\uparrow$ Replace n by n-1

$$\sum_{n=0}^{\infty} H_{n+1}(x, y) \frac{t^n}{n!} = 2x \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} - 2y \sum_{n=1}^{\infty} H_{n-1}(x, y) \frac{t^n}{(n-1)!}$$

$$\sum_{n=0}^{\infty} H_{n+1}(x, y) \frac{t^n}{n!} = 2x \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} - 2y \sum_{n=1}^{\infty} H_{n-1}(x, y) \frac{n t^n}{n!}$$

Finding $H_{n+1}(x, A)$ at $n = 0$ by using definition (2.1)

$$H_1(x, A) = \frac{(-1)^0 1!}{0! (1-0)!} (2x)^{(1-0)} y^0$$

$$H_1(x, A) = 2x$$

and since $H_0(x, A) = I$, then:

$$\sum_{n=1}^{\infty} H_{n+1}(x, y) \frac{t^n}{n!} = 2x \sum_{n=1}^{\infty} H_n(x, y) \frac{t^n}{n!} - 2y \sum_{n=1}^{\infty} H_{n-1}(x, y) \frac{n t^n}{n!}$$

By comparing t^n coefficients, we get the following third recurrence relation:

$$H_{n+1}(x, y) = 2x H_n(x, y) - 2yn H_{n-1}(x, y), \quad n \geq 1 \quad (2.5)$$

Replace n by n-1, then we have:

$$H_n(x, y) = 2x H_{n-1}(x, y) - 2y(n-1) H_{n-2}(x, y), \quad n \geq 2 \quad (2.6)$$

Now by substituting (2.3) and (2.4) in (2.6) we have:

$$n H_n(x, y) = x \frac{\partial}{\partial x} H_n(x, y) + 2y \frac{\partial}{\partial y} H_n(x, y) \quad (2.7)$$

2.2.2 First Order Differential Operators

From the recurrence relations as mentioned before, we introduce the following three operators: Lowering operator (J^-), increasing operator (J^+) and stationary operator (J^3).

Case (i) To find J^- :

From (2.3) we get: $J^- = t^{-1} \frac{\partial}{\partial x}$ such that

$$J^- [H_n(x, y) t^n] = 2n H_{n-1}(x, y) t^{n-1} \quad (2.8)$$

Case (ii) To find J^3 :

From (2.7)

$$x \frac{\partial}{\partial x} H_n(x, y) + 2y \frac{\partial}{\partial y} H_n(x, y) = n H_n(x, y)$$

So we get: $J^3 = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}$ such that

$$J^3 [H_n(x, y) t^n] = n H_n(x, y) t^n \quad (2.9)$$

Case (iii) To find J^+ :

From (2.5)

$$2x H_n(x, y) - 2yn H_{n-1}(x, y) = H_{n+1}(x, y)$$

Substitute (2.3) in the previous equation to get:

$$2x H_n(x, y) - y \frac{\partial}{\partial x} H_n(x, y) = H_{n+1}(x, y)$$

$$\left(2x - y \frac{\partial}{\partial x}\right) H_n(x, y) = H_{n+1}(x, y)$$

So we get: $J^+ = t(2x - y \frac{\partial}{\partial x})$ such that

$$J^+[H_n(x, y)t^n] = H_{n+1}(x, y)t^{n+1} \quad (2.10)$$

2.2.3 Commutative Relations of the Operators

We use the commutator notation $[A, B]$ with $[A, B]f = (AB - BA)f$ where $f = f(x, y; t) = H_n(x, y)t^n$ in order to get the following three relations:

Case (i) $[J^3, J^+] = J^+$

Proof: $[J^3, J^+]f = (J^3J^+ - J^+J^3)f$

$$\begin{aligned} &= \left[x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}\right] \left[t \left(2x - y \frac{\partial}{\partial x}\right)\right] f - \left[t \left(2x - y \frac{\partial}{\partial x}\right)\right] \left[x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}\right] f \\ &= \left[x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}\right] [2xtf - yt \frac{\partial}{\partial x} f] - \left[t \left(2x - y \frac{\partial}{\partial x}\right)\right] \left[x \frac{\partial}{\partial x} f + 2y \frac{\partial}{\partial y} f\right] \\ &= x \left[2xt \frac{\partial}{\partial x} f + 2tf - yt \frac{\partial^2}{\partial x^2} f\right] + 2y \left[2xt \frac{\partial}{\partial y} f - yt \frac{\partial^2}{\partial x \partial y} f - t \frac{\partial}{\partial x} f\right] - \\ &\quad \left[2x^2t \frac{\partial}{\partial x} f - yt \left[x \frac{\partial^2}{\partial x^2} f + \frac{\partial}{\partial x} f\right] + 4xty \frac{\partial}{\partial y} f - 2y^2t \frac{\partial^2}{\partial x \partial y} f\right] \\ &= 2x^2t \frac{\partial}{\partial x} f + 2xtf - xyt \frac{\partial^2}{\partial x^2} f + 4xty \frac{\partial}{\partial y} f - 2y^2t \frac{\partial^2}{\partial x \partial y} f - 2yt \frac{\partial}{\partial x} f - \\ &\quad \left[2x^2t \frac{\partial}{\partial x} f - xyt \frac{\partial^2}{\partial x^2} f - yt \frac{\partial}{\partial x} f + 4xty \frac{\partial}{\partial y} f - 2y^2t \frac{\partial^2}{\partial x \partial y} f\right] \\ &= 2xtf - 2yt \frac{\partial}{\partial x} f + yt \frac{\partial}{\partial x} f \\ &= 2xtf - yt \frac{\partial}{\partial x} f \end{aligned}$$

$$\begin{aligned}
&= t \left(2x - y \frac{\partial}{\partial x} \right) f \\
&= J^+ f
\end{aligned}$$

Case (ii) $[J^3, J^-] = -J^-$

Proof: $[J^3, J^-] f = (J^3 J^- - J^- J^3) f$

$$\begin{aligned}
&= \left[x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} \right] \left[t^{-1} \frac{\partial}{\partial x} \right] f - \left[t^{-1} \frac{\partial}{\partial x} \right] \left[x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} \right] f \\
&= \left[x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} \right] \left[t^{-1} \frac{\partial}{\partial x} f \right] - \left[t^{-1} \frac{\partial}{\partial x} \right] \left[x \frac{\partial}{\partial x} f + 2y \frac{\partial}{\partial y} f \right] \\
&= x t^{-1} \frac{\partial^2}{\partial x^2} f + 2y t^{-1} \frac{\partial^2}{\partial x \partial y} f - \left[t^{-1} x \frac{\partial^2}{\partial x^2} f + t^{-1} \frac{\partial}{\partial x} f + 2y t^{-1} \frac{\partial^2}{\partial x \partial y} f \right] \\
&= -t^{-1} \frac{\partial}{\partial x} f \\
&= -J^- f
\end{aligned}$$

Case (iii) $[J^+, J^-] = -2$

Proof: $[J^+, J^-] f = (J^+ J^- - J^- J^+) f$

$$\begin{aligned}
&= \left[t \left(2x - y \frac{\partial}{\partial x} \right) \right] \left[t^{-1} \frac{\partial}{\partial x} \right] f - \left[t^{-1} \frac{\partial}{\partial x} \right] \left[t \left(2x - y \frac{\partial}{\partial x} \right) \right] f \\
&= \left[t \left(2x - y \frac{\partial}{\partial x} \right) \right] \left[t^{-1} \frac{\partial}{\partial x} f \right] - \left[t^{-1} \frac{\partial}{\partial x} \right] \left[2x t f - y t \frac{\partial}{\partial x} f \right] \\
&= 2x \frac{\partial}{\partial x} f - y \frac{\partial^2}{\partial x^2} f - \left[2x \frac{\partial}{\partial x} f + 2f - y \frac{\partial^2}{\partial x^2} f \right] \\
&= -2f
\end{aligned}$$

From the following operator relations:

$$J^- [H_n(x, y) t^n] = 2n H_{n-1}(x, y) t^{n-1}$$

$$J^3 [H_n(x, y) t^n] = n H_n(x, y) t^n$$

$$\begin{aligned} J^+ J^- [H_n(x, y) t^n] &= J^+ [2n H_{n-1}(x, y) t^{n-1}] \\ &= 2n H_n(x, y) t^n \end{aligned}$$

We can form the differential operator $L = 2x \frac{\partial}{\partial x} - y \frac{\partial^2}{\partial x^2} - 2t \frac{\partial}{\partial t}$

$$L = J^+ J^- - 2n$$

$$L = t(2x - y \frac{\partial}{\partial x}) \cdot t^{-1} \frac{\partial}{\partial x} - 2n$$

$$L = 2x \frac{\partial}{\partial x} - y \frac{\partial^2}{\partial x^2} - 2n, \quad \text{where } n = t \frac{\partial}{\partial t} \quad (2.11)$$

Which commutes with J^+, J^- and J^3 , that is

$$[L, J^+] = 0, [L, J^-] = 0 \text{ and } [L, J^3] = 0$$

According to Weisner's group theoretic method, since the above relations are not isomorphic to any commutation relations obeyed by a well-known Lie algebra of a local Lie group, therefore we cannot link the set of differential operators $\{J^3, J^-, J^+\}$ with a particular Lie algebra and hence we are only bound to say that these differential operators generate three dimensional Lie algebra.

2.2.4 Extended Form of the Group Generated by Operators

Here we find the extended form of the group generated by J^3, J^- and J^+ as follows:

$$\text{Case (i)} \quad J^+ = t(2x - y \frac{\partial}{\partial x})$$

Let c be any arbitrary constant. Then,

$$\frac{dy}{dc} = 0$$

$$dy = 0$$

$$y = k$$

Applying the initial condition when $c = 0$, $y(0) = y_0 \Rightarrow y_0(0) = k$

$$\Rightarrow y = y_0$$

Similarly,

$$\frac{dt}{dc} = 0$$

$$\Rightarrow t = t_0$$

$$\frac{dx}{dc} = -y_0 t_0$$

$$dx = -y_0 t_0 dc$$

$$x = -y_0 t_0 c + k$$

when $c = 0$, $x(0) = x_0 \Rightarrow k = x_0$

$$\Rightarrow x = x_0 - y_0 t_0 c$$

$$\frac{dv}{dc} = 2xtv$$

$$\frac{dv}{v} = 2xt \, dc$$

Substituting the values of x, y and t

$$\frac{dv}{v} = 2(x_0 - y_0 t_0 c) t_0 \, dc$$

$$\ln v = 2x_0 c t_0 - y_0 t_0^2 c^2 + k$$

Applying the initial condition when $c = 0, v(0) = 1, \ln 1 = 0$

$$\Rightarrow k = 0$$

$$\Rightarrow \ln v = 2x_0 c t_0 - y_0 t_0^2 c^2$$

$$\Rightarrow v = \exp(2x_0 c t_0 - y_0 t_0^2 c^2)$$

$$e^{cJ^+} f(x, y; t) = \exp(2xct - yt^2 c^2) \cdot f(x - ytc, y, t) \quad (2.12)$$

$$\text{Case (ii)} \quad J^- = t^{-1} \frac{\partial}{\partial x} \quad .$$

Let b be any arbitrary constant. Then,

$$\frac{dy}{db} = 0$$

$$\Rightarrow y = y_0$$

$$\frac{dt}{db} = 0$$

$$\Rightarrow t = t_0$$

$$\frac{dv}{db} = 0$$

$$\Rightarrow v = 1$$

$$\frac{dx}{db} = t_0^{-1}$$

$$dx = t_0^{-1} db$$

$$x = t_0^{-1}b + k$$

$$\text{when } b = 0, k = x_0$$

$$\Rightarrow x = x_0 + t_0^{-1}b$$

$$e^{bJ^-} f(x, y; t) = f\left(x + \frac{b}{t}, y, t\right) \quad (2.13)$$

$$\text{Case (iii) } J^3 = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} ,$$

Let a be any arbitrary constant. Then,

$$\frac{dt}{da} = 0$$

$$\Rightarrow t = t_0$$

$$\frac{dv}{da} = 0$$

$$\Rightarrow v = 1$$

$$\frac{dx}{da} = x$$

$$\frac{dx}{x} = da$$

$$\ln x = a + k$$

$$\text{when } a = 0, x(0) = x_0 \Rightarrow \ln x_0 = k$$

$$\ln x = a + \ln x_0$$

$$\Rightarrow x = e^{(a + \ln x_0)}$$

$$\Rightarrow x = e^a e^{\ln x_0}$$

$$\Rightarrow x = x_0 e^a$$

$$\frac{dy}{da} = 2y$$

$$\frac{dy}{y} = 2 da$$

$$\ln y = 2a + k$$

$$\text{when } a = 0, y(0) = y_0 \Rightarrow k = \ln y_0$$

$$\ln y = 2a + \ln y_0$$

$$\Rightarrow y = e^{(2a + \ln y_0)}$$

$$\Rightarrow y = e^{2a} e^{\ln y_0}$$

$$\Rightarrow y = y_0 e^{2a}$$

$$e^{aJ^3} f(x, y; t) = f(xe^a, y e^{2a}, t) \quad (2.14)$$

2.2.5 Some Generating Functions

From the previous section, we have:

$$e^{cJ^+} f(x, y; t) = \exp(2xct - yt^2c^2) \cdot f(x - ytc, y, t)$$

$$e^{bJ^-} f(x, y; t) = f\left(x + \frac{b}{t}, y, t\right)$$

$$e^{aJ^3} f(x, y; t) = f(xe^a, ye^{2a}, t)$$

Now, we study the effect of the operators in the following three cases:

Case 1: The effect of the three operators: J^- , J^+ and J^3 on $f(x, y, t)$:

$$\begin{aligned} & e^{cJ^+} e^{bJ^-} e^{aJ^3} f(x, y, t) \\ &= \exp(2xct - yt^2c^2) e^{bJ^-} e^{aJ^3} f(x - ytc, y, t) \\ &= \exp(2xct - yt^2c^2) e^{aJ^3} f\left(x - ytc + \frac{b}{t}, y, t\right) \\ &= \exp(2xct - yt^2c^2) f\left(\left(x - ytc + \frac{b}{t}\right) e^a, ye^{2a}, t\right) \\ &= \exp[ct(2x - cty)] f\left(\left(x - ytc + \frac{b}{t}\right) e^a, ye^{2a}, t\right) . \end{aligned}$$

Now if we put $a = 0$; $b \neq 0$, $c \neq 0$ and writing $f(x, y, t) = H_n(x, y)t^n$, we get

$$e^{cJ^+} e^{bJ^-} (H_n(x, y)t^n) = \exp[ct(2x - cty)] \cdot H_n\left(x - ytc + \frac{b}{t}, y\right) t^n (t \neq 0) \quad (2.15)$$

Working on left hand side and by using the power series expansions, we get for $t \neq 0$:

$$\begin{aligned}
& e^{cJ^+} e^{bJ^-} (H_n(x, y) t^n) \\
&= \sum_{k=0}^{\infty} \frac{(cJ^+)^k}{k!} e^{bJ^-} (H_n(x, y) t^n) \\
&= \sum_{k=0}^{\infty} \frac{c^k}{k!} e^{bJ^-} (J^+)^k (H_n(x, y) t^n) \\
&= \sum_{k=0}^{\infty} \frac{c^k}{k!} e^{bJ^-} (H_{n+k}(x, y) t^{n+k}) \\
&= \sum_{k=0}^{\infty} \frac{c^k}{k!} \sum_{L=0}^{\infty} \frac{(bJ^-)^L}{L!} (H_{n+k}(x, y) t^{n+k}) \\
&= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{c^k}{k!} \frac{b^L (J^-)^L}{L!} (H_{n+k}(x, y) t^{n+k}) \\
&= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{c^k b^L}{k! L!} (J^-)^L (H_{n+k}(x, y) t^{n+k}) \\
&= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{c^k b^L}{k! L!} 2^L (n+k)(n+k-1)(n+k-2) \dots (n+k-L+1) (H_{n+k-L}(x, y) t^{n+k-L}) \\
& e^{cJ^+} e^{bJ^-} (H_n(x, y) t^n) = \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{c^k b^L}{k! L!} 2^L \frac{(n+k)!}{(n+k-L)!} (H_{n+k-L}(x, y) t^{n+k-L}) \quad (2.16)
\end{aligned}$$

From (2.15) and (2.16) we get for $t \neq 0$:

$$\begin{aligned}
& \exp[ct(2x - cty)] \cdot H_n\left(x - ytc + \frac{b}{t}, y\right) t^n \\
&= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{c^k b^L}{k! L!} 2^L \frac{(n+k)!}{(n+k-L)!} (H_{n+k-L}(x, y) t^{n+k-L}) \quad (2.17)
\end{aligned}$$

Case 2: The effect of one operator: J^- on $H_n(x, y)$:

Put $t = 1$ and $c = 0$ in (2.17), we get:

$$H_n(x + b, y) = \sum_{L=0}^{\infty} \frac{b^L}{L!} 2^L \frac{n!}{(n - L - 2)!} H_{n-L}(x, y) \quad (2.18)$$

Case 3: The effect of one operator: J^+ on $H_n(x, y)$:

Let $t = 1$ and $b = 0$ in (2.17), we get:

$$\exp(2xc - yc^2) H_n(x - yc, y) = \sum_{k=0}^{\infty} \frac{c^k}{k!} H_{n+k}(x, y) \quad (2.19)$$

2.3 Some Applications

Using the relation (2.2) in (2.19), we get:

$$\begin{aligned} \exp(2xc - yc^2) y^{\frac{n}{2}} H_n \left(\frac{x - yc}{\sqrt{y}} \right) &= \sum_{k=0}^{\infty} \frac{c^k}{k!} y^{\frac{n+k}{2}} H_{n+k} \left(\frac{x}{\sqrt{y}} \right) \\ \exp(2xc - yc^2) H_n \left(\frac{x - yc}{\sqrt{y}} \right) &= \sum_{k=0}^{\infty} \frac{c^k}{k!} y^{\frac{k}{2}} H_{n+k} \left(\frac{x}{\sqrt{y}} \right) \end{aligned} \quad (2.20)$$

If we put $y=1$, $c=t$, then we have:

$$\exp(2xt - t^2) H_n(x - t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_{n+k}(x) \quad (2.21)$$

This equation is a well-known result discussed by (Mc Bride 1971).

Putting $n=0$ in (2.21), then we have:

$$\begin{aligned} \exp(2xt - t^2) H_0(x - t) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x) \\ \exp(2xt - t^2) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x) \end{aligned} \quad (2.22)$$

This equation is a well-known generating function of Hermite polynomial $H_n(x)$ given by (Lebedev 1972).

Since $[J^+, J^-] = -2$, then the operators J^-, J^+ are non-commutative so $e^{cJ^+} e^{bJ^-} \neq e^{bJ^-} e^{cJ^+}$
i.e. $e^{cJ^+} e^{bJ^-} \neq e^{cJ^+ + bJ^-}$

If we interchange the order of the operators, we get the below new relation

$$\begin{aligned} &e^{bJ^-} e^{cJ^+} f(x, y, t) \\ &= e^{cJ^+} \left[f\left(x + \frac{b}{t}, y, t\right) \right] \\ &= \left[\exp\left(2\left(x + \frac{b}{t}\right)ct - yt^2c^2\right) \cdot f\left(x + \frac{b}{t} - ytc, y, t\right) \right] \end{aligned}$$

Then for $t \neq 0$,

$$e^{bJ^-} e^{cJ^+} H_n(x, y) t^n = \exp\left(2\left(x + \frac{b}{t}\right)ct - yt^2c^2\right) \cdot H_n\left(x + \frac{b}{t} - ytc, y\right) t^n \quad (2.23)$$

Working on left hand side and by using the power series expansions, for $t \neq 0$:

$$e^{bJ^-} e^{cJ^+} (H_n(x, y) t^n) = \sum_{L=0}^{\infty} \sum_{k=0}^{\infty} \frac{b^L}{L!} \frac{c^k}{k!} 2^L \frac{n!}{(n-L)!} (H_{n-L+k}(x, y) t^{n-L+k}) \quad (2.24)$$

So, the left-hand side of each of (2.23) and (2.24) are equal, we can get a new relation for $t \neq 0$ as follows:

$$\begin{aligned} & \exp[2xct + 2bc - yt^2c^2] \cdot H_n\left(x + \frac{b}{t} - ytc, y\right) t^n \\ &= \sum_{L=0}^{\infty} \sum_{k=0}^{\infty} \frac{b^L}{L!} \frac{c^k}{k!} 2^L \frac{n!}{(n-L)!} (H_{n-L+k}(x, y) t^{n-L+k}) \quad (2.25) \end{aligned}$$

This generating function that we derived cannot be obtained by series rearrangement techniques, but it is easy to generate it by using Weisner's method which is very useful in many cases.

Chapter 3: Generating Relations of Hermite Matrix Polynomials and Hermite Matrix Functions by Weisner's Group Theoretic Method

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Chapter 3: Generating relations of Hermite Matrix Polynomials and Hermite Matrix Functions by Weisner's group Theoretic Method

3.1 Introduction

The Hermite Polynomials and Functions play an important role in quantum mechanics and in many boundary value problems in parabolic cylindrical coordinates.

Hermitian matrix is a complex square matrix that is equal to its own conjugate transpose. In a Hermitian matrix all the elements of the main diagonal are real.

A Hermitian matrix M is said to be positive definite if the scalar $z^* M z$ is strictly positive for every non-zero column vector z of n complex numbers.

where z^* is the conjugate transpose of z .

B is a square root of A if $B^2 = A$ and $A = \sqrt{B}$.

An $n \times n$ matrix with n distinct nonzero eigenvalues has 2^n square roots.

A positive definite matrix has a unique positive definite square root.

Some matrices have no square roots. An example is the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

The absolute value of a matrix A equals to $(A^*A)^{\frac{1}{2}}$. And the exponential expression of it is represented by

$$e^{Ax} = \sum_{i=1}^n x_i y_i^T e^{x\lambda_i}, \quad A \in \mathbb{C}^{n \times n}$$

where x_i are the eigen vectors of A ,

y_i are the eigen vectors of A^T , and

λ_i are the eigen values of A and A^T .

(Jador and Company 1996) introduced a theory of Hermite matrix polynomials as finite series solutions of second order matrix deferential equation. These Hermite matrix polynomials $H_n(x, A)$ are defined by

$$H_n(x, A) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k (x\sqrt{2A})^{n-2k}}{k! (n-2k)!} \quad (3.1)$$

where $[n/2]$ is the greatest integer less than or equal to $\frac{n}{2}$, and

$\sqrt{2A}$ is the square root of the matrix $2A$ such that $\sqrt{2A} = \exp(\frac{1}{2}\log(2A))$.

Through our work we consider that A is a Hermitian positive definite matrix in $\mathbb{C}^{m \times m}$ and $\sigma(A)$ is the set of all eigen values of A .

Here

$$H_0(x, A) = I \quad (3.2)$$

Equation (3.1) is a solution for the second order differential equation

$$\frac{d^2v}{dx^2} - xA \frac{dv}{dx} + nAv = 0 \quad (3.3)$$

The scalar Hermite Polynomials can be expressed as (Rainville 1960),

$$H_n(x) = n! \sum_{k=0}^{[nl2]} \frac{(-1)^k (2x)^{n-2k}}{k! (n-2k)!} \quad (3.4)$$

By using (3.1) and (3.4) and special mapping theorem for each real number x , we get

$$\begin{aligned} \sigma(H_n(x, A)) &= \left\{ n! \sum_{k=0}^{[nl2]} \frac{(-1)^k (x \sqrt{2a})^{n-2k}}{k! (n-2k)!} ; a \in \sigma(A) \right\} \\ &= \left\{ H_n \left(x \sqrt{\frac{a}{2}} \right) ; a \in \sigma(A) \right\} . \end{aligned} \quad (3.5)$$

The Hermite matrix functions $T_n(x, A)$ which is used through this thesis was introduced by (Jador and Defez 1998) and defined by

$$T_n(x, A) = e^{\frac{-Ax^2}{4}} H_n(x, A), \quad n \geq 0 \quad (3.6)$$

where $H_n(x, A)$ is the Hermite matrix polynomial as defined in (3.1).

Equation (3.6) is a solution for the second order differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{2} \left(\frac{A^2x^2}{2} + A \right) y + (n+1)Ay = 0 \quad (3.7)$$

3.2 Generating Relations using Weisner's Method

3.2.1 Hermite Matrix Polynomial

3.2.1.1 Recurrence Relations

The following matrix generating function

$$G(x, t) = \exp (xt\sqrt{2A} - t^2I) : x, t \in C \quad (3.8)$$

has the Taylor series about $t = 0$ of the form:

$$\begin{aligned} G(x, t) &= \sum_{n \geq 0} \frac{(\sqrt{2A})^n x^n t^n}{n!} \sum_{n \geq 0} \frac{(-1)^n t^{2n}}{n!} \\ &= \sum_{n \geq 0} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (x\sqrt{2A})^{n-2k} t^n}{k! (n-2k)!} \\ G(x, t) &= \sum_{n \geq 0} \frac{H_n(x, A) t^n}{n!} \end{aligned} \quad (3.9)$$

Differentiating (3.8) partially with respect to x ,

$$\frac{\partial}{\partial x} G(x, t) = t\sqrt{2A} \exp(xt\sqrt{2A} - t^2I)$$

$$\frac{\partial}{\partial x} G(x, t) = t\sqrt{2A} G(x, t)$$

Using (3.9), we get:

$$\begin{aligned} \sum_{n \geq 1} \frac{H'_n(x, A) t^n}{n!} &= t\sqrt{2A} \sum_{n \geq 0} \frac{H_n(x, A) t^n}{n!} \\ \sum_{n \geq 1} \frac{H'_n(x, A) t^n}{n!} &= \sqrt{2A} \sum_{n \geq 0} \frac{H_n(x, A) t^{n+1}}{n!} \\ &\quad \uparrow \\ &\quad \text{Replace } n \text{ by } n-1 \end{aligned}$$

$$\sum_{n \geq 1} \frac{H'_n(x, A) t^n}{n!} = \sqrt{2A} \sum_{n \geq 1} \frac{H_{n-1}(x, A) t^n}{(n-1)!}$$

By comparing t^n coefficients, we get the first recurrence relation

$$H'_n(x, A) = \sqrt{2A} n H_{n-1}(x, A) \quad (3.10)$$

Differentiating (3.8) partially with respect to t ,

$$\frac{\partial}{\partial t} G(x, t) = (x\sqrt{2A} - 2t) \exp(xt\sqrt{2A} - t^2 I)$$

$$\frac{\partial}{\partial t} G(x, t) = (x\sqrt{2A} - 2t) G(x, t)$$

$$\frac{\partial}{\partial t} G(x, t) = x\sqrt{2A} G(x, t) - 2t G(x, t)$$

Using (3.9), we get,

$$\frac{\partial}{\partial t} \sum_{n \geq 0} \frac{H_n(x, A) t^n}{n!} = x\sqrt{2A} \sum_{n \geq 0} \frac{H_n(x, A) t^n}{n!} - 2t \sum_{n \geq 0} \frac{H_n(x, A) t^n}{n!}$$

Since $H_0(x, A) = I$, then $\frac{\partial}{\partial t} H_0(x, A) = 0$, so

$$\sum_{n \geq 1} \frac{n H_n(x, A) t^{n-1}}{n!} = x\sqrt{2A} \sum_{n \geq 0} \frac{H_n(x, A) t^n}{n!} - 2 \sum_{n \geq 0} \frac{H_n(x, A) t^{n+1}}{n!}$$

Replace n by $n+1$

Replace n by $n-1$

$$\sum_{n \geq 0} \frac{(n+1) H_{n+1}(x, A) t^n}{(n+1)!} = x\sqrt{2A} \sum_{n \geq 0} \frac{H_n(x, A) t^n}{n!} - 2 \sum_{n \geq 1} \frac{H_{n-1}(x, A) t^n}{(n-1)!}$$

$$\sum_{n \geq 0} \frac{H_{n+1}(x, A) t^n}{n!} = x\sqrt{2A} \sum_{n \geq 0} \frac{H_n(x, A) t^n}{n!} - 2 \sum_{n \geq 1} \frac{n H_{n-1}(x, A) t^n}{n!}$$

Finding $H_{n+1}(x, A)$ at $n=0$ by using definition (3.1)

$$H_1(x, A) = 1! \frac{(-1)^0 (x\sqrt{2A})^{1-2 \times 0}}{0! (1 - 2 \times 0)!}$$

$$H_1(x, A) = x\sqrt{2A}$$

and since $H_0(x, A) = I$, then:

$$\sum_{n \geq 1} \frac{H_{n+1}(x, A)t^n}{n!} = x\sqrt{2A} \sum_{n \geq 1} \frac{H_n(x, A)t^n}{n!} - 2 \sum_{n \geq 1} \frac{n H_{n-1}(x, A)t^n}{n!}$$

By comparing t^n coefficients, we get the second recurrence relation

$$H_{n+1}(x, A) = x\sqrt{2A} H_n(x, A) - 2nH_{n-1}(x, A), \quad n \geq 1 \quad (3.11)$$

3.2.1.2 First Order Differential Operators

From the recurrence relations, we find the following three operators:
Lowering operator (J^-), increasing operator (J^+) and stationary operator (J^3).

Case (i) To find J^- :

From (3.10) we get: $J^- = t^{-1} \frac{\partial}{\partial x}$ such that

$$J^- [H_n(x, A)t^n] = n\sqrt{2A} H_{n-1}(x, A) t^{n-1} \quad (3.12)$$

Case (ii) To find J^+ :

Multiplying (3.11) by $\sqrt{2A}$, we get:

$$\sqrt{2A} H_{n+1}(x, A) = x\sqrt{2A} \sqrt{2A} H_n(x, A) - 2n\sqrt{2A} H_{n-1}(x, A)$$

Substitute (3.10) in the previous equation, we have:

$$\sqrt{2A} H_{n+1}(x, A) = x2A H_n(x, A) - 2H'_n(x, A)$$

$$\sqrt{2A} H_{n+1}(x, A) = (x2A - 2 \frac{\partial}{\partial x}) H_n(x, A)$$

So we get: $J^+ = t(x2A - 2 \frac{\partial}{\partial x})$ such that

$$J^+[H_n(x, A)t^n] = \sqrt{2A} H_{n+1}(x, A) t^{n+1} \quad (3.13)$$

Case (iii) To find J^3 :

We define: $J^3 = t \frac{\partial}{\partial t}$ such that

$$J^3[H_n(x, A)t^n] = n H_n(x, A)t^n \quad (3.14)$$

3.2.1.3 Commutative Relations of the Operators

Consider $f = f(x, A; t) = H_n(x, A)t^n$ we use the commutator notation $[A, B]$ with $[A, B]f = (AB - BA)f$ in order to find the following relations:

Case (i) $[J^3, J^-] = -J^-$

Proof: $[J^3, J^-]f = (J^3J^- - J^-J^3)f$

$$\begin{aligned} &= \left[t \frac{\partial}{\partial t} \right] \left[t^{-1} \frac{\partial}{\partial x} \right] f - \left[t^{-1} \frac{\partial}{\partial x} \right] \left[t \frac{\partial}{\partial t} \right] f \\ &= \left[t \frac{\partial}{\partial t} \right] \left[t^{-1} \frac{\partial}{\partial x} f \right] - \left[t^{-1} \frac{\partial}{\partial x} \right] \left[t \frac{\partial}{\partial t} f \right] \\ &= \left[t (t^{-1} \frac{\partial^2}{\partial x \partial t} f - t^{-2} \frac{\partial}{\partial x} f) \right] - \left[t^{-1} t \frac{\partial^2}{\partial x \partial t} f \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{\partial^2}{\partial x \partial t} f - t^{-1} \frac{\partial}{\partial x} f - \frac{\partial^2}{\partial x \partial t} f \\
&= -t^{-1} \frac{\partial}{\partial x} f \\
&= -J^- f
\end{aligned}$$

Case (ii) $[J^3, J^+] = J^+$

Proof: $[J^3, J^+] f = (J^3 J^+ - J^+ J^3) f$

$$\begin{aligned}
&= \left[t \frac{\partial}{\partial t} \right] \left[t (2xA - 2 \frac{\partial}{\partial x}) \right] f - \left[t (2xA - 2 \frac{\partial}{\partial x}) \right] \left[t \frac{\partial}{\partial t} \right] f \\
&= \left[t \frac{\partial}{\partial t} \right] \left[2xAtf - 2t \frac{\partial}{\partial x} f \right] - \left[2xAt - 2t \frac{\partial}{\partial x} \right] \left[t \frac{\partial}{\partial t} f \right] \\
&= t \left[2xAt \frac{\partial}{\partial t} f + 2xAf - 2t \frac{\partial^2}{\partial x \partial t} f - 2 \frac{\partial}{\partial x} f \right] - \left[2xAt^2 \frac{\partial}{\partial t} f - 2t^2 \frac{\partial^2}{\partial x \partial t} f \right] \\
&= 2xAt^2 \frac{\partial}{\partial t} f + 2xtAf - 2t^2 \frac{\partial^2}{\partial x \partial t} f - 2t \frac{\partial}{\partial x} f - 2xAt^2 \frac{\partial}{\partial t} f + 2t^2 \frac{\partial^2}{\partial x \partial t} f \\
&= 2xAt^2 \frac{\partial}{\partial t} f + 2xtAf - 2t^2 \frac{\partial^2}{\partial x \partial t} f - 2t \frac{\partial}{\partial x} f - 2xAt^2 \frac{\partial}{\partial t} f + 2t^2 \frac{\partial^2}{\partial x \partial t} f \\
&= 2xtAf - 2t \frac{\partial}{\partial x} f \\
&= t \left(2xA - 2 \frac{\partial}{\partial x} \right) f \\
&= J^+ f
\end{aligned}$$

Case (iii) $[J^+, J^-] = -2A$

Proof: $[J^+, J^-]f = (J^+J^- - J^-J^+)f$

$$\begin{aligned}
&= \left[2xA t - 2t \frac{\partial}{\partial x} \right] \left[t^{-1} \frac{\partial}{\partial x} \right] f - \left[t^{-1} \frac{\partial}{\partial x} \right] \left[2xA t - 2t \frac{\partial}{\partial x} \right] f \\
&= \left[2xA t - 2t \frac{\partial}{\partial x} \right] \left[t^{-1} \frac{\partial}{\partial x} f \right] - \left[t^{-1} \frac{\partial}{\partial x} \right] \left[2xA t f - 2t \frac{\partial}{\partial x} f \right] \\
&= \left[2xA \frac{\partial}{\partial x} f - 2 \frac{\partial^2}{\partial x^2} f \right] - \left[2xA \frac{\partial}{\partial x} f + 2A f - 2 \frac{\partial^2}{\partial x^2} f \right] \\
&= -2A f
\end{aligned}$$

From the operator relations obtained in (3.12) and (3.13), we get:

$$\begin{aligned}
J^+ J^- [H_n(x, A) t^n] &= J^+ [n\sqrt{2A} H_{n-1}(x, A) t^{n-1}] \\
&= 2nA H_n(x, A) t^n
\end{aligned}$$

Now, we can form the following differential operator:

$$\begin{aligned}
L &= xA \frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2} - At \frac{\partial}{\partial t} \\
L &= J^+ J^- - 2nA \\
L &= t \left(2xA - 2 \frac{\partial}{\partial x} \right) \cdot t^{-1} \frac{\partial}{\partial x} - 2nA \\
&= 2xA \frac{\partial}{\partial x} - 2 \frac{\partial^2}{\partial x^2} - 2nA \\
L &= \frac{\partial^2}{\partial x^2} - xA \frac{\partial}{\partial x} + nA, \quad \text{where } n = t \frac{\partial}{\partial t} \tag{3.15}
\end{aligned}$$

which commutes with J^+ , J^- and J^3 . That is

$$[L, J^+] = 0, [L, J^-] = 0 \text{ and } [L, J^3] = 0$$

Remark: We observe that as in chapter 2 and according to Weisner's group theoretic method, the above relations are not isomorphic to any commutation relations obeyed by a well-known Lie algebra of a local Lie group. Therefore we cannot link the set of differential operators $\{J^3, J^-, J^+\}$ with a particular Lie algebra and hence we are only bound to say that these differential operators generate three dimensional Lie algebra.

3.2.1.4 Extended Form of the Group Generated by Operators

Here we find the extended form of the group generated by J^3, J^- and J^+ .

Case (i) $J^3 = t \frac{\partial}{\partial t}$,

Let b be any arbitrary constant. Then,

$$\frac{dx}{db} = 0, \text{ which gives: } x = x_0,$$

$$\frac{dv}{db} = 0, \text{ which gives: } v = 1,$$

$$\frac{dt}{db} = t, \text{ which gives: } \ln t = b + k$$

and if we let $e^k = t_0$, then we have: $t = t_0 e^b$

and

$$e^{bJ^3} f(x, A; t) = f(x, A, t e^b) \quad (3.16)$$

Case (ii) $J^- = t^{-1} \frac{\partial}{\partial x}$.

Let c be any arbitrary constant. Then, we have:

$$\frac{dt}{dc} = 0, \text{ which gives: } t = t_0,$$

$$\frac{dv}{dc} = 0, \text{ which gives: } v = 1,$$

$$\frac{dx}{dc} = t_0^{-1}, \text{ which gives: } x = t_0^{-1}c + k$$

and if we set $c = 0$ and $k = x_0$, we have: $x = x_0 + t_0^{-1}c$

and

$$e^{cJ^-} f(x, A; t) = f\left(x + \frac{c}{t}, A, t\right) \quad (3.17)$$

Case (iii) $J^+ = 2xtA - 2t \frac{\partial}{\partial x}$,

Let a be any arbitrary constant. Then,

$$\frac{dt}{da} = 0 \text{ which gives: } t = t_0,$$

$$\frac{dx}{da} = -2t_0, \text{ which gives } x = -2t_0 a + k$$

and if we set $a = 0$ and $k = x_0$, we have: $x = x_0 - 2t_0 a$

$$\frac{dv}{da} = 2xtA v, \text{ which gives: } \ln v = 2xtAa + k$$

and substituting the values of x and t as given above, we get

$$\ln v = 2x_0 a t_0 A - 2t_0^2 a^2 A + k.$$

When $a = 0$, $v(0) = 1$, we get $k = 0$ and hence,

$$v = \exp(2x_0at_0A - 2t_0^2 a^2A),$$

and

$$e^{aJ^+} f(x, A; t) = \exp(2xatA - 2t^2 a^2A) \cdot f(x - 2ta, A, t) \quad (3.18)$$

3.2.1.5 Some Generating Functions

From the extended forms (3.16), (3.17) and (3.18), we study the effect of the operators in the following three cases:

Case 1: The effect of the three operators: J^- , J^+ and J^3 on $f(x, A, t)$:

$$\begin{aligned} & e^{aJ^+} e^{cJ^-} e^{bJ^3} f(x, A, t) \\ &= \exp(2xatA - 2t^2 a^2A) e^{cJ^-} e^{bJ^3} f(x - 2ta, A, t) \\ &= [\exp(2xatA - 2t^2 a^2A) e^{bJ^3} f(x - 2ta + t^{-1}c, A, t)] \\ &= \exp(2xatA - 2t^2 a^2A) \cdot f(x - 2ta + t^{-1}c, A, te^b) . \end{aligned}$$

Now if we put $b = 0$; $a \neq 0$, $c \neq 0$ and writing $f(x, y, t) = H_n(x, A)t^n$, we get

$$e^{aJ^+} e^{cJ^-} (H_n(x, A, t^n)) = \exp(2xatA - 2t^2 a^2A) \cdot H_n(x - 2at + ct^{-1}, A, t). \quad (3.19)$$

Working on left hand side and by using the power series expansions, we get for $t \neq 0$:

$$\begin{aligned} & e^{aJ^+} e^{cJ^-} (H_n(x, A, t^n)) \\ &= \sum_{k=0}^{\infty} \frac{(aJ^+)^k}{k!} e^{cJ^-} (H_n(x, A)t^n) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{a^k}{k!} e^{cJ^-} J^{+k} (H_n(x, A) t^n) \\
&= \sum_{k=0}^{\infty} \frac{a^k}{k!} e^{cJ^-} ((\sqrt{2A})^k H_{n+k}(x, A) t^{n+k}) \\
&= \sum_{k=0}^{\infty} \frac{a^k}{k!} \sum_{L=0}^{\infty} \frac{(cJ^-)^L}{L!} ((\sqrt{2A})^k H_{n+k}(x, A) t^{n+k}) \\
&= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{a^k}{k!} \frac{c^L J^{-L}}{L!} ((\sqrt{2A})^k H_{n+k}(x, A) t^{n+k}) \\
&= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{a^k c^L}{k! L!} J^{-L} ((\sqrt{2A})^k H_{n+k}(x, A) t^{n+k}) \\
&= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{a^k c^L}{k! L!} (\sqrt{2A})^{k+L} (n+k)(n+k-1) \dots (n+k-(L-1)) H_{n+k-L}(x, A) t^{n+k-L} \\
&e^{aJ^+} e^{cJ^-} (H_n(x, A, t^n)) \\
&= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{a^k c^L}{k! L!} (\sqrt{2A})^{k+L} \frac{(n+k)!}{(n+k-L)!} H_{n+k-L}(x, A) t^{n+k-L}, \quad (3.20)
\end{aligned}$$

Now from (3.19) and (3.20) we get for $t \neq 0$:

$$\begin{aligned}
&\exp(2xatA - 2t^2 a^2 A) \cdot H_n(x - 2at + \frac{c}{t}, A, t) \\
&= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{a^k c^L}{k! L!} (\sqrt{2A})^{k+L} \frac{(n+k)!}{(n+k-L)!} H_{n+k-L}(x, A) t^{n+k-L} \\
&\hspace{25em} (3.21)
\end{aligned}$$

where $|\sqrt{2A}| < \infty, \ln|\sqrt{2A}| < \infty$

Case 2: The effect of one operator: J^- on $H_n(x, A)$:

Put $t = 1$ and $a = 0$ in (3.21), we get:

$$H_n(x + c, A) = \sum_{L=0}^{\infty} \frac{c^L}{L!} (\sqrt{2A})^L n(n-1) \dots (n-L+1) H_{n-L}(x, A)$$

$$H_n(x + c, A) = \sum_{L=0}^{\infty} \frac{c^L}{L!} (\sqrt{2A})^L \frac{n!}{(n-L)!} H_{n-L}(x, A) \quad (3.22)$$

Case 3: The effect of one operator: J^+ on $H_n(x, A)$:

Set $t = 1$ and $c = 0$ in (3.21), we get:

$$\exp(2xaA - 2a^2A) H_n(x - 2a, A) = \sum_{k=0}^{\infty} \frac{a^k}{k!} (\sqrt{2A})^k H_{n+k}(x, A) \quad (3.23)$$

3.2.1.6 Some Applications

By using the relation (3.5) in (3.23), we get:

$$\exp(2xaA - 2a^2A) H_n\left((x - 2a) \sqrt{\frac{A}{2}}\right) = \sum_{k=0}^{\infty} \frac{a^k (\sqrt{2A})^k}{k!} H_{n+k}\left(x \sqrt{\frac{A}{2}}\right) \quad (3.24)$$

where $|\sqrt{2A}| < \infty$.

This is a generalization of the generating functions obtained by (Weisner 1959, Rainville 1960 and Mc Bride 1971), and new generating function of Hermite polynomial $H_n(x)$ in terms of matrix argument can be obtained by putting $n=0$, or by using (3.4) and (3.5) in (3.23), so we get:

$$\exp(2xaA - 2a^2A) = \sum_{k=0}^{\infty} \frac{a^k (\sqrt{2A})^k}{k!} H_k\left(x \sqrt{\frac{A}{2}}\right) \quad (3.25)$$

with $|\sqrt{2A}| < \infty$.

which is a well-known generating function of Hermite polynomial $H_n(x)$ in terms of matrix argument obtained by (Rainville 1960).

3.2.2 Hermite Matrix Function

3.2.2.1 Recurrence Relations

Differentiating equation (3.6) with respect to x , gives:

$$\frac{\partial}{\partial x} T_n(x, A) = e^{\frac{-Ax^2}{4}} H'_n(x, A) - \frac{Ax}{2} T_n(x, A).$$

Using (3.10), we have:

$$\begin{aligned} \frac{\partial}{\partial x} T_n(x, A) &= e^{\frac{-Ax^2}{4}} n\sqrt{2A} H_{n-1}(x, A) - \frac{Ax}{2} T_n(x, A) \\ \frac{\partial}{\partial x} T_n(x, A) &= n\sqrt{2A} T_{n-1}(x, A) - \frac{Ax}{2} T_n(x, A) \\ \left(\frac{\partial}{\partial x} + \frac{Ax}{2}\right) T_n(x, A) &= n\sqrt{2A} T_{n-1}(x, A) \end{aligned} \quad (3.26)$$

Multiplying equation (3.11) by $e^{\frac{-Ax^2}{4}}$ gives:

$$T_{n+1}(x, A) = x\sqrt{2A} T_n(x, A) - 2nT_{n-1}(x, A), \quad n \geq 1 \quad (3.27)$$

3.2.2.2 First Order Differential Operators

From the recurrence relations, we find the following three operators:
Lowering operator (J^-), increasing operator (J^+) and stationary operator (J^3).

Case (i) To find J^- :

From (3.26) we get: $J^- = t^{-1} \left(\frac{Ax}{2} + \frac{\partial}{\partial x} \right)$ such that

$$J^- [T_n(x, A) t^n] = n\sqrt{2A} T_{n-1}(x, A) t^{n-1} \quad (3.28)$$

Case (ii) To find J^+ :

Multiply (3.27) by $\sqrt{2A}$, we get:

$$\sqrt{2A} T_{n+1}(x, A) = x \sqrt{2A} \sqrt{2A} T_n(x, A) - 2n\sqrt{2A} T_{n-1}(x, A) .$$

Substitute (3.23) in the previous equation, we get:

$$\sqrt{2A} T_{n+1}(x, A) = 2Ax T_n(x, A) - 2\left(\frac{\partial}{\partial x} + \frac{Ax}{2}\right) T_n(x, A)$$

$$\sqrt{2A} T_{n+1}(x, A) = 2Ax T_n(x, A) - 2\frac{\partial}{\partial x} T_n(x, A) - Ax T_n(x, A)$$

$$\sqrt{2A} T_{n+1}(x, A) = (xA - 2\frac{\partial}{\partial x}) T_n(x, A) .$$

So we get: $J^+ = t(xA - 2\frac{\partial}{\partial x})$ such that

$$J^+[T_n(x, A)t^n] = \sqrt{2A} T_{n+1}(x, A) t^{n+1} \quad (3.29)$$

Case (iii) To find J^3 :

We define: $J^3 = t\frac{\partial}{\partial t}$ such that

$$J^3[T_n(x, A)t^n] = n T_n(x, A)t^n \quad (3.30)$$

3.2.2.3 Commutative Relations of the Operators

We use the commutator notation $[A, B]$ with $[A, B]f = (AB - BA)f$ where $f = f(x, A; t) = T_n(x, A)t^n$, in order to get the following three relations:

Case (i) $[J^3, J^-] = -J^-$

Proof: $[J^3, J^-]f = (J^3J^- - J^-J^3)f$

$$\begin{aligned}
 &= \left[t \frac{\partial}{\partial t} \right] \left[t^{-1} \left(\frac{Ax}{2} + \frac{\partial}{\partial x} \right) \right] f - \left[t^{-1} \left(\frac{Ax}{2} + \frac{\partial}{\partial x} \right) \right] \left[t \frac{\partial}{\partial t} \right] f \\
 &= \left[t \frac{\partial}{\partial t} \right] \left[\frac{Axt^{-1}}{2} f + t^{-1} \frac{\partial}{\partial x} f \right] - \left[t^{-1} \left(\frac{Ax}{2} + \frac{\partial}{\partial x} \right) \right] \left[t \frac{\partial}{\partial t} f \right] \\
 &= \left[t \frac{Ax}{2} (t^{-1} \frac{\partial}{\partial t} f - t^{-2} f) \right] + \left[t (t^{-1} \frac{\partial^2}{\partial x \partial t} f - t^{-2} \frac{\partial}{\partial x} f) \right] - \left[\left(\frac{Ax}{2} \frac{\partial}{\partial t} f + \frac{\partial^2}{\partial x \partial t} f \right) \right] \\
 &= \left[\left(\frac{Ax}{2} \frac{\partial}{\partial t} f - \frac{Ax}{2} t^{-1} f \right) \right] + \left[\left(\frac{\partial^2}{\partial x \partial t} f - t^{-1} \frac{\partial}{\partial x} f \right) \right] - \left[\left(\frac{Ax}{2} \frac{\partial}{\partial t} f + \frac{\partial^2}{\partial x \partial t} f \right) \right] \\
 &= \frac{Ax}{2} \frac{\partial}{\partial t} f - \frac{Ax}{2} t^{-1} f + \frac{\partial^2}{\partial x \partial t} f - t^{-1} \frac{\partial}{\partial x} f - \frac{Ax}{2} \frac{\partial}{\partial t} f - \frac{\partial^2}{\partial x \partial t} f \\
 &= -\frac{Ax}{2} t^{-1} f - t^{-1} \frac{\partial}{\partial x} f \\
 &= -t^{-1} \left(\frac{Ax}{2} + \frac{\partial}{\partial x} \right) f \\
 &= -J^- f
 \end{aligned}$$

Case (ii) $[J^3, J^+] = J^+$

Proof: $[J^3, J^+]f = (J^3J^+ - J^+J^3)f$

$$= \left[t \frac{\partial}{\partial t} \right] \left[t (xA - 2 \frac{\partial}{\partial x}) \right] f - \left[t (xA - 2 \frac{\partial}{\partial x}) \right] \left[t \frac{\partial}{\partial t} \right] f$$

$$\begin{aligned}
&= \left[t \frac{\partial}{\partial t} \right] \left[xAtf - 2t \frac{\partial}{\partial x} f \right] - \left[xAt - 2t \frac{\partial}{\partial x} \right] \left[t \frac{\partial}{\partial t} f \right] \\
&= \left[xAt \left(t \frac{\partial}{\partial t} f + f \right) - 2t \left(t \frac{\partial^2}{\partial x \partial t} f + \frac{\partial}{\partial x} f \right) \right] - \left[xAt^2 \frac{\partial}{\partial t} f - 2t^2 \frac{\partial^2}{\partial x \partial t} f \right] \\
&= xAt^2 \frac{\partial}{\partial t} f + xAtf - 2t^2 \frac{\partial^2}{\partial x \partial t} f - 2t \frac{\partial}{\partial x} f - xAt^2 \frac{\partial}{\partial t} f + 2t^2 \frac{\partial^2}{\partial x \partial t} f \\
&= xAtf - 2t \frac{\partial}{\partial x} f \\
&= t(xA - 2 \frac{\partial}{\partial x})f \\
&= J^+ f
\end{aligned}$$

Case (iii) $[J^+, J^-] = -2A$

Proof: $[J^+, J^-]f = (J^+J^- - J^-J^+)f$

$$\begin{aligned}
&= \left[xAt - 2t \frac{\partial}{\partial x} \right] \left[\frac{Axt^{-1}}{2} + t^{-1} \frac{\partial}{\partial x} \right] f - \left[\frac{Axt^{-1}}{2} + t^{-1} \frac{\partial}{\partial x} \right] \left[xAt - 2t \frac{\partial}{\partial x} \right] f \\
&= \left[xAt - 2t \frac{\partial}{\partial x} \right] \left[\frac{Axt^{-1}}{2} f + t^{-1} \frac{\partial}{\partial x} f \right] - \left[\frac{Axt^{-1}}{2} + t^{-1} \frac{\partial}{\partial x} \right] \left[xAt f - 2t \frac{\partial}{\partial x} f \right] \\
&= \frac{A^2 x^2}{2} f - A \left(x \frac{\partial}{\partial x} f + f \right) + xA \frac{\partial}{\partial x} f - 2 \frac{\partial^2}{\partial x^2} f - \left[\frac{A^2 x^2}{2} f - \right. \\
&\quad \left. Ax \frac{\partial}{\partial x} f + A \left(x \frac{\partial}{\partial x} f + f \right) - 2 \frac{\partial^2}{\partial x^2} f \right] \\
&= -2Af
\end{aligned}$$

From the operator relations obtained in (3.28) and (3.29), we get:

$$\begin{aligned}
J^+ J^- [T_n(x, A) t^n] &= J^+ [n\sqrt{2A} T_{n-1}(x, A) t^{n-1}] \\
&= 2nA T_n(x, A) t^n
\end{aligned}$$

Now, we can form the following differential operator:

$$\begin{aligned}
L &= \frac{\partial^2}{dx^2} - \frac{1}{2} \left(\frac{x^2 A^2}{2} + A \right) + A \left(1 + t \frac{\partial}{\partial t} \right) \\
L &= J^+ J^- - 2nA \\
&= t \left(xA - 2 \frac{\partial}{\partial x} \right) \cdot t^{-1} \left(\frac{Ax}{2} + \frac{\partial}{\partial x} \right) - 2nA \\
&= \frac{x^2 A^2}{2} + xA \frac{\partial}{\partial x} - A - AX \frac{\partial}{\partial x} - 2 \frac{\partial^2}{dx^2} - 2nA \\
&= \frac{x^2 A^2}{2} - A - 2 \frac{\partial^2}{dx^2} - 2nA \\
&= \frac{x^2 A^2}{2} - 2A + A - 2 \frac{\partial^2}{dx^2} - 2nA \\
&= -2 \frac{\partial^2}{dx^2} + \frac{x^2 A^2}{2} - 2A + A - 2nA \\
&= -\frac{\partial^2}{dx^2} + \frac{x^2 A^2}{4} - A + \frac{A}{2} - nA \\
&= -\frac{\partial^2}{dx^2} + \frac{1}{2} \left(\frac{x^2 A^2}{2} + A \right) - A(1 + n) \\
L &= \frac{\partial^2}{dx^2} - \frac{1}{2} \left(\frac{x^2 A^2}{2} + A \right) + A(1 + n), \quad \text{where } n = t \frac{\partial}{\partial t} \quad (3.31)
\end{aligned}$$

which commutes with J^+, J^- and J^3 . That is

$$[L, J^+] = 0, [L, J^-] = 0 \text{ and } [L, J^3] = 0$$

Similar to previous case, these differential operators generate three-dimensional Lie algebra.

3.2.2.4 Extended Form of the Group Generated by Operators

Here we find the extended form of the group generated by J^3, J^- and J^+ .

Case (i) $J^3 = t \frac{\partial}{\partial t}$:

Follows similarly to the case of the stationary operator of Hermite matrix polynomial, we have:

$$e^{bJ^3} f(x, A; t) = f(x, A, t e^b) \quad (3.32)$$

Case (ii) $J^- = t^{-1} \left(\frac{Ax}{2} + \frac{\partial}{\partial x} \right)$:

Let c be any arbitrary constant. Then,

$$\frac{dt}{dc} = 0 \text{ which gives } t = t_0,$$

$$\frac{dx}{dc} = t_0^{-1} \text{ which gives } x = t_0^{-1}c + k$$

and here if $c = 0$ and $k = x_0$, we have $x = x_0 + t_0^{-1}c$

$$\frac{dv}{dc} = \frac{Ax}{2} t^{-1} v = \frac{A(x_0 + t_0^{-1}c)t_0^{-1}}{2} v, \text{ which gives}$$

$$\ln v = \frac{t_0^{-1}Ax_0c}{2} + \frac{t_0^{-2}Ac^2}{4} + k, \text{ and here}$$

$$\text{if } c = 0, v(0) = 1 \text{ then } k = 0 \text{ and } \ln v = \frac{t_0^{-1}Ax_0c}{2} + \frac{t_0^{-2}Ac^2}{4}$$

$$\text{thus, } v = \exp \left(\frac{t_0^{-1}Ax_0c}{2} + \frac{t_0^{-2}Ac^2}{4} \right)$$

and

$$e^{cJ^-} f(x, A; t) = \exp\left(\frac{Axc}{2t} + \frac{Ac^2}{4t^2}\right) f\left(x + \frac{c}{t}, A, t\right) \quad (3.33)$$

Case (iii) $J^+ = Axt - 2t \frac{\partial}{\partial x}$:

Let a be any arbitrary constant. Then,

$$\frac{dt}{da} = 0 \text{ which gives } t = t_0,$$

$$\frac{dx}{da} = -2t_0 \text{ which gives } x = -2t_0 a + k$$

and here if $a = 0, v(0) = 1$, then $k = 0$ and $x = x_0 - 2t_0 a$

$$\frac{dv}{da} = Axt v = A(x_0 - 2t_0 a)t_0, \text{ which gives}$$

$$\ln v = x_0 at_0 A - t_0^2 a^2 A + k$$

hence, $v = L \exp(x_0 at_0 A - t_0^2 a^2 A)$ and if $a = 0, v(0) = 1$ then

$k = 0$ and we get:

$$v = \exp(x_0 at_0 A - t_0^2 a^2 A)$$

and

$$e^{aJ^+} f(x, A; t) = \exp(Axta - At^2 a^2) \cdot f(x - 2t a, A, t) \quad (3.34)$$

3.2.2.5 Some Generating Functions

From the extended forms (3.32), (3.33) and (3.34), we study the effect of the operators in the following three cases:

Case 1: The effect of the three operators: J^- , J^+ and J^3 on $f(x, A, t)$:

$$\begin{aligned}
 e^{aJ^+} e^{cJ^-} e^{bJ^3} f(x, A, t) &= \\
 \exp(Axta - At^2 a^2) e^{cJ^-} e^{bJ^3} [f(x - 2at, A, t)] \\
 &= \exp\left(\frac{Axc}{2t} + \frac{Ac^2}{4t^2}\right) \exp(Axta - At^2 a^2) e^{bJ^3} [f(x - 2at + \frac{c}{t}, A, t)] \\
 &= \exp\left(\frac{Axc}{2t} + \frac{Ac^2}{4t^2}\right) \exp(Axta - At^2 a^2) \cdot f(x - 2at + \frac{c}{t}, A, te^b) \\
 &= \exp\left(\frac{Axc}{2t} + \frac{Ac^2}{4t^2} + xatA - a^2t^2 A\right) f(x - 2at + \frac{c}{t}, A, te^b)
 \end{aligned}$$

Now, if we put $b = 0$; $a \neq 0$, $c \neq 0$ and letting $f(x, y, t) = T_n(x, A)t^n$, we get:

$$e^{aJ^+} e^{cJ^-} (T_n(x, A, t^n)) = \exp\left(\frac{Axc}{2t} + \frac{Ac^2}{4t^2} + Axta - Aa^2t^2\right) \cdot T_n(x - 2ta + \frac{c}{t}, A, t^n) \quad (3.35)$$

Working on left hand side and by using the power series expansions, we get for $t \neq 0$:

$$e^{aJ^+} e^{cJ^-} (T_n(x, A, t^n))$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{(aJ^+)^k}{k!} e^{cJ^-} (T_n(x, A) t^n) \\
&= \sum_{k=0}^{\infty} \frac{a^k}{k!} e^{cJ^-} J^{+k} (T_n(x, A) t^n) \\
&= \sum_{k=0}^{\infty} \frac{a^k}{k!} e^{cJ^-} ((\sqrt{2A})^k T_{n+k}(x, A) t^{n+k}) \\
&= \sum_{k=0}^{\infty} \frac{a^k}{k!} \sum_{L=0}^{\infty} \frac{(cJ^-)^L}{L!} ((\sqrt{2A})^k T_{n+k}(x, A) t^{n+k}) \\
&= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{a^k}{k!} \frac{c^L J^{-L}}{L!} ((\sqrt{2A})^k T_{n+k}(x, A) t^{n+k}) \\
&= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{a^k c^L}{k! L!} J^{-L} ((\sqrt{2A})^k T_{n+k}(x, A) t^{n+k}) \\
&= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{a^k c^L}{k! L!} ((\sqrt{2A})^{k+L} (n+k)(n+k-1) \dots (n+k-(L+1)) T_{n+k-L}(x, A) t^{n+k-L}) \\
&e^{aJ^+} e^{cJ^-} (T_n(x, A, t^n)) \\
&= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{a^k c^L (\sqrt{2A})^{k+L}}{k! L!} \frac{(n+k)!}{(n+k-L-2)!} T_{n+k-L}(x, A) t^{n+k-L}, \quad (3.36)
\end{aligned}$$

Now, from (3.35) and (3.36), we get:

$$\begin{aligned}
&\exp\left(\frac{Axc}{2t} + \frac{Ac^2}{4t^2} + Axta - At^2 a^2\right) \cdot T_n\left(x - 2ta + \frac{c}{t}, A, t^n\right) \\
&= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{a^k c^L (\sqrt{2A})^{k+L}}{k! L!} \frac{(n+k)!}{(n+k-L-2)!} T_{n+k-L}(x, A) t^{n+k-L}, \\
&\hspace{25em} (3.37)
\end{aligned}$$

where $|\sqrt{2A}| < \infty$, $\ln|\sqrt{2A}| < \infty$ and $t \neq 0$.

Case 2: The effect of one operator: J^- on $T_n(x, A)$:

Put $t = 1$ and $a = 0$ in (3.37), we get:

$$\exp\left(\frac{Axc}{2} + \frac{Ac^2}{4}\right) T_n(x + c, A) = \sum_{L=0}^{\infty} \frac{c^L}{L!} (\sqrt{2A})^L \frac{n!}{(n-L-2)!} T_{n-L}(x, A) \quad (3.38)$$

Case 3: The effect of one operator: J^+ on $T_n(x, A)$:

Let $t = 1$ and $c = 0$ in (3.37), we get:

$$\exp(xaA - a^2A) T_n(x - 2a, A) = \sum_{k=0}^{\infty} \frac{a^k}{k!} (\sqrt{2A})^k T_{n+k}(x, A) \quad (3.39)$$

3.2.2.6 Some Applications

Using the relation (3.6) in (3.39), we get:

$$\begin{aligned} \exp(xaA - a^2A) e^{\frac{-A(x-2a)^2}{4}} H_n(x - 2a, A) &= \sum_{k=0}^{\infty} \frac{a^k}{k!} (\sqrt{2A})^k e^{\frac{-Ax^2}{4}} H_{n+k}(x, A) \\ \exp(xaA - a^2A - \frac{Ax^2}{4} + axA - a^2A) H_n(x - 2a, A) &= \sum_{k=0}^{\infty} \frac{a^k}{k!} (\sqrt{2A})^k e^{\frac{-Ax^2}{4}} H_{n+k}(x, A) \\ \exp(2xaA - 2a^2A) H_n(x - 2a, A) &= \sum_{k=0}^{\infty} \frac{a^k}{k!} (\sqrt{2A})^k H_{n+k}(x, A) \end{aligned} \quad (3.40)$$

where $|\sqrt{2A}| < \infty$. This relation is a special case of (Shahwan 2009).

Also, by using the relation (3.5) in (3.39), we get:

$$\exp(2xaA - 2a^2A) H_n\left((x - 2a)\sqrt{\frac{A}{2}}\right) = \sum_{k=0}^{\infty} \frac{a^k}{k!} (\sqrt{2A})^k H_{n+k}\left(x\sqrt{\frac{A}{2}}\right) \quad (3.41)$$

where $|\sqrt{2A}| < \infty$

This is also a special case of (Shahwan 2009) which has been derived earlier by (Weisner 1955, Mc. Bride 1971 and Rainville 1960).

If we put $n=0$ in (3.41), we have:

$$\exp(2xaA - 2a^2A) = \sum_{k=0}^{\infty} \frac{a^k}{k!} (\sqrt{2A})^k H_k\left(x\sqrt{\frac{A}{2}}\right) \quad (3.42)$$

where $|\sqrt{2A}| < \infty$

This is the well-known generating relation for Hermite polynomial given by (Rainville 1960), and it is also a special case of (Shahwan 2009).

Chapter 4: Generating Relations of Generalized Hermite Polynomials Associated with Functions of Parabolic Cylinder

4.1 Introduction

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Generating Relations of Generalized Hermite Polynomials Associated with Functions of Parabolic Cylinder

4.1 Introduction

The associated Hermite polynomials $He_n^v(z)$ that generalize the scaled Hermite polynomial $He_n(z)$ corresponding to $v = 0$ are introduced to represent the raising and lowering of the indices of functions of the parabolic cylinder $D_v(z)$ in finite integer steps.

The functions of parabolic cylinder $D_v(z)$ are the solutions of Weber equation, an ordinary linear second order differential equation for a complex variable z with a complex parameter v (Wunsche 2003).

$$\left(\frac{d^2}{dz^2} - \frac{z^2}{4} + v + \frac{1}{2} \right) D_v(z) = 0 \quad (4.1)$$

The function of parabolic cylinder $D_v(z)$ satisfies the Whittaker standardization:

$$\begin{aligned} D_v(0) &= \frac{2^{v/2} \sqrt{\pi}}{\Gamma\left(\frac{1-v}{2}\right)} \\ \frac{\partial D_v}{\partial z}(0) &= -\frac{2^{\frac{v+1}{2}} \sqrt{\pi}}{\Gamma\left(-\frac{v}{2}\right)} \end{aligned} \quad (4.2)$$

The explicit series representation of $D_v(z)$ is given by:

$$D_v(z) = \exp\left(-\frac{z^2}{4}\right) \frac{2^{v/2}\sqrt{\pi}}{\Gamma(-v)} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k-v)}{k! \Gamma\left(\frac{k-v+1}{2}\right)} \left(\frac{z}{\sqrt{2}}\right)^k \quad (4.3)$$

For nonnegative integer n , the function can be expressed as follows:

$$D_n(z) = \exp\left(\frac{-z^2}{4}\right) He_n(z), \quad (n = 0, 1, 2, \dots) \quad (4.4)$$

where $He_n(z)$ are the well-known Hermite polynomials in scaled form defined by Rodrigues type.

$$He_n(z) = \exp\left(-\frac{1}{2} \frac{\partial^2}{\partial z^2}\right) z^n = (-1)^n \exp\left(\frac{z^2}{2}\right) \frac{\partial^n}{\partial z^n} \exp\left(-\frac{z^2}{2}\right) \quad (4.5)$$

$He_n(z)$ can be expressed using the usual Hermite polynomials $H_n(x)$ by:

$$He_n(z) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!}{k! (n-2k)! 2^k} z^{n-2k} = \frac{1}{\sqrt{2^n}} H_n\left(\frac{z}{\sqrt{2}}\right) \quad (4.6)$$

4.2 Generating Relations using Weisner's Method

4.2.1 Recurrence Relations

Using Weber equation as in (4.1), we have:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{z^2}{4} + v + \frac{1}{2} \right) D_v(z) = 0$$

$$\frac{\partial^2}{\partial z^2} D_v(z) - \frac{z^2}{4} D_v(z) + v D_v(z) + \frac{1}{2} D_v(z) = 0 \quad (4.7)$$

Also by using the equation as in (4.3), we have:

$$D_v(z) = \exp\left(-\frac{z^2}{4}\right) \frac{2^{\frac{v}{2}} \sqrt{\pi}}{\Gamma(-v)} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k-v)}{k! \Gamma\left(\frac{k-v+1}{2}\right)} \left(\frac{z}{\sqrt{2}}\right)^k$$

$$D_v(z) = \exp\left(-\frac{z^2}{4}\right) \frac{2^{\frac{v}{2}} \sqrt{\pi}}{\Gamma(-v)} \left[\frac{\Gamma(-v)}{\Gamma\left(\frac{-v+1}{2}\right)} + \sum_{k=1}^{\infty} \frac{(-1)^k \Gamma(k-v)}{k! \Gamma\left(\frac{k-v+1}{2}\right)} \left(\frac{z}{\sqrt{2}}\right)^k \right]$$

$$\frac{\partial}{\partial z} D_v(z) = \exp\left(-\frac{z^2}{4}\right) \frac{2^{v/2} \sqrt{\pi}}{\Gamma(-v)} \sum_{k=1}^{\infty} \frac{(-1)^k \Gamma(k-v)}{k! \Gamma\left(\frac{k-v+1}{2}\right)} k \frac{z^{k-1}}{\sqrt{2}^k} - \frac{2z}{4} D_v(z)$$

$$= \exp\left(-\frac{z^2}{4}\right) \frac{2^{v/2} \sqrt{\pi}}{\Gamma(-v)} \sum_{k=1}^{\infty} \frac{(-1)^k \Gamma(k-v)}{(k-1)! \Gamma\left(\frac{k-v+1}{2}\right)} \frac{z^{k-1}}{\sqrt{2}^k} - \frac{z}{2} D_v(z)$$

↑
replace k by k+1

$$= \exp\left(-\frac{z^2}{4}\right) \frac{2^{v/2} \sqrt{\pi}}{\Gamma(-v)} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \Gamma(k+1-v)}{k! \Gamma\left(\frac{k-v+2}{2}\right)} \frac{z^k}{\sqrt{2}^{k+1}} - \frac{z}{2} D_v(z)$$

$$= \exp\left(-\frac{z^2}{4}\right) \frac{2^{v/2} \sqrt{\pi}}{\Gamma(-v)} \sum_{k=0}^{\infty} \frac{(-1)(-1)^k (k-v) \Gamma(k-v)}{k! \frac{k-v}{2} \Gamma\left(\frac{k-v}{2}\right)} \frac{z^k}{\sqrt{2}^1} \left(\frac{z}{\sqrt{2}}\right)^k - \frac{z}{2} D_v(z)$$

$$= \exp\left(-\frac{z^2}{4}\right) \frac{2^{\frac{v}{2}}\sqrt{\pi}}{\Gamma(-v)} (-\sqrt{2}) \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k-v)}{k! \Gamma\left(\frac{k-v}{2}\right)} \left(\frac{z}{\sqrt{2}}\right)^k - \frac{z}{2} D_v(z).$$

Thus,

$$\frac{\partial}{\partial z} D_v(z) = v D_{v-1}(z) - \frac{z}{2} D_v(z) \quad (4.8)$$

If we replace v by $v - 1$ in (4.8), we get:

$$\frac{\partial}{\partial z} D_{v-1}(z) = (v-1) D_{v-2}(z) - \frac{z}{2} D_{v-1}(z) \quad (4.9)$$

$$\frac{\partial^2}{\partial z^2} D_v(z) = \frac{\partial}{\partial z} [v D_{v-1}(z)] - \frac{\partial}{\partial z} \left[\frac{z}{2} D_v(z) \right]$$

$$\frac{\partial^2}{\partial z^2} D_v(z) = v \frac{\partial}{\partial z} D_{v-1}(z) - \frac{1}{2} D_v(z) - \frac{z}{2} \frac{\partial}{\partial z} D_v(z)$$

Now, from (4.9) and (4.8), we have:

$$\frac{\partial^2}{\partial z^2} D_v(z) = v(v-1) D_{v-2}(z) - zv D_{v-1}(z) - \frac{1}{2} D_v(z) + \frac{z^2}{4} D_v(z)$$

Substitute the last equation in (4.7), we have:

$$v(v-1) D_{v-2}(z) - zv D_{v-1}(z) - \frac{1}{2} D_v(z) + \frac{z^2}{4} D_v(z) - \frac{z^2}{4} D_v(z) + v D_v(z) + \frac{1}{2} D_v(z) = 0$$

$$v(v-1) D_{v-2}(z) - zv D_{v-1}(z) + v D_v(z) = 0, \quad (v \neq 0)$$

Dividing by v , we get:

$$(v-1) D_{v-2}(z) - z D_{v-1}(z) + D_v(z) = 0$$

Replace v by $v + 1$, we get:

$$\begin{aligned} vD_{v-1}(z) - zD_v(z) + D_{v+1}(z) &= 0 \\ D_{v+1}(z) - zD_v(z) + vD_{v-1}(z) &= 0 \end{aligned} \quad (4.10)$$

$$D_{v+1}(z) - \frac{z}{2}D_v(z) - \frac{z}{2}D_v(z) + vD_{v-1}(z) = 0$$

$$\begin{aligned} D_{v+1}(z) + \frac{\partial}{\partial z}D_v(z) - \frac{z}{2}D_v(z) &= 0 \\ D_{v+1}(z) + \frac{\partial}{\partial z}D_v(z) &= \frac{z}{2}D_v(z) \end{aligned} \quad (4.11)$$

From (4.8), we have:

$$\frac{\partial}{\partial z}D_v(z) - vD_{v-1}(z) = -\frac{z}{2}D_v(z) \quad (4.12)$$

Now, adding (4.11) to (4.12), we have:

$$D_{v+1}(z) + 2\frac{\partial}{\partial z}D_v(z) - vD_{v-1}(z) = 0 \quad (4.13)$$

4.2.2 First Order Differential Operators

From the recurrence relations, we find the following three operators:
Lowering operator (J^-), increasing operator (J^+) and stationary operator (J^3).

Case (i) To find J^+ :

By adding (4.10) and (4.13) together, we get:

$$2D_{v+1} + (2\frac{\partial}{\partial z} - z)D_v = 0$$

$$2 D_{v+1} = (z - 2 \frac{\partial}{\partial z}) D_v$$

$$D_{v+1} = (\frac{z}{2} - \frac{\partial}{\partial z}) D_v .$$

So we get: $J^+ = (\frac{z}{2} - \frac{\partial}{\partial z}) t$ such that

$$J^+[D_v(z) t^v] = D_{v+1}(z) t^{v+1} \quad (4.14)$$

Case (ii) To find J^- :

By subtracting (4.10) from (4.13), we get:

$$(2 \frac{\partial}{\partial z} + z) D_v - 2 v D_{v-1} = 0$$

$$(2 \frac{\partial}{\partial z} + z) D_v = 2 v D_{v-1}$$

$$(\frac{\partial}{\partial z} + \frac{z}{2}) D_v = v D_{v-1}$$

So we get: $J^- = (\frac{z}{2} + \frac{\partial}{\partial z}) t^{-1}$ such that

$$J^-[D_v(z) t^v] = v D_{v-1}(z) t^{v-1} \quad (4.15)$$

Case (iii) To find J^3 :

We define: $J^3 = t \frac{\partial}{\partial t}$ such that

$$J^3[D_v(z) t^v] = v D_v(z) t^v \quad (4.16)$$

4.2.3 Commutative Relations of the Operators

We use the commutator notation $[A, B]$ with $[A, B] f = (AB - BA)f$ where $f = f(z; t) = D_v(z)t^v$ in order to get the following relations:

Case (i) $[J^3, J^-] = -J^-$

Proof: $[J^3, J^-]f = (J^3J^- - J^-J^3)f$

$$\begin{aligned}
 &= \left[t \frac{\partial}{\partial t} \right] \left[t^{-1} \left(\frac{z}{2} + \frac{\partial}{\partial z} \right) \right] f - \left[t^{-1} \left(\frac{z}{2} + \frac{\partial}{\partial z} \right) \right] \left[t \frac{\partial}{\partial t} \right] f \\
 &= \left[t \frac{\partial}{\partial t} \right] \left[t^{-1} \frac{z}{2} f + t^{-1} \frac{\partial}{\partial z} f \right] - \left[t^{-1} \frac{z}{2} + t^{-1} \frac{\partial}{\partial z} \right] \left[t \frac{\partial}{\partial t} f \right] \\
 &= \left[t \left(t^{-1} \frac{z}{2} \frac{\partial}{\partial t} f - t^{-2} \frac{z}{2} f + t^{-1} \frac{\partial^2}{\partial z \partial t} f - t^{-2} \frac{\partial}{\partial z} f \right) \right] - \left[\frac{z}{2} \frac{\partial}{\partial t} f + \frac{\partial^2}{\partial z \partial t} f \right] \\
 &= \left[\frac{z}{2} \frac{\partial}{\partial t} f - t^{-1} \frac{z}{2} f + \frac{\partial^2}{\partial z \partial t} f - t^{-1} \frac{\partial}{\partial z} f \right] - \left[\frac{z}{2} \frac{\partial}{\partial t} f + \frac{\partial^2}{\partial z \partial t} f \right] \\
 &= -t^{-1} \frac{z}{2} f - t^{-1} \frac{\partial}{\partial z} f \\
 &= -t^{-1} \left(\frac{z}{2} + \frac{\partial}{\partial z} \right) f \\
 &= -J^- f
 \end{aligned}$$

Case (ii) $[J^3, J^+] = J^+$

Proof: $[J^3, J^+]f = (J^3J^+ - J^+J^3)f$

$$\begin{aligned}
&= \left[t \frac{\partial}{\partial t} \right] \left[t \left(\frac{z}{2} - \frac{\partial}{\partial z} \right) \right] f - \left[t \left(\frac{z}{2} - \frac{\partial}{\partial z} \right) \right] \left[t \frac{\partial}{\partial t} \right] f \\
&= \left[t \frac{\partial}{\partial t} \right] \left[t \left(\frac{z}{2} f - \frac{\partial}{\partial z} f \right) \right] - \left[t \left(\frac{z}{2} - \frac{\partial}{\partial z} \right) \right] \left[t \frac{\partial}{\partial t} f \right] \\
&= \left[t \left(t \frac{z}{2} \frac{\partial}{\partial t} f + \frac{z}{2} f - t \frac{\partial^2}{\partial z \partial t} f - \frac{\partial}{\partial z} f \right) \right] - \left[t^2 \frac{z}{2} \frac{\partial}{\partial t} f - t^2 \frac{\partial^2}{\partial z \partial t} f \right] \\
&= \left[t^2 \frac{z}{2} \frac{\partial}{\partial t} f + t \frac{z}{2} f - t^2 \frac{\partial^2}{\partial z \partial t} f - t \frac{\partial}{\partial z} f \right] - \left[t^2 \frac{z}{2} \frac{\partial}{\partial t} f - t^2 \frac{\partial^2}{\partial z \partial t} f \right] \\
&= t \frac{z}{2} f - t \frac{\partial}{\partial z} f \\
&= t \left(\frac{z}{2} - \frac{\partial}{\partial z} \right) f \\
&= J^+ f
\end{aligned}$$

Case (iii) $[J^+, J^-] = 1$

Proof: $[J^+, J^-]f = (J^+ J^- - J^- J^+)f$

$$\begin{aligned}
&= \left[\frac{z}{2} - \frac{\partial}{\partial z} \right] \left[\frac{z}{2} + \frac{\partial}{\partial z} \right] f - \left[\frac{z}{2} + \frac{\partial}{\partial z} \right] \left[\frac{z}{2} - \frac{\partial}{\partial z} \right] f \\
&= \left[\frac{z}{2} - \frac{\partial}{\partial z} \right] \left[\frac{z}{2} f + \frac{\partial}{\partial z} f \right] - \left[\frac{z}{2} + \frac{\partial}{\partial z} \right] \left[\frac{z}{2} f - \frac{\partial}{\partial z} f \right] \\
&= \frac{z^2}{2^2} f + \frac{z}{2} \frac{\partial}{\partial z} f - \left(\frac{z}{2} \frac{\partial}{\partial z} f + \frac{f}{2} \right) - \frac{\partial^2}{\partial z^2} f - \left[\frac{z^2}{2^2} f - \frac{z}{2} \frac{\partial}{\partial z} f + \left(\frac{z}{2} \frac{\partial}{\partial z} f + \frac{f}{2} \right) - \frac{\partial^2}{\partial z^2} f \right] \\
&= -\frac{z}{2} \frac{\partial}{\partial z} f + \frac{z}{2} \frac{\partial}{\partial z} f + \frac{f}{2} - \frac{z}{2} \frac{\partial}{\partial z} f + \frac{z}{2} \frac{\partial}{\partial z} f + \frac{f}{2} \\
&= f
\end{aligned}$$

From the operator relations obtained in (4.14) and (4.15), we get:

$$\begin{aligned} J^+ J^- [D_v(z) t^n] &= J^+ [v D_{v-1}(z) t^{v-1}] \\ &= v D_v(z) t^v \end{aligned}$$

We can form the following differential operator: $L = -\frac{\partial^2}{\partial z^2} + \frac{z^2}{4} - \frac{1}{2} - v$

$$\begin{aligned} L &= J^+ J^- - v \\ &= \left(\frac{z}{2} - \frac{\partial}{\partial z}\right) t \cdot \left(\frac{z}{2} + \frac{\partial}{\partial z}\right) t^{-1} - v \\ &= \frac{z^2}{4} + \frac{z}{2} \frac{\partial}{\partial z} - \frac{1}{2} - \frac{z}{2} \frac{\partial}{\partial z} - \frac{\partial^2}{\partial z^2} - v \\ &= \frac{z^2}{4} - \frac{1}{2} - \frac{\partial^2}{\partial z^2} - v \\ L &= -\frac{\partial^2}{\partial z^2} + \frac{z^2}{4} - \frac{1}{2} - v, \quad \text{where } v = t \frac{\partial}{\partial t} \end{aligned} \tag{4.17}$$

which commutes with J^+, J^- and J^3 . That is:

$$[L, J^+] = 0, [L, J^-] = 0 \text{ and } [L, J^3] = 0$$

4.2.4 Extended Form of the Group Generated by Operators

Here we find the extended form of the group generated by J^3, J^- and J^+ .

Case (i) $J^3 = t \frac{\partial}{\partial t}$:

Let a be any arbitrary constant. Then,

$$\frac{dz}{da} = 0 \text{ and we get } z = z_0 ,$$

$$\frac{dv}{da} = 0 \text{ and we get } v = 1 ,$$

$$\frac{dt}{da} = t \text{ and we get } \ln t = a + k$$

which implies that: $t = e^a e^k = t_0 e^a$

Thus,

$$e^{aJ^3} f(z; t) = f(z; t e^a) \quad (4.18)$$

Case (ii) $J^- = t^{-1} \frac{z}{2} + t^{-1} \frac{\partial}{\partial z}$:

Let b be any arbitrary constant. Then,

$$\frac{dt}{db} = 0 \text{ and we get } t = t_0 ,$$

$$\frac{dz}{db} = t_0^{-1} \text{ and we get } z = z_0 + t_0^{-1} b ,$$

$$\frac{dv}{db} = \frac{z}{2t} v$$

$$\frac{dv}{v} = \frac{z}{2t} db$$

Substituting the expressions of z and t as given above, we get:

$$\frac{dv}{v} = \frac{1}{2t} \left(z_0 + \frac{b}{t_0} \right) db$$

$$\frac{dv}{v} = \left(\frac{z_0}{2t_0} + \frac{b}{2t_0^2} \right) db$$

$$\ln v = \frac{z_0 b}{2t_0} + \frac{b^2}{4t_0^2} + k$$

If $b = 0, v(0) = 1$ then $k = 0$

$$\text{Thus, } v = \exp \left(\frac{z_0 b}{2t_0} + \frac{b^2}{4t_0^2} \right)$$

and

$$e^{bJ^-} f(z; t) = \exp \left(\frac{zb}{2t} + \frac{b^2}{4t^2} \right) \cdot f \left(z + \frac{b}{t}; t \right) \quad (4.19)$$

$$\text{Case (iii) } J^+ = t \frac{z}{2} - t \frac{\partial}{\partial z} ,$$

Let c be any arbitrary constant. Then,

$$\frac{dt}{dc} = 0 \text{ and we get } t = t_0 ,$$

$$\frac{dz}{dc} = -t_0 \text{ and we get } z = z_0 - t_0 c ,$$

$$\frac{dv}{dc} = t \frac{z}{2} v$$

$$\frac{dv}{v} = t \frac{z}{2} dc$$

Substituting the values of z and t as given above, we get:

$$\frac{dv}{v} = \frac{t_0}{2} (z_0 - t_0 c) dc$$

$$\frac{dv}{v} = \left(\frac{z_0 t_0}{2} - \frac{c t_0^2}{2} \right) dc$$

$$\ln v = \frac{z_0 t_0 c}{2} - \frac{c^2 t_0^2}{4} + k$$

If $c = 0, v(0) = 1$ then $k = 0$

$$\text{Thus, } v = \exp \left(\frac{z_0 t_0 c}{2} - \frac{c^2 t_0^2}{4} \right)$$

and

$$e^{cJ^+} f(z; t) = \exp \left(\frac{ztc}{2} - \frac{c^2 t^2}{4} \right) \cdot f(z - tc; t) \quad (4.20)$$

4.2.5 Some Generating Functions

From the extended forms (4.18), (4.19) and (4.20), we study the effect of the operators in the following three cases:

Case 1: The effect of the three operators: J^- , J^+ and J^3 on $D_v(z)t$:

$$\begin{aligned} & e^{cJ^+} e^{bJ^-} e^{aJ^3} f(z; t) \\ &= \exp \left(\frac{ztc}{2} - \frac{c^2 t^2}{4} \right) e^{bJ^-} e^{aJ^3} [f(z - tc, t)] \\ &= \exp \left(\frac{ztc}{2} - \frac{c^2 t^2}{4} + \frac{zb}{2t} + \frac{b^2}{4t^2} \right) e^{aJ^3} \left[f \left(z - tc + \frac{b}{t}, t \right) \right] \end{aligned}$$

$$= \exp\left(\frac{ztc}{2} - \frac{c^2 t^2}{4} + \frac{zb}{2t} + \frac{b^2}{4t^2}\right) f\left(z - t c + \frac{b}{t}, te^a\right)$$

Now if we put $a = 0$; $b \neq 0$, $c \neq 0$ and writing $f(z; t) = D_v(z)t^v$, we get:

$$e^{cJ^+} e^{bJ^-} D_v(z)t^v = \exp\left(\frac{ztc}{2} - \frac{c^2 t^2}{4} + \frac{zb}{2t} + \frac{b^2}{4t^2}\right) D_v\left(z - t c + \frac{b}{t}\right)t^v \quad (4.21)$$

Working on left hand side and by using the power series expansions, we get for $t \neq 0$:

$$\begin{aligned} & e^{cJ^+} e^{bJ^-} D_v(z)t^v \\ &= \sum_{k=0}^{\infty} \frac{(cJ^+)^k}{k!} e^{bJ^-} (D_v(z)t^v) \\ &= \sum_{k=0}^{\infty} \frac{c^k}{k!} e^{bJ^-} (J^+)^k (D_v(z)t^v) \\ &= \sum_{k=0}^{\infty} \frac{c^k}{k!} e^{bJ^-} (D_{v+k}(z)t^{v+k}) \\ &= \sum_{k=0}^{\infty} \frac{c^k}{k!} \sum_{L=0}^{\infty} \frac{(bJ^-)^L}{L!} (D_{v+k}(z)t^{v+k}) \\ &= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{c^k b^L}{k! L!} (J^-)^L (D_{v+k}(z)t^{v+k}) \\ &= \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{c^k b^L}{k! L!} (v+k)(v+k-1) \dots (v+k-(L-1)) D_{v+k-L}(z)t^{v+k-L} \\ & e^{cJ^+} e^{bJ^-} D_v(z)t^v = \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{c^k b^L}{k! L!} \frac{(v+k)!}{(v+k-L)!} D_{v+k-L}(z)t^{v+k-L} \quad (4.22) \end{aligned}$$

Now, from (4.21) and (4.22) we get:

$$\begin{aligned} \exp\left(\frac{ztc}{2} - \frac{c^2 t^2}{4} + \frac{zb}{2t} + \frac{b^2}{4t^2}\right) D_v\left(z - t c + \frac{b}{t}\right) t^v \\ = \sum_{k=0}^{\infty} \sum_{L=0}^{\infty} \frac{c^k b^L}{k! L!} \frac{(v+k)!}{(v+k-L)!} D_{v+k-L}(z) t^{v+k-L} \end{aligned} \quad (4.23)$$

Case 2: The effect of one operator: J^- on $D_v(z)$:

Put $t = 1$ and $c = 0$ in (4.23), then we have:

$$\exp\left(\frac{2bz + b^2}{4}\right) D_v(z + b) = \sum_{L=0}^{\infty} \frac{b^L}{L!} \frac{v!}{(v-L)!} D_{v-L}(z) \quad (4.24)$$

Case 3: The effect of one operator: J^+ on $D_v(z)$:

Put $t = 1$ and $b = 0$ in (4.23), then we have:

$$\exp\left(\frac{2cz - c^2}{4}\right) D_v(z - c) = \sum_{k=0}^{\infty} \frac{c^k}{k!} D_{v+k}(z) \quad (4.25)$$

4.3 Some Applications

Applying the relation (4.4) in (4.25) with $v = n$, we get:

$$\begin{aligned}
 \exp\left(\frac{2cz - c^2}{4}\right) \exp\left(\frac{-(z - c)^2}{4}\right) He_n(z - c) &= \sum_{k=0}^{\infty} \frac{c^k}{k!} \exp\left(\frac{-z^2}{4}\right) He_{n+k}(z) \\
 \exp\left(\frac{cz}{2} - \frac{c^2}{4} - \frac{z^2}{4} + \frac{2cz}{4} - \frac{c^2}{4}\right) He_n(z - c) &= \sum_{k=0}^{\infty} \frac{c^k}{k!} \exp\left(\frac{-z^2}{4}\right) He_{n+k}(z) \\
 \exp\left(\frac{cz}{2} - \frac{c^2}{4} + \frac{cz}{2} - \frac{c^2}{4}\right) He_n(z - c) &= \sum_{k=0}^{\infty} \frac{c^k}{k!} He_{n+k}(z) \\
 \exp\left(cz - \frac{c^2}{2}\right) He_n(z - c) &= \sum_{k=0}^{\infty} \frac{c^k}{k!} He_{n+k}(z) \tag{4.26}
 \end{aligned}$$

Also, by applying the relation (4.6) in (4.26), we get:

$$\begin{aligned}
 \exp\left(cz - \frac{c^2}{2}\right) \frac{1}{\sqrt{2^n}} H_n\left(\frac{z - c}{\sqrt{2}}\right) &= \sum_{k=0}^{\infty} \frac{c^k}{k!} \frac{1}{\sqrt{2^{n+k}}} H_{n+k}\left(\frac{z}{\sqrt{2}}\right) \\
 \exp\left(cz - \frac{c^2}{2}\right) H_n\left(\frac{z - c}{\sqrt{2}}\right) &= \sum_{k=0}^{\infty} \frac{c^k}{k!} \frac{1}{\sqrt{2^k}} H_{n+k}\left(\frac{z}{\sqrt{2}}\right) \tag{4.27}
 \end{aligned}$$

Now if we replace c by $\sqrt{2}t$ and z by $\sqrt{2}z$ and put $n=0$ in (4.27), we get:

$$\begin{aligned}
 \exp\left(\sqrt{2}t \cdot \sqrt{2}z - \frac{(\sqrt{2}t)^2}{2}\right) H_0\left(\frac{\sqrt{2}z - \sqrt{2}t}{\sqrt{2}}\right) &= \sum_{k=0}^{\infty} \frac{(\sqrt{2}t)^k}{k!} \frac{1}{\sqrt{2^k}} H_{0+k}\left(\frac{\sqrt{2}z}{\sqrt{2}}\right) \\
 \exp(2zt - t^2) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(z)
 \end{aligned}$$

This is the well-known generating function for Hermite polynomials $H_n(x)$ (Lebedev 1972).

4.4 Conclusion and Summary

Through the work in this thesis, we have seen that Weisner's group theoretic method have proved to be very useful for obtaining the generating functions of the generalized Hermite polynomials with two variables $H_n(x, y)$, Hermite matrix polynomials $H_n(x, A)$ and Hermite matrix functions $T_n(x, A)$. The new generating relations for generalized Hermite polynomials associated with functions of parabolic cylinder $D_\nu(z)$ are obtained and the well-known generating relations for Hermite polynomials $H_n(x)$ are derived as special cases.

This technique proved how it is useful to be used in finding generating relations for different Hermite polynomials and functions. It is also can be used to find generating functions of other polynomials such as Bessel polynomials, generalized Laguerre polynomials, Gegenbauer polynomials.

For future work, it deserve some interest to find generating relations for other functions such as Dexon polynomials which are very useful in digital communications and technology, and Chebyshev polynomials too.

Beside that, we may try to apply a similar technique and work in finding generating functions with fractional order differential equations.

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ملخص

في هذه الرسالة تم استخدام طريقة ويزنر لإيجاد العديد من العلاقات المولدة لكثيرات الحدود و الاقترانات المرتبطة بدالة هيرمايت.

الحالات التي تم دراستها في هذا العمل تشمل كثيرة الحدود هيرمايت ذات متغيرين، كثيرة الحدود و دالة هيرمايت المعرفة بدلالة المصفوفات، و كثيرة الحدود هيرمايت المعرفة بدلالة الأسطوانة المكافئة.

تم توضيح كافة الخطوات بتفاصيلها المرتبطة بطريقة ويزنر بدءاً من إيجاد العلاقات التكرارية ثم الحصول على العلاقات المؤثرة و العلاقات التبادلية بينهم، إضافة إلى الصيغة الممتدة للمجموعات الناتجة من هذه العلاقات المؤثرة، لننتهي بالحصول على علاقات مولدة جديدة و التي هي تعميم لعلاقات أساسية معروفة سابقاً.

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أطروحة مقدمة كجزء من متطلبات الحصول على درجة الماجستير في الرياضيات

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