

UNIVERSITY OF BAHRAIN
COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS



**CHARACTERIZATIONS
OF INNER PRODUCT SPACES
USING
ORTHOGONALITY RELATIONS**

BY

SAWSAN JUMA ALI HELAL

A Thesis Submitted to the Department of Mathematics as
a Partial Fulfillment of the Requirements for the Degree

of

MASTER OF SCIENCE

Under the Supervision

of

Dr. Mahmood Al Abbas

JUNE 2001

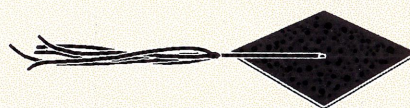
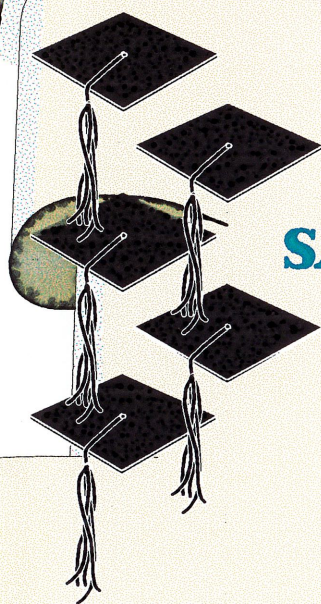
**UNIVERSITY OF BAHRAIN
COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS**



**CHARACTERIZATIONS
OF INNER PRODUCT SPACES
USING
ORTHOGONALITY RELATIONS**

BY

SAWSAN JUMA ALI HELAL



MASTER OF SCIENCE

JUNE 2001



13th June 2001

The Thesis Defence Committee considers the thesis entitled:

“Characterization of Inner product spaces using orthogonality relations”

Submitted by *Sawsan Juma Ali Helal* (I.D. No. 911482) to the Department of Mathematics is satisfactory and acceptable for the Master of Science Degree in Mathematics.

Thesis Defence Committee:

Dr. Mahmood Al-Abbas
Supervisor

Department of Mathematics
University of Bahrain

Prof. Anton von Bachhaus
Internal Examiner

Department of Mathematics
University of Bahrain

Prof. Zeghib Abdelghani
External Examiner

CNRS*
Lyon - France

Dr. A. Abusafiya
Chairman

Department of Mathematics
University of Bahrain

* National Center of Scientific Research

ACKNOWLEDGMENT

A research like this, however simple, is never the work of one person. While I prepared the material and wrote it, I am indebted to my advisor Dr. Mahmood Al Abbas who has supplied what has become my "background", and in the collection of facts and attitudes toward them that are a part of my general knowledge. To him I owe more than I can ever say.

Special mention, for my parents for their support, to whom this piece of work is dedicated.

ABSTRACT

Inner product spaces are special normed linear spaces. An inner product space is a vector space X with an inner product defined on X . Here, an inner product on X is a mapping of $X \times X$ into the real field $\mathbf{R}^*$ of X ; that is, with every pair of vectors x and y there is associated a real number which is written $\langle x, y \rangle$ and is called the inner product of x and y , such that for all vectors x, y, z and scalars r we have

$$(IP1) \quad \langle x, y \rangle = \langle y, x \rangle$$

$$(IP2) \quad \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

$$(IP3) \quad \langle rx, y \rangle = r \langle x, y \rangle$$

$$(IP4) \quad \langle x, x \rangle \geq 0, \langle x, x \rangle = 0 \Leftrightarrow x = 0.$$

An inner product on X defines a norm on X given by

$$\|x\| = \sqrt{\langle x, x \rangle}$$

But normed linear spaces need not be inner product spaces. As an example the space of continuous functions $C[a, b]$ with norm defined by

$$\|f\| = \max_{t \in J} |f(t)| \quad J = [a, b] \text{ and } f \in C[a, b].$$

A normed linear space $(X, \|\cdot\|)$, or simply X , is called an inner product space if there is an inner product $\langle x, y \rangle$ defined in it such that $\|x\| = \sqrt{\langle x, x \rangle}$.

Since inner product spaces have been, up to now, the most useful spaces in practical applications of functional analysis, a problem of characterizing these spaces is of most importance. Different concepts were used for an effective unifying approach toward this problem. One of these concepts is using parameters. For example defining a formula, then verifying that if this formula satisfies certain conditions, the space is an inner product space. Indeed there are several known characterizations of inner product spaces which can be stated in terms of parameters. Geometry is another approach to characterize inner product spaces, which is related to most other concepts. However, throughout our research we are dealing with the concept of **orthogonality**, which is of basic importance of such studying.

* All our normed spaces will be over reals. The real case sufficient for most purposes, since if X

is a normed space over the complex field which, as a space over the reals, has the inner product

product $\langle x, y \rangle_{\mathbf{R}}$, then $\langle x, y \rangle = \langle x, y \rangle_{\mathbf{R}} + i \langle x, iy \rangle_{\mathbf{R}}$.

The most natural notion of orthogonality arises in the case where there is an inner product compatible with the norm on a space X is $\perp$, defined by

$$x \perp y \text{ if and only if } \langle x, y \rangle = 0.$$

Some of the major properties of this relation are listed as follows:

- (1) **Nondegeneracy :** If $x \perp x$, then $x = 0$;
- (2) **Symmetry :** If $x \perp y$, then $y \perp x$;
- (3) **Homogeneity :** If $x \perp y$, then $ax \perp by$ ($a, b \in \mathbf{R}$);
- (4) **(Right -) Additivity :** If $x \perp y$ and $x \perp z$ then $x \perp (y + z)$;
- (5) **Resolvability :** If $x, y \in X$ then there exists $a \in \mathbf{R}$ such that

$$x \perp (ax + y).$$

For general normed linear spaces, there are different known definitions of orthogonality which do not require the existence of an inner product. But, apparently there is no general study of this family of definitions. Therefore we will study some of them, and assuming that they adequate to the aim of this research. These orthogonalities will be covered in chapter I, and defined as follows:

ISOSCELES – ORTHOGONALITY [18] :

$$x \perp^i y \text{ if and only if } \|x - y\| = \|x + y\|.$$

PYTHAGOREAN – ORTHOGONALITY [18] :

$$x \perp^P y \text{ if and only if } \|x - y\|^2 = \|x\|^2 + \|y\|^2.$$

BIRKHOFF – ORTHOGONALITY [19] :

$$x \perp^B y \text{ if and only if } \|x\| \leq \|x + ky\| \text{ for all } k \in \mathbf{R}.$$

α – ORTHOGONALITY [2] :

If $\alpha \neq 1$, $(x \perp y)(\alpha)$ if and only if

$$(1 + \alpha^2) \|x - y\|^2 = \|x - \alpha y\|^2 + \|y - \alpha x\|^2.$$

(α, β) – ORTHOGONALITY [1] :

If $\alpha, \beta \neq 1$, $(x \perp y)(\alpha, \beta)$ if and only if

$$\|x - y\|^2 + \|\alpha x - \beta y\|^2 = \|x - \beta y\|^2 + \|\alpha x - y\|^2.$$

It should be noted that pythagorean and isosceles orthogonalities are special cases of α -orthogonality, which is in turn a special case of (α, β) -orthogonality. The usual approach in studying these concepts consists of verifying which of the

properties (1)-(5) are satisfied and then proving that some of the remaining properties are sufficient to force X to be an inner product space.

In chapter II, we are listing different characterizations of inner product spaces by proving that mostly equivalence between any two orthogonality relations, defined in chapter I, is a characteristic property of inner product spaces.

Finally, in chapter III, we define a new orthogonality relation based on two real numbers α and β , where α and β are not both zero and satisfying the condition $\alpha \cdot \beta \geq 0$, and denoted by $[\alpha, \beta]$ -orthogonality. This new definition is different from the definition of (α, β) -orthogonality, in using multiplication sign instead of addition. That is for x and $y \in X$, we have

$$(x \perp y)[\alpha, \beta] \text{ if and only if } \|x - \alpha y\| \cdot \|x - \beta y\| = \|x + \alpha y\| \cdot \|x + \beta y\|.$$

You may see that this definition is a generalization of the isosceles orthogonality, e.g. if $\alpha = \beta = 1$, then

$$(x \perp y)[\alpha, \beta] \Leftrightarrow x \overset{i}{\perp} y.$$

But of course they are not equivalent, (§1). However, Same procedure used in studying the previous orthogonalities, in chapter I, will be applied for $[\alpha, \beta]$ -orthogonality. At the end of chapter III, we obtain some characterizations of inner product spaces using this new orthogonality relation.

TABLE OF CONTENTS

Preface	Page
NORM POSTULATES	1
CHAPTER I:	
MAIN PROPERTIES OF ORTHOGONALY RELATIONS	
Introduction	2
§1· Isosceles orthogonality	3
§2· Pythagorean orthogonality	11
§3· Birkhoff orthogonality	22
§4· α – orthogonality	43
§5· (α, β) – orthogonality	51
§6· Equivalence of orthogonality relations	62
SUMMARY	68
CHAPTER II:	
RELATIONS BETWEEN ORTHOGONALITIES AND CHARACTERIZATIONS OF INNER PRODUCT SPACES	
Introduction	69
$\left[x \perp^i y \Rightarrow (\Leftrightarrow) x \perp^P y \right] \Leftrightarrow X \text{ is an i.p.s.}$	73
$\left[x \perp^P y \Rightarrow (\Leftrightarrow) x \perp^B y \right] \Leftrightarrow X \text{ is an i.p.s.}$	75
$\left[x \perp^i y \Rightarrow (\Leftrightarrow) x \perp^B y \right] \Leftrightarrow X \text{ is an i.p.s.}$	79
If $ \alpha , \beta \neq 1$ then $\left[x \perp^B y \Rightarrow x \perp_{\mathcal{M}(\alpha, \beta)} \right] \Leftrightarrow X \text{ is an i.p.s.}$	81
If $ \alpha , \beta \neq 1$ then $\left[x \perp_{\mathcal{M}(\alpha, \beta)} \Rightarrow x \perp^B y \right] \Leftrightarrow X \text{ is an i.p.s.}$	90
If $\alpha = \beta$ and $ \alpha < 1$, then $\left[x \perp^i y \Rightarrow (\Leftrightarrow) x \perp_{\mathcal{M}(\alpha, \alpha)} \right] \Leftrightarrow X \text{ is an i.p.s.}$	93
SUMMARY	97

CHAPTER III:

$[\alpha, \beta]$ – ORTHOGONALITY

Introduction	98
§1• Definition of $[\alpha, \beta]$ – orthogonality	99
§2• Properties of $[\alpha, \beta]$ – orthogonality	103
§3• Consequences of homogeneity of $[\alpha, \beta]$ – orthogonality	112
§4• The Characterization Theorem	116
§5• Set of open problems related to $[\alpha, \beta]$ – orthogonality	119
SUMMARY	120
REFERENCES	121

NORM POSTULATES

There is a long list of conditions on the norm, called **norm postulates**, strong enough to characterize inner product spaces amongst normed linear spaces. The best known postulate is by *Jordan and von Neumann*

(JN) $x, y \in X$, $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$. This condition is called *the parallelogram law*. [16]

The other norm postulates are here :

(D1) If $x, y \in X$, $\|x\| = \|y\| = 1$, then $\|x+y\|^2 + \|x-y\|^2 = 4$. [12]

(D, ~) If $\|x\| = \|y\| = 1$, then there exists a and b with $0 < a < 1$, $0 < b < 1$, such that

$$b(1-b)\|ax + (1-a)y\|^2 + a(1-a)\|bx - (1-b)y\|^2 \sim [a + b - 2ab][ab + (1-a)(1-b)] \text{ where } \sim \text{ means } = \text{ or } \geq \text{ or } \leq. \quad [14]$$

(L 1) If there is a fixed real $\alpha \neq 0, \pm 1$ such that $x, y \in X$ and $\|x-y\| = \|x+y\|$ imply $\|x-\alpha y\| = \|x+\alpha y\|$. That is, if $x \perp^i y \Rightarrow x \perp^i \alpha y$ for fixed α . [4]

(L 2) If $x, y \in X$ such that $\|x\| = \|y\|$ and α is a real number such that $\alpha \neq 0$, then $\|\alpha x + (1/\alpha)y\| \geq \|x+y\|$. [4]

(F) If α is a real number and x and y are vectors in X such that $\|x\| = \|y\|$, then $\|\alpha x + y\| = \|x + \alpha y\|$. [6]

CHAPTER I

MAIN PROPERTIES OF ORTHOGONALITY RELATIONS

INTRODUCTION	2
§1• Isosceles orthogonality	3
§2• Pythagorean orthogonality	11
§3• Birkhoff orthogonality	22
§4• α -orthogonality	43
§5• (α, β) -orthogonality	51
§6• Equivalence of orthogonality relations	62
SUMMARY	68

CHAPTER I

MAIN PROPERTIES OF ORTHOGONALITY RELATIONS

INTRODUCTION

In this chapter we study five types of orthogonality relations including Isosceles orthogonality (§1), Pythagorean orthogonality (§2), Birkhoff orthogonality (§3), α -orthogonality (§4), and (α, β) -orthogonality (§5), which have been studied extensively by James [17], [18], [19], and the present authors [1], [2], among others. Here, we are going to summarize their results to use them as a background for our new orthogonality relation defined in chapter III. At the end of this chapter we show that the five relations are different concepts of orthogonality unless the space is an inner product space. Considering some simple examples will show that.

§1. JAMES ISOSCELES ORTHOGONALITY

James had introduced the following orthogonality relation:

$$x \perp^i y \text{ if } \|x + y\| = \|x - y\| \quad [18]$$

Clearly by definition isosceles-orthogonality is symmetric. Further properties are discussed as follows:

THEOREM I.1

If x and y are elements of a normed linear space X , then there exists a number a such that

$$\|x + (ax + y)\| = \|x - (ax + y)\|, \text{ or } x \perp^i ax + y.$$

For the proof we need

LEMMA I.2

If x and y are elements of a normed linear space, then

$$\lim_{n \rightarrow \infty} [\|(n+a)x + y\| - \|nx + y\|] = a\|x\|, \quad a \in \mathbf{R}.$$

Proof. Since $\frac{n}{(n+a)} + \frac{a}{(n+a)} = 1$ identically,

$$\begin{aligned} \|(n+a)x + y\| &= \left[\frac{n}{(n+a)} + \frac{a}{(n+a)} \right] \|(n+a)x + y\| \\ &= \frac{n}{n+a} \|(n+a)x + y\| + \frac{a}{n+a} \|(n+a)x + y\| \\ &= \left\| nx + \frac{ny}{(n+a)} \right\| + a \left\| x + \frac{y}{(n+a)} \right\|, \end{aligned}$$

that is if n is large enough so that $(n+a) > 0$. Thus

$$\|(n+a)x + y\| - \|nx + y\| = \left[\left\| nx + \frac{ny}{n+a} \right\| - \|nx + y\| \right] + a \left\| x + \frac{y}{n+a} \right\|,$$

Chapter I

Isosceles orthogonality

$$\text{but } \left\| nx + \frac{ny}{n+a} \right\| - \|nx + y\| \leq \frac{a}{n+a} \|y\|, \text{ (using the triangle inequality)}$$

which approaches zero as n becomes infinite. Also,

$$\lim_{n \rightarrow \infty} \left\| x + \frac{y}{n+a} \right\| = \|x\|. \text{ Hence}$$

$$\lim_{n \rightarrow \infty} \left[\|(n+a)x + y\| - \|nx + y\| \right] = a\|x\|.$$

$$\text{One can show that } \lim_{n \rightarrow -\infty} \left[\|(n+a)x + y\| - \|nx + y\| \right] = -a\|x\|.$$

Let $h = -n$. Then

$$\lim_{h \rightarrow -\infty} \left[\|(-h+a)x + y\| - \|-hx + y\| \right] = a\|x\|.$$

Simplify,

$$\lim_{h \rightarrow -\infty} \left[\|(h-a)x - y\| - \|hx - y\| \right] = a\|x\|.$$

Now putting $b = -a$,

$$\lim_{h \rightarrow -\infty} \left[\|(h+b)x - y\| - \|hx - y\| \right] = -b\|x\|.$$

But replacing y by $-y$ does not affect the limit, hence we get

$$\lim_{h \rightarrow -\infty} \left[\|(h+b)x + y\| - \|hx + y\| \right] = -b\|x\|, \text{ or}$$

$$\lim_{n \rightarrow -\infty} \left[\|(n+a)x + y\| - \|nx + y\| \right] = -a\|x\|.$$

We are ready now to prove Theorem I.1. Define the real function of a real

$$\begin{aligned} \text{variable, } f(n) &= \|x + (nx + y)\| - \|x - (nx + y)\| \\ &= \|(n+1)x + y\| - \|(n-1)x + y\|. \end{aligned}$$

Chapter I
Isosceles orthogonality

$$\begin{aligned} \text{Then } \lim_{n \rightarrow \infty} f(n) &= \lim_{n \rightarrow \infty} \left[\|(n-1)x + y\| - \|(n-2)x + y\| \right] \\ &= 2\|x\| \text{ by the previous Lemma.} \end{aligned}$$

Likewise, Lemma I.2 gives $\lim_{n \rightarrow -\infty} f(n) = -2\|x\|$. It now follows from continuity of the norm and the Intermediate Value Theorem that there exists a number a such that

$$f(a) = 0, \text{ or } \|x + (ax + y)\| = \|x - (ax + y)\|.$$

THEOREM I.3

If $x (\neq 0)$ and y are two elements of a normed linear space and $\|y\| \leq \|x\|$, then $\|a\| \leq \|y\|/\|x\|$ if $x \perp^i ax + y$.

$$\begin{aligned} \text{Proof. Define } f(n) &= \|x + (nx + y)\| - \|x - (nx + y)\| \\ &= \|(n+1)x + y\| - \|(n-1)x + y\|. \end{aligned}$$

Now, $x \perp^i ax + y \Rightarrow f(a) = 0$. Therefore $\|a\| \leq \|y\|/\|x\|$ if and only if $f(n) \neq 0$ outside the interval $[-\|y\|/\|x\|, \|y\|/\|x\|]$; that is for $|n| > \|y\|/\|x\|$.

From the triangle inequality of the norm,

$$\begin{aligned} \|(n+1)x + y\| &\geq |n+1|\|x\| - \|y\|, \text{ and} \\ \|(n-1)x + y\| &\leq |n-1|\|x\| + \|y\|. \end{aligned}$$

Thus

$$\begin{aligned} f(n) &\geq [|n+1| \cdot \|x\| - \|y\|] - [|n-1| \cdot \|x\| + \|y\|], \\ &= [|n+1| - |n-1|] \cdot \|x\| - 2\|y\|. \end{aligned}$$

Now, for $n > \|y\|/\|x\|$, we consider two cases :

$$\|y\|/\|x\| < n \leq 1, \text{ or } n > 1$$

(1) If $\frac{\|y\|}{\|x\|} < n \leq 1$, then

$$f(n) \geq [(n+1) + (n-1)]\|x\| - 2\|y\| = 2n\|x\| - 2\|y\| = 2[n\|x\| - \|y\|] > 0.$$

(2) If $n > 1$, then

$$f(n) > [(n+1) - (n-1)]\|x\| - 2\|y\| = 2[\|x\| - \|y\|] \geq 0 \quad \text{since } \|y\| \leq \|x\|.$$

Thus from (1) and (2), we have $f(n) > 0$ for $n > \frac{\|y\|}{\|x\|}$.

$$\begin{aligned} \text{Now consider } f(-n) &= \|x - nx + y\| - \|x + nx - y\| \\ &= \|(1-n)x + y\| - \|(1+n)x - y\| \\ &= \|(n-1)x - y\| - \|(n+1)x - y\| \\ &= -[\|(n+1)x - y\| - \|(n-1)x - y\|]. \end{aligned}$$

Thus $f(-n) = -f(n)$ with y replaced by $-y$. Since the above argument is valid if y replaced by $-y$, it follows that $f(-n) > 0$ if $-n > \frac{\|y\|}{\|x\|}$. That is $f(n) < 0$ if $n < -\frac{\|y\|}{\|x\|}$. Hence $f(n)$ cannot be zero if $|n| > \frac{\|y\|}{\|x\|}$.

Now, $f(a) = 0$ implies that $|a| \leq \frac{\|y\|}{\|x\|}$.

DEFINITION I.4

A normed linear space X is called strictly convex if

$$0 < k < 1, x \neq y \in X \text{ imply } \|kx + (1-k)y\| < k\|x\| + (1-k)\|y\|. [20], \text{Def. 5.1.1.}$$

THEOREM I.5

Isosceles orthogonality in X is unique if and only if X is strictly convex.

Proof. (i)- Suppose that X is strictly convex and isosceles orthogonality is not unique. Then $\exists x, z \in X$ and $a < b$ real numbers such that

$$x \perp^i ax + z \text{ and } x \perp^i bx + z.$$

Let $-y = (bx + z)$ and $\alpha = b - a$. Then $\alpha > 0$, $x \perp^i y$ and $x \perp^i -(\alpha x + y)$.

Thus $x \perp^i y$ and $x \perp^i \alpha x + y$ for some positive real α and element $y \in X$.

Now, define a function $q(t) = \|y + tx\|$, $-\infty < t < \infty$.

$$\begin{aligned}
 \text{Then } q(k \cdot t_1 + (1 - k) \cdot t_2) &= \|y + [k \cdot t_1 + (1 - k) \cdot t_2]x\| \\
 &= \|y + k \cdot t_1 x + (1 - k) \cdot t_2 x\| \\
 &= \|[k + (1 - k)]y + k \cdot t_1 x + (1 - k) \cdot t_2 x\| \\
 &= \|k[y + t_1 x] + (1 - k)[y + t_2 x]\| \\
 &< k\|y + t_1 x\| + (1 - k)\|y + t_2 x\|, \text{ for } 0 < k < 1 \\
 &\quad \text{(since the space is strictly convex)} \\
 &= kq(t_1) + (1 - k)q(t_2).
 \end{aligned}$$

Thus $q(t)$ is a strictly convex function. Furthermore,

$$q(1) = \|y + x\| = \|y - x\| = q(-1) \text{ since } x \perp^i y, \text{ and}$$

$$q(\alpha + 1) = \|y + (\alpha + 1)x\| = \|y + (\alpha - 1)x\| = q(\alpha - 1) \text{ since } x \perp^i \alpha x + y.$$

In the case $0 < \alpha < 2$ we see that

$$\begin{aligned}
 q(\alpha - 1) &= q\left[\frac{2 - \alpha}{2} \cdot (-1) + \frac{\alpha}{2} \cdot (1)\right] < q(1) = q\left[\frac{\alpha}{2} \cdot (\alpha - 1) + \left(1 - \frac{\alpha}{2}\right) \cdot (\alpha + 1)\right] \\
 &< \frac{\alpha}{2} \cdot q(\alpha - 1) + \left(1 - \frac{\alpha}{2}\right) \cdot q(\alpha + 1) \\
 &= \left(\frac{\alpha}{2} + \left(1 - \frac{\alpha}{2}\right)\right)q(\alpha + 1) = q(\alpha + 1).
 \end{aligned}$$

Thus $q(\alpha - 1) < q(\alpha + 1)$ which is a contradiction.

In the case $\alpha = 2$, then we have $q(-1) = q(1) = q(3)$. But for $t \in (-1, 3)$,
 $t = (-1) \cdot k + 3 \cdot (1 - k) = 3 - 4k$ where $0 < k < 1$ (indeed $k = (3 - t)/4$).

Therefore $q(t) < kq(-1) + (1 - k)q(3) = q(-1)$, a contradiction.

In the case $\alpha > 2$, then $\alpha - 1 > 1$, so $[-1, 1] \cap [\alpha - 1, \alpha + 1] = \emptyset$ (disjoint).

Thus $q(t)$ will have two distinct local minima - one each in the intervals $[-1, 1]$ and $[\alpha - 1, \alpha + 1]$. But $q(t)$ is strictly convex and thus have only one point of minimum - a global minimum. Again a contradiction is seen.

(ii) - To prove the other way, we start with the assumption that X is not strictly convex, so let $x \neq y \in X$ such that $\|x\| = \|y\| = \left\| \frac{x+y}{2} \right\| = 1$,
 (see [20], proposition 5.1.2).

$$\text{Then } \|x + y\| = \|(x + y) + (x - y)\| = \|(x + y) - (x - y)\| = 2, \quad \text{or}$$

$$\|x'\| = \|x' + y'\| = \|x' - y'\|,$$

where $x' = x + y$ and $y' = x - y \neq 0$. Then we have

$$\|x' + y'/2 - y'/2\| = \|x' + y'/2 + y'/2\| = \|x' - y'/2 - y'/2\|.$$

Thus $x' + (y'/2) \perp^i (y'/2)$ and $x' - (y'/2) \perp^i (y'/2)$.

i.e. $x' + a(y'/2) \perp^i (y'/2)$ where $a = \pm 1$.

Therefore isosceles orthogonality is not unique. This completes the proof.

Example : If X is not strictly convex, isosceles orthogonality need not be

unique. i.e. there exists $a (\neq 0) \in \mathbf{R}$ such that $x \perp^i y$ and $x \perp^i ax + y$.

It is known that the space X is not strictly convex if there exist $x \neq y \in X$

such that $\|x\| = \|y\| = \left\| \frac{x+y}{2} \right\| = 1$.

Chapter I
Isosceles orthogonality

Therefore $C[a, b]$, the space of all continuous functions defined on $J = [a, b]$, with $\|f\| = \max_{t \in J} |f(t)|$ where $f \in C[a, b]$ is not strictly convex.

$$\left[\text{Define } f(t) = 1, \quad g(t) = \frac{t-a}{b-a}, \quad \text{then } \|f\| = \|g\| = \left\| \frac{f+g}{2} \right\| = 1. \right]$$

In particular, the space $C(I)$ where $I = [0, 1]$ is not strictly convex, and we show that isosceles orthogonality in the space $C(I)$ is not unique.

Define $f(t) = t - t^2$, $g(t) = t + t^2$, then $f - g = -2t^2$, and $f + g = 2t$,

$\|f - g\| = \|f + g\| = 2$. Thus $f \perp^i g$. Furthermore

$$f - (f + g) = t - t^2 - 2t = -(t + t^2) \Rightarrow \|f - (f + g)\| = 2, \text{ and}$$

$$f + (f + g) = t - t^2 + 2t = 3t - t^2 \Rightarrow \|f + (f + g)\| = (3 - 1) = 2.$$

Thus $f \perp^i (f + g)$ ($a = 1$).

The effect of assuming homogeneity and additivity of such orthogonality will now be investigated, it being shown that isosceles orthogonality is homogeneous or additive only for inner product spaces.

THEOREM 1.6

If isosceles orthogonality is homogeneous in a normed linear space X , then X is an inner product space.

Proof. It has been proved by Ficken [6] that a normed linear space is an inner product space if and only if $\|ax + y\| = \|x + ay\|$ for all real numbers a and x and y for which $\|x\| = \|y\|$. Let x and y be any two elements of X , such that $\|x\| = \|y\|$, then $\|(x + y) + (x - y)\| = \|(x + y) - (x - y)\|$; that is $(x + y) \perp^i (x - y)$.

If isosceles orthogonality is homogeneous in X , then

$$\|(a+1)(x+y) + (a-1)(x-y)\| = \|(a+1)(x+y) - (a-1)(x-y)\|,$$

or $\|ax + y\| = \|x + ay\|$. Thus X is an inner product space.

COROLLARY I.7

If for every element x of a normed linear space X there can be found a nonzero element orthogonal to x by Roberts' definition⁽¹⁾ in each two-dimensional linear subspace containing x , then X is an inner product space. ([18], p.298 Corollary 4.7).

THEOREM I.8

If isosceles orthogonality is additive in a normed linear space X , then X is an inner product space.

Proof. If $x \perp^i y$, then $x \perp^i -y$ and $y \perp^i x$ because of the nature of the condition for isosceles orthogonality. Hence if isosceles orthogonality is additive,

then $nx \perp^i my$ for all integers n and m .

Thus $\|nx + my\| = \|nx - my\|$ and $\|x + my/n\| = \|x - my/n\|$,

or $\|x + qy\| = \|x - qy\|$ where q is a rational number.

Using density of rationals in $\mathbf{R}$ and continuity of the norm it follows that

$$\|x + ry\| = \|x - ry\| \text{ for all real numbers } r.$$

Thus we have proved that isosceles orthogonality is homogeneous if it is additive, and Theorem I.6 gives that the space is an inner product space.

(1) Roberts -orthogonality:

$$x \perp^R y \Leftrightarrow \|x - ry\| = \|x + ry\| \text{ for every } r \in \mathbf{R}.$$

§2. PYTHAGOREAN ORTHOGONALITY

An element x in a normed linear space is said to be orthogonal to an element y in the pythagorean sense if

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 \quad [18] \quad \text{and written } x \perp^P y.$$

This type of orthogonality is obviously symmetric, and it can be shown that for any elements x and y , $x \perp^P ax + y$ for some number a .

That is

$$\|x - (ax + y)\|^2 = \|x\|^2 + \|ax + y\|^2.$$

As for isosceles orthogonality, the assumption of homogeneity or additivity of pythagorean orthogonality in a normed linear space implies that the space is an inner product space. Thus homogeneity and additivity are equivalent.

THEOREM I.9

If x and y are elements of a normed linear space X , then there exists a number a such that

$$\|x\|^2 + \|ax + y\|^2 = \|x - (ax + y)\|^2, \text{ or } x \perp^P ax + y.$$

Proof. Define the real-valued function of a real variable, $f(n)$, as

$$\begin{aligned} f(n) &= \|x\|^2 + \|nx + y\|^2 - \|x - (nx + y)\|^2, \text{ or} \\ &= \|x\|^2 + \|nx + y\|^2 - \|(1 - n)x - y\|^2, \text{ or} \\ &= \|x\|^2 + \|nx + y\|^2 - \|(n - 1)x + y\|^2. \end{aligned}$$

Using the identity $\frac{(2n - 1)}{n^2} + \frac{(n - 1)^2}{n^2} = 1$, we get

$$\begin{aligned}
f(n) &= \|x\|^2 + \left\{ \frac{(2n-1)}{n} + \frac{(n-1)^2}{n} \right\} \|nx + y\|^2 - \|(n-1)x + y\|^2 \\
&= \left\{ \|x\|^2 + \frac{(2n-1)}{n} \|nx + y\|^2 \right\} + \left\{ \frac{(n-1)^2}{n} \|nx + y\|^2 - \|(n-1)x + y\|^2 \right\} \\
&= \|x\|^2 + \frac{(2n-1)}{n} \|nx + y\|^2 + \left\| \frac{(n-1)}{n} [nx + y] \right\|^2 - \|(n-1)x + y\|^2 \\
&= \|x\|^2 + \frac{(2n-1)}{n} \|nx + y\|^2 + \left\| (n-1)x + \frac{(n-1)}{n} y \right\|^2 - \|(n-1)x + y\|^2 \\
&= \|x\|^2 + (2n-1) \left\| x + \frac{1}{n} y \right\|^2 + \left[\left\| (n-1)x + \frac{(n-1)}{n} y \right\| - \|(n-1)x + y\| \right] \\
&\quad \times \left[\left\| (n-1)x + \frac{(n-1)}{n} y \right\| + \|(n-1)x + y\| \right].
\end{aligned}$$

Using triangle inequality, $\left| \|w\| - \|z\| \right| \leq \|w - z\|$, so that

$$\begin{aligned}
\left| \left\| (n-1)x + \frac{(n-1)}{n} y \right\| - \|(n-1)x + y\| \right| &\leq \left\| (n-1)x + \frac{(n-1)}{n} y - (n-1)x - y \right\| \\
&= \left\| -\frac{1}{n} y \right\| = \frac{1}{n} \|y\|.
\end{aligned}$$

Thus

$$\begin{aligned}
f(n) &\geq \|x\|^2 + (2n-1) \left\| x + \frac{1}{n} y \right\|^2 - \frac{1}{n} \|y\| \times \\
&\quad \left[\left\| (n-1)x + \frac{(n-1)}{n} y \right\| + \|(n-1)x + y\| \right]
\end{aligned}$$

If $n \geq 1$, it follows by using the triangle inequality that

$$\left\|x + \frac{1}{n}y\right\| \geq \left|\left\|x\right\| - \frac{1}{n}\left\|y\right\|\right|. \text{ Squaring,}$$

$$\left\|x + \frac{1}{n}y\right\|^2 \geq \left[\left\|x\right\| - \frac{1}{n}\left\|y\right\|\right]^2, \text{ and}$$

$$\left\|(n-1)x + \frac{n-1}{n}y\right\| + \left\|(n-1)x + y\right\| \leq$$

$$\left\|(n-1)x\right\| + \left\|\frac{n-1}{n}y\right\| + \left\|(n-1)x + y\right\|$$

$$= (n-1)\left\|x\right\| + \frac{n-1}{n}\left\|y\right\| + (n-1)\left\|x\right\| + \left\|y\right\|, \text{ since } n \geq 1$$

$$= 2(n-1)\left\|x\right\| + \frac{2n-1}{n}\left\|y\right\|. \text{ Hence}$$

$$f(n) \geq \left\|x\right\|^2 + (2n-1)\left[\left\|x\right\| - \frac{1}{n}\left\|y\right\|\right]^2 - \frac{1}{n}\left\|y\right\| \cdot \left[2(n-1)\left\|x\right\| + \frac{2n-1}{n}\left\|y\right\|\right]$$

$$= \left\|x\right\|^2 + (2n-1)\left\{\left\|x\right\|^2 - 2\frac{1}{n}\left\|x\right\| \cdot \left\|y\right\| + \frac{1}{n^2}\left\|y\right\|^2\right\}$$

$$- 2\frac{(n-1)}{n}\left\|x\right\| \cdot \left\|y\right\| - \frac{(2n-1)}{n^2}\left\|y\right\|^2$$

$$= \left\{1 + (2n-1)\right\} \cdot \left\|x\right\|^2 - 2\left\{\frac{(2n-1)}{n} + \frac{(n-1)}{n}\right\} \cdot \left\|x\right\| \cdot \left\|y\right\|$$

$$= 2n\left\|x\right\|^2 - 2\left(\frac{3n-2}{n}\right)\left\|x\right\| \cdot \left\|y\right\| = 2\left\|x\right\| \cdot \left[n\left\|x\right\| - \left(\frac{3n-2}{n}\right)\left\|y\right\|\right]$$

$$= 2\left\|x\right\| \cdot \left[n\left\|x\right\| - \left(3 - \frac{2}{n}\right)\left\|y\right\|\right].$$

which can be made greater than zero by taking n sufficiently large.

Now consider

$$\begin{aligned} f(-n) &= \|x\|^2 + \|-nx + y\|^2 - \|(n-1)x + y\|^2 \\ &= \|x\|^2 + \|nx - y\|^2 - \|(n+1)x - y\|^2. \end{aligned}$$

Using the identity $\frac{(n+1)^2}{n} - \frac{(2n+1)}{n} = 1$, we get

$$\begin{aligned} f(-n) &= \|x\|^2 + \left\{ \frac{(n+1)^2}{n} - \frac{(2n+1)}{n} \right\} \|nx - y\|^2 - \|(n+1)x - y\|^2 \\ &= \|x\|^2 - \frac{(2n+1)}{n} \|nx - y\|^2 + \frac{(n+1)^2}{n} \|nx - y\|^2 - \|(n+1)x - y\|^2 \\ &= \|x\|^2 - (2n+1) \left\| x - \frac{1}{n}y \right\|^2 + \left\| (n+1)x - \frac{(n+1)}{n}y \right\|^2 - \|(n+1)x - y\|^2 \\ &= \|x\|^2 - (2n+1) \left\| x - \frac{1}{n}y \right\|^2 + \left[\left\| (n+1)x - \frac{(n+1)}{n}y \right\| - \|(n+1)x - y\| \right] \\ &\quad \times \left[\left\| (n+1)x - \frac{(n+1)}{n}y \right\| + \|(n+1)x - y\| \right]. \end{aligned}$$

Let $n > 0$ and using triangle inequality, we have

$$\left\| x - \frac{1}{n}y \right\|^2 \geq \left[\left\| x \right\| - \frac{1}{n} \left\| y \right\| \right]^2,$$

$$\left\| (n+1)x - \frac{(n+1)}{n}y \right\| - \|(n+1)x - y\| \leq \left\| -\frac{1}{n}y \right\| = \frac{1}{n} \|y\|, \text{ and}$$

$$\left\| (n+1)x - \frac{(n+1)}{n}y \right\| + \|(n+1)x - y\| \leq 2(n+1) \|x\| + \frac{(2n+1)}{n} \|y\|.$$

Thus

$$\begin{aligned}
 f(-n) &\leq \|x\|^2 - (2n+1) \left[\|x\| - \frac{1}{n} \|y\| \right]^2 + \frac{1}{n} \|y\| \cdot \left[2(n+1) \|x\| + \frac{(2n+1)}{n} \|y\| \right] \\
 &= \|x\|^2 - (2n+1) \left[\|x\| - \frac{1}{n} \|y\| \right]^2 + \left[2 \left(\frac{n+1}{n} \right) \cdot \|x\| \cdot \|y\| + \left(\frac{2n+1}{n^2} \right) \cdot \|y\|^2 \right] \\
 &= -2n \|x\|^2 + 2 \left(\frac{3n+2}{n} \right) \cdot \|x\| \cdot \|y\| \\
 &= -2 \|x\| \cdot \left[n \|x\| - \left(3 + \frac{2}{n} \right) \cdot \|y\| \right].
 \end{aligned}$$

Thus $f(-n)$ is negative if n is sufficiently large. Therefore $f(n)$ is negative for n be sufficiently small, and it follows by continuity of $f(n)$ that $f(a) = 0$ for some number a . That is

$$\|x\|^2 + \|ax + y\|^2 = \|x - (ax + y)\|^2.$$

Now let us prove that such number is unique.

THEOREM 1.10

Pythagorean orthogonality is unique.

Proof. Let us assume that pythagorean orthogonality is not unique. Then we must have elements $x \neq 0$ and $y \in X$, and a number $\alpha > 0$ such that

$$x \perp^P y \text{ and } x \perp^P (\alpha x + y).$$

Thus

$$(1) \quad \|x + y\|^2 = \|x\|^2 + \|y\|^2 \text{ and } \|x + (\alpha x + y)\|^2 = \|x\|^2 + \|\alpha x + y\|^2.$$

Setting $g(t) = \|y + tx\|^2$ we have, by (1),

$$(2) \quad g(1) = \|x\|^2 + g(0)$$

$$(3) \quad g(1 + \alpha) = \|x\|^2 + g(\alpha)$$

$$\begin{aligned} \text{Now } g(k t_1 + (1 - k) t_2) &= \|y + [k t_1 + (1 - k) t_2] x\|^2 \\ &= \|[k + (1 - k)] y + [k t_1 + (1 - k) t_2] x\|^2 \\ &= \|k[y + t_1 x] + (1 - k)[y + t_2 x]\|^2 \\ &\leq \left\{ \|k[y + t_1 x]\| + \|(1 - k)[y + t_2 x]\| \right\}^2 \quad (\text{using triangle inequality}) \\ &= \left\{ k\|y + t_1 x\| + (1 - k)\|y + t_2 x\| \right\}^2 \quad (0 < k < 1) \\ &= k^2\|y + t_1 x\|^2 + (1 - k)^2\|y + t_2 x\|^2 + 2k(1 - k)\|y + t_1 x\| \cdot \|y + t_2 x\| \\ &= k\|y + t_1 x\|^2 + (k^2 - k)\|y + t_1 x\|^2 + \\ &\quad (1 - k)\|y + t_2 x\|^2 + [(1 - k)^2 - (1 - k)]\|y + t_2 x\|^2 \\ &\quad + 2k(1 - k)\|y + t_1 x\| \cdot \|y + t_2 x\| \\ &= k g(t_1) + (1 - k) g(t_2) + (k^2 - k)\|y + t_1 x\|^2 + (k^2 - k)\|y + t_2 x\|^2 \\ &\quad - 2(k^2 - k)\|y + t_1 x\| \cdot \|y + t_2 x\| \\ &= k g(t_1) + (1 - k) g(t_2) + \\ &\quad (k^2 - k) \left[\|y + t_1 x\|^2 - 2\|y + t_1 x\| \cdot \|y + t_2 x\| + \|y + t_2 x\|^2 \right] \\ &= k g(t_1) + (1 - k) g(t_2) + (k^2 - k) \left| \|y + t_1 x\| - \|y + t_2 x\| \right|^2 \\ &= k g(t_1) + (1 - k) g(t_2) - k(1 - k) \left| \|y + t_1 x\| - \|y + t_2 x\| \right|^2. \end{aligned}$$

$$(4) \quad \text{Thus } g(k t_1 + (1 - k) t_2) \leq k g(t_1) + (1 - k) g(t_2), \quad -\infty < t < \infty.$$

On the other hand, for $0 < k < 1$, the triangle inequality in (4) will be strict if $g(t_1) \neq g(t_2)$. i.e.

$$(5) \quad g(k t_1 + (1 - k) t_2) < k g(t_1) + (1 - k) g(t_2), \quad g(t_1) \neq g(t_2).$$

Now suppose $0 < \alpha < 1$, we get

$$(6) \quad g(\alpha) = g(\alpha \cdot 1 + (1 - \alpha) \cdot 0) < \alpha g(1) + (1 - \alpha)g(0)$$

(since from (2), $g(1) \neq g(0)$ and $x \neq 0$).

$$(7) \quad g(1) = g(\alpha \cdot \alpha + (1 - \alpha) \cdot (1 + \alpha)) < \alpha g(\alpha) + (1 - \alpha)g(1 + \alpha)$$

(since from (3), $g(\alpha) \neq g(1 + \alpha)$ and $x \neq 0$).

Also by using (2) and (3) we get,

$$\begin{aligned} g(1) &< \alpha g(\alpha) + (1 - \alpha) [\|x\|^2 + g(\alpha)] \\ &= \alpha g(\alpha) + (1 - \alpha) [g(1) - g(0) + g(\alpha)] \\ &= g(1) - \alpha g(1) - (1 - \alpha)g(0) + g(\alpha) \end{aligned}$$

$$\text{or} \quad 0 < -\alpha g(1) - (1 - \alpha)g(0) + g(\alpha)$$

$$\text{or} \quad \alpha g(\alpha) + (1 - \alpha)g(0) < g(\alpha), \text{ contradicts (6).}$$

In the case $\alpha > 1$, we use convexity of g , and (2) and (3) to obtain

$g(0) \neq g(\alpha)$ and $g(1) \neq g(1 + \alpha)$, and then by (5) we get

$$(8) \quad g(1) = g\left(\frac{\alpha - 1}{\alpha} \cdot 0 + \frac{1}{\alpha} \cdot \alpha\right) < \frac{\alpha - 1}{\alpha} \cdot g(0) + \frac{1}{\alpha} \cdot g(\alpha)$$

$$\begin{aligned} (9) \quad g(\alpha) &= g\left(\frac{1}{\alpha} \cdot 1 + \frac{\alpha - 1}{\alpha} \cdot (1 + \alpha)\right) < \frac{1}{\alpha} \cdot g(1) + \frac{\alpha - 1}{\alpha} \cdot g(1 + \alpha) \\ &= \frac{1}{\alpha} \cdot g(1) + \frac{\alpha - 1}{\alpha} [g(1) - g(0) + g(\alpha)] \end{aligned}$$

Chapter I
Pythagorean orthogonality

$$g(\alpha) < g(1) - \frac{\alpha - 1}{\alpha} \cdot g(0) + g(\alpha) - \frac{1}{\alpha} \cdot g(\alpha)$$

$$\text{or } 0 < g(1) - \frac{\alpha - 1}{\alpha} \cdot g(0) - \frac{1}{\alpha} \cdot g(\alpha)$$

$$\text{or } \frac{\alpha - 1}{\alpha} \cdot g(0) + \frac{1}{\alpha} \cdot g(\alpha) < g(1), \text{ contradicts (8).}$$

In the case $\alpha = 1$, we have

From (2),

$$g(1) = g(0) + \|x\|^2$$

$$\text{or } g(0) = g(1) - \|x\|^2.$$

From (3),

$$g(2) = g(1) + \|x\|^2$$

Therefore

$$\begin{aligned} g(2) + g(0) &= \left[g(1) + \|x\|^2 \right] + \left[g(1) - \|x\|^2 \right] \\ &= 2g(1) \end{aligned}$$

$$\text{or } g(1) = \frac{1}{2} [g(2) + g(0)].$$

On the other hand, convexity of g , with $g(2) \neq g(0)$, give

$$g(1) = g\left(\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0\right) < \frac{1}{2} \cdot g(2) + \frac{1}{2} \cdot g(0) = \frac{1}{2} [g(2) + g(0)],$$

which is false. Hence pythagorean orthogonality is unique in any normed linear space.

Chapter I
Pythagorean orthogonality

Note that $x \perp^P y$ does not imply $x \perp^P -y$, and if such happened for every $x, y \in X$, then X must be an inner product space.

THEOREM I.11

If X is a normed linear space, and for every $x, y \in X$, we have $x \perp^P y$ implies $x \perp^P -y$, then X is an inner product space.

Proof. Given $x \perp^P y$, then by assumption $x \perp^P -y$.

Therefore $\|x - y\|^2 = \|x\|^2 + \|y\|^2$, and

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Thus $\|x - y\|^2 = \|x + y\|^2$ or $\|x - y\| = \|x + y\|$.

i.e., $x \perp^i y$. We have proved that $x \perp^P y \Rightarrow x \perp^i y$, which is sufficient condition to make X an inner product space, as we will see in chapter II.

However, we have the following corollary.

COROLLARY I.12

For x and y be elements of a normed linear space X , $\exists b \in \mathbf{R}$ such that $\|x\|^2 + \|b \cdot x + y\|^2 = \|x + (bx + y)\|^2$ and that number is unique.

Now as in the previous section for isosceles orthogonality, we will prove that homogeneity or additivity of pythagorean orthogonality implies that X is an inner product space.

THEOREM I.13

If pythagorean orthogonality is homogeneous in a normed linear space X , then X is an inner product space.

Chapter I
Pythagorean orthogonality

Proof. It will be shown that the condition given by Jordan and Neumann for the existence of an inner product space in a normed linear space is satisfied, namely :

$$\|x + y\|^2 + \|x - y\|^2 = 2\left[\|x\|^2 + \|y\|^2\right] \text{ for all } x \text{ and } y.$$

It is known that for any elements x and y a number a exists such that $x \perp^P a \cdot x + y$.

Assuming homogeneity of the orthogonality gives that $k \cdot x \perp^P a \cdot x + y$ for all real k ,

$$\text{or } k^2\|x\|^2 + \|a \cdot x + y\|^2 = \|k \cdot x - (a \cdot x + y)\|^2.$$

If $k = (a \pm 1)$, then

$$(a \pm 1)^2\|x\|^2 + \|a \cdot x + y\|^2 = \|(a \pm 1)x - (a \cdot x + y)\|^2, \text{ or}$$

$$(a \pm 1)^2\|x\|^2 + \|a \cdot x + y\|^2 = \|\pm 1 \cdot x - y\|^2, \text{ or}$$

$$(a \pm 1)^2\|x\|^2 + \|a \cdot x + y\|^2 = \|x \mp y\|^2.$$

Hence

$$\begin{aligned} \|x + y\|^2 + \|x - y\|^2 &= \left[(a - 1)^2\|x\|^2 + \|a \cdot x + y\|^2 \right] + \left[(a + 1)^2\|x\|^2 + \|a \cdot x + y\|^2 \right] \\ &= 2(a^2 + 1)\|x\|^2 + 2\|a \cdot x + y\|^2 \\ &= 2\left[(a^2 + 1)\|x\|^2 + \|a \cdot x + y\|^2 \right]. \end{aligned}$$

If $k = a$, then

$$a^2\|x\|^2 + \|a \cdot x + y\|^2 = \|a \cdot x - (a \cdot x + y)\|^2, \text{ or}$$

$$a^2\|x\|^2 + \|a \cdot x + y\|^2 = \|y\|^2, \text{ or}$$

$$\|a \cdot x + y\|^2 = \|y\|^2 - a^2\|x\|^2, \text{ and hence}$$

$$\begin{aligned}\|x + y\|^2 + \|x - y\|^2 &= 2 \left[(a^2 + 1) \|x\|^2 + \|y\|^2 - a^2 \|x\|^2 \right] \\ &= 2 \left[\|x\|^2 + \|y\|^2 \right].\end{aligned}$$

Thus X is an inner product space.

THEOREM I.14

The properties of homogeneity and additivity of pythagorean orthogonality are equivalent for normed linear space.

Proof. If pythagorean orthogonality is homogeneous then X is an inner product space by Theorem I.13. The orthogonality is then additive because such property valid in inner product spaces. Conversely, suppose that pythagorean orthogonality is additive and $x \perp^P y$. Also there exists a number a such that $x \perp^P a \cdot x - y$, and additivity of the orthogonality implies that $x \perp^P a \cdot x$ and hence $a = 0$ if $x \neq 0$; that is $x \perp^P -y$. Now using symmetry of pythagorean orthogonality with additivity, it follows that $nx \perp^P my$ for all integers n and m . Thus

$$\begin{aligned}\|nx\|^2 + \|my\|^2 &= \|nx - my\|^2, \text{ or} \\ \|x\|^2 + \left\| \frac{m}{n} y \right\|^2 &= \left\| x - \frac{m}{n} y \right\|^2. \quad (n \neq 0)\end{aligned}$$

From continuity of the norm it follows that

$$\|x\|^2 + \|r \cdot y\|^2 = \|x - r \cdot y\|^2 \quad \text{for all } r \in \mathbf{R},$$

and so pythagorean orthogonality is homogeneous.

§3. BIRKHOFF ORTHOGONALITY

Birkhoff orthogonality is defined in a different manner. We say that x is orthogonal to y in the sense of Birkhoff orthogonality, and written $x \perp^B y$ if

$$\|x + ky\| \geq \|x\| \quad \forall k \in \mathbf{R}. \quad [19]$$

Geometrically, this means that the line through x in the y - direction supports the ball $\{z : \|z\| \leq \|x\|\}$ at x .

Simple facts about Birkhoff orthogonality are :

(I) $x \perp^B y \Rightarrow \alpha x \perp^B \beta y \quad \forall \alpha, \beta \in \mathbf{R}$ (homogeneity).

That is $\|\alpha x + k\beta y\| \geq \|\alpha x\| \quad \forall k$. If $\alpha = 0$, then it is trivial, thus let $\alpha \neq 0$.

Then $\|\alpha x + k\beta y\| \geq \|\alpha x\|$ if and only if $\|x + k\beta/\alpha y\| \geq \|x\|$, which is true since $x \perp^B y$.

(II) $x \perp^B y \Leftrightarrow$ there exists a nonzero bounded linear functional f such that $f(x) = \|f\| \|x\|$ and $f(y) = 0$. ($x \neq 0$)

Proof. Say that there is such functional f . Then linearity of f implies that

$$f(x + ky) = f(x) + kf(y) = f(x) = \|f\| \cdot \|x\|.$$

But f is bounded, therefore $f(x + ky) \leq \|f\| \cdot \|x + ky\|$, and so we have

$$\|f\| \cdot \|x\| \leq \|f\| \cdot \|x + ky\| \quad \text{or} \quad \|x\| \leq \|x + ky\| \quad (\because \|f\| \neq 0).$$

Conversely, if $x \perp^B y$, let $Z = \text{span}(x, y)$, that means every element $z \in Z$ can be written $z = \alpha x + \beta y$ for some real numbers α and β . Now define f on Z by $f(\alpha x + \beta y) = \alpha \|x\|$.

Clearly f is a linear functional. Further $|f(\alpha x + \beta y)| = |\alpha| \cdot \|x\| \leq |\alpha| \cdot \|x + ky\|$ for all numbers k . ($\because x \perp^B y$). Or $|f(\alpha x + \beta y)| \leq \|\alpha x + \alpha ky\|$. Then with $\alpha \neq 0$ and $k = \beta/\alpha$, we get $|f(z)| \leq \|z\|$. If $\alpha = 0$, then $|f(z)| = 0 \leq \|z\|$, and so f is bounded. Indeed $\|f\| = 1$. Thus $f(x) = \|x\| = \|f\| \cdot \|x\|$ and $f(y) = 0$. Now Hahn – Banach Theorem [5], p.213, implies that there exists a linear extension functional $\tilde{f}$ defined on the space X with $\tilde{f}|_Z = f$ and $\|\tilde{f}\| = \|f\|$. Other properties are discussed in the following.

EXISTENCE OF ORTHOGONAL ELEMENTS

DEFINITION I.15

A hyperplane of a normed linear space X is any closed maximal linear subset H of X , or any translation $x + H$ of H . [17]

THEOREM I.16

If f be a nonzero bounded linear functional on a normed linear space X , then $|f(x)| = \|f\| \cdot \|x\|$ if and only if $x \perp^B H$, where H is the hyperplane of all h for which $f(h) = 0$. (In Algebra, H is called the kernel of f .)

If x is orthogonal to a linear subset $L \subset X$, then there exists a linear functional f for which $f(x) = \|f\| \cdot \|x\|$ and $f(y) = 0$ if $y \in L$, and there is a hyperplane H through the origin with $x \perp^B H$ and $L \subset H$.

Proof. For $x = 0$ theorem is trivial, so let $x \neq 0$. Now let H be the hyperplane consisting of all elements h for which $f(h) = 0$. Suppose that

$|f(x)| = \|f\| \cdot \|x\|$. Then $f(h) = 0$ and linearity of f imply that $|f(x + h)|$

$|f(x)| = \|f\| \cdot \|x\|$. But f is bounded, so $|f(x+h)| \leq \|f\| \cdot \|x+h\|$, and hence $\|x\| \leq \|x+h\|$ for all elements $h \in H$. f is linear, thus $kh \in H$, and $x \perp^B H$.

Conversely, suppose $x \perp^B H$ and $|f(x)| = p\|x\|$, for $p \in \mathbf{R}$. Then homogeneity of Birkhoff orthogonality and linearity of f give :

* For any $\alpha \in \mathbf{R}$, $\alpha x \perp^B H$; that is $\|\alpha x\| \leq \|\alpha x + h\|$ if $h \in H$,

* $|f(\alpha x)| = p\|\alpha x\|$.

Hence $|f(\alpha x + h)| = |f(\alpha x)| = p\|\alpha x\| \leq p\|\alpha x + h\|$, or $|f(\alpha x + h)| \leq p\|\alpha x + h\|$

for all $h \in H$. Since H is a hyperplane through the origin, it follows that

$|f(y)| \leq p\|y\|$ for all y of X . Thus $\|f\| = p$ and $|f(x)| = \|f\| \cdot \|x\|$. (Definition of $\|f\| \Rightarrow \|f\| \leq p$, and $|f(x)| = p\|x\| \Rightarrow p \leq \|f\|$.)

Now if L is a linear subset of X . Then $Z = \langle x \rangle \oplus L$ is a subspace of X .

Define $f : Z \rightarrow \mathbf{R}$ by $f(\alpha x + y) = \alpha\|x\|$. Then f is a linear functional with

$\|f\| = 1$, $f(x) = \|x\|$ and $f(y) = 0$ for $y \in L$. But Hahn - Banach Theorem

gives that $\exists \tilde{f} : X \rightarrow \mathbf{R}$, a linear extension functional of f with same norm.

Therefore first part of the theorem implies that for $H = \{h : \tilde{f}(h) = 0\}$ then

$x \perp^B H$, and clearly $L \subseteq H$.

The following theorem is a direct consequence of Theorem I.16 and the fact

that for any element x of a normed linear space, there is a linear functional

f such that $f(x) = \|f\| \cdot \|x\|$.

THEOREM I.17

Any element of a normed linear space is orthogonal to some hyperplane through the origin. This hyperplane being unique for a given element if and only if for any $x (\neq 0)$ and $y \in X$ there is a unique number a with $x \perp^B ax + y$.

However, for a given hyperplane H through the origin there does not necessarily exist an element x with $x \perp^B H$. This problem is equivalent to finding an element x with $f(x) = \frac{\|f\|}{\|x\|}$ for a given linear functional f . (See [19], Theorem 7.1.)

COROLLARY I.18

- (1) If $x (\neq 0)$ and y be two elements of a normed linear space, then there exists a number a such that $x \perp^B ax + y$.
- (2) If $x \perp^B ax + y$, then $|a| \leq \frac{\|y\|}{\|x\|}$. ($x \neq 0$)
- (3) $x \perp^B ax + y$ if and only if there is a linear functional f with $f(x) = \frac{\|f\|}{\|x\|}$ and $a = -f(y)/f(x)$.

Proof.

- (1) It is well known that for $x \in X$ there exists a linear functional f such that $f(x) = \frac{\|f\|}{\|x\|}$. Then Theorem I.16 gives that $x \perp^B H$, where H is the hyperplane of all h for which $f(h) = 0$. Thus $ax + y \in H$ if and only if $f(ax + y) = 0$; that is $af(x) + f(y) = 0$ or $a = -f(y)/f(x)$. Thus there exists a number a for which $x \perp^B ax + y$.
- (2) Let $x \perp^B ax + y$. Then using definition of Birkhoff orthogonality, we have $\|x + k(ax + y)\| \geq \|x\|$ for all real k , and letting $k = -1/a$ gives

$$\| -1/a \ y \| = \| 1/a \| \| y \| \geq \| x \| \quad \text{or that} \quad \| a \| \leq \| y \| / \| x \|.$$

(3) $x \perp^B ax + y \Leftrightarrow \exists f (\neq 0)$ a bounded linear functional defined on X , such that $f(x) = \| f \| \cdot \| x \|$ and $f(ax + y) = 0$; that is $a = -f(y)/f(x)$.

Note that in inner product spaces, $x \perp^B ax + y \Leftrightarrow \langle x, ax + y \rangle = 0$; that is $a = -\langle x, y \rangle / \| x \|^2$ ($x \neq 0$). Thus the linear functional is $f(z) = \langle x, z \rangle$ for $z \in X$.

Unlike isosceles and pythagorean orthogonality, $x \perp^B y$ does not imply $y \perp^B x$. Thus Corollary I.18 does not give the existence of a number b such that $bx + y \perp^B x$.

THEOREM I .19

For any elements x and y of a normed linear space X , there exists a number b such that $bx + y \perp^B x$. This number b is the value of k for which $\| kx + y \|$ takes on its absolute minimum. If $Ax + y \perp^B x$ and $Bx + y \perp^B x$, then $nx + y \perp^B x$ if n is a number between A and B .

Proof.

By definition, $bx + y \perp^B x$ if and only if $\| (bx + y) + kx \| \geq \| bx + y \|$ for all k . That is, if and only if $\| (b + k)x + y \| = \| \lambda x + y \| \geq \| bx + y \|$ for all real λ , or that $\| bx + y \|$ is the smallest value of $\| \lambda x + y \|$. Since $\| \lambda x + y \|$ is continuous in λ , it must take on its minimum. Suppose $Ax + y \perp^B x$ and $Bx + y \perp^B x$. If $A = B$ then the proof is trivial.

Therefore let $A < B$. Now using the triangle inequality we can show that

$f(n) = \|nx + y\|$, $n \in \mathbf{R}$ is a convex function. Also, $Ax + y \perp^B x$ and $Bx + y \perp^B x$ imply that $f(A) = f(B)$; that is $\|Ax + y\| = \|Bx + y\|$, being the minimum. Hence $f(n) \geq f(B)$ for every n . Furthermore, for every n , with $A \leq n \leq B$ there is k , $0 \leq k \leq 1$ such that $n = kA + (1 - k)B$, indeed

$k = \frac{(B - n)}{(B - A)}$. Then using this fact and convexity of f we get $f(n) \leq f(B)$

for n , $A \leq n \leq B$. Thus $f(n) = f(A) = f(B)$ or that $\|nx + y\| = \|Ax + y\| = \|Bx + y\|$ if $A \leq n \leq B$. That is, $\|nx + y\|$ is the minimum or $nx + y \perp^B x$.

This completes the proof.

GATEAUX DIFFERENTIABILITY OF THE NORM

DEFINITION 1.20

We say that the norm is Gateaux – differentiable at a point x if and only

if $G(x; y) = \lim_{h \rightarrow 0} \frac{\|x + hy\| - \|x\|}{h}$ exists for all elements y of X .

The norm is said to be Gateaux – differentiable if and only if, it is Gateaux – differentiable at all points of the space.

Let $N_+(x; y) = \lim_{h \rightarrow +0} \frac{\|x + hy\| - \|x\|}{h}$,

$N_-(x; y) = \lim_{h \rightarrow -0} \frac{\|x + hy\| - \|x\|}{h}$. Clearly $N_-(x; y) = -N_+(x; -y)$.

$N_+(x; y)$ is called the right – Gateaux differential, and $N_-(x; y)$ is the left – Gateaux differential.

Then the norm is Gateaux -differentiable if $N_+(x; y) = N_-(x; y)$. However, in general normed linear spaces $N_+(x; y)$ and $N_-(x; y)$ always exist, even though $G(x; y)$ does not exist. This is because from [20], Lemma 5.4.14, p.483, we have for $x, y \in X$ and $h > 0$ the real - valued function $R(h; y)$

$$\text{where } R(h; y) = \frac{\|x + hy\| - \|x\|}{h} \text{ then}$$

(i) $R(h; y)$ is nondecreasing in h , and

$$(ii) -\|y\| \leq R(h; y) \leq \|y\|.$$

$$\text{It follows that } N_+(x; y) = \lim_{h \rightarrow +0} R(h; y) = \inf_{h > 0} \frac{\|x + hy\| - \|x\|}{h} \text{ exists.}$$

$$\text{Also } N_-(x; y) = \sup_{h < 0} \frac{\|x + hy\| - \|x\|}{h} \text{ exists.}$$

$$\begin{aligned} \text{Now } \lim_{n \rightarrow +\infty} \|nx + y\| - \|nx\| &= \lim_{1/n \rightarrow +0} \frac{\|x + 1/n y\| - \|x\|}{1/n} \\ &= \lim_{h \rightarrow +0} \left[\frac{\|x + hy\| - \|x\|}{h} \right] = N_+(x; y). \end{aligned}$$

Similarly,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \|nx\| - \|nx - y\| &= - \lim_{n \rightarrow +\infty} \|nx - y\| - \|nx\| \\ &= -N_+(x; -y) \\ &= N_-(x; y). \end{aligned}$$

Therefore $\lim_{n \rightarrow \infty} \|nx + y\| - \|nx\|$ and $\lim_{n \rightarrow \infty} \|nx\| - \|nx - y\|$ exist, and their

values are determined from the following theorem.

THEOREM I . 21

$$\lim_{n \rightarrow \infty} \|nx + y\| - \|nx\| = -A\|x\| \quad \text{and} \quad \lim_{n \rightarrow \infty} \|nx\| - \|nx - y\| = -B\|x\|, \quad x \neq 0,$$

where A and B are the smallest and largest numbers a for which $x \perp^B ax + y$.

Proof. Since $\lim_{n \rightarrow \infty} \|nx + y\| - \|nx\|$ exists then let $-r\|x\|$ for $r \in \mathbf{R}$ be the

limit point. Thus $\lim_{n \rightarrow \infty} \|nx + y\| - \|nx\| = -r\|x\|$. It is known that for every

real number a , $\lim_{n \rightarrow \infty} \|(n + a)x + y\| - \|nx + y\| = a\|x\|$, (Lemma I.2). Add

and subtract $\|nx\|$, to get

$$\lim_{n \rightarrow \infty} [\|(n + a)x + y\| - \|nx\|] - [\|nx + y\| - \|nx\|] = a\|x\|,$$

or

$$\lim_{n \rightarrow \infty} [\|(n + a)x + y\| - \|nx\|] + r\|x\| = a\|x\|,$$

or

$$\lim_{n \rightarrow \infty} [\|(n + a)x + y\| - \|nx\|] = (a - r)\|x\|.$$

If $x \perp^B ax + y$, then homogeneity of Birkhoff orthogonality gives

$nx \perp^B ax + y$. Therefore

$$\|(n + a)x + y\| \geq \|nx\|, \text{ or that } \|(n + a)x + y\| - \|nx\| \geq 0.$$

Thus

$$(1) \quad (a - r)\|x\| \geq 0; \text{ that is } r \leq a \text{ and so } r \leq A.$$

Now using definition of limit $\lim_{n \rightarrow \infty} \|nx + y\| - \|nx\| = -r\|x\|$ and let ε be

any positive number, then $\exists N$ such that $\|\|nx + y\| - \|nx\| + r\|x\|\| < \varepsilon\|x\|$ if

$n > N$. Simplify, $\|\|nx + y\| - (n - r)\|x\|\| < \varepsilon\|x\|$ if $n > N$. We may choose

N , such that $N > |r|$, so that $n > |r|$ and $(n - r) > 0$ for all $n > N$.

Then there is a number λ_n , $|\lambda_n| < \varepsilon$ for which

$$\|nx + y\| - (n - r)\|x\| = \lambda_n \|x\|,$$

or $\|nx + y\| = (n - r + \lambda_n)\|x\| \geq 0$.

Thus $\|nx + y\| = \|(n - r + \lambda_n)x\|$.

But this can be written

$$\begin{aligned} & \left\| \left[(n - 2^{-1}r + 2^{-1}\lambda_n)x + 2^{-1}y \right] + \left[(2^{-1}r - 2^{-1}\lambda_n)x + 2^{-1}y \right] \right\| \\ &= \left\| \left[(n - 2^{-1}r + 2^{-1}\lambda_n)x + 2^{-1}y \right] - \left[(2^{-1}r - 2^{-1}\lambda_n)x + 2^{-1}y \right] \right\|. \end{aligned}$$

That is

$$\left[(n - 2^{-1}r + 2^{-1}\lambda_n)x + 2^{-1}y \right] \perp^i \left[(2^{-1}r - 2^{-1}\lambda_n)x + 2^{-1}y \right].$$

Then it follows that for $|k| \geq 1$, ([18], Theorem 4.1),

$$\begin{aligned} & \left\| \left[(n - 2^{-1}r + 2^{-1}\lambda_n)x + 2^{-1}y \right] + k \left[(2^{-1}r - 2^{-1}\lambda_n)x + 2^{-1}y \right] \right\| \\ & \geq \left\| (n - 2^{-1}r + 2^{-1}\lambda_n)x + 2^{-1}y \right\|. \end{aligned}$$

If k is replaced by $2m$ and this inequality divided by $|m|$, it

$$\text{becomes } \left\| \left[\frac{(2n - r + \lambda_n)x + y}{2m} \right] + (r - \lambda_n)x + y \right\| \geq \left\| \frac{(2n - r + \lambda_n)x + y}{2m} \right\|.$$

Let p be any number, but with $m = n/p$, then we have

$$\left\| p \left[\left(1 - \frac{r}{2n} + \frac{\lambda_n}{2n}\right)x + \frac{y}{2n} \right] + (r - \lambda_n)x + y \right\| \geq \left\| p \left[\left(1 - \frac{r}{2n} + \frac{\lambda_n}{2n}\right)x + \frac{y}{2n} \right] \right\|.$$

Since $|\lambda_n| < \varepsilon$, it follows that $\lim_{n \rightarrow \infty} \lambda_n = 0$. Hence as n becomes infinite it

gives that

$$\|px + (rx + y)\| \geq \|px\|.$$

Since $-r\|x\|$ is the limit point of $\|nx + y\| - \|nx\|$ as n becomes infinite, then we have

$$(2) \quad x \perp^B rx + y, \text{ and so } r \geq A.$$

By (1) and (2), $r = A$ and $\lim_{n \rightarrow \infty} \|nx + y\| - \|nx\| = -A\|x\|$.

If B is the largest number a for which $x \perp^B ax + y$, then $-B$ is the smallest for which $x \perp^B -Bx - y$ ($\because x \perp^B Bx + y \Rightarrow x \perp^B -(Bx + y)$).

Hence

$$\lim_{n \rightarrow \infty} \|nx - y\| - \|nx\| = -(-B)\|x\| = B\|x\|,$$

or $\lim_{n \rightarrow \infty} \|nx\| - \|nx - y\| = -B\|x\|$. This completes the proof.

SOME PROPERTIES OF THE FUNCTIONALS $N_+(x; y)$ AND $N_-(x; y)$

The norm of a general normed linear space is not differentiable at all nonzero points, although the differentials $N_{\pm}(x; y)$ exist. Let $x \neq 0$, then we have :

$$(1) \quad N_+(x; y) = -A\|x\| \text{ and } N_-(x; y) = -B\|x\|,$$

where A and B are the smallest and largest of the numbers a for which $x \perp^B ax + y$ (Theorem I.21).

$$(2) \quad N_+(x; y + z) \leq N_+(x; y) + N_+(x; z),$$

$$N_-(x; y + z) \geq N_-(x; y) + N_-(x; z).$$

Proof. For $h > 0$ we have

$$\begin{aligned} \left[\frac{\|x + hy\| - \|x\|}{h} \right] + \left[\frac{\|x + hz\| - \|x\|}{h} \right] &\geq \frac{\|2x + h(y + z)\| - 2\|x\|}{h} \\ &\geq 2 \left[\frac{\|x + h/2 (y + z)\| - \|x\|}{h} \right] \\ &= \frac{\|x + h/2 (y + z)\| - \|x\|}{h/2}. \end{aligned}$$

Then taking limit as $h \rightarrow +0$ we get (2) – part 1.

For $h < 0$ we have

$$\left[\frac{\|x + hy\| - \|x\|}{h} \right] + \left[\frac{\|x + hz\| - \|x\|}{h} \right] \leq \frac{\|x + h/2 (y + z)\| - \|x\|}{h/2}.$$

Then taking limit as $h \rightarrow -0$ we get (2) – part 2.

(3) $N_{\pm}(x; ty) = t \cdot N_{\pm}(x; y)$ for $t \geq 0$.

Proof.

For $t = 0$, (3) is clearly true from definition of $N_{\pm}$, so let $t > 0$. Then

by (1) $N_{+}(x; y) = -A\|x\|$ and $N_{+}(x; ty) = -A'\|x\|$, where A and A'

are the smallest of the numbers a and b respectively, for which

$x \perp^B ax + y$ and $x \perp^B bx + ty$. We have to show that $A' = tA$. Now

homogeneity of Birkhoff orthogonality gives $x \perp^B t(Ax + y)$ or

$x \perp^B tAx + ty$. Thus $A' \leq tA$. Similarly, we have $x \perp^B 1/t (A'x + ty)$

or $x \perp^B A'/t x + y$. Thus $A \leq A'/t$ or $tA \leq A'$. Therefore $A' = tA$.

In a similar manner one can show that

$$N_-(x; ty) = t \cdot N_-(x; y) \text{ for } t \geq 0.$$

Note that for $t < 0$,

$$\begin{aligned} N_{\pm}(x; ty) &= N_{\pm}(x; (-t) \cdot -y) = -t \cdot N_{\pm}(x; -y) & (\text{by (3)}) \\ &= -t \cdot -N_{\mp}(x; y) = t \cdot N_{\mp}(x; y). \end{aligned}$$

Thus, if the Gateaux differential of the norm exists at x , then

$$N_{\pm}(x; ty) = t \cdot N_{\pm}(x; y) \text{ for } t \in \mathbf{R}.$$

Hence $N_{\pm}(x; ty)$ are linear functionals in y .

$$(4) \quad N_{\pm}(x; x) = \|x\|; \quad |N_{\pm}(x; y)| \leq \|y\|.$$

Proof.

$$N_{\pm}(x; x) = \lim_{h \rightarrow \pm 0} \frac{\|x + hx\| - \|x\|}{h} = \lim_{h \rightarrow \pm 0} \frac{((1 + h) - 1)\|x\|}{h}.$$

Since h is very small we may assume that $1 + h > 0$, then

$$N_{\pm}(x; x) = \lim_{h \rightarrow \pm 0} \frac{h\|x\|}{h} = \|x\|.$$

Now, since $N_+(x; y) = -A\|x\|$ and $N_-(x; y) = -B\|x\|$, where A

and B are the smallest and largest of the numbers a for which $x \perp^B ax + y$. Thus we have

$$x \perp^B \left[-N_{\pm}(x; y) / \|x\| \right] x + y. \quad (x \neq 0)$$

Therefore the inequality (4) follows from this result and Corollary I.18;

that is $|a| \leq \|y\| / \|x\|$.

$$(5) \quad N_{\pm}(x; rx + sy) = r\|x\| + s \cdot N_{\pm}(x; y), \text{ for } s \geq 0 \text{ and all } r.$$

Proof.

If A is the smallest of the numbers a for which $x \perp^B ax + sy$ and B is the largest, then $A - r$ and $B - r$ are the smallest and largest of the numbers b for which $x \perp^B bx + (rx + sy)$, since

$$(A - r)x + (rx + sy) = Ax + sy \quad \text{and} \quad (B - r)x + (rx + sy) = Bx + sy.$$

Thus from (1)

$$\begin{aligned} N_+(x; rx + sy) &= -(A - r)\|x\| = r\|x\| - A\|x\| \\ &= r\|x\| + N_+(x; sy) \\ &= r\|x\| + s N_+(x; y) \quad (\text{using (3)}) \end{aligned}$$

and

$$\begin{aligned} N_-(x; rx + sy) &= -(B - r)\|x\| = r\|x\| - B\|x\| \\ &= r\|x\| + N_-(x; sy) \\ &= r\|x\| + s N_-(x; y) \quad (\text{using (3)}) \end{aligned}$$

$$(6) \quad x \perp^B ax + y \text{ if and only if } N_-(x; y) \leq -a\|x\| \leq N_+(x; y).$$

Proof.

By (1) $-N_+(x; y) = A\|x\|$ and $-N_-(x; y) = B\|x\|$, where A and B are the smallest and largest of the numbers a for which $x \perp^B ax + y$. Thus

$$N_-(x; y) \leq -a\|x\| \leq N_+(x; y)$$

$$\text{if and only if} \quad -B\|x\| \leq -a\|x\| \leq -A\|x\|$$

$$\text{or} \quad A \leq a \leq B. \quad (x \neq 0)$$

Now Theorem I.19 gives that $x \perp^B ax + y$ if and only if $A \leq a \leq B$. Hence it follows from (6) that

$$x \perp^B y \text{ if and only if } N_-(x; y) \leq 0 \leq N_+(x; y),$$

and

$$\text{if } N_+(x; y) = 0 \text{ or } N_-(x; y) = 0, \text{ then } x \perp^B y.$$

If the normed linear space X is Gateaux differentiable, then

$$x \perp^B ax + y \text{ if and only if } N_+(x; y) = N_-(x; y) = -a\|x\|.$$

TYPES OF UNIQUENESS OF ORTHOGONALITY

For any elements x and y of a normed linear space, there is a number a for which $x \perp^B ax + y$ (Corollary I.18), and a number b for which $bx + y \perp^B x$ (Theorem I.19). These numbers are not unique in general, their uniqueness being related to the concepts of differentiability of the norm and strict convexity. Since Birkhoff orthogonality is not symmetric, this uniqueness can take the following form :

DEFINITION I.22

Birkhoff orthogonality is right – unique if and only if for no elements $x (\neq 0)$ and y there is more than one number a for which $x \perp^B ax + y$.

DEFINITION I.23

Birkhoff orthogonality is left – unique if and only if for no elements $x (\neq 0)$ and y there is more than one number b for which $bx + y \perp^B x$.

If orthogonality is symmetric, then it is right – unique if and only if it is left – unique. This was the case for isosceles and pythagorean orthogonality, but not for Birkhoff orthogonality.

DEFINITION I.24

A functional f defined on a normed linear space X is Gateaux differentiable at a point x if and only if for all elements y of X , then

$$\lim_{h \rightarrow 0} \frac{f(x + hy) - f(x)}{h} \text{ exists. [19], Def.4.3.}$$

In this case the limit denoted by $f(x; y)$.

LEMMA I.25

The norm in a normed linear space X possesses a Gateaux differential at each nonzero point if and only if it is right – additive.

Proof. Assume that the norm is Gateaux differentiable at each nonzero point

and $x \perp^B y$ and $x \perp^B z$. Then from (6), $N(x; y) = N(x; z) = 0$. But it

is known that $N(x; y)$ is a linear functional of y if it exists, thus

$$N(x; y + z) = N(x; y) + N(x; z) = 0.$$

This implies that $x \perp^B y + z$. Conversely, say the norm is right – additive.

Then assume that $N(x; y)$ does not exist at some nonzero point x_0 in the direction y_0 . That is $N_+(x_0; y_0) \neq N_-(x_0; y_0)$. But it is known from

(6) that $x_0 \perp^B ax_0 + y_0 \Leftrightarrow N_-(x_0; y_0) \leq -a \|x_0\| \leq N_+(x_0; y_0)$ or

$$\Leftrightarrow \frac{-A}{\|x_0\|} \leq a \leq \frac{-B}{\|x_0\|},$$

where $A = N_+(x_o; y_o)$ and $B = N_-(x_o; y_o)$. Thus

$$x_o \perp^B \left(-A/\|x_o\| \right) x_o + y_o \quad \text{and} \quad x_o \perp^B \left(-B/\|x_o\| \right) x_o + y_o.$$

Then by homogeneity of Birkhoff orthogonality and right-additivity of the

norm we have $x_o \perp^B \left(\frac{A-B}{\|x_o\|} \right) x_o$, which means that

$$\|x_o + k(A-B)x_o\| \geq \|x_o\| \quad \text{for all real number } k. \text{ This is true only if}$$

$A = B$; that is $N_+(x_o; y_o) = N_-(x_o; y_o)$, a contradiction.

LEMMA I.26

Birkhoff orthogonality in a normed linear space X is right-additive if and only if it is right-unique.

Proof. Suppose orthogonality is right-additive and $x \perp^B ax + y$ and $x \perp^B bx + y$ for $x \neq 0$. Then $x \perp^B (bx + y)$ and right-additivity gives that $x \perp^B (a-b)x$. But this means that $\|x + k(a-b)x\| \geq \|x\|$ for all $k \in \mathbf{R}$, which is true only if $a = b$. Therefore orthogonality is right-unique. Conversely, if orthogonality is right-unique and $x \perp^B y$ and $x \perp^B z$, then y and z both belong to the hyperplane H through the origin for which $x \perp^B H$ (Theorem I.17), H being unique if for all y there is a unique number a for which $x \perp^B ax + y$. Then $y + z \in H$ and so $x \perp^B y + z$.

Then Lemma I.25 and Lemma I.26 give the following theorem

THEOREM I.27

Birkhoff orthogonality in a normed linear space X is right – unique if and only if it is Gateaux differentiable.

THEOREM I.28

A necessary and sufficient condition that a normed linear space X be strictly convex is that orhtogonality be left – unique.

Proof. If X is not strictly convex then there are elements x and y ($y \neq 0$) such that $\|x\| + \|y\| = \|x + y\|$ and $x \neq ty$ for any t ([20] , Proposition 5.1.11).

From $x + y = [(1 - k)x + ky] + kx + (1 - k)y$, it follows that

$$\|x + y\| \leq \|(1 - k)x + ky\| + k\|x\| + (1 - k)\|y\| \text{ if } 0 \leq k \leq 1.$$

Since $\|x\| + \|y\| = \|x + y\|$, then

$$\|x\| + \|y\| \leq \|(1 - k)x + ky\| + k\|x\| + (1 - k)\|y\|$$

or

$$(1 - k)\|x\| + k\|y\| \leq \|(1 - k)x + ky\|,$$

and the triangle inequality gives that

$$(1 - k)\|x\| + k\|y\| \geq \|(1 - k)x + ky\|.$$

Thus $(1 - k)\|x\| + k\|y\| = \|(1 - k)x + ky\|$ for $0 \leq k \leq 1$.

If $k = \|x\| / (\|x\| + \|y\|)$, this becomes

$$\frac{\|y\|x + \|x\|y\|}{\|x\| + \|y\|} = \frac{\|y\| \cdot \|x\| + \|x\| \cdot \|y\|}{\|x\| + \|y\|} = \frac{2\|y\| \cdot \|x\|}{\|x\| + \|y\|}, \text{ or}$$

$$(1) \quad \left\| \left\| y \right\| x + \left\| x \right\| y \right\| = 2 \left\| y \right\| \cdot \left\| x \right\|$$

$$\text{Setting } x' = \left\| y \right\| x + \left\| x \right\| y \quad \text{and} \quad y' = \left\| y \right\| x - \left\| x \right\| y.$$

Then $y' \neq 0$, since otherwise we have $x = \frac{\left\| x \right\|}{\left\| y \right\|} \cdot y$ a contradiction.

Similarly $x' \neq 0$. Then it follows

$$\left\| x' + y' \right\| = \left\| 2 \left\| y \right\| x \right\| = 2 \left\| y \right\| \cdot \left\| x \right\| = \left\| x' \right\| \quad \text{by (1)}$$

$$\left\| x' - y' \right\| = \left\| -2 \left\| x \right\| y \right\| = 2 \left\| x \right\| \cdot \left\| y \right\| = \left\| x' \right\|$$

Thus $\left\| x' + y' \right\| = \left\| x' - y' \right\| = \left\| x' \right\|$ ($x' \perp^i y'$). Therefore

$$\left\| x' + ay' \right\| = \left\| x' \right\| \quad \text{if } |a| \leq 1,$$

and $\left\| x' + ay' \right\| \geq \left\| x' \right\|$ for $|a| \geq 1$ ([18], Theorem 4.1).

Thus $f(a) = \left\| x' + ay' \right\|$, $a \in \mathbf{R}$ takes its minimum at every a for which

$|a| \leq 1$. By Theorem I.19 we have $x' + ay' \perp^B y'$ for $|a| \leq 1$, which

contradicts the left – uniqueness of orthogonality. Thus the space must

be strictly convex if orthogonality is left – unique. Conversely, suppose

that the space is strictly convex and show that Birkhoff orthogonality

in this space is left – unique. Define a function

$$f(n) = \left\| nx + y \right\|, \quad -\infty < n < \infty.$$

Then f is a convex function (by the triangle inequality), and by

strict convexity of the space f will be strictly convex. Thus $f(n)$ has

only one point of minimum – a global minimum. Suppose that

$ax + y \perp^B x$ and $bx + y \perp^B x$ with $a < b$. Then Theorem I.19 gives

that $\|ax + y\|$ and $\|bx + y\|$ are the minimum values of $f(n)$. Hence $f(a) = f(b)$ and $f(n)$ is constant for $a \leq n \leq b$. But f is strictly convex, thus for $n = 1/2(a + b)$, we get $f(n) < f(a)$, a contradiction. Therefore orthogonality is left – unique.

CHARACTERIZATIONS OF INNER PRODUCT SPACES

Various necessary and sufficient conditions for the existence of an inner product in normed linear spaces of two or more dimensions are known. Some of them were studied in the previous sections.

A famous characterization of inner product spaces of three or more dimensions is the one given by **Kakutani** ([21], Theorem 3) is that **for every two – dimensional subspace E of X there is a norm – 1 linear projection P of X onto E** . Other characterizations valid for three or more dimensions will be given here, expressed by means of orthogonality and hyperplanes.

THEOREM I.29

If the real – normed linear space has at least three dimensions and Birkhoff orthogonality is symmetric, then X is an inner product space.
Proof. Let x_1 and x_2 be any two linearly independent elements of a three – dimensional subspace X_0 of X , and $H = \text{span}(x_1, x_2)$. Then there is an element $y \in X_0$ orthogonal to H , and the space X_0 can be

written as a direct sum, $X_o = H \oplus \langle y \rangle$. That is each $x \in X_o$ has a unique representation $x = h + ay$ for some $h \in H$ and $a \in R$. Define the projection P of X_o onto H by $P : X_o \rightarrow H$; $P(x) = h$. Since orthogonality is symmetric, we have $H \perp^B y$. Then $\|P(x)\| \leq \|x\|$ for all x in X_o and $\|P\| = 1$. Thus it follows from Kakutani's characterization that inner product can be defined in X_o and hence in X itself [16].

However, symmetry of Birkhoff orthogonality is not a sufficient condition for the existence of an inner product in a two – dimensional normed linear spaces. Consider the following example :

Let B_r ($r > 2$) be the normed linear space of pairs $(x_1, x_2) = x$ of real numbers,

$$\text{where } \left\{ \begin{array}{l} \|x\|^r = |x_1|^r + |x_2|^r \text{ if } x_1 \text{ and } x_2 \text{ are of the same sign} \\ \|x\|^s = |x_1|^s + |x_2|^s \text{ otherwise, where } s = r/(r-1). \end{array} \right\}$$

Now for $x = (1, 0)$ and $y = (0, 1)$ we have $\|x\| = \|y\| = 1$ and

$x + y = (1, 1)$, $x - y = (1, -1)$. Then $\|x + y\|^r = \|x - y\|^s = 2$.

Thus $\|x + y\|^2 + \|x - y\|^2 \neq 2 \left[\|x\|^2 + \|y\|^2 \right]$ and so $(B_r, \|\cdot\|)$ is not an inner

product space. It can easily be verified that Birkhoff orthogonality in this space is symmetric.

THEOREM I.30

An inner product can be defined in a normed linear space of three or more dimensions if orthogonality (Birkhoff's) is additive on the left.

Proof. We may assume $\dim X = 3$. Let $x_1, x_2 \in X$ and $H = \text{span}(x_1, x_2)$. Then there exist hyperplanes H_1 and H_2 such that $x_1 \perp^B H_1$ and $x_2 \perp^B H_2$. Let $0 \neq x_0 \in H_1 \cap H_2$, then $x_1 \perp^B x_0$ and $x_2 \perp^B x_0$. But homogeneity and left-additivity of Birkhoff orthogonality give that $ax_1 + bx_2 \perp^B x_0$ for all real numbers a and b or that $H \perp^B x_0$. Thus the space X can be written as $X = H \oplus M$ where $M = \langle x_0 \rangle$, and hence every element $x \in X$ has a unique representation $x = h + \alpha x_0$ with $h \in H$ and $\alpha \in \mathbf{R}$. Define the projection P of X onto H by

$$P : X \rightarrow H \quad P(x) = h.$$

Then $\|P(x)\| \leq \|x\|$ for all x ($\because H \perp^B x_0$), and $\|P\| = 1$. Since there is a projection of norm one on any given two-dimensional linear subspace of X , it follows that an inner product can be defined in the space X .

The conclusion of the above theorem is not valid without the assumption that the space of more than two dimensions, since it is clear that for a two-dimensional normed linear space orthogonality is additive on the left if and only if for any $x (\neq 0)$ there is a unique nonzero element orthogonal to x . It therefore follows that orthogonality is additive on the left in a two-dimensional normed linear space if and only if the space is strictly convex (Theorem I.28).

§4. α -ORTHOGONALITY

Let X denote a normed linear space over the reals with $\|\cdot\|$, and let $\alpha \neq 1$ be a fixed real number. We define the concept of α -orthogonality as follows :

For $x, y \in X$, we say x is α -orthogonal to y , denoted $(x \perp y)(\alpha)$, if and only if

$$(1 + \alpha^2)\|x - y\|^2 = \|x - \alpha y\|^2 + \|y - \alpha x\|^2. \quad [2]$$

It is clear that α -orthogonality is symmetric, i.e., if $x, y \in X$, then

$$(x \perp y)(\alpha) \Leftrightarrow (y \perp x)(\alpha).$$

It is also easy to see that if $(x \perp x)(\alpha)$, that is

$$0 = 2\|x - \alpha x\|^2 = 2(1 - \alpha)^2\|x\|^2, \quad \text{then } x = 0.$$

Furthermore, pythagorean orthogonality and isosceles orthogonality are special cases of α -orthogonality for $\alpha = 0$ and $\alpha = -1$, respectively.

However, a simple example, the two-dimensional vector $\mathbf{R}^2$ space with $\|(x_1, x_2)\| = \max\{|x_i|\}$, shows that for $\alpha \neq 0, -1$, α -orthogonality is distinct from pythagorean orthogonality and isosceles orthogonality.

Let $x = (1, 1)$, $y = (1, 0)$ and consider the case when $1/2 < \alpha < 1$.

Then $\|x\| = \|y\| = \|x - y\| = 1$, and $\|x + y\| = 2$. Thus clearly $x \perp^i y$ and $x \perp^P y$. Furthermore,

$$x - \alpha y = (1 - \alpha, 1) \Rightarrow \|x - \alpha y\| = 1,$$

$$y - \alpha x = (1 - \alpha, -\alpha) \Rightarrow \|y - \alpha x\| = \alpha.$$

Thus $1 + \alpha^2 = \|x - \alpha y\|^2 + \|y - \alpha x\|^2$ or $(x \perp y)(\alpha)$.

$$\begin{aligned} \text{If } \alpha \neq 0, 1, \text{ then } (x \perp y)(1/\alpha) &\Leftrightarrow \left(1 + \frac{1}{\alpha^2}\right) \|x - y\|^2 = \left\|x - \frac{1}{\alpha} y\right\|^2 + \left\|y - \frac{1}{\alpha} x\right\|^2 \\ &\Leftrightarrow \left(\frac{\alpha^2 + 1}{\alpha^2}\right) \|x - y\|^2 = \left\|\frac{\alpha x - y}{\alpha}\right\|^2 + \left\|\frac{\alpha y - x}{\alpha}\right\|^2 \\ &\Leftrightarrow (\alpha^2 + 1) \|x - y\|^2 = \|\alpha x - y\|^2 + \|\alpha y - x\|^2 \\ &\Leftrightarrow (x \perp y)(\alpha). \end{aligned}$$

Therefore, there is no loss of generality if we assume that $\alpha \in [-1, 1)$. Also recall that for any real k , Lemma I.2 implies that

$$\begin{aligned} \lim_{t \rightarrow \infty} \left[\| (t + k)x + y \| - \| tx + y \| \right] &= k \|x\|, \text{ and} \\ \lim_{t \rightarrow -\infty} \left[\| (t + k)x + y \| - \| tx + y \| \right] &= -k \|x\|. \end{aligned}$$

Now we are ready to prove the following theorem :

THEOREM I.31 (Resolvability)

If $\alpha \neq 1$, then for any x and y there is a real number a such that

$$(x \perp ax + y)(\alpha).$$

Proof. Since the statement clearly is true if $x = 0$, we will assume that $x \neq 0$. Define

$$g(t) = \|x - \alpha(tx + y)\|^2 + \|tx + y - \alpha x\|^2 - (1 + \alpha^2) \|x - (tx + y)\|^2.$$

We will show that $\lim_{t \rightarrow +\infty} g(t) = +\infty$ and $\lim_{t \rightarrow -\infty} g(t) = -\infty$, so that

continuity of the norm will imply the existence of a real number a for which $g(a) = 0$ and hence $(x \perp ax + y)(\alpha)$.

Since

$$\begin{aligned}
 g(t) &= \left[\|x - \alpha(tx + y)\|^2 - \alpha^2 \|x - (tx + y)\|^2 \right] + \\
 &\quad \left[\|tx + y - \alpha x\|^2 - \|x - (tx + y)\|^2 \right] \\
 &= \left[\|(\alpha t - 1)x + \alpha y\|^2 - \alpha^2 \|tx + (y - x)\|^2 \right] + \\
 &\quad \left[\|(t - \alpha)x + y\|^2 - \|tx + (y - x)\|^2 \right] \\
 &= \alpha^2 \left[\|(t - 1/\alpha)x + y\|^2 - \|tx + (y - x)\|^2 \right] + \\
 &\quad \left[\|(t - \alpha)x + y\|^2 - \|tx + (y - x)\|^2 \right] \\
 &= \alpha^2 \left[\left\| \left[t + (1 - 1/\alpha) \right] x + (y - x) \right\|^2 - \|tx + (y - x)\|^2 \right] + \\
 &\quad \left[\left\| \left[t + (1 - \alpha) \right] x + (y - x) \right\|^2 - \|tx + (y - x)\|^2 \right]
 \end{aligned}$$

Using Lemma I.2, with $k = (1 - 1/\alpha)$, $(1 - \alpha)$ respectively, we get

$$\begin{aligned}
 \lim_{t \rightarrow +\infty} \left[\left\| \left[t + (1 - 1/\alpha) \right] x + (y - x) \right\| - \|tx + (y - x)\| \right] &= (1 - 1/\alpha) \|x\|, \\
 \lim_{t \rightarrow +\infty} \left[\left\| \left[t + (1 - \alpha) \right] x + (y - x) \right\| - \|tx + (y - x)\| \right] &= (1 - \alpha) \|x\|.
 \end{aligned}$$

It follows that there are functions $\varepsilon_1, \varepsilon_2$ which approach 0 as $t \rightarrow +\infty$.

Thus

$$\begin{aligned}
 g(t) &= \alpha^2 \left[\varepsilon_1 + (1 - 1/\alpha) \|x\| \right] \left[\left\| \left(t + (1 - 1/\alpha) \right) x + (y - x) \right\| + \|tx + (y - x)\| \right] \\
 &\quad + \left[\varepsilon_2 + (1 - \alpha) \|x\| \right] \left[\left\| \left(t + (1 - \alpha) \right) x + (y - x) \right\| + \|tx + (y - x)\| \right].
 \end{aligned}$$

$$g(t) = \left[\varepsilon_1 + \alpha(\alpha - 1) \|x\| \right] \times \\
\left[\left\| \left(t + (1 - 1/\alpha) \right) x + (y - x) \right\| - \left\| tx + (y - x) \right\| + 2 \left\| tx + (y - x) \right\| \right] \\
+ \left[\varepsilon_2 + (1 - \alpha) \|x\| \right] \times \\
\left[\left\| \left(t + (1 - \alpha) \right) x + (y - x) \right\| - \left\| tx + (y - x) \right\| + 2 \left\| tx + (y - x) \right\| \right].$$

Then again, there are functions $\varepsilon_3, \varepsilon_4$ such that $\varepsilon_3, \varepsilon_4 \rightarrow 0$ as $t \rightarrow +\infty$, and

$$g(t) = \left[\varepsilon_1 + \alpha(\alpha - 1) \|x\| \right] \times \left[\varepsilon_3 + (1 - 1/\alpha) \|x\| + 2 \left\| tx + (y - x) \right\| \right] + \\
\left[\varepsilon_2 + (1 - \alpha) \|x\| \right] \times \left[\varepsilon_4 + (1 - \alpha) \|x\| + 2 \left\| tx + (y - x) \right\| \right] \\
= 2 \left\| tx + (y - x) \right\| \left[\varepsilon_1 + \alpha(\alpha - 1) \|x\| + \varepsilon_2 + (1 - \alpha) \|x\| \right] + \\
\left[\varepsilon_1 + \alpha(\alpha - 1) \|x\| \right] \left[\varepsilon_3 + (1 - 1/\alpha) \|x\| \right] + \\
\left[\varepsilon_2 + (1 - \alpha) \|x\| \right] \left[\varepsilon_4 + (1 - \alpha) \|x\| \right] \\
= 2 \left\| tx + (y - x) \right\| \left[\varepsilon_1 + \varepsilon_2 + \{ \alpha(\alpha - 1) + (1 - \alpha) \} \|x\| \right] + \\
\left[\varepsilon_1 + \alpha(\alpha - 1) \|x\| \right] \left[\varepsilon_3 + (1 - 1/\alpha) \|x\| \right] + \\
\left[\varepsilon_2 + (1 - \alpha) \|x\| \right] \left[\varepsilon_4 + (1 - \alpha) \|x\| \right] \\
= 2 \left\| tx + (y - x) \right\| \left[\varepsilon_1 + \varepsilon_2 + (\alpha - 1)^2 \|x\| \right] + \\
\left[\varepsilon_1 + \alpha(\alpha - 1) \|x\| \right] \left[\varepsilon_3 + (1 - 1/\alpha) \|x\| \right] + \\
\left[\varepsilon_2 + (1 - \alpha) \|x\| \right] \left[\varepsilon_4 + (1 - \alpha) \|x\| \right].$$

Now since $\alpha \neq 1$, then $g(t) \rightarrow +\infty$ as $t \rightarrow +\infty$.

Similarly, we can use Lemma I.2 to rewrite $g(t)$ as

$$g(t) = \left[\delta_1 - \alpha(\alpha - 1) \|x\| \right] \times \left[\delta_3 - (1 - 1/\alpha) \|x\| + 2 \left\| tx + (y - x) \right\| \right] + \\
\left[\delta_2 - (1 - \alpha) \|x\| \right] \times \left[\delta_4 - (1 - \alpha) \|x\| + 2 \left\| tx + (y - x) \right\| \right],$$

where $\delta_1, \delta_2, \delta_3, \delta_4 \rightarrow 0$ as $t \rightarrow -\infty$.

Simplify,

$$g(t) = 2\|tx + (y - x)\| \left[\delta_1 + \delta_2 - (\alpha - 1)^2 \|x\| \right] + \\
\left[\delta_1 - \alpha(\alpha - 1) \|x\| \right] \left[\delta_3 - (1 - 1/\alpha) \|x\| \right] + \\
\left[\delta_2 - (1 - \alpha) \|x\| \right] \left[\delta_4 - (1 - \alpha) \|x\| \right].$$

With $\alpha \neq 1$, $g(t)$ approaches $-\infty$ as $t \rightarrow -\infty$.

This completes the proof of the theorem.

We make additional assumptions about our α -orthogonality relation and derive characterizations of inner product spaces. We say that α -orthogonality is homogeneous if for all $x, y \in X$, λ, μ real numbers, $(x \perp y)(\alpha)$ implies $(\lambda x \perp \mu y)(\alpha)$, and that α -orthogonality is additive (to the right) if for all $x, y, z \in X$, if $(x \perp y)(\alpha)$ and $(x \perp z)(\alpha)$, then $(x \perp (y + z))(\alpha)$.

Now, we will show that if α -orthogonality is homogeneous or additive, then the space $(X, \|\cdot\|)$ is a real inner-product space. Since the result is known if $\alpha = -1$ (isosceles orthogonality), we assume that $|\alpha| < 1$.

THEOREM I.32 (The Characterization Theorem)

If α -orthogonality is homogeneous, then $(x \perp y)(\alpha)$ if and only if $\|x - y\|^2 = \|x\|^2 + \|y\|^2$, i.e., α -orthogonality is equivalent to pythagorean orthogonality.

Proof. Since the statement is obvious if $x = 0$ or $y = 0$, we will assume in the following that x and y are nonzero. Also, as noted previously, it suffices to consider $|\alpha| < 1$.

1) Let $(x \perp y)(\alpha)$. We begin by establishing the following statement :

$$P(n) : (1 + \alpha^{2n}) \|x - y\|^2 = \|x - \alpha^{2n-1} y\|^2 + \|y - \alpha^{2n-1} x\|^2.$$

$P(1)$ is equivalent to $(x \perp y)(\alpha)$. If $P(n)$ is true and $(x \perp y)(\alpha)$, then letting $\beta = \alpha^{2n-1}$, we get

$$(1 + \beta^2) \|x - y\|^2 = \|x - \beta y\|^2 + \|y - \beta x\|^2.$$

Using homogeneity, we may replace x by βx in $P(n)$ and then y by βy in $P(n)$ to obtain :

$$(1) \quad (1 + \beta^2) \|\beta x - y\|^2 = \beta^2 \|x - y\|^2 + \|y - \beta^2 x\|^2.$$

$$(2) \quad (1 + \beta^2) \|x - \beta y\|^2 = \|x - \beta^2 y\|^2 + \beta^2 \|x - y\|^2.$$

Now add (1) and (2) to obtain

$$(1 + \beta^2) \left[\|\beta x - y\|^2 + \|x - \beta y\|^2 \right] = 2\beta^2 \|x - y\|^2 + \|y - \beta^2 x\|^2 + \|x - \beta^2 y\|^2.$$

Then use $P(n)$ to get

$$(1 + \beta^2) \left[(1 + \beta^2) \|x - y\|^2 \right] = (1 + \beta^2)^2 \|x - y\|^2 = \\ 2\beta^2 \|x - y\|^2 + \|y - \beta^2 x\|^2 + \|x - \beta^2 y\|^2.$$

Simplify,

$$(1 + 2\beta^2 + \beta^4) \|x - y\|^2 = 2\beta^2 \|x - y\|^2 + \|y - \beta^2 x\|^2 + \|x - \beta^2 y\|^2, \quad \text{or}$$

$$(1 + \beta^4) \|x - y\|^2 = \|y - \beta^2 x\|^2 + \|x - \beta^2 y\|^2. \quad \text{Putting again } \beta = \alpha^{2^{n-1}},$$

$$(1 + \alpha^{2^2 \times 2^{n-1}}) \|x - y\|^2 = \|y - \alpha^{2 \times 2^{n-1}} x\|^2 + \|x - \alpha^{2 \times 2^{n-1}} y\|^2, \quad \text{or}$$

$$(1 + \alpha^{2^{n+1}}) \|x - y\|^2 = \|y - \alpha^{2^n} x\|^2 + \|x - \alpha^{2^n} y\|^2, \quad \text{which is } P(n+1).$$

Thus $P(n)$ is true for $n \geq 1$. Since $|\alpha| < 1$, we get the desired result by

taking the limit as $n \rightarrow \infty$ in $P(n)$

$$\left[\begin{array}{l} \text{Since } |\alpha| < 1, \text{ then as } n \rightarrow \infty, \quad \alpha^{2^n}, \quad \alpha^{2^{n-1}} \rightarrow 0. \text{ Furthermore} \\ \left| \|x\| - \alpha^{2^{n-1}} \|y\| \right| \leq \|x - \alpha^{2^{n-1}} y\| \leq \|x\| + \alpha^{2^{n-1}} \|y\|, \\ \text{thus } \|x - \alpha^{2^{n-1}} y\| \rightarrow \|x\|. \text{ Similarly } \|y - \alpha^{2^{n-1}} x\| \rightarrow \|y\|. \end{array} \right]$$

2) Conversely, assume that x and y satisfy the condition

$$(3) \quad \|x - y\|^2 = \|x\|^2 + \|y\|^2.$$

By Theorem I.31, there is a real number a such that $(x \perp ax + y)(\alpha)$.

By homogeneity of α -orthogonality, we also have $(\beta x \perp ax + y)(\alpha)$ for any real β , and hence by part-1 of this proof,

$$(4) \quad \|(\beta - a)x - y\|^2 = \beta^2 \|x\|^2 + \|ax + y\|^2.$$

Applying (4) for $\beta = a$ and $\beta = a + 1$, we get

$$\|y\|^2 = a^2 \|x\|^2 + \|ax + y\|^2, \text{ and}$$

$$\|x - y\|^2 = (a + 1)^2 \|x\|^2 + \|ax + y\|^2. \text{ Then combining these results with (3),}$$

$$\begin{aligned} \left[\|x\|^2 + \|y\|^2 \right] &= (a + 1)^2 \|x\|^2 + \left[\|y\|^2 - a^2 \|x\|^2 \right] \\ &= (2a + 1) \|x\|^2 + \|y\|^2. \end{aligned}$$

Thus $2a \|x\|^2 = 0$, and hence $a = 0$. This completes the proof of the theorem.

Now, since α -orthogonality and pythagorean orthogonality are equivalent if α -orthogonality is homogeneous, then homogeneity of α -orthogonality gives homogeneity of pythagorean orthogonality, which in turn implies that the space is an inner product space. And, by using a proof similar to that used in Theorem I.14, the main result now follows easily.

THEOREM I.33

If α -orthogonality is homogeneous or additive, then the space X is a real - inner product space.

§5. (α, β) -ORTHOGONALITY

Here we study a relation of orthogonality in a normed linear space which generalizes isosceles and pythagorean orthogonality, as well as α -orthogonality. Also, we will show that the space is a real inner product space if and only if this relation of orthogonality is homogeneous or additive.

DEFINITION I.34

Let X denote a normed linear space over reals, and α, β be fixed real numbers, $\alpha, \beta \neq 1$. If $x, y \in X$ we say x is (α, β) -orthogonal to y , denote $(x \perp y)(\alpha, \beta)$ provided

$$\|x - y\|^2 + \|\alpha x - \beta y\|^2 = \|x - \beta y\|^2 + \|\alpha x - y\|^2. [1]$$

It is noted that $(0, 0)$ -orthogonality is pythagorean orthogonality, while $(0, -1)$, $(-1, 0)$, and $(-1, -1)$ -orthogonality are isosceles orthogonality. If $\alpha = \beta$, then (α, β) -orthogonality is just α -orthogonality.

PROPERTIES OF (α, β) -ORTHOGONALITY

We derive some consequences of (α, β) -orthogonality which will be useful in the sequel. The following lemma follows immediately from the definition of (α, β) -orthogonality.

LEMMA I.35 Let $x, y \in X$.

- i) If $\alpha, \beta \neq 0$, $(x \perp y)(\alpha, \beta)$ if and only if $(\alpha x \perp \beta y)(1/\alpha, 1/\beta)$.
- ii) If $\beta \neq 0$, $(x \perp y)(\alpha, \beta)$ if and only if $(x \perp \beta y)(\alpha, 1/\beta)$.
- iii) If $\alpha \neq 0$, $(x \perp y)(\alpha, \beta)$ if and only if $(\alpha x \perp y)(1/\alpha, \beta)$.

The following corollary is an immediate consequence of the above lemma and the assumption $(\lambda x \perp \mu y)(\alpha, \beta)$ for any $\lambda, \mu \in \mathbf{R}$.

COROLLARY I. 36 Suppose that

$$(\lambda x \perp \mu y)(\alpha, \beta) \text{ for any } \lambda, \mu \in \mathbf{R}.$$

- i) If $\alpha, \beta \neq 0$, $(x \perp y)(\alpha, \beta)$ if and only if $(x \perp y)(1/\alpha, 1/\beta)$.
- ii) If $\beta \neq 0$, $(x \perp y)(\alpha, \beta)$ if and only if $(x \perp y)(\alpha, 1/\beta)$.
- iii) If $\alpha \neq 0$, $(x \perp y)(\alpha, \beta)$ if and only if $(x \perp y)(1/\alpha, \beta)$.

Hence, this corollary holds if (α, β) -orthogonality is homogeneous. The following lemma and corollary are also immediate from definition.

LEMMA I. 37

For all $x, y \in X$, $(x \perp y)(\alpha, \beta)$ if and only if $(y \perp x)(\beta, \alpha)$.

COROLLARY I. 38

(α, β) -orthogonality is homogeneous if and only if (β, α) -orthogonality is homogeneous.

Also, the following lemmas are needed.

LEMMA I. 39 If $x \in X$, $x \neq 0$,

$$\lim_{h \rightarrow \pm 0} \left[\frac{\|x + hy\|^2 - \|x\|^2}{h} \right] = 2\|x\| \cdot N_{\pm}(x; y).$$

Where $N_{\pm}(x; y)$ are the left and right Gateaux differentials. Since the proof easily follows from definition of the Gateaux differentials and elementary facts, it will not be mentioned here.

LEMMA I.40

If $k > 0$ and $x \in X$, $x \neq 0$,

$$(i) \quad \lim_{t \rightarrow +\infty} \left[\left\| (t \pm k)x + y \right\|^2 - \left\| tx + y \right\|^2 \right] = \pm \infty$$

$$(ii) \quad \lim_{t \rightarrow -\infty} \left[\left\| (t \pm k)x + y \right\|^2 - \left\| tx + y \right\|^2 \right] = \mp \infty.$$

Proof.

$$(i) \quad \lim_{t \rightarrow +\infty} \left[\left\| (t \pm k)x + y \right\|^2 - \left\| tx + y \right\|^2 \right]$$

$$= \lim_{t \rightarrow +\infty} \left[\left\| (t \pm k)x + y \right\| - \left\| tx + y \right\| \right] \cdot \left[\left\| (t \pm k)x + y \right\| + \left\| tx + y \right\| \right]$$

$$= \pm k \left\| x \right\| \cdot \lim_{t \rightarrow +\infty} \left[\left\| (t \pm k)x + y \right\| + \left\| tx + y \right\| \right]. \quad (\text{using Lemma I.2})$$

$$\text{But } \left\| (t \pm k)x + y \right\| + \left\| tx + y \right\| \geq (t \pm k) \left\| x \right\| - \left\| y \right\| + t \left\| x \right\| - \left\| y \right\|$$

$$= (2t \pm k) \left\| x \right\| - 2 \left\| y \right\|,$$

which tends to $+\infty$ as $t \rightarrow +\infty$. Hence

$$\lim_{t \rightarrow +\infty} \left[\left\| (t \pm k)x + y \right\|^2 - \left\| tx + y \right\|^2 \right] = (\pm k \left\| x \right\|) \cdot (+\infty) = \pm \infty.$$

$$(ii) \quad \lim_{t \rightarrow -\infty} \left[\left\| (t \pm k)x + y \right\|^2 - \left\| tx + y \right\|^2 \right]$$

$$= \lim_{t \rightarrow +\infty} \left[\left\| (t \mp k)x - y \right\|^2 - \left\| tx - y \right\|^2 \right] = \mp \infty. \quad (\text{by part (i)})$$

Now we can prove the following theorem..

THEOREM I.41

For any $x, y \in X$, there exists a real number a such that

$(x \perp ax + y)(\alpha, \beta)$. That is

$$\|x - (ax + y)\|^2 + \|\alpha x - \beta(ax + y)\|^2 = \|x - \beta(ax + y)\|^2 + \|\alpha x - (ax + y)\|^2$$

Proof. Set $\beta \neq 0$ and

$$g(t) = \|x - tx - y\|^2 + \|\alpha x - \beta tx - \beta y\|^2 - \|x - \beta tx - \beta y\|^2 - \|\alpha x - tx - y\|^2,$$

where $-\infty < t < \infty$. Then

$$\begin{aligned} g(t) &= \|x - tx - y\|^2 - \|\alpha x - tx - y\|^2 + \|\alpha x - \beta tx - \beta y\|^2 - \|x - \beta tx - \beta y\|^2 \\ &= \left\| t \left(\frac{x}{t} - \frac{y}{t} - x \right) \right\|^2 - \left\| t \left(\frac{\alpha x}{t} - \frac{y}{t} - x \right) \right\|^2 + \\ &\quad \left\| t \left(\frac{\alpha x}{t} - \frac{\beta y}{t} - \beta x \right) \right\|^2 - \left\| t \left(\frac{x}{t} - \frac{\beta y}{t} - \beta x \right) \right\|^2 \\ &= t^2 \left[\left\| x + \frac{1}{t}(y - x) \right\|^2 - \left\| x + \frac{1}{t}(y - \alpha x) \right\|^2 + \right. \\ &\quad \left. \left\| \beta x + \frac{1}{t}(\beta y - \alpha x) \right\|^2 - \left\| \beta x + \frac{1}{t}(\beta y - x) \right\|^2 \right]. \end{aligned}$$

Add and subtract $t^2\|x\|^2$, $t^2\|\beta x\|^2$. Therefore

$$g(t) = t^2 \left[\left\{ \left\| x + \frac{1}{t}(y - x) \right\|^2 - \|x\|^2 \right\} - \left\{ \left\| x + \frac{1}{t}(y - \alpha x) \right\|^2 - \|x\|^2 \right\} + \right. \\ \left. \left\{ \left\| \beta x + \frac{1}{t}(\beta y - \alpha x) \right\|^2 - \|\beta x\|^2 \right\} - \left\{ \left\| \beta x + \frac{1}{t}(\beta y - x) \right\|^2 - \|\beta x\|^2 \right\} \right]$$

Then for $t \neq 0$,

$$\begin{aligned} \frac{g(t)}{t} = & \left[\frac{\left\| x + \frac{1}{t}(y - x) \right\|^2 - \|x\|^2}{1/t} \right] - \left[\frac{\left\| x + \frac{1}{t}(y - \alpha x) \right\|^2 - \|x\|^2}{1/t} \right] \\ & + \left[\frac{\left\| \beta x + \frac{1}{t}(\beta y - \alpha x) \right\|^2 - \|\beta x\|^2}{1/t} \right] - \left[\frac{\left\| \beta x + \frac{1}{t}(\beta y - x) \right\|^2 - \|\beta x\|^2}{1/t} \right]. \end{aligned}$$

Then in limit as $t \rightarrow \pm\infty$ ($1/t \rightarrow \pm 0$), we get by Lemma I.39,

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} \frac{g(t)}{t} = & 2\|x\| \cdot N_{\pm}(x; y - x) - 2\|x\| \cdot N_{\pm}(x; y - \alpha x) \\ & + 2\|\beta x\| \cdot N_{\pm}(\beta x; \beta y - \alpha x) - 2\|\beta x\| \cdot N_{\pm}(\beta x; \beta y - x). \end{aligned}$$

Here $N_+(x; y)$ and $N_-(x; y)$ are respectively the right and left - Gateaux derivatives of the norm at x in the direction of y . Thus using the following property

$$N_{\pm}(x; rx + sy) = r\|x\| + s \cdot N_{\pm}(x; y) \quad \forall r, s \geq 0,$$

we have

$$N_{\pm}(x; y - x) = N_{\pm}(x; y) - \|x\|;$$

$$N_{\pm}(x; y - \alpha x) = N_{\pm}(x; y) - \alpha\|x\|;$$

$$N_{\pm}(\beta x; \beta y - \alpha x) = N_{\pm}(\beta x; \beta y - \alpha/\beta \cdot \beta x) = N_{\pm}(\beta x; \beta y) - \alpha/\beta \|\beta x\|;$$

$$N_{\pm}(\beta x; \beta y - x) = N_{\pm}(\beta x; \beta y - 1/\beta \cdot \beta x) = N_{\pm}(\beta x; \beta y) - 1/\beta \|\beta x\|.$$

Thus

$$\begin{aligned}
 \lim_{t \rightarrow \pm\infty} \frac{g(t)}{t} &= 2\|x\| \cdot \left[N_{\pm}(x; y) - \|x\| - N_{\pm}(x; y) + \alpha\|x\| \right] \\
 &\quad + 2\|\beta x\| \cdot \left[N_{\pm}(\beta x; \beta y) - \alpha/\beta\|\beta x\| - N_{\pm}(\beta x; \beta y) + 1/\beta\|\beta x\| \right] \\
 &= 2(\alpha - 1)\|x\|^2 + 2/\beta(1 - \alpha)\|\beta x\|^2 \\
 &= 2(\alpha - 1)\|x\|^2 + 2\beta(1 - \alpha)\|x\|^2 \\
 &= 2(1 - \alpha)(\beta - 1)\|x\|^2 \neq 0.
 \end{aligned}$$

(I) If $(1 - \alpha)(\beta - 1) > 0$, that is $\lim_{t \rightarrow \pm\infty} \frac{g(t)}{t} > 0$, then

$$g(t) \rightarrow +\infty \text{ as } t \rightarrow +\infty \quad \text{and} \quad g(t) \rightarrow -\infty \text{ as } t \rightarrow -\infty.$$

(II) If $(1 - \alpha)(\beta - 1) < 0$, that is $\lim_{t \rightarrow \pm\infty} \frac{g(t)}{t} < 0$, then

$$g(t) \rightarrow -\infty \text{ as } t \rightarrow +\infty \quad \text{and} \quad g(t) \rightarrow +\infty \text{ as } t \rightarrow -\infty.$$

Hence in both cases $g(t)$ changes sign $(-\infty < t < +\infty)$ and thus continuity of the norm gives that there is a number $a \in \mathbf{R}$ such that $g(a) = 0$, which was to be proved.

Now if $\beta = 0$, then consider the following cases :

Case1. $\alpha = -1$. Then Theorem I.1 proves the existence of a .

Case2. $|\alpha| < 1$. Then the result is immediate from Lemma I.40.

$$\text{Since } g(t) = \|(t-1)x + y\|^2 - \|(t-\alpha)x + y\|^2 + (\alpha^2 - 1)\|x\|^2,$$

$$\begin{aligned}
 g(t) &= \|(t-1)x + y\|^2 - \left\| \left((t-1) + (1-\alpha) \right) x + y \right\|^2 + (\alpha^2 - 1)\|x\|^2 \\
 &= - \left\{ \left\| \left((t-1) + (1-\alpha) \right) x + y \right\|^2 - \|(t-1)x + y\|^2 \right\} + (\alpha^2 - 1)\|x\|^2.
 \end{aligned}$$

Chapter I
(α, β)-orthogonality

Then from Lemma I.40 $\lim_{t \rightarrow \pm\infty} g(t) = \mp\infty$, and hence a exists.

Case3. $|\alpha| > 1$. Then $|1/\alpha| < 1$, and the argument of case2 produces a real number a such that $(\alpha x \perp a(\alpha x) + y)(1/\alpha, \beta)$. By Lemma I.35, this implies that $(x \perp a\alpha x + y)(\alpha, \beta)$. This completes the proof.

COROLLARY I.42

For any $x, y \in X$ there exists a real number a such that $(ax + y \perp x)(\alpha, \beta)$.

CONSEQUENCES OF HOMOGENEITY

If we assume that (α, β) – orthogonality is homogeneous, then we obtain the result that (α, β) – orthogonality implies the isosceles orthogonality and (for $\alpha, \beta \neq -1$) pythagorean orthogonality.

LEMMA I.43

If $\alpha \neq -1$ and $(\lambda x \perp \mu y)(\alpha, \beta)$ for any $\lambda, \mu \in \mathbf{R}$, then

$$\|x - y\|^2 = (1 - \beta^2) \|y\|^2 + \|x - \beta y\|^2.$$

Hence, in particular, if (α, β) – orthogonality is homogeneous, then the above statement holds.

Proof.

By corollary to Lemma I.35 $(x \perp y)(\alpha, \beta)$ if and only if $(x \perp y)(1/\alpha, \beta)$.

Thus it suffices to assume $|\alpha| < 1$. Suppose $(x \perp y)(\alpha, \beta)$. Then

$$\|x - y\|^2 + \|\alpha x - \beta y\|^2 = \|x - \beta y\|^2 + \|\alpha x - y\|^2.$$

Chapter I
 (α, β) -orthogonality

Since the result is immediate for $\alpha = 0$, we may suppose $\alpha \neq 0$. Denote by $P(n)$ the statement

$$P(n) : \quad \|x - y\|^2 + \|\alpha^n x - \beta y\|^2 = \|x - \beta y\|^2 + \|\alpha^n x - y\|^2.$$

Clearly $P(1)$ is true. By hypothesis, $(\alpha^n x \perp y)(\alpha, \beta)$, thus

$$\|\alpha^n x - y\|^2 + \|\alpha^{n+1} x - \beta y\|^2 = \|\alpha^n x - \beta y\|^2 + \|\alpha^{n+1} x - y\|^2.$$

Now if $P(n)$ is true for $n > 0$, then adding this to $P(n)$ we obtain

$$\|x - y\|^2 + \|\alpha^{n+1} x - \beta y\|^2 = \|x - \beta y\|^2 + \|\alpha^{n+1} x - y\|^2,$$

which is $P(n+1)$. Thus $P(n)$ is true for all positive integers n , but

$\lim_{n \rightarrow +\infty} \alpha^n = 0$, so in the limit, by continuity of the norm, we have

$$\|x - y\|^2 + \beta^2 \|y\|^2 = \|x - \beta y\|^2 + \|y\|^2,$$

and the conclusion of the Lemma follows.

THEOREM I.44

If $\alpha, \beta \neq -1$ and $(\lambda x \perp \mu y)(\alpha, \beta)$ for any $\lambda, \mu \in \mathbf{R}$, then $x \perp^P y$.

Hence, in particular, if (α, β) -orthogonality is homogeneous, then

$(x \perp y)(\alpha, \beta)$ implies $x \perp^P y$.

Proof. As before, by the corollary to Lemma I.35 we may suppose $|\beta| < 1$.

Assume $(x \perp y)(\alpha, \beta)$ and let $P(n)$ denote the statement

$$P(n) : \quad \|x - y\|^2 = (1 - \beta^{2n}) \|y\|^2 + \|x - \beta^n y\|^2.$$

The statement $P(1)$ is Lemma I.43. Since $(x \perp \beta^n y)(\alpha, \beta)$ by hypothesis, we have by Lemma I.43

$$\|x - \beta^n y\|^2 = (1 - \beta^2) \|\beta^n y\|^2 + \|x - \beta^{n+1} y\|^2,$$

or
$$\|x - \beta^n y\|^2 = (1 - \beta^2) \beta^{2n} \|y\|^2 + \|x - \beta^{n+1} y\|^2.$$

Now if we assume $P(n)$, $(n > 0)$ is true, then substituting this in $P(n)$ gives

$$\begin{aligned} \|x - y\|^2 &= (1 - \beta^{2n}) \|y\|^2 + \left[(1 - \beta^2) \beta^{2n} \|y\|^2 + \|x - \beta^{n+1} y\|^2 \right] \\ &= (1 - \beta^{2(n+1)}) \|y\|^2 + \|x - \beta^{n+1} y\|^2 \end{aligned}$$

or $P(n+1)$. Hence $P(n)$ holds for all positive integers n , and since

$\beta^n \rightarrow 0$ as $n \rightarrow +\infty$, taking limits in $P(n)$ implies that

$$\|x - y\|^2 = \|y\|^2 + \|x\|^2, \text{ i.e., } x \perp^P y.$$

THEOREM I.45

If $(\lambda x \perp \mu y)(\alpha, \beta)$ for any $\lambda, \mu \in \mathbf{R}$, then $x \perp^i y$.

Hence, in particular, if (α, β) -orthogonality is homogeneous, then

$(x \perp y)(\alpha, \beta)$ implies $x \perp^i y$.

Proof. If $\alpha = \beta = -1$ the result is immediate. Otherwise,

* If $\alpha \neq -1$ consider two cases :

Case 1. If $\beta = -1$, the result is immediate from Lemma I.43.

Case 2. If $\beta \neq -1$, by Theorem I.44 we have $\|x - y\|^2 = \|x\|^2 + \|y\|^2$. But by hypothesis, $(x \perp -y)(\alpha, \beta)$ holds, so $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. Hence $\|x - y\| = \|x + y\|$ and the result again follows.

* If $\alpha = -1$, then $(x \perp y)(-1, \beta)$ implies by Lemma I.37 that $(y \perp x)(\beta, -1)$, with $\beta \neq -1$, and applying the above result we get $y \perp^i x$. But symmetry of isosceles orthogonality gives that $(x \perp y)(-1, \beta)$ implies $x \perp^i y$. This completes the proof.

THEOREM I.46

If (α, β) - orthogonality is homogeneous then the space X is strictly convex. (Refer to [1], Theorem 2.3 for the proof.)

THE CHARACTERIZATION THEOREM

We now ready obtain the main characterization theorem.

THEOREM I.47

If (α, β) - orthogonality is homogeneous in a normed linear space X then X is an inner product space.

Proof. If $(\alpha, \beta) = (-1, -1)$, $(0, 0)$, $(0, -1)$, or $(-1, 0)$ the result is known from Theorem I.6 and Theorem I.13. Otherwise, it suffices by Lorch (L1), page 1, to show that for some fixed constant $a \neq \pm 1, 0$,

$$\|x - y\| = \|x + y\| \quad \text{implies} \quad \|x - ay\| = \|x + ay\|.$$

Assume $x, y \in X$ satisfy $\|x - y\| = \|x + y\|$, and fix $a \neq \pm 1, 0$. Then there exists $b \in \mathbf{R}$ such that $(by + x \perp y)(\alpha, \beta)$. It follows by homogeneity that $(by + x \perp ky)(\alpha, \beta)$ for all $k \in \mathbf{R}$ and by Theorem I.45 we have

$$\|by + x - ky\| = \|by + x + ky\|.$$

Letting $k = b + 1$, $b - 1$ respectively we obtain

$$\|x - y\| = \|x + (2b + 1)y\| \quad \text{and} \quad \|x + y\| = \|x + (2b - 1)y\|.$$

Thus $\|x + (2b + 1)y\| = \|x + (2b - 1)y\|$ or $(x + 2by) \perp^i y$.

But by Theorem I.46 X is strictly convex and hence $b = 0$. Thus $(x \perp y)(\alpha, \beta)$ from which it follows that $(x \perp ay)(\alpha, \beta)$, and by Theorem I.45 $x \perp^i ay$, and X is an inner product space.

In a manner exactly analogous to that in Theorem I.8 or Theorem I.14 we may show that if (α, β) -orthogonality is both left and right additive on space X , then it is homogeneous and hence X is an inner product space. Since the proof is almost identical, the details will be omitted. The result may be summarized as follows.

THEOREM I.48

Let X be a normed linear space. The following are equivalent :

- i) (α, β) - orthogonality is symmetric and left - additive,
- ii) (α, β) - orthogonality is symmetric and right - additive,
- iii) (α, β) - orthogonality is left and right - additive,
- iv) (α, β) - orthogonality is homogeneous,
- v) X is an inner product space.

§6. EQUIVALENCE OF ORTHOGONALITY RELATIONS

We left with important property for each of the following orthogonalities in inner product spaces. Let X be an inner product space. Using that $\|x\|^2 = \langle x, x \rangle$, we have the following results:

$$(1) \quad x \perp^B y \Leftrightarrow \langle x, y \rangle = 0.$$

$$\begin{aligned} \text{Consider } \|x + ky\|^2 &= \langle x + ky, x + ky \rangle \\ &= \langle x, x \rangle + k\langle x, y \rangle + k\langle y, x \rangle + k^2\langle y, y \rangle \\ &= \|x\|^2 + 2k\langle x, y \rangle + k^2\|y\|^2. \end{aligned}$$

Then $\|x + ky\|^2 \geq \|x\|^2 \Leftrightarrow 2k\langle x, y \rangle + k^2\|y\|^2 \geq 0$ for all k , which is true only if $\langle x, y \rangle = 0$.

$$(2) \quad (x \perp y)(\alpha, \beta) \Leftrightarrow \langle x, y \rangle = 0.$$

$$\begin{aligned} (x \perp y)(\alpha, \beta) &\Leftrightarrow \|x - y\|^2 + \|\alpha x - \beta y\|^2 = \|x - \beta y\|^2 + \|\alpha x - y\|^2 \\ &\Leftrightarrow \left[\|x\|^2 - 2\langle x, y \rangle + \|y\|^2 \right] + \left[\alpha^2\|x\|^2 - 2\alpha\beta\langle x, y \rangle + \beta^2\|y\|^2 \right] = \\ &\quad \left[\|x\|^2 - 2\beta\langle x, y \rangle + \beta^2\|y\|^2 \right] + \left[\alpha^2\|x\|^2 - 2\alpha\langle x, y \rangle + \|y\|^2 \right] \\ &\Leftrightarrow -2\langle x, y \rangle[1 + \alpha\beta] = -2\langle x, y \rangle[\alpha + \beta] \\ &\Leftrightarrow \langle x, y \rangle[1 + \alpha\beta - \alpha - \beta] = 0 \\ &\Leftrightarrow \langle x, y \rangle[(1 - \beta) + \alpha(\beta - 1)] = (1 - \alpha)(1 - \beta)\langle x, y \rangle = 0. \end{aligned}$$

But $\alpha \neq 1$, and $\beta \neq 1$. Thus $\langle x, y \rangle = 0$.

Chapter I
Equivalence of Orthogonality Relations

But isosceles, pythagorean and α – orthogonality are just special cases of

(α, β) – orthogonality. Thus

$$x \perp^i y \Leftrightarrow \langle x, y \rangle = 0$$

$$x \perp^P y \Leftrightarrow \langle x, y \rangle = 0$$

$$(x \perp y)(\alpha) \Leftrightarrow \langle x, y \rangle = 0$$

THEOREM I.49

All orthogonalities are equivalent in inner product spaces.

However, in general normed linear spaces, this is not true.

Example 1. Isosceles and pythagorean orthogonality are not equivalent.

(i) Let $f(t) = t$, $g(t) = -2t + \frac{1}{4}\sqrt{65}t^2 \in C[0, 1]$.

Then $\|f\| = 1$,

$$\|g\| = \left\| g\left(\frac{4}{\sqrt{65}}\right) \right\| = \left| -2 \times \frac{4}{\sqrt{65}} + \frac{1}{4}\sqrt{65} \times \frac{16}{65} \right| = \left| \frac{-8}{\sqrt{65}} + \frac{4}{\sqrt{65}} \right| = \frac{4}{\sqrt{65}},$$

(Note that $\frac{4}{\sqrt{65}}$ is the critical point where $g' = 0$.)

$$\begin{aligned} \|f - g\| &= \left\| 3t - \frac{1}{4}\sqrt{65}t^2 \right\| = \left| (f - g)\left(\frac{6}{\sqrt{65}}\right) \right| = \left| 3 \times \frac{6}{\sqrt{65}} - \frac{1}{4}\sqrt{65} \times \frac{36}{65} \right| \\ &= \left| \frac{18}{\sqrt{65}} - \frac{9}{\sqrt{65}} \right| = \frac{9}{\sqrt{65}}, \end{aligned}$$

(Again $\frac{6}{\sqrt{65}}$ is the critical point where $(f - g)' = 0$.)

$$\|f + g\| = \left\| -t + \frac{1}{4}\sqrt{65}t^2 \right\| = \left| (f + g)(1) \right| = \left| -1 + \frac{1}{4}\sqrt{65} \right|.$$

Chapter I

Equivalence of Orthogonality Relations

Thus $\|f\|^2 + \|g\|^2 = \|f - g\|^2$, but

$\|f - g\| \neq \|f + g\|$; that is $f \perp^i g$.

(ii) Conversely, let $f(t) = t + t^2$, $g(t) = t - t^2 \in C[0,1]$.

Then $\|f - g\| = \|2t^2\| = 2$, and $\|f + g\| = \|2t\| = 2$.

But $\|f\| = \|(t + t^2)(1)\| = 2$, $\|g\| = \|(t - t^2)(1/2)\| = 1/4$.

Thus $\|f\|^2 + \|g\|^2 \neq \|f - g\|^2$.

Example 2. Isosceles and Birkhoff orthogonality are not equivalent

(i) Again choose $f(t) = t$, $g(t) = -2t + \frac{1}{4}\sqrt{65}t^2$. Then

$$\|f(1) + kg(1)\| = \left| (1 - 2k) + k \frac{\sqrt{65}}{4} \right| > |1 - 2k + 2k| = 1.$$

That is $\|f(1) + kg(1)\| > \|f\|$, and hence $\|f + kg\| > \|f\| \forall k$.

But as it proved in Example 1 - (i), $f \perp^i g$.

(ii) Let X be the normed linear space consisting of all pairs (a, b)

of real numbers, with $\|(a, b)\|$ as the larger of $|a|$ and $|b|$.

Let $x = (1, 0)$ and $y = (n - 1, n)$, where n is a positive integer.

Then $\|x\| = 1$, $(x - y) = (2 - n, -n)$ and $(x + y) = (n, n)$.

If $n > 1$, then $\|x - y\| = \|x + y\| = n$, and so $x \perp^i y$.

Furthermore,

$$x - \frac{1}{2n}y = \left(1 - \frac{n-1}{2n}, -\frac{n}{2n}\right) = \left(\frac{n+1}{2n}, -\frac{1}{2}\right) \Rightarrow \left\|x - \frac{1}{2n}y\right\| = \frac{n+1}{2n}.$$

Chapter I
Equivalence of Orthogonality Relations

Now $\frac{n+1}{2n} < 1 \Leftrightarrow n+1 < 2n \Leftrightarrow 1 < n$, which is true.

Thus

$$\left\| x - \frac{1}{2n} y \right\| < \|x\|, \text{ or that } x \perp^B y.$$

That is given the element $x = (1, 0)$ and $n > 1$, there can be found an element y orthogonal to x ($x \perp^i y$) and $\|x + ky\| < \|x\|$.

Example 3. Pythagorean orthogonality and Birkhoff orthogonality are not equivalent.

(i) Let $f(t) = t + t^2$ and $g(t) = t - t^2 \in C[0, 1]$.

Then $f \perp^P g$ as in example 1 – (ii).

Furthermore, $\|f + kg\| = \|(t + t^2) + k(t - t^2)\| \geq 2 = \|f\|$. Thus $f \perp^B g$.

(ii) Consider the space $X = \mathbf{R}^2$ with $\|(x_1, x_2)\| = |x_1| + |x_2|$, and let

$x = (7, 1)$, $y = (2, -4)$. Then $\|x\| = 8$, $\|y\| = 6$ and

$$x - y = (5, 5) \Rightarrow \|x - y\| = 10. \text{ Thus clearly } \|x - y\|^2 = \|x\|^2 + \|y\|^2.$$

That is $x \perp^P y$. On the other hand,

$$x \perp^B y \Leftrightarrow \|x + ky\| = \|(7 + 2k, 1 - 4k)\| = |7 + 2k| + |1 - 4k| \geq 8 \text{ for all } k.$$

But at $k = 1/4$ we have $\|x + ky\| = |7 + 1/2| = 7\frac{1}{2} < 8$. Thus $x \not\perp^B y$.

Example 4. Isosceles and (α, β) – orthogonality are not equivalent.

(i) Let $f(t) = t$, $g(t) = t^2 \in C[-1, 1]$, and $\lambda, \mu \in \mathbf{R}$ such that $|\lambda| \leq |\mu|$, then

$$\|\lambda f + \mu g\| = \max_t |\lambda t + \mu t^2| = \begin{cases} |\lambda(1) + \mu(1)^2| = |\lambda + \mu| & \text{if } \lambda, \mu \text{ have same sign} \\ |\lambda(-1) + \mu(-1)^2| = |-\lambda + \mu| & \text{otherwise} \end{cases}$$

Therefore in general $\|\lambda f + \mu g\| = |\lambda| + |\mu|$, and so $\|f - g\| = \|f + g\|$.

Now with α and β be real numbers such that $|\alpha| < 1$, $|\beta| > 1$, then

$$(f \perp g)(\alpha, \beta) \text{ if and only if } \|t - t^2\|^2 + \|\alpha t - \beta t^2\|^2 = \|t - \beta t^2\|^2 + \|\alpha t - t^2\|^2.$$

That is

$$4 + (|\alpha| + |\beta|)^2 = (1 + |\beta|)^2 + (|\alpha| + 1)^2, \text{ simplify,}$$

$$1 + |\alpha| \cdot |\beta| = |\alpha| + |\beta| \text{ or } 1 + |\alpha| \cdot |\beta| - |\alpha| - |\beta| = 0,$$

$$[1 - |\alpha|][1 - |\beta|] = 0 \text{ which yields } |\alpha| = 1 \text{ or } |\beta| = 1.$$

Thus $f \perp g(\alpha, \beta)$ if $|\alpha| \neq 1$ and $|\beta| \neq 1$.

(ii) Let $f(t) = 1$, $g(t) = t \in C[0, 1]$. Then

$$\|f - g\| = \|1 - t\| = 1, \quad \|f + g\| = \|1 + t\| = 2; \text{ that is } f \perp^i g.$$

But with $\alpha, \beta \in \mathbf{R}$ such that $\alpha > 1$, $0 < \beta < 1$, we have

$$\|\alpha f - \beta g\| = \|\alpha - \beta t\| = \alpha \quad [-\beta \leq -\beta t \leq 0, \text{ so } (\alpha - \beta) \leq \alpha - \beta t \leq \alpha],$$

$$\|f - \beta g\| = \|1 - \beta t\| = 1 \quad [(1 - \beta) \leq 1 - \beta t \leq 1],$$

$$\|\alpha f - g\| = \|\alpha - t\| = \alpha \quad [\alpha - 1 \leq \alpha - t \leq \alpha].$$

Thus $\|f - g\|^2 + \|\alpha f - \beta g\|^2 = \|f - \beta g\|^2 + \|\alpha f - g\|^2$; that is $(f \perp g)(\alpha, \beta)$.

Example 5. Pythagorean and (α, β) – orthogonality are not equivalent.

(i) Consider the space $X = \mathbf{R}^2$ with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$.

Let $x = (7, 1)$, $y = (2, -4)$, as in example 3 – (ii). Then $\|x\| = 8$, $\|y\| = 6$, and

$\|x - y\| = \|(5, 5)\| = 10$. Thus $x \perp^P y$.

Now $(x \perp y)(\alpha, \beta)$ if and only if

$$100 + \left[|7\alpha - 2\beta| + |\alpha + 4\beta| \right]^2 = \left[|7 - 2\beta| + |1 + 4\beta| \right]^2 + \left[|7\alpha - 2| + |\alpha + 4| \right]^2$$

Choose α and β such that $\alpha > 1$ and $1 < \beta < 7/2$, so that

$(x \perp y)(\alpha, \beta)$ if and only if

$$100 + \left[(7\alpha - 2\beta) + (\alpha + 4\beta) \right]^2 = \left[(7 - 2\beta) + (1 + 4\beta) \right]^2 + \left[(7\alpha - 2) + (\alpha + 4) \right]^2.$$

$$\Leftrightarrow 100 + (7\alpha - 2\beta)^2 + (\alpha + 4\beta)^2 + 2(7\alpha - 2\beta)(\alpha + 4\beta) =$$

$$(7 - 2\beta)^2 + (1 + 4\beta)^2 + 2(1 - 2\beta)(1 + 4\beta) +$$

$$(7\alpha - 2)^2 + (\alpha + 4)^2 + 2(7\alpha - 2)(\alpha + 4)$$

$$\Leftrightarrow 100 + \left(49\alpha^2 - 28\alpha\beta + 4\beta^2 \right) + \left(\alpha^2 + 8\alpha\beta + 16\beta^2 \right) + \left(14\alpha^2 + 52\alpha\beta - 16\beta^2 \right) =$$

$$\left(49 - 28\beta + 4\beta^2 \right) + \left(1 + 8\beta + 16\beta^2 \right) + \left(14 + 52\beta - 16\beta^2 \right) +$$

$$\left(49\alpha^2 - 28\alpha + 4 \right) + \left(\alpha^2 + 8\alpha + 16 \right) + \left(14\alpha^2 + 52\alpha - 16 \right)$$

$$\Leftrightarrow 32 + 32\alpha\beta = 32\alpha + 32\beta$$

$$\Leftrightarrow 1 + \alpha\beta = \alpha + \beta$$

$$\Leftrightarrow (1 - \alpha)(1 - \beta) = 0 \Leftrightarrow \alpha = 1 \text{ or } \beta = 1, \text{ which is impossible. Thus } (x \perp y)(\alpha, \beta).$$

(i) Easily we can show the converse, that $(x \perp y)(\alpha, \beta)$ but $x \not\perp^P y$, as the one discussed in Example 4 – (ii).

SUMMARY OF CHAPTER I

Orthogonality Relation	Properties of the orthogonality relations						
	Nondegeneracy	Symmetry	Homogeneity	Additivity		Resolvability	Uniqueness
ISOSCELES	always	always	i . p . s.	i . p . s.		always	rotund
PYTHAGOREAN	always	always	i . p . s.	i . p . s.		always	always
BIRKHOFF	always	$\frac{\dim X > 3}{\text{i . p . s.}}$	always	$\frac{\dim X = 2}{\text{rotund}}$	$\frac{\dim X > 3}{\text{i . p . s.}}$	always	<u>Left</u> rotund <u>Right</u> smooth
α - ORTHOGONALITY	always	always	i . p . s.	i . p . s.		always	
(α, β) - ORTHOGONALITY	always	In general, Not symmetric	i . p . s.	i . p . s.		always	

- Rotund space : strictly convex space.
- Smooth space : Gateaux – differentiable space.
- i . p . s. = inner product space

CHAPTER II

RELATIONS BETWEEN ORTHOGONALITIES AND CHARACTERIZATIONS OF INNER PRODUCT SPACES

INTRODUCTION	69
$\left[x \perp^i y \Rightarrow (\Leftrightarrow) x \perp^P y \right] \Leftrightarrow X \text{ is an i.p.s.}$	73
$\left[x \perp^P y \Rightarrow (\Leftrightarrow) x \perp^B y \right] \Leftrightarrow X \text{ is an i.p.s.}$	75
$\left[x \perp^i y \Rightarrow (\Leftrightarrow) x \perp^B y \right] \Leftrightarrow X \text{ is an i.p.s.}$	79
If $ \alpha , \beta \neq 1$ then $\left[x \perp^B y \Rightarrow x \perp \mathcal{Y}(\alpha, \beta) \right] \Leftrightarrow X \text{ is an i.p.s.}$	81
If $ \alpha , \beta \neq 1$ then $\left[x \perp \mathcal{Y}(\alpha, \beta) \Rightarrow x \perp^B y \right] \Leftrightarrow X \text{ is an i.p.s.}$	90
If $\alpha = \beta$ and $ \alpha < 1$, then $\left[x \perp^i y \Rightarrow (\Leftrightarrow) x \perp \mathcal{Y}(\alpha, \alpha) \right] \Leftrightarrow X \text{ is an i.p.s.}$	93
SUMMARY	97

CHAPTER II

RELATIONS BETWEEN ORTHOGONALITIES AND CHARACTERIZATIONS OF INNER PRODUCT SPACES

INTRODUCTION

As we have pointed in the previous chapter (§6) all the orthogonality relations (Isosceles, Pythagorean, Birkhoff, and (α, β) -orthogonality) mean same when X is an inner product space, but, in general, they do not mean same in normed linear spaces. Moreover, equivalence between almost any two of them is a characteristic property of inner product spaces. Throughout this chapter we provide some of these characterizations of inner product spaces.

There are new characterizations of inner product spaces discussed in this chapter.

Let us start with the following Lemma, which will be needed in the proof of many theorems.

LEMMA II.1

(i) If $x \neq y \in X$ such that $\|x\| = \|y\| = \left\| \frac{x+y}{2} \right\| = 1$, then

$$\|\lambda x + \mu y\| = |\lambda| + |\mu| \text{ for } \lambda, \mu \in \mathbf{R}, \text{ such that } \lambda \cdot \mu \geq 0.$$

Proof. We have $\|\lambda x + \mu y\| \leq |\lambda| \cdot \|x\| + |\mu| \cdot \|y\| = |\lambda| + |\mu|$.

(by the triangle inequality)

$$\begin{aligned} \text{Now } \|\lambda x + \mu y\| &= \|\lambda x + \lambda y - \lambda y - \mu x + \mu x + \mu y\| \\ &= \|(\lambda + \mu)(x + y) - (\lambda y + \mu x)\| \\ &\geq |\lambda + \mu| \cdot \|x + y\| - \|\lambda y + \mu x\| \\ &\geq |\lambda + \mu| \cdot \|x + y\| - [|\lambda| + |\mu|] \\ &= 2[|\lambda| + |\mu|] - [|\lambda| + |\mu|] \quad (\lambda \cdot \mu \geq 0 \Rightarrow |\lambda + \mu| = |\lambda| + |\mu|) \\ &= |\lambda| + |\mu| \end{aligned}$$

Therefore $\|\lambda x + \mu y\| = |\lambda| + |\mu|$.

(ii) If $\|x - y\| = \|x + y\| = 2$, for some $x, y \in S_X$ (the unit sphere of X),

then $\|\lambda x + \mu y\| = |\lambda| + |\mu|$ for any $\lambda, \mu \in \mathbf{R}$.

Proof.

Case 1. $\lambda \cdot \mu \geq 0$. The result follows from (i).

Case 2. $\lambda \cdot \mu < 0$.

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

Again by the triangle inequality $\|\lambda x + \mu y\| \leq |\lambda| + |\mu|$.

$$\begin{aligned}\text{Now } \|\lambda x + \mu y\| &= \|\lambda x - \lambda y + \lambda y + \mu x - \mu x + \mu y\| \\ &= \|(\lambda - \mu)(x - y) + (\lambda y + \mu x)\| \\ &\geq |\lambda - \mu| \cdot \|x - y\| - \|\lambda y + \mu x\| \\ &\geq 2|\lambda - \mu| - [|\lambda| + |\mu|] \\ &= 2[|\lambda| + |\mu|] - [|\lambda| + |\mu|] \quad (\lambda \cdot \mu < 0 \Rightarrow |\lambda - \mu| = |\lambda| + |\mu|) \\ &= |\lambda| + |\mu|\end{aligned}$$

Thus $\|\lambda x + \mu y\| = |\lambda| + |\mu|$. This completes the proof.

Indeed the following results in this chapter are modification of results of O.P.Kapoor and Jagadish Prasad. That is some of them are reformulated, as we assumed they will be more clear to the reader. However, others are corrected. Therefore we prefer to mention the reference beside the main theorems discussed throughout this chapter.

LEMMA II.2

Let X be a normed linear space.

If for x and $y \in X$, $x \perp^P y \Rightarrow x \perp^i y$, then X is strictly convex.

Proof. On contrary, say that X is not strictly convex. Then there

exist $x \neq y \in X$ such that $\|x\| = \|y\| = \left\| \frac{x+y}{2} \right\| = 1$. Then $\frac{x+y}{2} \perp^P -y$. Since,

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

otherwise, we have $\left\| \frac{x+3y}{2} \right\|^2 = \left\| \frac{x+y}{2} \right\|^2 + \|y\|^2 = 2$. But Lemma II.1 gives that

$\left\| \frac{x+3y}{2} \right\|^2 = 4$, a contradiction. Hence, by Theorem I.9 there exists a number

$\alpha \neq 0$ such that $\frac{x+y}{2} \perp^p -(\alpha \frac{x+y}{2} + y)$. Then by assumption we have

$\frac{x+y}{2} \perp^i -(\alpha \frac{x+y}{2} + y)$; that is

$$(1) \quad \left\| (1+\alpha) \frac{x+y}{2} + y \right\| = \left\| (1-\alpha) \frac{x+y}{2} - y \right\|$$

$$\text{or} \quad \left\| \frac{1+\alpha}{2} x + \frac{3+\alpha}{2} y \right\| = \left\| \frac{1-\alpha}{2} x - \frac{1+\alpha}{2} y \right\|.$$

But $|\alpha| \leq 1$, by Theorem I.3, and so by Lemma II.1 $\|(1+\alpha)x + y\| = (2+\alpha)$.

Thus $(2+\alpha) = \left\| \frac{1-\alpha}{2} x - \frac{1+\alpha}{2} y \right\| \leq \frac{1-\alpha}{2} + \frac{1+\alpha}{2} = 1$. That means $\alpha \leq -1$,

combining this inequality with $|\alpha| \leq 1$, we get $\alpha = -1$. Then $\frac{x+y}{2} \perp^p \frac{x-y}{2}$.

That is

$$(2) \quad \|y\|^2 = \left\| \frac{x+y}{2} \right\|^2 + \left\| \frac{x-y}{2} \right\|^2$$

$$\text{or} \quad 1 = 1 + \left\| \frac{x-y}{2} \right\|^2 \quad \text{or} \quad \|x-y\| = 0, \text{ which implies that } x = y.$$

Contradicting the assumption that $x \neq y$. Thus X is strictly convex.

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

THEOREM II.3 ([15], Theorem 4)

For a normed linear space X the following are equivalent :

- (i) X is an inner product space;
- (ii) $x, y \in X, x \perp^P y \Rightarrow x \perp^i y$;
- (iii) $x, y \in X, x \perp^i y \Rightarrow x \perp^P y$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) Let (ii) holds, then by Lemma II.2 X is strictly convex.

Now suppose that (ii) does not imply (iii). There exist points x and y

such that $x \perp^i y$ but $x \not\perp^P y$. Choose $\alpha \neq 0$ such that $x \perp^P \alpha x + y$.

But then, by (ii), $x \perp^i \alpha x + y$. Thus $x \perp^i y$ and $x \perp^i \alpha x + y$,

contradicting the uniqueness of isosceles orthogonality in strictly convex spaces (Theorem I.5).

(iii) $\Rightarrow$ (i) We employ the reasoning of Mahlon M. Day, (D1), see

page 1. Let $\|x\| = \|y\| = 1$. Then $(x + y) \perp^i (x - y)$, since

$$\|(x + y) + (x - y)\| = \|2x\| = \|2y\| = \|(x + y) - (x - y)\|. \text{ Therefore, by (iii)}$$

$(x + y) \perp^P (x - y)$ and so

$$\|x + y\|^2 + \|x - y\|^2 = \|(x + y) - (x - y)\|^2 = \|2y\|^2 = 4.$$

Then by (D1) it follows that X is an inner product space.

LEMMA II.4

Let X be a normed linear space.

If for $x, y \in X$ and $x \perp^P y \Rightarrow x \perp^B y$. Then X is strictly convex.

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

Proof. Let $x \perp^P y \Rightarrow x \perp^B y$ in the space X , and X is not

strictly convex. Thus $\exists x \neq y \in X$ such that $\|x\| = \|y\| = \left\| \frac{x+y}{2} \right\| = 1$.

Then

$$\frac{x+y}{2} \perp^P -y, \quad \text{since} \quad \left\| \frac{x+y}{2} + y \right\|^2 = (2)^2 = 4, \quad (\text{using Lemma II.1})$$

while $\left\| \frac{x+y}{2} \right\|^2 + \|y\|^2 = 2$. Therefore there exists $\alpha \neq 0$ such that

$$\frac{x+y}{2} \perp^P -(\alpha \frac{x+y}{2} + y); \quad \text{that is}$$

$$(1) \quad \left\| (1+\alpha) \frac{x+y}{2} + y \right\|^2 = \left\| \frac{x+y}{2} \right\|^2 + \left\| \alpha \frac{x+y}{2} + y \right\|^2 = 1 + \left\| \alpha \frac{x+y}{2} + y \right\|^2.$$

But the assumption gives $\frac{x+y}{2} \perp^B -(\alpha \frac{x+y}{2} + y)$; that is

$$(2) \quad \left\| \frac{x+y}{2} + k \left(\alpha \frac{x+y}{2} + y \right) \right\| \geq \left\| \frac{x+y}{2} \right\| = 1 \quad \text{for all real } k.$$

With $k = -\frac{1}{\alpha}$, equation (2) will be

$$\left\| \frac{x+y}{2} - \left(\frac{x+y}{2} \right) - \frac{1}{\alpha} y \right\| = \left\| -\frac{1}{\alpha} y \right\| = \left| \frac{1}{\alpha} \right| \cdot \|y\| = \left| \frac{1}{\alpha} \right| \geq 1,$$

yields $|\alpha| \leq 1$. Again putting $k = -\frac{1}{\alpha+2}$, we have

$$\left\| \frac{x+y}{2} - \frac{\alpha}{\alpha+2} \left(\frac{x+y}{2} \right) - \frac{y}{\alpha+2} \right\| = \left\| \left(1 - \frac{\alpha}{\alpha+2} \right) \frac{x+y}{2} - \frac{y}{\alpha+2} \right\|$$

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

$$\begin{aligned} \left\| \left(1 - \frac{\alpha}{\alpha + 2} \right) \left(\frac{x + y}{2} \right) - \frac{y}{\alpha + 2} \right\| &= \left\| \frac{2}{\alpha + 2} \left(\frac{x + y}{2} \right) - \frac{y}{\alpha + 2} \right\| \\ &= \left| \frac{1}{\alpha + 2} \right| \cdot \|x\| = \left| \frac{1}{\alpha + 2} \right| \geq 1, \end{aligned}$$

yields $|\alpha + 2| \leq 1$. Then combining these two inequalities we get $\alpha = -1$.

Substituting for α in (1) gives $\|y\|^2 = 1 + \left\| -\left(\frac{x + y}{2} \right) + y \right\|^2 = 1 + \left\| \frac{y - x}{2} \right\|^2$.

Therefore $1 = 1 + \left\| \frac{y - x}{2} \right\|^2$ or $\|y - x\| = 0$, that is $x = y$ which is

a contradiction. X must therefore strictly convex.

THEOREM II.5 ([15], Theorem 5)

For a normed linear space X the following are equivalent :

- (i) X is an inner product space;
- (ii) $x, y \in X$, $x \perp^P y \Rightarrow x \perp^B y$;
- (iii) $x, y \in X$, $x \perp^B y \Rightarrow x \perp^P y$.

Proof.

(i) $\Rightarrow$ (ii) Say X is an inner product space. Then pythagorean and Birkhoff orthogonality are equivalent

(ii) $\Rightarrow$ (iii) Suppose that (ii) does not imply (iii). Let x and y be elements of X such that $x \perp^B y$ and $x \not\perp^P y$. But then there exists a number $\alpha \neq 0$ such that $\alpha y + x \perp^P y$. By (ii), $\alpha y + x \perp^B y$, which contradicts the left uniqueness of Birkhoff orthogonality in strictly convex spaces.

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

(iii) $\Rightarrow$ (i) Let $\|x\| = \|y\| = 1$.

* If $x \perp^B y$, then homogeneity of Birkhoff orthogonality gives

$x \perp^B -y$ and by assumption we have

$$x \perp^P y \Rightarrow \|x - y\|^2 = \|x\|^2 + \|y\|^2 = 2,$$

$$x \perp^P -y \Rightarrow \|x + y\|^2 = \|x\|^2 + \|y\|^2 = 2.$$

$$\text{Thus } \|x + y\|^2 + \|x - y\|^2 = 4.$$

* If $x \perp^B y$, then choose $z \in X$ such that $\|z\| = 1$ and $x \perp^B z$.

This choice is possible from homogeneity of Birkhoff orthogonality and Theorem I.18. Now let α and β be numbers, such that

$y = \alpha x + \beta z$. Then using (iii) and the fact that $\lambda x \perp^B \mu z$ for all λ, μ

we have

$$\|y\|^2 = \|\alpha x + \beta z\|^2 = \|\alpha x\|^2 + \|\beta z\|^2 = \alpha^2 + \beta^2.$$

Similarly,

$$\|x + y\|^2 = \|(1 + \alpha)x + \beta z\|^2 = (1 + \alpha)^2 + \beta^2,$$

$$\|x - y\|^2 = \|(1 - \alpha)x - \beta z\|^2 = (1 - \alpha)^2 + \beta^2.$$

Thus

$$\begin{aligned} \|x + y\|^2 + \|x - y\|^2 &= (1 + \alpha)^2 + (1 - \alpha)^2 + 2\beta^2 \\ &= 2 + 2\alpha^2 + 2\beta^2 \\ &= 2[\alpha^2 + \beta^2] + 2 = 2 + 2 = 4. \end{aligned}$$

Then Day's criterion (D1) gives the result.

COROLLARY II.6

A normed linear space X is an inner product space if and only if

(L') for some $\alpha \neq 0, -1$,

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 \Rightarrow \|x + \alpha y\|^2 = \|x\|^2 + \|\alpha y\|^2.$$

Proof. The case $\alpha = 1$ has been proved. That is when $x \perp^P y$ implies

$x \perp^P -y$. Suppose x and $y \in X$ are such that $\|x - y\|^2 = \|x\|^2 + \|y\|^2$.

Then by (L') $\|x + \alpha y\|^2 = \|x\|^2 + \|\alpha y\|^2$. Repeated application of (L')

we have

$$\|x + \alpha(\alpha y)\|^2 = \|x\|^2 + \|\alpha(\alpha y)\|^2, \quad \text{or} \quad \|x + \alpha^2 y\|^2 = \|x\|^2 + \|\alpha^2 y\|^2,$$

again

$$\|x + \alpha(\alpha^2 y)\|^2 = \|x\|^2 + \|\alpha(\alpha^2 y)\|^2, \quad \text{or} \quad \|x + \alpha^3 y\|^2 = \|x\|^2 + \|\alpha^3 y\|^2,$$

and for $n \geq 1$, we have

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 \Rightarrow \|x + \alpha^n y\|^2 = \|x\|^2 + \|\alpha^n y\|^2.$$

Rewrite this

$$(*) \quad \|x + \alpha^n y\|^2 - \|x\|^2 = \alpha^{2n} \|y\|^2.$$

Now if $|\alpha| < 1$, putting $t = \alpha^n$ in (*) yields

$$\|x + ty\|^2 - \|x\|^2 = t^2 \|y\|^2, \quad \text{or} \quad \frac{\|x + ty\|^2 - \|x\|^2}{t} = t \|y\|^2.$$

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

If $|\alpha| > 1$, then using symmetry of pythagorean orthogonality, we may

interchange x and y in (*) to get $\|y + \alpha^n x\|^2 - \|y\|^2 = \alpha^{2n} \|x\|^2$.

Dividing by α^{2n} , we get

$$\left\| \frac{1}{\alpha^n} y + x \right\|^2 - \frac{\|y\|^2}{\alpha^{2n}} = \|x\|^2, \quad \text{or} \quad \left[\left\| x + \frac{1}{\alpha^n} y \right\|^2 - \|x\|^2 \right] = \frac{\|y\|^2}{\alpha^{2n}},$$

it may written

$$\left[\left\| x + \frac{1}{\alpha^n} y \right\|^2 - \|x\|^2 \right] \cdot \alpha^{2n} = \frac{\|y\|^2}{\alpha^{2n}}, \quad \text{or} \quad \frac{\left\| x + \frac{1}{\alpha^n} y \right\|^2 - \|x\|^2}{\left(\frac{1}{\alpha^n} \right)} = \frac{\|y\|^2}{\alpha^{2n}}.$$

Putting $t = \frac{1}{\alpha^n}$ gives $\frac{\|x + ty\|^2 - \|x\|^2}{t} = t\|y\|^2$.

In the limit (as $t \rightarrow +0$, that is $n \rightarrow +\infty$) we obtain

$$\begin{aligned} \lim_{t \rightarrow +0} \left[\frac{\|x + ty\|^2 - \|x\|^2}{t} \right] &= \lim_{t \rightarrow +0} \left[\frac{\|x + ty\| - \|x\|}{t} \right] [\|x + ty\| + \|x\|] \\ &= N_+(x; y) [2\|x\|] = 0. \end{aligned}$$

Thus $N_+(x; y) = 0$ for $x \neq 0$. This shows that $x \perp^B y$. Thus $x \perp^P y$

$\Rightarrow x \perp^B y$. Therefore X is an inner product space.

The necessity of (L') can be proved easily, using that $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$

in inner product spaces.

THEOREM II.7 ([15], Theorem 7)

For a normed linear space X the following are equivalent :

- (i) X is an inner product space ;
- (ii) $x, y \in X, x \perp^B y \Rightarrow x \perp^i y$;
- (iii) $x, y \in X, x \perp^i y \Rightarrow x \perp^B y$.

Proof. (i) $\Rightarrow$ (ii) and (iii) is straight.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds. Let $x \in X$ and M denote any two - dimensional subspace containig x . Then M contains an element y which is linearly independent with x . Choose α such that $x \perp^B (\alpha x + y)$, and so $x \perp^B k(\alpha x + y)$ for all $k \in \mathbf{R}$. Then by (ii), $x \perp^i k(\alpha x + y)$ for all k ; that is

$$\|x - k(\alpha x + y)\| = \|x + k(\alpha x + y)\| \quad \text{or} \quad x \perp^R (\alpha x + y).$$

Thus we have proved that every two - dimensional subspace containing a point x , contains a point $y_o = (\alpha x + y) \neq 0$, orthogonal to x in the Roberts sense. Then Corollary I.7 shows that X is an inner product space.

(iii) $\Rightarrow$ (i) Let $x, y \in X$ such that $\|x\| = \|y\|$. Then $(x + y) \perp^i (x - y)$,

and therefore $(x + y) \perp^B (x - y)$. Thus we have

$$\|x + y + k(x - y)\| \geq \|x + y\| \quad \text{for all real numbers } k.$$

In particular for all $\alpha \neq 0$ we have

$$\left\| x + y + \frac{\alpha^2 - 1}{\alpha^2 + 1} (x - y) \right\| \geq \|x + y\|.$$

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

Simplify, we get

$$\left\| (\alpha^2 + 1)(x + y) + (\alpha^2 - 1)(x - y) \right\| \geq (\alpha^2 + 1) \|x + y\|$$

or

$$\begin{aligned} \left\| (\alpha^2 + 1 + \alpha^2 - 1)x + (\alpha^2 + 1 - \alpha^2 + 1)y \right\| &= \left\| 2\alpha^2 x + 2y \right\| \\ &= 2 \left\| \alpha^2 x + y \right\| \geq (\alpha^2 + 1) \|x + y\|. \end{aligned}$$

Dividing by 2α yields

$$\left\| \alpha x + \alpha^{-1} y \right\| \geq \frac{\alpha^2 + 1}{2\alpha} \|x + y\| \geq \|x + y\| \quad (\text{if } \alpha > 0),$$

and

$$\left\| \alpha x + \alpha^{-1} y \right\| \geq \frac{\alpha^2 + 1}{-2\alpha} \|x + y\| \geq \|x + y\|. \quad (\text{if } \alpha < 0)$$

(Note that $\frac{\alpha^2 + 1}{2\alpha} \geq 1 \Leftrightarrow (\alpha^2 + 1) \geq 2\alpha \Leftrightarrow \alpha^2 - 2\alpha + 1 = (\alpha - 1)^2 \geq 0$,

similarly, $\frac{\alpha^2 + 1}{-2\alpha} \geq 1 \Leftrightarrow (\alpha^2 + 1) \geq -2\alpha \Leftrightarrow \alpha^2 + 2\alpha + 1 = (\alpha + 1)^2 \geq 0$.)

Thus (iii) implies the following criterion of inner product spaces by

Lorch (L2)

$$\|x\| = \|y\| \Rightarrow \left\| \alpha x + \alpha^{-1} y \right\| \geq \|x + y\| \quad \text{for all } \alpha \neq 0.$$

This completes the proof.

LEMMA II.8

Let X be a normed linear space, and for $x, y \in X$, $x \perp^B y \Leftrightarrow$

$(x \perp y)(\alpha, \beta)$.

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

- (i) If $\alpha \neq 0$, then $x \perp^B y \Leftrightarrow (x \perp y)(1/\alpha, \beta)$.
- (ii) If $\beta \neq 0$, then $x \perp^B y \Leftrightarrow (x \perp y)(\alpha, 1/\beta)$.
- (iii) If $\alpha, \beta \neq 0$, then $x \perp^B y \Leftrightarrow (x \perp y)(1/\alpha, 1/\beta)$.

Proof. Easily follows from homogeneity of Birkhoff orthogonality and Corollary I.36.

Now you may compare the following results Theorem II.9, Lemma II.11, and Theorem II.12 with Theorem 9, [15].

THEOREM II.9

Let X be a normed linear space, and $\|\alpha\|, \|\beta\| \neq 1$.

Then the following are equivalent :

- (i) X is an inner product space;
- (ii) $x, y \in X$, $x \perp^B y \Rightarrow (x \perp y)(\alpha, \beta)$.

Proof. (i) $\Rightarrow$ (ii) is straight.

(ii) $\Rightarrow$ (i) Let $x \perp^B y$. By (ii) and the fact that $\lambda x \perp^B \mu y$ for all $\lambda, \mu \in \mathbf{R}$, we get :

$$\begin{aligned}
 \|x - y\|^2 &= \|x - \beta y\|^2 + \|\alpha x - y\|^2 - \|\alpha x - \beta y\|^2 \\
 &= \left[\|x - \beta^2 y\|^2 + \|\alpha x - \beta y\|^2 - \|\alpha x - \beta^2 y\|^2 \right] + \\
 &\quad \left[\|\alpha x - \beta y\|^2 + \|\alpha^2 x - y\|^2 - \|\alpha^2 x - \beta y\|^2 \right] - \|\alpha x - \beta y\|^2 \\
 &= \left[\|x - \beta^2 y\|^2 + \|\alpha^2 x - y\|^2 \right] - \|\alpha x - \beta^2 y\|^2 - \|\alpha^2 x - \beta y\|^2 + \|\alpha x - \beta y\|^2.
 \end{aligned}$$

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

$$\begin{aligned}
 \text{Thus } \|x - y\|^2 &= \left[\|x - \beta^2 y\|^2 + \|\alpha^2 x - y\|^2 \right] - \|\alpha x - \beta^2 y\|^2 - \|\alpha^2 x - \beta y\|^2 \\
 &\quad + \left[\|\alpha x - \beta^2 y\|^2 + \|\alpha^2 x - \beta y\|^2 - \|\alpha^2 x - \beta^2 y\|^2 \right] \\
 &= \|x - \beta^2 y\|^2 + \|\alpha^2 x - y\|^2 - \|\alpha^2 x - \beta^2 y\|^2.
 \end{aligned}$$

Now suppose for $n \leq k$, where k is a fixed positive integer with $k \geq 2$, that

$$(*) \quad x \perp^B y \Rightarrow \|x - y\|^2 = \|x - \beta^n y\|^2 + \|\alpha^n x - y\|^2 - \|\alpha^n x - \beta^n y\|^2.$$

Homogeneity of Birkhoff orthogonality implies that :

$$x \perp^B \beta^k y \stackrel{(iii)}{\Rightarrow} \|x - \beta^k y\|^2 = \|x - \beta^{k+1} y\|^2 + \|\alpha x - \beta^k y\|^2 - \|\alpha x - \beta^{k+1} y\|^2,$$

$$\alpha^k x \perp^B y \stackrel{(iii)}{\Rightarrow} \|\alpha^k x - y\|^2 = \|\alpha^k x - \beta y\|^2 + \|\alpha^{k+1} x - y\|^2 - \|\alpha^{k+1} x - \beta y\|^2.$$

$$\begin{aligned}
 \text{Then } \|x - y\|^2 &= \left[\|x - \beta^{k+1} y\|^2 + \|\alpha x - \beta^k y\|^2 - \|\alpha x - \beta^{k+1} y\|^2 \right] \\
 &\quad + \left[\|\alpha^k x - \beta y\|^2 + \|\alpha^{k+1} x - y\|^2 - \|\alpha^{k+1} x - \beta y\|^2 \right] - \|\alpha^k x - \beta^k y\|^2 \\
 &= \left[\|x - \beta^{k+1} y\|^2 + \|\alpha^{k+1} x - y\|^2 - \|\alpha^{k+1} x - \beta^{k+1} y\|^2 \right] \\
 &\quad + \left[\|\alpha x - \beta^k y\|^2 + \|\alpha^k x - \beta y\|^2 - \|\alpha^k x - \beta^k y\|^2 \right] \\
 &\quad - \left[\|\alpha x - \beta^{k+1} y\|^2 + \|\alpha^{k+1} x - \beta y\|^2 - \|\alpha^{k+1} x - \beta^{k+1} y\|^2 \right].
 \end{aligned}$$

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

Applying (*) for $n = k - 1$ and $n = k$ respectively, give

$$\alpha x \perp^B \beta y \Rightarrow \|\alpha x - \beta y\|^2 = \|\alpha x - \beta^k y\|^2 + \|\alpha^k x - \beta y\|^2 - \|\alpha^k x - \beta^k y\|^2,$$

and

$$\alpha x \perp^B \beta y \Rightarrow \|\alpha x - \beta y\|^2 = \|\alpha x - \beta^{k+1} y\|^2 + \|\alpha^{k+1} x - \beta y\|^2 - \|\alpha^{k+1} x - \beta^{k+1} y\|^2.$$

$$\text{Hence, } \|x - y\|^2 = \|x - \beta^{k+1} y\|^2 + \|\alpha^{k+1} x - y\|^2 - \|\alpha^{k+1} x - \beta^{k+1} y\|^2.$$

Thus by induction we have proved that

$$x \perp^B y \Rightarrow \|x - y\|^2 = \|x - \beta^n y\|^2 + \|\alpha^n x - y\|^2 - \|\alpha^n x - \beta^n y\|^2, \quad n \geq 1.$$

Since by Lemma II.8 it suffices to assume that $|\alpha|, |\beta| < 1$, then this in the

limit yields $x \perp^B y \Rightarrow \|x - y\|^2 = \|x\|^2 + \|y\|^2$, which is sufficient for X

to be an inner product space (Theorem II.5).

The following lemma is needed and will be stated without proof. (Refer to [11], page 42, Lemma 2.5.)

LEMMA II.10

Define a real valued function $f : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$, by

$$f(a, b) = 4 + \left[\frac{|a|}{2} + \frac{|a|}{2} + b \right]^2 - \left[\frac{1}{2} + \frac{1}{2} + b \right]^2 - \left[\frac{|a|}{2} + \frac{|a|}{2} + 1 \right]^2,$$

for $a, b \in \mathbf{R}$. Then $f(a, b) = 0$ if and only if $a = 1$ or $b = 1$.

LEMMA II.11

Let X be a normed linear space and $|\alpha|, |\beta| \neq 1$. If for $x, y \in X$

and $(x \perp y)(\alpha, \beta) \Rightarrow x \perp^B y$, then X is strictly convex.

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

Proof. Assume on the contrary that X is not strictly convex. Choose

$x \neq y$ as extreme points of the unit ball of X such that

$$\|x\| = \|y\| = \left\| \frac{x+y}{2} \right\| = 1.$$

We claim that $(\frac{x+y}{2} \perp -y)(\alpha, \beta)$. For, otherwise, this implies that

$\frac{x+y}{2} \perp -y$; that is

$$\left\| \frac{x+y}{2} - ky \right\| \geq \left\| \frac{x+y}{2} \right\| = 1 \text{ for any } k \in \mathbf{R},$$

or

$$(1) \quad \left\| \frac{1}{2}x + \left(\frac{1}{2} - k\right)y \right\| \geq 1 \text{ for any } k \in \mathbf{R}.$$

Putting $k = 1$ in (1) yields $\|x - y\| \geq 2$. But $\|x - y\| \leq 2$ by the triangle inequality, thus $\|x - y\| = 2$. Hence, Lemma II.1 gives

$$(2) \quad \|\lambda x + \mu y\| = |\lambda| + |\mu| \text{ for any } \lambda, \mu \in \mathbf{R}.$$

Since $(\frac{x+y}{2} \perp -y)(\alpha, \beta)$, we have

$$\left\| \frac{x+y}{2} + y \right\|^2 + \left\| \alpha \frac{x+y}{2} + \beta y \right\|^2 = \left\| \frac{x+y}{2} + \beta y \right\|^2 + \left\| \alpha \frac{x+y}{2} + y \right\|^2,$$

or

$$\left\| \frac{1}{2}x + \frac{3}{2}y \right\|^2 + \left\| \frac{\alpha}{2}x + \left(\frac{\alpha}{2} + \beta\right)y \right\|^2 = \left\| \frac{1}{2}x + \left(\frac{1}{2} + \beta\right)y \right\|^2 + \left\| \frac{\alpha}{2}x + \left(\frac{\alpha}{2} + 1\right)y \right\|^2.$$

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

Use (2) to get

$$4 + \left[\left| \frac{\alpha}{2} \right| + \left| \frac{\alpha}{2} + \beta \right| \right]^2 = \left[\left| \frac{1}{2} \right| + \left| \frac{1}{2} + \beta \right| \right]^2 + \left[\left| \frac{\alpha}{2} \right| + \left| \frac{\alpha}{2} + 1 \right| \right]^2,$$

which implies by Lemma II.10 that $\alpha = 1$ or $\beta = 1$, a contradiction.

Hence $\left(\frac{x+y}{2} \perp -y \right) (\alpha, \beta)$. Then there exists a real number $a \neq 0$ such

that $\left(\frac{x+y}{2} \perp - \left(a \frac{x+y}{2} + y \right) \right) (\alpha, \beta)$; that is

$$\begin{aligned} (3) \quad & \left\| \frac{x+y}{2} + \alpha \left(\frac{x+y}{2} \right) + y \right\|^2 + \left\| \alpha \left(\frac{x+y}{2} \right) + \beta \left[a \frac{x+y}{2} + y \right] \right\|^2 \\ &= \left\| \frac{x+y}{2} + \beta \left[a \frac{x+y}{2} + y \right] \right\|^2 + \left\| \alpha \left(\frac{x+y}{2} \right) + a \frac{x+y}{2} + y \right\|^2. \end{aligned}$$

Then by assumption $\frac{x+y}{2} \perp - \left(a \frac{x+y}{2} + y \right)$; that is

$$(4) \quad \left\| \frac{x+y}{2} - ka \frac{x+y}{2} - ky \right\| \geq \left\| \frac{x+y}{2} \right\| = 1, \text{ for all real } k.$$

Putting $k = \frac{1}{a}$ then $k = \frac{1}{a+2}$ in (4) yields $\left| \frac{1}{\alpha} \right| \geq 1$, or $|\alpha| \leq 1$, and

$\left| \frac{1}{a+2} \right| \geq 1$, or $|a+2| \leq 1$, respectively. These two together give $a = -1$.

Then (3) gives

$$\begin{aligned} & \|y\|^2 + \left\| \alpha \left(\frac{x+y}{2} \right) + \beta \left[- \left(\frac{x+y}{2} \right) + y \right] \right\|^2 = \\ & \left\| \frac{x+y}{2} + \beta \left[- \left(\frac{x+y}{2} \right) + y \right] \right\|^2 + \left\| \alpha \left(\frac{x+y}{2} \right) - \left(\frac{x+y}{2} \right) + y \right\|^2, \end{aligned}$$

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

or

$$(5) \quad 1 + \left\| \frac{\alpha - \beta}{2}x + \frac{\alpha + \beta}{2}y \right\|^2 = \left\| \frac{1 - \beta}{2}x + \frac{1 + \beta}{2}y \right\|^2 + \left\| \frac{\alpha - 1}{2}x + \frac{\alpha + 1}{2}y \right\|^2.$$

Now by Lemma II.8 it suffices to assume that $|\alpha|, |\beta| < 1$. Therefore we are going to consider the following cases.

Case1. $\alpha \cdot \beta = 0$. There are two possibilities :

(i) If $\alpha = 0$, then from (5) we have

$$1 + \beta^2/4 \|x + y\|^2 = \left\| \frac{1 - \beta}{2}x + \frac{1 + \beta}{2}y \right\|^2 + 1/4 \|x + y\|^2,$$

or by Lemma II.1 (since $|\beta| < 1$),

$$1 + \beta^2/4 \|x + y\|^2 = \left[\frac{1 - \beta}{2} + \frac{1 + \beta}{2} \right]^2 + 1/4 \|x + y\|^2, \text{ or}$$

$$1 + (\beta^2 - 1)/4 \|x + y\|^2 = 1 \quad \text{or} \quad (\beta^2 - 1)/4 \|x + y\|^2 = 0.$$

Since $\beta \neq 1$, then $\|x + y\| = 0$ or $x = y$, a contradiction.

(ii) If $\beta = 0$, and $\alpha \neq 0$, then from (5) we have

$$1 + \alpha^2 \left\| \frac{x + y}{2} \right\|^2 = \left\| \frac{x + y}{2} \right\|^2 + \left\| \frac{\alpha - 1}{2}x + \frac{\alpha + 1}{2}y \right\|^2.$$

Thus

$$\left\| \frac{\alpha - 1}{2}x + \frac{\alpha + 1}{2}y \right\|^2 = \alpha^2 \quad \text{or} \quad \left\| \frac{\alpha - 1}{2\alpha}x + \frac{\alpha + 1}{2\alpha}y \right\|^2 = 1.$$

Since $|\alpha| < 1$, therefore there are two possibilities :

$$0 < \alpha < 1 \quad \text{or} \quad -1 < \alpha < 0.$$

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

In the case $0 < \alpha < 1$, writing

$$y = \left[\frac{1-\alpha}{1+\alpha} \right] x + \left[1 - \frac{1-\alpha}{1+\alpha} \right] \cdot \left[\frac{\alpha-1}{2\alpha} x + \frac{\alpha+1}{2\alpha} y \right],$$

contradicts the fact that y is an extreme point.

In the case $-1 < \alpha < 0$, writing

$$x = \left[\frac{1+\alpha}{1-\alpha} \right] y + \left[1 - \frac{1+\alpha}{1-\alpha} \right] \cdot \left[\frac{\alpha-1}{2\alpha} x + \frac{\alpha+1}{2\alpha} y \right],$$

contradicts the fact that x is an extreme point.

Case 2. $\alpha \cdot \beta \neq 0$. Then with $|\alpha|, |\beta| < 1$, we have two possibilities :

$$|\alpha| \geq |\beta| \quad \text{or} \quad |\alpha| \leq |\beta|.$$

(i) If $|\alpha| \geq |\beta|$. Then $\alpha \geq \mp\beta$ or $\alpha \leq \mp\beta$, thus $\alpha \pm \beta \geq 0$ or $\alpha \pm \beta \leq 0$.

Therefore by Lemma II.1, equation(5) becomes

$$(6) \quad 1 + \left[\frac{\alpha-\beta}{2} + \frac{\alpha+\beta}{2} \right]^2 = \left[\frac{1-\beta}{2} + \frac{1+\beta}{2} \right]^2 + \left\| \frac{\alpha-1}{2} x + \frac{\alpha+1}{2} y \right\|^2,$$

$$\text{or} \quad \left\| \frac{\alpha-1}{2\alpha} x + \frac{\alpha+1}{2\alpha} y \right\| = 1.$$

The rest of the argument is the same as above.

(ii) $|\alpha| \leq |\beta|$. Then $|1/\alpha| \geq |1/\beta|$ and $|1/\alpha|, |1/\beta| > 1$.

As before, we claim that $(\frac{x+y}{2} \perp -y)(1/\alpha, 1/\beta)$. For, otherwise,

by Lemma II.8 we have $\frac{x+y}{2} \perp^B -y$. Hence again we have

$$\|\lambda x + \mu y\| = |\lambda| + |\mu| \quad \text{for all } \lambda, \mu \in \mathbf{R}. \quad (\text{Equation (2).})$$

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

Now, $(\frac{x+y}{2} \perp -y)(1/\alpha, 1/\beta)$ gives

$$\left\| \frac{x+y}{2} + y \right\|^2 + \left\| \frac{1}{\alpha} \cdot \frac{x+y}{2} + \frac{1}{\beta} \cdot y \right\|^2 = \left\| \frac{x+y}{2} + \frac{1}{\beta} \cdot y \right\|^2 + \left\| \frac{1}{\alpha} \cdot \frac{x+y}{2} + y \right\|^2,$$

or

$$\left\| \frac{1}{2}x + \frac{3}{2}y \right\|^2 + \left\| \frac{1}{2\alpha}x + \left(\frac{1}{2\alpha} + \frac{1}{\beta}\right)y \right\|^2 = \left\| \frac{1}{2}x + \left(\frac{1}{2} + \frac{1}{\beta}\right)y \right\|^2 + \left\| \frac{1}{2\alpha}x + \left(\frac{1}{2\alpha} + 1\right)y \right\|^2,$$

and (2) gives

$$4 + \left[\left| \frac{1}{2\alpha} \right| + \left| \frac{1}{2\alpha} + \frac{1}{\beta} \right| \right]^2 = \left[\left| \frac{1}{2} \right| + \left| \frac{1}{2} + \frac{1}{\beta} \right| \right]^2 + \left[\left| \frac{1}{2\alpha} \right| + \left| \frac{1}{2\alpha} + 1 \right| \right]^2, \text{ which implies}$$

by Lemma II.10, that $1/\alpha = 1$ or $1/\beta = 1$; that is $\alpha = 1$ or $\beta = 1$,

a contradiction. Thus $(\frac{x+y}{2} \perp -y)(1/\alpha, 1/\beta)$.

There is a real number $b \neq 0$ such that $(\frac{x+y}{2} \perp - (b \frac{x+y}{2} + y))(1/\alpha, 1/\beta)$.

That is

$$\begin{aligned} (7) \quad & \left\| \frac{x+y}{2} + (b \frac{x+y}{2} + y) \right\|^2 + \left\| \frac{1}{\alpha} \cdot \frac{x+y}{2} + \frac{1}{\beta} \cdot (b \frac{x+y}{2} + y) \right\|^2 \\ &= \left\| \frac{x+y}{2} + \frac{1}{\beta} \cdot (b \frac{x+y}{2} + y) \right\|^2 + \left\| \frac{1}{\alpha} \cdot \frac{x+y}{2} + (b \frac{x+y}{2} + y) \right\|^2. \end{aligned}$$

Then using Lemma II.8 we have $\frac{x+y}{2} \perp^B - (b \frac{x+y}{2} + y)$.

That is

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

$$(8) \quad \left\| \frac{x+y}{2} - k(b \frac{x+y}{2} + y) \right\| \geq \left\| \frac{x+y}{2} \right\| = 1.$$

As before putting $k = 1/b$ then $k = 1/(b+2)$ in (8) respectively give

$|b| \leq 1$ and $|b+2| \leq 1$. Thus $b = -1$. Substituting this into (7) we get

$$(9) \quad 1 + \left\| \left(\frac{1}{\alpha} - \frac{1}{\beta} \right) \cdot \frac{x}{2} + \left(\frac{1}{\alpha} + \frac{1}{\beta} \right) \cdot \frac{y}{2} \right\|^2 \\ = \left\| \left(1 - \frac{1}{\beta} \right) \cdot \frac{x}{2} + \left(1 + \frac{1}{\beta} \right) \cdot \frac{y}{2} \right\|^2 + \left\| \left(\frac{1}{\alpha} - 1 \right) \cdot \frac{x}{2} + \left(\frac{1}{\alpha} + 1 \right) \cdot \frac{y}{2} \right\|^2.$$

Since $|1/\alpha| \geq |1/\beta|$ and $|1/\alpha|, |1/\beta| > 1$, therefore

$$(10) \quad 1 + \frac{1}{4} \left[\left(\frac{1}{\alpha} - \frac{1}{\beta} \right) + \left(\frac{1}{\alpha} + \frac{1}{\beta} \right) \right]^2 = \left\| \left(1 - \frac{1}{\beta} \right) \cdot \frac{x}{2} + \left(1 + \frac{1}{\beta} \right) \cdot \frac{y}{2} \right\|^2 \\ + \frac{1}{4} \left[\left(\frac{1}{\alpha} - 1 \right) + \left(\frac{1}{\alpha} + 1 \right) \right]^2,$$

$$\text{or} \quad \left\| \left(1 - \frac{1}{\beta} \right) \cdot \frac{x}{2} + \left(1 + \frac{1}{\beta} \right) \cdot \frac{y}{2} \right\| = 1.$$

There are two possibilities :

$$1/\beta > 1 \quad \text{or} \quad 1/\beta < -1.$$

In the case $1/\beta > 1$, writing

$$y = \left(\frac{1/\beta - 1}{1/\beta + 1} \right) x + \left[1 - \left(\frac{1/\beta - 1}{1/\beta + 1} \right) \right] \cdot \left[\left(1 - \frac{1}{\beta} \right) \cdot \frac{x}{2} + \left(1 + \frac{1}{\beta} \right) \cdot \frac{y}{2} \right],$$

contradicts the fact that y is an extreme point.

In the case $1/\beta < -1$, writing

$$x = \left(\frac{1/\beta + 1}{1/\beta - 1} \right) y + \left[1 - \left(\frac{1/\beta + 1}{1/\beta - 1} \right) \right] \cdot \left[\left(1 - \frac{1}{\beta} \right) \cdot \frac{x}{2} + \left(1 + \frac{1}{\beta} \right) \cdot \frac{y}{2} \right],$$

contradicts the fact that x is an extreme point.

Hence X is strictly convex, and the proof is completed.

THEOREM II.12

Let X be a normed linear space, and $|\alpha|, |\beta| \neq 1$.

Then the following are equivalent :

(i) X is an inner product space ;

(ii) $x, y \in X$, $(x \perp y)(\alpha, \beta) \Rightarrow x \perp^B y$.

Proof. (i) $\Rightarrow$ (ii) is straight.

(ii) $\Rightarrow$ (i) We show that (ii) of this theorem implies the statement (ii) of Theorem II.9, and hence X is an inner product space. Suppose not then there are points x and $y \in X$ such that $x \perp^B y$ and $(x \perp y)(\alpha, \beta)$. Choose $a \neq 0$ such that $(ay + x \perp y)(\alpha, \beta)$. But then $ay + x \perp^B y$, which violates the left uniqueness of Birkhoff orthogonality in strictly convex spaces.

LEMMA II.13

Let X be a normed linear space and $|\alpha|, |\beta| < 1$ or > 1 .

If for $x, y \in X$ and $(x \perp y)(\alpha, \beta) \Rightarrow x \perp^i y$, then X is strictly convex.

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

Proof. Assume on contrary that X is not strictly convex. Choose

$x \neq y$ as extreme points of the unit ball of X such that

$$\|x\| = \|y\| = \left\| \frac{x+y}{2} \right\| = 1$$

We claim that $(\frac{x+y}{2} \perp -y)(\alpha, \beta)$. For, otherwise, we have

$$\frac{x+y}{2} \perp -y \text{ by hypothesis.}$$

That is

$$\left\| \frac{x+y}{2} + y \right\| = \left\| \frac{x+y}{2} - y \right\| \quad \text{or} \quad \left\| \frac{1}{2}x + \frac{3}{2}y \right\| = \left\| \frac{1}{2}x - \frac{1}{2}y \right\|.$$

$$\text{Thus (by Lemma II.1)} \quad 2 = \left\| \frac{x-y}{2} \right\| \leq 1 \quad (\text{by the triangle inequality}),$$

which is impossible. Hence there exists a real number $a \neq 0$ such that

$$(\frac{x+y}{2} \perp - (a \frac{x+y}{2} + y))(\alpha, \beta). \text{ This, in turn, implies that}$$

$$\frac{x+y}{2} \perp - (a \frac{x+y}{2} + y).$$

That is

$$(1) \quad \left\| (1+a) \frac{x+y}{2} + y \right\| = \left\| (1-a) \frac{x+y}{2} - y \right\|$$

$$\left\| \frac{1+a}{2}x + \frac{3+a}{2}y \right\| = \left\| \frac{1-a}{2}x - \frac{1+a}{2}y \right\|.$$

But $|a| \leq 1$. Hence equation (1) becomes

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

$$\left\| \left(\frac{1-a}{2} \right)x - \left(\frac{1+a}{2} \right)y \right\| = \left[\frac{1+a}{2} + \frac{3+a}{2} \right] = 2+a.$$

This implies that

$$\begin{aligned} (2+a) &= \left\| \left(\frac{1-a}{2} \right)x - \left(\frac{1+a}{2} \right)y \right\| \leq \left| \frac{1-a}{2} \right| + \left| \frac{1+a}{2} \right| \\ &= \frac{1-a}{2} + \frac{1+a}{2} \\ &= 1. \end{aligned}$$

Hence $a = -1$ and $\left(\frac{x+y}{2} \perp \frac{x-y}{2} \right) (\alpha, \beta)$.

That is

$$(2) \quad 1 + \left\| \frac{\alpha - \beta}{2}x + \frac{\alpha + \beta}{2}y \right\|^2 = \left\| \frac{1 - \beta}{2}x + \frac{1 + \beta}{2}y \right\|^2 + \left\| \frac{\alpha - 1}{2}x + \frac{\alpha + 1}{2}y \right\|^2.$$

Consider the following cases :

Case1. $|\alpha| \geq |\beta|$

There are two possibilities : $|\alpha|, |\beta| < 1$ or $|\alpha|, |\beta| > 1$.

In the case $|\alpha|, |\beta| < 1$, then the proof is similar to that in Lemma II.11.

In the case $|\alpha|, |\beta| > 1$, then equation (2) becomes

$$1 + \left[\frac{\alpha - \beta}{2} + \frac{\alpha + \beta}{2} \right]^2 = \left\| \frac{1 - \beta}{2}x + \frac{1 + \beta}{2}y \right\|^2 + \left[\frac{\alpha - 1}{2} + \frac{\alpha + 1}{2} \right]^2.$$

Thus $\left\| \frac{1 - \beta}{2}x + \frac{1 + \beta}{2}y \right\| = 1.$

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

Then with $\beta > 1$, witting

$$y = \left[\frac{\beta - 1}{\beta + 1} \right] x + \left[1 - \frac{\beta - 1}{\beta + 1} \right] \cdot \left[\frac{1 - \beta}{2} x + \frac{1 + \beta}{2} y \right],$$

contradicts the fact that y is an extreme point.

With $\beta < -1$, witting

$$x = \left[\frac{\beta + 1}{\beta - 1} \right] y + \left[1 - \frac{\beta + 1}{\beta - 1} \right] \cdot \left[\frac{1 - \beta}{2} x + \frac{1 + \beta}{2} y \right],$$

contradicts the fact that x is an extreme point. Thus X is strictly convex.

Case 2. $|\alpha| \leq |\beta|$

Then we have $x \perp^i y \Leftrightarrow y \perp^i x$ by symmetry of isosceles orthogonality

$$\Leftrightarrow (y \perp x)(\alpha, \beta) \quad \text{by hypothesis}$$

$$\Leftrightarrow (x \perp y)(\beta, \alpha).$$

Hence X is strictly convex by the previous case, since $|\beta| \geq |\alpha|$.

THEOREM II.14 ([15], Theorem 11)

Let X be a normed linear space and $|\alpha|, |\beta| < 1$ or > 1 .

Consider the following statements :

(i) $x, y \in X$ and $(x \perp y)(\alpha, \beta) \Rightarrow x \perp^i y$.

(ii) $x, y \in X$ and $x \perp^i y \Rightarrow (x \perp y)(\alpha, \beta)$.

Then (i) implies (ii).

Proof. Suppose (i) does not imply (ii).

Then there exist points x and y such that $\|x - y\| = \|x + y\|$, but

$$\|x - y\|^2 + \|\alpha x - \beta y\|^2 \neq \|x - \beta y\|^2 + \|\alpha x - y\|^2.$$

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

Thus there exists a number $a \neq 0$ such that

$$\|x - (ax + y)\|^2 + \|\alpha x - \beta(ax + y)\|^2 = \|x - \beta(ax + y)\|^2 + \|\alpha x - (ax + y)\|^2.$$

By (i) we have

$$\|x - (ax + y)\| = \|x + (ax + y)\|.$$

Thus $x \perp^i y$ and $x \perp^i ax + y$, which contradicts the uniqueness of isosceles orthogonality in strictly convex spaces.

THEOREM II.15

Let X be a normed linear space and α, β be any real numbers such that

(i) $\alpha \neq \beta$ with $\alpha = -1$ or $\beta = -1$, and $\alpha \cdot \beta \neq 0$.

Or

(ii) $\alpha = \beta$ with $|\alpha| < 1$.

If $x \perp^i y$ implies $(x \perp y)(\alpha, \beta)$ for $x, y \in X$, then X is an inner product space.

Proof.

(i) $\alpha \neq \beta$ with $\alpha = -1$ or $\beta = -1$, and $\alpha \cdot \beta \neq 0$.

If $\alpha = -1, \beta \neq 0$ let $x \perp^i y$. That is

$$(1) \quad \|x - y\| = \|x + y\|.$$

Hence by hypothesis we have $(x \perp y)(-1, \beta)$. That is

$$(2) \quad \|x - y\|^2 + \|x + \beta y\|^2 = \|x - \beta y\|^2 + \|x + y\|^2.$$

Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

From (1) and (2) we get

$$\|x + \beta y\| = \|x - \beta y\|, \text{ or simply } x \perp^i \beta y.$$

Thus by Lorch (L1), (see page 1) X is an inner product space.

If $\beta = -1$ and $\alpha \neq 0$, then one can show, in a similar manner to the

above proof, that $x \perp^i y$ implies $x \perp^i \alpha y$. Hence X is an inner product space by (L1).

(ii) $\alpha = \beta$ and $|\alpha| < 1$.

Let $\|x\| = \|y\| = 1$. Then

$$\|(x + y) - (x - y)\| = \|x + y + x - y\|; \text{ that is } (x + y) \perp^i (x - y).$$

Therefore by hypothesis $((x + y) \perp (x - y))(\alpha, \alpha)$. That is

$$(1 + \alpha^2) \|(x + y) - (x - y)\|^2 = \|(x + y) - \alpha(x - y)\|^2 + \|\alpha(x + y) - (x - y)\|^2,$$

or

$$(1 + \alpha^2) \|2y\|^2 = \|(1 - \alpha)x + (1 + \alpha)y\|^2 + \|(\alpha - 1)x + (\alpha + 1)y\|^2.$$

Simplify,

$$4(1 + \alpha^2) = \|(1 - \alpha)x + (1 + \alpha)y\|^2 + \|(1 - \alpha)x - (1 + \alpha)y\|^2,$$

or

$$1 + \alpha^2 = \left\| \frac{1 - \alpha}{2} x + \frac{1 + \alpha}{2} y \right\|^2 + \left\| \frac{1 - \alpha}{2} x - \frac{1 + \alpha}{2} y \right\|^2.$$

The result follows from Day's characterization $(D, \sim)$ with $a = b = \frac{1 - \alpha}{2}$,

and $\sim$ replaced by $=$. The details are as follows :

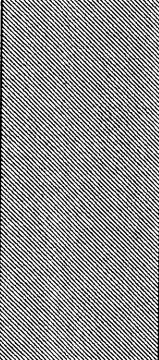
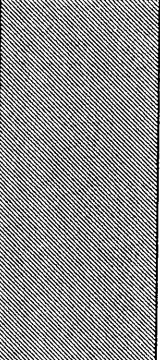
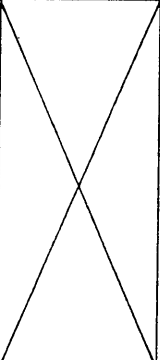
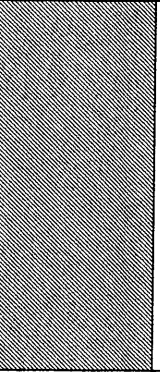
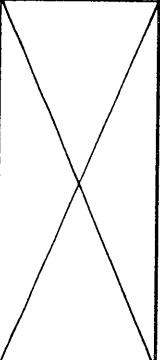
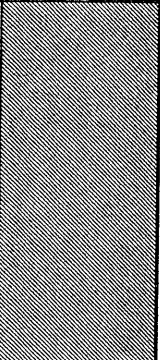
Chapter II

Relations between Orthogonalities and Characterizations of Inner Product Spaces

$$\begin{aligned}
 LHS : & b(1-b)\left\|ax + (1-a)y\right\|^2 + a(1-a)\left\|bx - (1-b)y\right\|^2 \\
 &= \frac{1-\alpha}{2} \left(\frac{1+\alpha}{2} \right) \left[\left\| \frac{1-\alpha}{2}x + \frac{1+\alpha}{2}y \right\|^2 + \left\| \frac{1-\alpha}{2}x - \frac{1+\alpha}{2}y \right\|^2 \right] \\
 &= \frac{1-\alpha^2}{4} \left[\left\| \frac{1-\alpha}{2}x + \frac{1+\alpha}{2}y \right\|^2 + \left\| \frac{1-\alpha}{2}x - \frac{1+\alpha}{2}y \right\|^2 \right].
 \end{aligned}$$

$$\begin{aligned}
 RHS : & [a+b-2ab][ab+(1-a)(1-b)] \\
 &= \left[(1-\alpha) - 2\left(\frac{1-\alpha}{2}\right)^2 \right] \left[\left(\frac{1-\alpha}{2}\right)^2 + \left(\frac{1+\alpha}{2}\right)^2 \right] \\
 &= \left[\frac{1-\alpha^2}{2} \right] \left[\frac{1+\alpha^2}{2} \right] = \left[\frac{1-\alpha^2}{4} \right] [1+\alpha^2].
 \end{aligned}$$

SUMMARY OF CHAPTER II

	Isosceles	Pythagorean	Birkhoff	(α, β) - orthogonality
Isosceles		Only in i . p . s.	Only in i . p . s.	$\alpha = \beta, \alpha < 1$ Only in i . p . s.
Pythagorean	Only in i . p . s.		Only in i . p . s.	
Birkhoff	Only in i . p . s.	Only in i . p . s.		$ \alpha , \beta \neq 1$ Only in i . p . s.
(α, β) - orthogonality	$\alpha = \beta, \alpha < 1$ Only in i . p . s.		$ \alpha , \beta \neq 1$ Only in i . p . s.	

 Trivial case

 Not covered in chapter II

* i . p . s . : Inner product space

CHAPTER III

$[\alpha, \beta]$ – ORTHOGONALITY

INTRODUCTION	98
§1· Definition of $[\alpha, \beta]$ -orthogonality	99
§2· Properties of $[\alpha, \beta]$ -orthogonality	103
§3· Consequences of homogeneity of $[\alpha, \beta]$ -orthogonality	112
§4· The characterization theorem	116
§5· Set of open problems related to $[\alpha, \beta]$ -orthogonality	119
SUMMARY	120

CHAPTER III

$[\alpha, \beta]$ – ORTHOGONALITY

INTRODUCTION

Orthogonality in inner product spaces is a binary relation that can be expressed in many ways without explicit mention to inner product of the space.

According to this observation we are going to define a new orthogonality relation denoted by $[\alpha, \beta]$ -orthogonality, where α and β are fixed real numbers, not both zero and $\alpha, \beta \geq 0$. Throughout this chapter we study the properties of this relation, as we did for orthogonality relations in chapter I, and ending with characterizations of inner product spaces.

§1. DEFINITION OF $[\alpha, \beta]$ -ORTHOGONALITY

Let X be a real inner product space, and α, β be fixed real numbers, not both zero, with $\alpha \cdot \beta \geq 0$.

If $x, y \in X$ such that $\|x - \alpha y\| \cdot \|x - \beta y\| = \|x + \alpha y\| \cdot \|x + \beta y\|$, then

$$\begin{aligned}
 & \|x - \alpha y\|^2 \cdot \|x - \beta y\|^2 = \\
 & \left[\|x\|^2 - 2\alpha \langle x, y \rangle + \alpha^2 \|y\|^2 \right] \cdot \left[\|x\|^2 - 2\beta \langle x, y \rangle + \beta^2 \|y\|^2 \right] \\
 & = \|x\|^4 + \alpha^2 \beta^2 \|y\|^4 + (\alpha^2 + \beta^2) \|x\|^2 \|y\|^2 + 4\alpha\beta \langle x, y \rangle^2 \\
 & \quad - 2(\alpha + \beta) \|x\|^2 \langle x, y \rangle - 2(\alpha^2 \beta + \alpha \beta^2) \|y\|^2 \langle x, y \rangle \\
 & = \|x\|^4 + \alpha^2 \beta^2 \|y\|^4 + (\alpha^2 + \beta^2) \|x\|^2 \|y\|^2 + 4\alpha\beta \langle x, y \rangle^2 \\
 & \quad - 2\langle x, y \rangle \left\{ (\alpha + \beta) \|x\|^2 + (\alpha^2 \beta + \alpha \beta^2) \|y\|^2 \right\} \\
 & = \|x\|^4 + \alpha^2 \beta^2 \|y\|^4 + (\alpha^2 + \beta^2) \|x\|^2 \|y\|^2 + 4\alpha\beta \langle x, y \rangle^2 \\
 & \quad - 2\langle x, y \rangle \left\{ (\alpha + \beta) \|x\|^2 + \alpha\beta(\alpha + \beta) \|y\|^2 \right\} \\
 & = \|x\|^4 + \alpha^2 \beta^2 \|y\|^4 + (\alpha^2 + \beta^2) \|x\|^2 \|y\|^2 + 4\alpha\beta \langle x, y \rangle^2 \\
 & \quad - 2(\alpha + \beta) \langle x, y \rangle \left\{ \|x\|^2 + \alpha\beta \|y\|^2 \right\}.
 \end{aligned}$$

Similarly, $\|x + \alpha y\|^2 \cdot \|x + \beta y\|^2 =$

$$\begin{aligned}
 & \|x\|^4 + \alpha^2 \beta^2 \|y\|^4 + (\alpha^2 + \beta^2) \|x\|^2 \|y\|^2 + 4\alpha\beta \langle x, y \rangle^2 \\
 & + 2(\alpha + \beta) \langle x, y \rangle \left\{ \|x\|^2 + \alpha\beta \|y\|^2 \right\}.
 \end{aligned}$$

Chapter III
Definition of $[\alpha, \beta]$ -orthogonality

Thus

$$\begin{aligned} & \|x - \alpha y\|^2 \cdot \|x - \beta y\|^2 - \|x + \alpha y\|^2 \cdot \|x + \beta y\|^2 = \\ & - 4(\alpha + \beta) \langle x, y \rangle \left\{ \|x\|^2 + \alpha\beta \|y\|^2 \right\} = 0 \quad \text{if and only if} \quad \langle x, y \rangle = 0. \end{aligned}$$

On the other hand, $x \perp y$ if and only if $\langle x, y \rangle = 0$ for $x, y \in X$.

This means the definition of orthogonality in an inner product space may be reformulated in the following way:

$$x \perp y \text{ if } \|x - \alpha y\| \cdot \|x - \beta y\| = \|x + \alpha y\| \cdot \|x + \beta y\|.$$

DEFINITION III.1

Let X be a normed linear space over $\mathbf{R}$ and α, β be fixed numbers, not both zero, and satisfying the condition $\alpha \cdot \beta \geq 0$. An element x of X is said to be orthogonal to an element y of X , denoted by $(x \perp y)[\alpha, \beta]$, if

$$\|x - \alpha y\| \cdot \|x - \beta y\| = \|x + \alpha y\| \cdot \|x + \beta y\|.$$

Without loss of generality we may assume that $\alpha, \beta \geq 0$ (nonnegative). Then it is noted that $[1, 1]$, $[1, 0]$ and $[0, 1]$ are isosceles orthogonality. However, in general normed linear spaces and if $[\alpha, \beta] \neq [1, 1]$, $[1, 0]$ and $[0, 1]$ then $[\alpha, \beta]$ -orthogonality is distinct from isosceles orthogonality as shown in the following example.

Example 1. Consider the space $\mathbf{R}^2$ with $\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$.

Case 1. If $0 \leq \beta \leq \alpha < 1$.

Let $x = (1, 1 + \alpha)$, $y = (1, 0)$. Then

$$x - y = (0, 1 + \alpha) \Rightarrow \|x - y\| = 1 + \alpha,$$

$$x + y = (2, 1 + \alpha) \Rightarrow \|x + y\| = 2. \text{ Thus } x \perp^i y.$$

$$x - \alpha y = (1 - \alpha, 1 + \alpha), \quad x - \beta y = (1 - \beta, 1 + \alpha),$$

$$x + \alpha y = (1 + \alpha, 1 + \alpha), \quad x + \beta y = (1 + \beta, 1 + \alpha).$$

Where the norm of each equals $1 + \alpha$. Thus clearly $(x \perp y)[\alpha, \beta]$.

Case 2. If $\alpha \geq \beta > 1$. ($\Rightarrow 0 \leq 1/\alpha, 1/\beta < 1$)

Note that $\beta \leq \alpha$ implies that $1/\alpha \leq 1/\beta$.

Let $x = (1, 1 + 1/\beta)$, $y = (1, 0)$. Then

$$x - y = (0, 1 + 1/\beta) \Rightarrow \|x - y\| = 1 + 1/\beta,$$

$$x + y = (2, 1 + 1/\beta) \Rightarrow \|x + y\| = 2. \text{ Thus } x \perp^i y \text{ (or } y \perp^i x).$$

Now, we will show that $(y \perp x)[\alpha, \beta]$, and here are the steps :

$$x - 1/\alpha y = (1 - 1/\alpha, 1 + 1/\beta), \quad x - 1/\beta y = (1 - 1/\beta, 1 + 1/\beta),$$

$$x + 1/\alpha y = (1 + 1/\alpha, 1 + 1/\beta), \quad x + 1/\beta y = (1 + 1/\beta, 1 + 1/\beta),$$

where the norm of each is $1 + 1/\beta$.

Thus $(x \perp y)[1/\alpha, 1/\beta]$. But one of the important properties of $[\alpha, \beta]$

– orthogonality is discussed in Lemma III.4, next section, is that :

$$(x \perp y)[1/\alpha, 1/\beta] \Leftrightarrow (y \perp x)[\alpha, \beta]. \text{ Therefore } (y \perp x)[\alpha, \beta].$$

Geometrically, $[\alpha, \beta]$ – orthogonality can be discussed as follows :

$(x \perp y)[\alpha, \beta]$ if and only if $\text{area A} = \text{area B}$.

Where area A is the area of the rectangle of length $\|x + \beta y\|$ and width $\|x + \alpha y\|$,
and area B is the area of the rectangle of length $\|x - \beta y\|$ and width $\|x - \alpha y\|$.

Consider the following figures where we assumed that $\beta > \alpha > 0$.

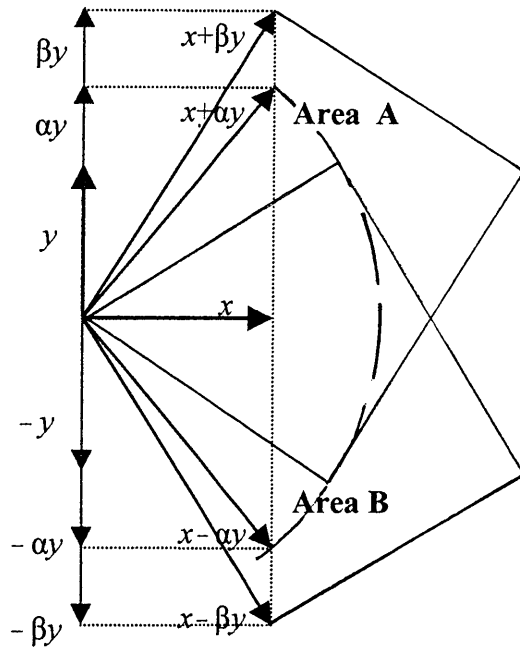


Fig. 1
 $(x \perp y)[\alpha, \beta]$

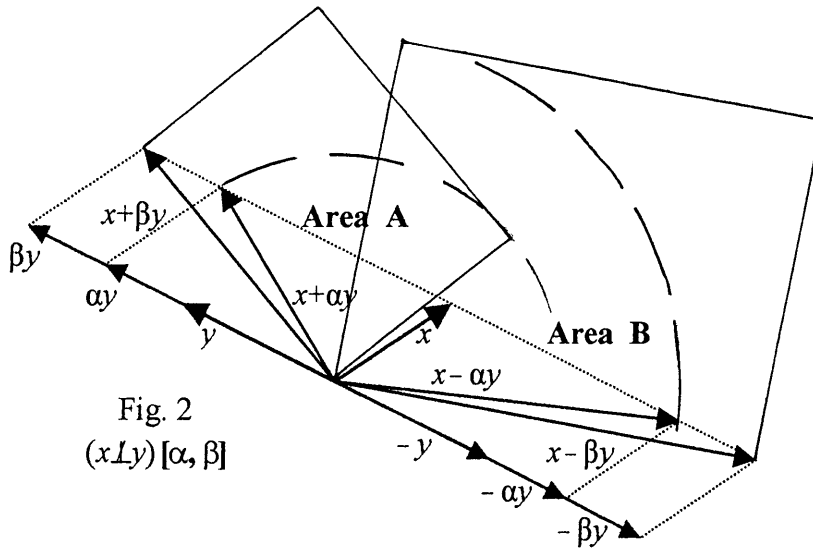


Fig. 2
 $(x \perp y)[\alpha, \beta]$

Illustration of $[\alpha, \beta]$ -orthogonality in Euclidean plane $\mathbf{R}^2$

§2. PROPERTIES OF $[\alpha, \beta]$ -ORTHOGONALITY

Here are some important properties of the new definition: -

- (I) $(x \perp y)[\alpha, \beta] \Leftrightarrow (x \perp -y)[\alpha, \beta] \Leftrightarrow (-x \perp y)[\alpha, \beta]$.
- (II) $(x \perp y)[\alpha, \beta] \Leftrightarrow (\lambda x \perp \lambda y)[\alpha, \beta]$ for all real $\lambda \neq 0$.
- (III) $(\lambda x \perp y)[\alpha, \beta] \Leftrightarrow (x \perp y)[\alpha/|\lambda|, \beta/|\lambda|]$ for all real $\lambda \neq 0$.
- (IV) $(x \perp \mu y)[\alpha, \beta] \Leftrightarrow (x \perp y)[|\mu|\alpha, |\mu|\beta]$ for all real $\mu \neq 0$.
- (V) $[\alpha, \beta]$ -orthogonality, in general, is neither additive nor homogeneous.

Example 2. Let x and y same as those in example1 - case1, and choose

$\alpha = 1/2, \beta = 1/4$. Then $(x \perp y)[1/2, 1/4]$.

For $z = 2y = (2, 0)$, we have

$$x - \alpha z = (1 - 2\alpha, 1 + \alpha) = (0, 3/2) \Rightarrow \|x - \alpha z\| = 3/2,$$

$$x - \beta z = (1 - 2\beta, 1 + \alpha) = (1/2, 3/2) \Rightarrow \|x - \beta z\| = 3/2,$$

$$x + \alpha z = (1 + 2\alpha, 1 + \alpha) = (2, 3/2) \Rightarrow \|x + \alpha z\| = 2,$$

$$x + \beta z = (1 + 2\beta, 1 + \alpha) = (3/2, 3/2) \Rightarrow \|x + \beta z\| = 3/2.$$

Clearly $(x \perp z)[1/2, 1/4]$; that is $(x \perp 2y)[1/2, 1/4]$.

(VI) $[\alpha, \beta]$ -orthogonality is not symmetric

Example 3. Let $\alpha = 1/2$, $\beta = 1/4$, $x = (1, 3/2)$, and $y = (1, 0)$.

Then $(x \perp y)[\alpha, \beta]$ by example 1. But

$$y - \alpha x = (1 - \alpha, -3/2 \times \alpha) = (1/2, -3/4) \Rightarrow \|y - \alpha x\| = 3/4,$$

$$y - \beta x = (1 - \beta, -3/2 \times \beta) = (3/4, -3/8) \Rightarrow \|y - \beta x\| = 3/4,$$

$$y + \alpha x = (1 + \alpha, 3/2 \times \alpha) = (3/2, 3/4) \Rightarrow \|y + \alpha x\| = 3/2,$$

$$y + \beta x = (1 + \beta, 3/2 \times \beta) = (5/4, 3/8) \Rightarrow \|y + \beta x\| = 5/4.$$

Clearly, $3/4 \times 3/4 \neq 3/2 \times 5/4$; that is $(y \perp x)[\alpha, \beta]$.

(VII) PROPOSITION III.2

For any nonzero real numbers λ, μ we have

$$(\lambda x \perp \mu x)[\alpha, \beta] \Leftrightarrow x = 0.$$

Proof. $(\lambda x \perp \mu x)[\alpha, \beta] \Leftrightarrow \|\lambda x - \alpha \mu x\| \cdot \|\lambda x - \beta \mu x\| = \|\lambda x + \alpha \mu x\| \cdot \|\lambda x + \beta \mu x\|$

or $\Leftrightarrow |\lambda - \alpha \mu| \cdot |\lambda - \beta \mu| \cdot \|x\|^2 = |\lambda + \alpha \mu| \cdot |\lambda + \beta \mu| \cdot \|x\|^2$. Squaring,

$$\Leftrightarrow [(\lambda - \alpha \mu)(\lambda - \beta \mu)]^2 \cdot \|x\|^4 = [(\lambda + \alpha \mu)(\lambda + \beta \mu)]^2 \cdot \|x\|^4$$

$$\Leftrightarrow \left[\lambda^2 - (\alpha + \beta)\lambda\mu + \alpha^2\beta^2\mu^2 \right]^2 \cdot \|x\|^4 = \left[\lambda^2 + (\alpha + \beta)\lambda\mu + \alpha^2\beta^2\mu^2 \right]^2 \cdot \|x\|^4$$

$$\Leftrightarrow \left\{ \left[\lambda^2 - (\alpha + \beta)\lambda\mu + \alpha^2\beta^2\mu^2 \right]^2 - \left[\lambda^2 + (\alpha + \beta)\lambda\mu + \alpha^2\beta^2\mu^2 \right]^2 \right\} \|x\|^4 = 0$$

$$\Leftrightarrow (-2(\alpha + \beta)\lambda\mu) \left(2\lambda^2 + 2\alpha^2\beta^2\mu^2 \right) \cdot \|x\|^4 = 0$$

$$\Leftrightarrow -4(\alpha + \beta)\lambda\mu \left(\lambda^2 + \alpha^2\beta^2\mu^2 \right) \cdot \|x\|^4 = 0 \quad \Leftrightarrow \|x\| = 0 \text{ or } x = 0.$$

COROLLARY III.3

For $x, y \in X$, and $x, y \neq 0$, then

$(x \perp y)[\alpha, \beta]$ implies that x and y are linearly independent.

(VIII) LEMMA III.4

$(x \perp y)[\alpha, \beta] \Leftrightarrow (y \perp x)[1/\alpha, 1/\beta]$ for α and $\beta \neq 0$.

Proof. If $(x \perp y)[\alpha, \beta]$ then $\|x - \alpha y\| \cdot \|x - \beta y\| = \|x + \alpha y\| \cdot \|x + \beta y\|$.

Divide by $\alpha\beta$ (since $\alpha, \beta \neq 0$) we get

$$\frac{\|x - \alpha y\|}{\alpha} \cdot \frac{\|x - \beta y\|}{\beta} = \frac{\|x + \alpha y\|}{\alpha} \cdot \frac{\|x + \beta y\|}{\beta},$$

$$\|1/\alpha x - y\| \cdot \|1/\beta x - y\| = \|1/\alpha x + y\| \cdot \|1/\beta x + y\|$$

$$\text{or } \|y - 1/\alpha x\| \cdot \|y - 1/\beta x\| = \|y + 1/\alpha x\| \cdot \|y + 1/\beta x\|.$$

Thus $(y \perp x)[1/\alpha, 1/\beta]$.

COROLLARY III.5

For $\alpha, \beta \neq 0$, $[\alpha, \beta]$ – orthogonality is symmetric, right – additive, left – additive, homogeneous if and only if $[1/\alpha, 1/\beta]$ – orthogonality is symmetric, left – additive, right – additive, homogeneous respectively.

(IX) THEOREM III.6 (RESOLVABILITY)

For $x, y \in X \exists a \in \mathbf{R}$ such that $(ax + y) \perp x$ $[\alpha, \beta]$.

Proof. Since the result is trivial when $x = 0$, we assume that $x \neq 0$.

Define a real valued function $f(t)$ by

$$\begin{aligned} f(t) &= \|(t + \alpha)x + y\| \cdot \|(t + \beta)x + y\| - \|(t - \alpha)x + y\| \cdot \|(t - \beta)x + y\| \\ &= t^2 \|x + 1/t(\alpha x + y)\| \cdot \|x + 1/t(\beta x + y)\| \\ &\quad - t^2 \|x + 1/t(-\alpha x + y)\| \cdot \|x + 1/t(-\beta x + y)\| \quad (t \neq 0) \\ &= t \|x + 1/t(\alpha x + y)\| \cdot t \|x + 1/t(\beta x + y)\| \\ &\quad - t \|x + 1/t(-\alpha x + y)\| \cdot t \|x + 1/t(-\beta x + y)\| \\ &= \left[\frac{\|x + 1/t(\alpha x + y)\| - \|x\|}{1/t} + t\|x\| \right] \cdot \left[\frac{\|x + 1/t(\beta x + y)\| - \|x\|}{1/t} + t\|x\| \right] \\ &\quad - \left[\frac{\|x + 1/t(-\alpha x + y)\| - \|x\|}{1/t} + t\|x\| \right] \cdot \left[\frac{\|x + 1/t(-\beta x + y)\| - \|x\|}{1/t} + t\|x\| \right] \\ &= [M(\alpha) + t\|x\|] \cdot [M(\beta) + t\|x\|] - [M(-\alpha) + t\|x\|] \cdot [M(-\beta) + t\|x\|], \end{aligned}$$

$$\text{where } M(\gamma) = \frac{\|x + 1/t(\gamma x + y)\| - \|x\|}{1/t}.$$

Simplify, we get

$$\begin{aligned} f(t) &= M(\alpha)M(\beta) + [M(\alpha) + M(\beta)]t\|x\| + t^2\|x\|^2 \\ &\quad - M(-\alpha)M(-\beta) - [M(-\alpha) + M(-\beta)]t\|x\| - t^2\|x\|^2 \\ &= [M(\alpha)M(\beta) - M(-\alpha)M(-\beta)] + [M(\alpha) + M(\beta) - M(-\alpha) - M(-\beta)]t\|x\| \\ &= A + B\|x\|t, \text{ with} \end{aligned}$$

$$A = [M(\alpha)M(\beta) - M(-\alpha)M(-\beta)] \quad , \quad B = [M(\alpha) + M(\beta) - M(-\alpha) - M(-\beta)].$$

Chapter III
Properties of $[\alpha, \beta]$ -orthogonality

Now, $\lim_{t \rightarrow \pm\infty} M(\gamma) = N_{\pm}(x; \gamma x + y) = N_{\pm}(x; y) + \gamma \|x\|.$

Thus $\lim_{t \rightarrow \pm\infty} A = [N_{\pm}(x; y) + \alpha \|x\|] \cdot [N_{\pm}(x; y) + \beta \|x\|]$
 $- [N_{\pm}(x; y) - \alpha \|x\|] \cdot [N_{\pm}(x; y) - \beta \|x\|]$
 $= N_{\pm}^2(x; y) + (\alpha + \beta) \|x\| N_{\pm}(x; y) + \alpha \beta \|x\|^2$
 $- N_{\pm}^2(x; y) + (\alpha + \beta) \|x\| N_{\pm}(x; y) - \alpha \beta \|x\|^2$
 $= 2(\alpha + \beta) \|x\| N_{\pm}(x; y).$

Also $\lim_{t \rightarrow \pm\infty} B = N_{\pm}(x; y) + \alpha \|x\| + N_{\pm}(x; y) + \beta \|x\|$
 $- N_{\pm}(x; y) + \alpha \|x\| - N_{\pm}(x; y) + \beta \|x\|$
 $= 2(\alpha + \beta) \|x\|.$

Therefore

$$\lim_{t \rightarrow \pm\infty} f(t) = 2(\alpha + \beta) \|x\| \cdot \left[N_{\pm}(x; y) + \|x\| \cdot \lim_{t \rightarrow \pm\infty} t \right] = \pm\infty. \quad (x \neq 0)$$

Then continuity of the norm implies that $f(a) = 0$ for some number a , or that $(ax + y) \perp x$ $[\alpha, \beta]$.

COROLLARY III.7

For $x, y \in X$, $\exists a \in \mathbf{R}$ such that $(x \perp ax + y) [\alpha, \beta]$.

Proof. Again we assume that $x \neq 0$.

Case1. $\alpha, \beta \neq 0$.

Theorem III.6 shows that there is a real number a such that $(ax + y) \perp x$ $[1/\alpha, 1/\beta]$,

then using Lemma III.4 we have $(x \perp ax + y) [\alpha, \beta]$.

Case 2. $\alpha = 0$, thus $\beta \neq 0$.

$$\begin{aligned} \text{Then } (x \perp ax + y)[\alpha, \beta] &\Leftrightarrow \|x\| \cdot \|x - \beta(ax + y)\| = \|x\| \cdot \|x + \beta(ax + y)\| \\ &\Leftrightarrow \|x - \beta(ax + y)\| = \|x + \beta(ax + y)\| \quad (x \neq 0) \\ &\Leftrightarrow x \perp^i \beta(ax + y) \\ &\Leftrightarrow x \perp^i (a\beta) \cdot x + (\beta y), \end{aligned}$$

and Theorem I.1 assures the existence of such number a .

Case 3. $\beta = 0$, thus $\alpha \neq 0$.

Then $(x \perp ax + y)[\alpha, \beta] \Leftrightarrow x \perp^i \alpha(ax + y)$, and again the number a exists by Theorem I.1.

THEOREM III.8

If $\alpha = \beta$ then $[\alpha, \alpha]$ - orthogonality is right - unique if and only if X is strictly convex.

Proof. Let X be a strictly convex space. Suppose that $[\alpha, \alpha]$ - orthogonality is not right - unique. Then there exist $x (\neq 0)$ and $y \in X$ and a nonzero real number $a (> 0)$, such that $(x \perp y)[\alpha, \alpha]$ and $(x \perp ax + y)[\alpha, \alpha]$. Thus we have respectively,

$$\begin{aligned} (1) \quad &\|\alpha y - x\| = \|\alpha y + x\| \\ (2) \quad &\|\alpha y + (\alpha a - 1)x\| = \|\alpha y + (\alpha a + 1)x\|. \end{aligned}$$

Now define a real - valued function:

$$q(t) = \|\alpha y + tx\| \quad \text{for } -\infty < t < \infty.$$

$$\text{Then } q(k \cdot t_1 + (1 - k) \cdot t_2) = \|\alpha y + [k \cdot t_1 + (1 - k) \cdot t_2] \cdot x\|$$

Chapter III

Properties of $[\alpha, \beta]$ -orthogonality

$$\begin{aligned}
 &= \left\| [k + (1 - k)] \cdot \alpha y + [k \cdot t_1 + (1 - k) \cdot t_2] \cdot x \right\| \\
 &= \left\| k[\alpha y + t_1 \cdot x] + (1 - k)[\alpha y + t_2 \cdot x] \right\| \\
 &< k \left\| \alpha y + t_1 \cdot x \right\| + (1 - k) \left\| \alpha y + t_2 \cdot x \right\| \quad (\text{Since } X \text{ is strictly convex.})
 \end{aligned}$$

Thus $q(t)$ is a strictly convex function. Furthermore, from (1) and (2) we have

$$(3) \quad q(-1) = q(1)$$

$$(4) \quad q(\alpha a - 1) = q(\alpha a + 1).$$

Now consider the following cases :

Case 1. If $0 < \alpha a < 2$, then

$$\begin{aligned}
 q(\alpha a - 1) &= q \left[\frac{2 - \alpha a}{2} \cdot (-1) + \frac{\alpha a}{2} \cdot (1) \right] \\
 &< \left(\frac{2 - \alpha a}{2} \right) \cdot q(-1) + \left(\frac{\alpha a}{2} \right) \cdot q(1) \quad (\text{Since } q(t) \text{ is a strictly convex.}) \\
 &= q(1) \quad (\text{From (3).}) \\
 &= q \left[\frac{\alpha a}{2} \cdot (\alpha a - 1) + \left(1 - \frac{\alpha a}{2} \right) \cdot (\alpha a + 1) \right] \\
 &< \left(\frac{\alpha a}{2} \right) \cdot q(\alpha a - 1) + \left(1 - \frac{\alpha a}{2} \right) \cdot q(\alpha a + 1) \\
 &\quad (\text{Since } q(t) \text{ is a strictly convex.}) \\
 &= q(\alpha a + 1) \quad (\text{From (4).})
 \end{aligned}$$

Thus $q(\alpha a - 1) < q(\alpha a + 1)$ which contradicts (4).

Case 2. If $\alpha a = 2$, then from (3) and (4), we have $q(-1) = q(1) = q(3)$. But

for $t = 1 = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot (3)$, and by strict convexity of the function $q(t)$ we

have $q(1) < \frac{1}{2}q(-1) + \frac{1}{2}q(3) = q(3)$, a contradiction.

Case 3. If $\alpha a > 2$, then $\alpha a - 1 > 1$, so $[-1, 1] \cap [\alpha a - 1, \alpha a + 1] = \Phi$. Thus $q(t)$ will have two distinct local minima - one each in the intervals $[-1, 1]$ and $[\alpha a - 1, \alpha a + 1]$. But $q(t)$ is strictly convex and thus have only one point of minimum - a global minimum. Again a contradiction is seen. Therefore $[\alpha, \alpha]$ -orthogonality is right - unique in strictly convex spaces.

Conversely, assume that $[\alpha, \alpha]$ -orthogonality is right - unique. Show that X is strictly convex. We will prove that right - uniqueness of $[\alpha, \alpha]$ -orthogonality implies uniqueness of isosceles orthogonality, which in turn implies the strict convexity of the space X . On contrary, assume that isosceles is not unique, and so there exist $x (\neq 0)$ and $y \in X$ such that $x \perp^i y$ and for

some nonzero real number a , then $x \perp^i ax + y$. That is

$$(1) \quad \|x - y\| = \|x + y\|$$

$$(2) \quad \|x - (ax + y)\| = \|x + (ax + y)\|.$$

Let $z = y/\alpha$ and $w = (ax + y)/\alpha$ ($\because \alpha \neq 0$). Then (1) and (2) become

$$\|x - \alpha z\| = \|x + \alpha z\| \quad \text{and} \quad \|x - \alpha w\| = \|x + \alpha w\|.$$

But $w = (a/\alpha) \cdot x + y/\alpha = \lambda x + z$ ($\lambda = a/\alpha \neq 0$). Therefore we get

$$\|x - \alpha z\| = \|x + \alpha z\| \quad \text{and} \quad \|x - \alpha(\lambda x + z)\| = \|x + \alpha(\lambda x + z)\|.$$

This contradicts right - uniqueness of $[\alpha, \alpha]$ -orthogonality. Thus isosceles orthogonality is unique and so the space X is strictly convex.

COROLLARY III.9

Right and left uniqueness of $[\alpha, \alpha]$ – orthogonality are equivalent.

Proof. From Lemma III.4, we have

(1) $[\alpha, \alpha]$ – orthogonality is left – unique if and only if $[1/\alpha, 1/\alpha]$ – orthogonality is right – unique.

Furthermore, from Theorem III.8,

(2) $[1/\alpha, 1/\alpha]$ – orthogonality is right – unique if and only if X is strictly convex.

Thus from (1) and (2) we get $[\alpha, \alpha]$ – orthogonality is left – unique if and only if X is strictly convex, and so right and left uniqueness of $[\alpha, \alpha]$ – orthogonality are equivalent. This completes the proof.

Note that when either $\alpha = 0$ or $\beta = 0$, then Theorem III.8 and its corollary are obviously true.

§3. CONSEQUENCES OF HOMOGENEITY OF $[\alpha, \beta]$ -ORTHOGONALITY

$(x \perp y)[\alpha, \beta]$ does not imply that x and y are orthogonal in the isosceles sense in general, but if $[\alpha, \beta]$ -orthogonality is homogeneous it does. That will be a corollary of the following theorem.

THEOREM III.10

Let $x, y \in X$. If $(\lambda x \perp \mu y)[\alpha, \beta] \quad \forall \lambda, \mu \in \mathbf{R}$, then $x \perp^i y$.

Proof. Let x and $y \in X$ such that $(x \perp y)[\alpha, \beta]$. We may assume that $\beta \leq \alpha$, and $\alpha \neq 0$. Show that $x \perp^i y$. We have to consider two cases :

Case 1. $\beta = \alpha$. Then by assumption we have $(\alpha x \perp \alpha y)[\alpha, \alpha]$; that is

$$\|\alpha x - \alpha y\|^2 = \|\alpha x + \alpha y\|^2$$

$$\text{or} \quad \|\alpha x - \alpha y\| = \|\alpha x + \alpha y\|$$

$$\text{or} \quad \alpha \cdot \|x - y\| = \alpha \cdot \|x + y\|, \quad \alpha \neq 0$$

$$\text{Thus} \quad \|x - y\| = \|x + y\|, \text{ or that } x \perp^i y.$$

Case 2. $\beta < \alpha$. We use induction over integers in the proof of this case.

Define the statement

$$P(n) : \|x - y\| \cdot \left\| x + \left(-\beta/\alpha\right)^n y \right\| = \|x + y\| \cdot \left\| x - \left(-\beta/\alpha\right)^n y \right\|, \quad n \in \mathbf{Z}.$$

Now, $(x \perp y)[\alpha, \beta]$ means

$$(1) \quad \|x - \alpha y\| \cdot \|x - \beta y\| = \|x + \alpha y\| \cdot \|x + \beta y\|.$$

By assumption, we have $(x \perp 1/\alpha y)[\alpha, \beta]$, which implies that

$$P(1) \quad \|x - y\| \cdot \|x - \beta/\alpha y\| = \|x + y\| \cdot \|x + \beta/\alpha y\|.$$

Suppose that $P(k)$ is true; that is

$$\|x - y\| \cdot \left\| x + \left(-\beta/\alpha\right)^k y \right\| = \|x + y\| \cdot \left\| x - \left(-\beta/\alpha\right)^k y \right\|.$$

But by assumption $(\lambda x \perp \mu y)[\alpha, \beta]$ for every real numbers λ and μ , then

$P(k)$ still true if y replaced by $\beta/\alpha y$, and hence

$$(*) \quad \left\| x - \beta/\alpha y \right\| \cdot \left\| x - \left(-\beta/\alpha\right)^{k+1} y \right\| = \left\| x + \beta/\alpha y \right\| \cdot \left\| x + \left(-\beta/\alpha\right)^{k+1} y \right\|$$

Now, divide $P(1)$ by $(*)$ we get

$$\frac{P(1)}{(*)} : \frac{\|x - y\|}{\left\| x - \left(-\beta/\alpha\right)^{k+1} y \right\|} = \frac{\|x + y\|}{\left\| x + \left(-\beta/\alpha\right)^{k+1} y \right\|}. \text{ Simplify,}$$

$$\|x - y\| \cdot \left\| x + \left(-\beta/\alpha\right)^{k+1} y \right\| = \|x + y\| \cdot \left\| x - \left(-\beta/\alpha\right)^{k+1} y \right\|$$

which is $P(k+1)$. Thus $P(n)$ is true for all integers n . Let $r = -\beta/\alpha$,

then $|r| < 1$, and $P(n)$ will be

$$\|x - y\| \cdot \left\| x + r^n y \right\| = \|x + y\| \cdot \left\| x - r^n y \right\|.$$

Taking limit as $n \rightarrow \infty$, and noting that the triangle inequality gives

$$\left| \|x\| - |r|^n \cdot \|y\| \right| \leq \left\| x \pm r^n y \right\| \leq \|x\| + |r|^n \cdot \|y\|, \text{ then}$$

$$\|x\| \leq \lim_{n \rightarrow \infty} \left\| x \pm r^n y \right\| \leq \|x\|, \text{ or } \lim_{n \rightarrow \infty} \left\| x \pm r^n y \right\| = \|x\|.$$

Thus $\lim_{n \rightarrow \infty} P(n) : \|x - y\| \cdot \|x\| = \|x + y\| \cdot \|x\|$ or

$$\|x - y\| = \|x + y\|, \text{ i.e. } x \perp^i y. \quad (x \neq 0)$$

COROLLARY III.11

If $[\alpha, \beta]$ – orthogonality is homogeneous and $(x \perp y)[\alpha, \beta]$ then $x \perp^i y$.

REMARK III.12

We feel that if $[\alpha, \beta]$ – orthogonality is homogeneous and $(x \perp y)[\alpha, \beta]$ then $x \perp^P y$.

Using these results we can prove the following.

CONJECTURE III.13

If $[\alpha, \beta]$ – orthogonality is homogeneous then the space X is strictly convex.

Proof. Assume that X is not strictly convex. Then there exist $x \neq y \in X$

such that $\|x\| = \|y\| = \left\| \frac{x+y}{2} \right\| = 1$.

We claim that $(\frac{x+y}{2} \perp y)[\alpha, \beta]$. For, otherwise, we have $\frac{x+y}{2} \perp^i y$

or that $\left\| \frac{x-y}{2} \right\| = \left\| \frac{x+3y}{2} \right\|$. But Lemma II.1 implies that $\left\| \frac{x+3y}{2} \right\| = 2$,

and so $\left\| \frac{x-y}{2} \right\| = 2$ which is impossible, since $\left\| \frac{x-y}{2} \right\| \leq \left\| \frac{x}{2} \right\| + \left\| \frac{y}{2} \right\| = 1$.

Hence there exists a nonzero real number a such that

$(\frac{x+y}{2} \perp a(\frac{x+y}{2}) + y)[\alpha, \beta]$. This, in turn, implies that

$\frac{x+y}{2} \perp^i a(\frac{x+y}{2}) + y$. That is

(1) $\left\| (1-a)\frac{x+y}{2} - y \right\| = \left\| (1+a)\frac{x+y}{2} + y \right\|$

$$\text{or} \quad \left\| \frac{1-a}{2}x - \frac{1+a}{2}y \right\| = \left\| \frac{1+a}{2}x + \frac{3+a}{2}y \right\|.$$

But $\|a\| \leq 1$. Hence equation (1) becomes

$$\left\| \frac{1-a}{2}x - \frac{1+a}{2}y \right\| = \frac{1+a}{2} + \frac{3+a}{2} = 2+a. \quad (\text{By Lemma II.1.})$$

$$\text{But} \quad \left\| \frac{1-a}{2}x - \frac{1+a}{2}y \right\| \leq \frac{1-a}{2} + \frac{1+a}{2} = 1. \quad \text{Thus } 2+a \leq 1, \text{ and}$$

hence $a = -1$. Then we have $\frac{x+y}{2} \perp \frac{x-y}{2}$ and Remark III.12 gives

$\frac{x+y}{2} \perp^p \frac{x-y}{2}$. That is

$$(2) \quad \|y\|^2 = \left\| \frac{x+y}{2} \right\|^2 + \left\| \frac{x-y}{2} \right\|^2$$

$$\text{or} \quad 1 = 1 + \left\| \frac{x-y}{2} \right\|^2$$

$$\text{or} \quad \|x-y\| = 0, \text{ i.e., } x = y.$$

Contradicting the assumption that $x \neq y$. Thus X is strictly convex.

§4. THE CHARACTERIZATION THEOREM

Throughout this section we try to get new characterizations of inner product spaces. First characterization obtained as a direct consequence of Theorem III.10.

THEOREM III.14

Let X be a normed linear space, and for $x, y \in X$, if $x \perp^B y$ implies that $(x \perp y)[\alpha, \beta]$, then X is an inner product space.

Proof. Let $x \perp^B y$. Then by assumption and the fact that $\lambda x \perp^B \mu y$ for all $\lambda, \mu \in \mathbf{R}$, we see that $(\lambda x \perp \mu y)[\alpha, \beta]$ for all $\lambda, \mu \in \mathbf{R}$, and hence by Theorem III.10, $x \perp^i y$. Thus $x \perp^B y$ implies $x \perp^i y$, which is sufficient for X to be an inner product space.

THEOREM III.15

Let X be a normed linear space, and α, β be any nonnegative real numbers (not both zero) such that either

(i) $\alpha = 0$ and $\beta \neq 1$

or

(ii) $\beta = 0$ and $\alpha \neq 1$

or

(iii) $\alpha = \beta$ and $\alpha \neq 1$. Then the following statement is valid.

If $x \perp^i y \Rightarrow (x \perp y)[\alpha, \beta]$ for $x, y \in X$, then X is an inner product space.

Proof.

(i) If $\alpha = 0$ and $\beta \neq 1$, then for $x, y \in X$ we have

$$\begin{aligned}
 (x \perp y)[0, \beta] &\Leftrightarrow \|x\| \cdot \|x - \beta y\| = \|x\| \cdot \|x + \beta y\| \\
 &\Leftrightarrow \|x - \beta y\| = \|x + \beta y\| \quad (x \neq 0) \\
 &\Leftrightarrow x \perp^i \beta y.
 \end{aligned}$$

Thus by hypothesis we have $x \perp^i y$ implies $x \perp^i \beta y$, and by Lorch L1 (see page 1), X is an inner product space.

(ii) Similarly, if $\beta = 0$ and $\alpha \neq 1$, we have $(x \perp y)[\alpha, 0] \Leftrightarrow x \perp^i \alpha y$, so X is an inner product space.

(iii) If $\alpha = \beta$ and $\alpha \neq 1$, then for $x, y \in X$, $(x \perp y)[\alpha, \alpha] \Leftrightarrow x \perp^i \alpha y$, and again X is an inner product space.

THEOREM III.16

If $[\alpha, \beta]$ – orthogonality is homogeneous, then for every element x of the space X there can be found a nonzero element orthogonal to x by Roberts' definition in each two - dimensional subspace containing x . Hence X is an inner product space by Corollary I.1.

Proof. Let $x \in X$ and M be any two - dimensional subspace of the space X containing x . Easily we can prove existence of a nonzero element z of M , which is linearly independent of x . Now using Theorem III.6 we get $(x \perp ax + z)[\alpha, \beta]$ for some real number a . Let $y = ax + z$, then $y \neq 0$, since otherwise we get $z = -ax$ which means that x and z are linearly dependent, a contradiction. Furthermore, since y is a linear combination of two elements of M , then $y \in M$. Therefore, $y(\neq 0) \in M$ and $(x \perp y)[\alpha, \beta]$. But homogeneity of $[\alpha, \beta]$ – orthogonality implies that $(x \perp ry)[\alpha, \beta]$, $r \in \mathbf{R}$,

then Theorem III.10 gives that $x \perp^i ry$ for every $r \in \mathbf{R}$ or that $x \perp^R y$. This completes the proof.

We need the following lemmas for the next theorem.

LEMMA III.17

If $[\alpha, \beta]$ – orthogonality is additive, then it is homogeneous.

Proof. Say $[\alpha, \beta]$ – orthogonality is additive (left and right), and $(x \perp y)[\alpha, \beta]$.

Then using property (I); that is $(x \perp - y)[\alpha, \beta]$ and follow same procedure used in the proof of Theorem I.8 to get that $(\lambda x \perp \mu y)[\alpha, \beta]$ for all λ and $\mu \in \mathbf{R}$.

LEMMA III.18

If $[\alpha, \beta]$ – orthogonality is symmetric and left or right – additive, then it is homogeneous.

Proof. If $[\alpha, \beta]$ – orthogonality is symmetric and left or right – additive, then it is additive, and hence by Lemma III.17 $[\alpha, \beta]$ – orthogonality is homogeneous.

Now we summarize the main results in the following theorem :

THEOREM III.19

If X is a real normed linear space, then the following are equivalent :

- (i) $[\alpha, \beta]$ – orthogonality is symmetric and left- additive;
- (ii) $[\alpha, \beta]$ – orthogonality is symmetric and right- additive;
- (iii) $[\alpha, \beta]$ – orthogonality is additive;
- (iv) $[\alpha, \beta]$ – orthogonality is homogeneous;
- (v) X is an inner product space .

§5 SET OF OPEN PROBLEMS RELATED TO $[\alpha, \beta]$ -ORTHOGONALITY

Since we define a new orthogonality relation, $[\alpha, \beta]$ – orthogonality, obviously we left with some open problems.

Among of them some of the most interesting are the following ones :

Problem 1. It is known that if $[\alpha, \beta]$ – orthogonality is homogeneous, then it implies isosceles orthogonality (Theorem III.10). However we can not prove a similar result with pythagorean orthogonality (Remark III.12).

Problem 2. Since Remark III.12 is just a conjecture, the next result is an open problem. That is proving the space X is strictly convex if $[\alpha, \beta]$ – orthogonality is homogeneous, without refering to Remark III.12.

Problem 3. Discussing uniqueness of $[\alpha, \beta]$ – orthogonality when $\alpha \neq \beta$.

Beside these open problems, still we got good results expressed in relation with $[\alpha, \beta]$ – orthogonality, and I hope that I succeed in presenting these results.

SUMMARY OF CHAPTER III

Let α, β be fixed nonnegative real numbers (not both zero) and x and $y \in X$ (a normed linear space), then $(x \perp y) [\alpha, \beta] \Leftrightarrow \ x - \alpha y\ \cdot \ x - \beta y\ = \ x + \alpha y\ \cdot \ x + \beta y\ .$					
Properties of $[\alpha, \beta]$ – orthogonality					
Nondegeneracy	Symmetry	Additivity	Homogeneity	Existence of orthogonal elements	Uniqueness
always	In general, it is not symmetric.	Only in inner product spaces.	Only in inner product spaces.	$\forall x (\neq 0)$ and $y \in X$ $\exists a, b \in \mathbf{R}$ such that $(bx + y \perp x) [\alpha, \beta]$ and $(x \perp ax + y) [\alpha, \beta].$	For $\alpha = \beta$ and X be s.c. then there exist unique real numbers a, b such that $(bx + y \perp x) [\alpha, \beta]$ and $(x \perp ax + y) [\alpha, \beta]$ for every $x (\neq 0)$ and $y \in X$.

- s. c. : strictly convex space.

REFERENCES

REFERENCES

1. C.R.Diminnie, R.W.Freese, and E.Z.Andalafte, (1985) " (α, β) -orthogonality and a characterization of inner product spaces", Math. Japonica **30**, No.3, pp. 341-349.
2. C.R.Diminnie, R.W.Freese, and E.Z.Andalafte, (1983) "*An extension of pythagorean and isosceles orthogonality and a characterization of inner product spaces*", Journal of Approximation Theory **39**, pp. 295-298.
3. Dan Amir, (1986) "*Characterizations of Inner Product Spaces*", Birkhäuser, Boston.
4. E.R.Lorch, (1948) "*On certain implications which characterize Hilbert space*", Ann. of Math. **49**, No. 2, pp. 523-532.
5. Erwin Kreyszig, (1978) "*Introductory Functional Analysis with Applications*", Canada, John Wiley & Sons. Inc.
6. F.A.Ficken, (1944) "*Note on the existence of scalar products in normed linear spaces*", Annals of Mathematics **45**, pp. 362-366.
7. G.Birkhoff, (1935) "*Orthogonality in linear metric spaces*", Duke Math. J. **1** pp. 169-172.
8. Javier Alonso and Carlos Benitez, (1988) "*Orthogonality in normed linear spaces. A survey. Part I: Main properties*", Extracta Mathematicae **3**, No.1, pp. 1-15.
9. Javier Alonso and Carlos Benitez, (1988) "*Orthogonality in normed linear spaces. A survey. Part II: Relations between main orthogonalities*", Extracta Mathematicae **4** No.3, pp. 121-131.
10. Javier Alonso, (1994) "*Uniqueness properties of isosceles orthogonality in normed linear spaces*", Ann. Sci. Math. Québec **18**, No.1, pp. 25-38.
11. M.A.Al-Abbas, (1988) "*Orthogonality in normed linear spaces and introduction to quasi- semi-inner product (r,p) spaces*", (Msc Thesis).
12. M.M.Day, (1947) "*Some characterizations of inner product spaces*", Trans.Amer.Math.Soc. **62**, pp. 320-337.

13. M.M.Day, (1973) "*Normed Linear Spaces*", 3rd edition (Ergebnisse der Mathematik und ihrer Grenzgebiete, 21. Springer-Verlag, Berlin, Heidelberg, New York).
14. M.M.Day, (1959) "*On criteria of Kasahara and Blumenthal for inner-product spaces*", Proc. Amer. Math. Soc. **10**, pp. 92-100.
15. O.P.Kapoor and Jagadish Prasad, (1978) "*Orthogonality and characterizations of inner product spaces*", Bull. Austral. Math. Soc. **19**, pp. 403-416.
16. P.Jordan and J.von Neumann, (1935) "*Inner products in linear metric spaces*", Ann. of Math. **36**, pp. 291-302.
17. R.C.James, (1947) "*Inner products in normed linear spaces*", Bull. Amer. Math. Soc. **53**, pp. 559-566.
18. R.C.James, (1945) "*Orthogonality in normed linear spaces*", Duke Math. J. **12**, pp. 291-302.
19. R.C.James, (1947) "*Orthogonality and linear functionals in normed linear spaces*", Trans. Amer. Math. Soc. **61**, pp. 265-292.
20. S.Axler, F.W.Gehring and K.A.Ribet, (1998) "*An introduction to Banach space theory*", Graduate texts in Mathematics **183**, Editorial Board.
21. Shizuo Kakutani, (1940) "*Some characterizations of Euclidean space*", Jap. J. Math. **16**, pp. 93-97.

طرق تمييز فضاءات حاصل الضرب الداخلي باستخدام علاقات التعامد

فضاء حاصل الضرب الداخلي (inner product space) هو فضاء متجهي (vector space) X يُعرف عليه حاصل ضرب داخلي . و هنا حاصل الضرب الداخلي ليس إلا دالة معرفة على المجموعة $X \times X$ ، حيث تُعين لكل زوج مرتب (x, y) عدد حقيقي يُرمز له بالرمز $\langle x, y \rangle$ و يسمى حاصل الضرب الداخلي للمتجهين x و y ، كما يتميز حاصل الضرب الداخلي بالخواص التالية :-

(IP1) حاصل الضرب الداخلي تبديلي (symmetric)

(IP2) حاصل الضرب الداخلي تجميعي (additive)

(IP3) حاصل الضرب الداخلي متجانس (homogeneous)

(IP4) حاصل الضرب الداخلي لمتجه في نفسه دائماً أكبر من الصفر ، و يساوي صفراً إذا كان المتجه هو المتجه الصفري .

فضاءات حاصل الضرب الداخلي (inner product spaces) تمثل حالاتاً خاصة من الفضاءات القياسية (normed linear spaces) ، حيث يمكن تعريف طول المتجه x بالعلاقة التالية :

$$\|x\| = \sqrt{\langle x, x \rangle}$$

إلا أن العكس ليس دائماً صحيح ، بمعنى ليس كل فضاء قياسي (normed linear space) هو فضاء حاصل ضرب داخلي (inner product space) . و حيث إن فضاءات حاصل الضرب الداخلي لما تتمتع به من خواص تلعب دوراً هاماً في تطبيقات تحليل الدوال (Functional Analysis) ، فإن تمييز هذه الفضاءات في غاية الأهمية . و قد شغلت هذه المشكلة كثيراً من الباحثين ، فتعددت نتيجة لذلك طرق تمييز فضاءات حاصل الضرب الداخلي و من ضمنها استخدام علاقات التعامد (orthogonality relations) .

علاقة التعامد هنا هي علاقة معرفة على $X \times X$ ، حيث يُقال لمتجهين x و y أنهما متعامدان ويُرمز لهما بالرمز $x \perp y$ إذا كان طوليهما يحققان شروطاً معينة . و إذا كان X يُمثل فضاءً لحاصل الضرب الداخلي عندها يتعامد المتجهان في X إذا كان حاصل الضرب الداخلي لهما يساوي صفراً ،

بمعنى

$$x \perp y \Leftrightarrow \langle x, y \rangle = 0$$

تُحقق علاقة التعامد المعرفة على فضاء حاصل الضرب الداخلي (inner product space)

(الخواص التالية :

● لا يمكن أن يكون المتجه عمودي على نفسه عدا المتجه الصفري

(nondegeneracy property)

● علاقة التعامد علاقة تبديلية (symmetric)

● علاقة التعامد علاقة تجميعية (additive)

● علاقة التعامد علاقة متجانسة (homogeneous)

● دائماً يمكننا إيجاد متجهات متعامدة (*resolvability property*)

إلا أن هذه الصفات ليست متحققة دائماً لكل علاقة تعامد تُعرف على فضاء قياسي (normed linear space) .

تبعاً لمفهوم التعامد المذكور سابقاً فإن هناك عدد كبير من علاقات التعامد التي يمكن تعريفها على فضاءات قياسية (normed linear spaces) ولا تستدعي بالضرورة وجود حاصل الضرب الداخلي . حتى الآن لا توجد دراسة تضم جميع هذه التعريفات نظراً لكثرتها ، لذا فإننا سنكتفي بدراسة خمسة أنواع من علاقات التعامد لتمثل الخلفية العلمية للبحث . هذه العلاقات هي :

$$\text{Isosceles orthogonality} \quad x \perp^i y \Leftrightarrow \|x - y\| = \|x + y\| .$$

$$\text{Pythagorean orthogonality} \quad x \perp^P y \Leftrightarrow \|x - y\|^2 = \|x\|^2 + \|y\|^2 .$$

$$\text{Birkhoff orthogonality} \quad x \perp^B y \Leftrightarrow \|x\| \leq \|x + ky\| \quad \forall \quad k \in \mathbf{R} .$$

α - orthogonality ($\alpha \neq 1$)

$$(x \perp y)(\alpha) \Leftrightarrow (1 + \alpha^2) \|x - y\|^2 = \|x - \alpha y\|^2 + \|y - \alpha x\|^2 .$$

(α, β) - orthogonality ($\alpha, \beta \neq 1$)

$$(x \perp y)(\alpha, \beta) \Leftrightarrow \|x - y\|^2 + \|\alpha x - \beta y\|^2 = \|x - \beta y\|^2 + \|\alpha x - y\|^2 .$$

الطريقة المثلى لدراسة هذه العلاقات المعرفة على فضاء قياسي (normed linear space) X ، تكون من دراسة أياً من الخواص السابقة متحققة بالنسبة لعلاقة التعامد ، ثم إثبات أن بعض الخواص المتبقية كافية لجعل الفضاء X فضاءً لحاصل الضرب الداخلي . الوحدة الأولى من البحث ستختص بهذه الدراسة .

بينما في الوحدة الثانية سنقوم بإثبات أن التكافؤ على الغالب بين أي علاقتي تعامد في X يعني أن X هو فضاء حاصل ضرب داخلي .

وأخيراً في الوحدة الثالثة، سنعرف علاقة تعامد جديدة على فضاء قياسي (normed linear space) X نسميها $[\alpha, \beta]$ - orthogonality حيث α, β عددان حقيقيان ليس كلاهما صفراً ولهما نفس الإشارة و حيث يكون المتجهان x, y متعمدان بالنسبة لهذه العلاقة إذا حققا الشرط التالي :

$$\|x - \alpha y\| \cdot \|x - \beta y\| = \|x + \alpha y\| \cdot \|x + \beta y\| .$$