

UNIVERSITY OF BAHRAIN

**UNIFORMLY DOMINATED SETS OF P-SUMMING
OPERATORS**

BY

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**A THESIS SUBMITTED TO THE DEPARTMENT OF
MATHEMATICS**

AS

**A PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE**

OF

MASTER OF SCIENCE

**UNDER THE SUPERVISION OF
PROFESSOR ROSHDI KHALIL**

MARCH 1996



9 June 1996

The Thesis Defence Committee considers the Thesis titled:

"Uniformly Dominated Sets of P-summing Operators"

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ACKNOWLEDGEMENTS

This study can not be accomplished without great mercy and invaluable support from Allah . So, I must express my sincere thanks and record my gratitude to him.

I am very grateful to my parents, may God bless them, for their forbearance and encouragement, for their sacrifices and endless support, and for their kindness and patience.

My special gratitude is extended to my wife who has given me encouragement and moral support throughout our years together, especially during the preparation of the master program.

My deepest appreciation and thanks is due to my supervisor, Professor Roshdi Khalil. I would like to express my great debt for his kind guidance, invaluable support, and helpful suggestions. Again a special word of thanks for him for fruitful conversations and due to his generosity and patience during this study.

INTRODUCTION

Operator theory has become a sufficiently large area of functional analysis of mathematics. One of the most interested and well-known part of this area is the so-called the theory of operator ideals on Banach spaces. It emerged as a by-product of the theory of tensor products that was produced in the period 1950-1956 by Schatten, Ruston, and Grothendieck. However certain ideals like finite , approximable , and compact operators had been investigated and studied much earlier .

In the period from 1965 to 1978 , the theory of operator ideals has grown rapidly and it was modified by many mathematicians . An extensive treatment was given by A. Pietsch in his monograph [27] .

The thesis has two major objectives . The first is to cover the basic theory of tensor products and to study some advanced level material of some operator ideals. The main operator ideals that we discussed in details are those of absolutely p -summing operators with $1 \leq p < \infty$. Of course , the selection of these ideals reflects our interests and illustrates the direction of the thesis . At the same time , we attempted to give a sufficient number of examples to give the reader a feel for the subject .

The second objective is to give a further study , which is an original work , in the absolutely p -summing operators . We study what is called uniformly dominated sets of p -summing operators.

The thesis consists of three Chapters .

Chapter one discusses the theory of tensor products and studies some sequence and function spaces that we need in the subsequent chapters . The distinction between the injective and projective tensor products is carefully expressed and some basic properties of them are also treated .

Chapter two introduces special operator ideals. Namely those operators that are compact, weakly compact , p -nuclear , completely continuous , and absolutely p -summing operators . The chapter mainly studies the absolutely p -summing operators and treats their algebraic properties and introduces some useful characterizations . It presents some basic examples and discusses the duality of these operators in the finite dimensional case . At the end of this chapter , there is an application of the p -summing operators to the theory of Hilbert spaces .

Chapter three creates a further study on certain types of subsets of the absolutely p -summing operators . These subsets are called , in our study , the uniformly , summability , sequentially , totally sequentially , finitely , and finitely pointwise dominated subsets . We have tried to include many examples . Further properties and some implications of these concepts are discussed also .

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22.3.1996

NOTATIONS

$\ f\ _{\alpha(p)}$	The norm of p-Bochner's integrable functions .
a.e	Almost everywhere .
$B_1(X)$	The closed unit ball of a Banach space X .
$\mathbb{C}$	The field of complex numbers .
card M	The cardinal number of a set M .
$C(K)$	Space of continuous functions on K .
$C(K, Y)$	Space of continuous vector valued functions .
$C_2(X, Y)$	Space of Hilbert-Schmidt operators from X into Y .
$\mathcal{C}(X, Y)$	Space of completely continuous operators from X into Y .
c_0	Space of sequences converging to zero .
$\dim X$	Dimension of a space X .
δ_x	The point mass measure at x .
$E \times K$	The cross product of E and K .
(e_n)	The standard basis of ℓ^2 .
$E(P, \lambda, \mu)$	The largest λ -uniformly dominated subset determined by μ which contains E .
$E\{p, \lambda, (x_n^*)\}$	The largest λ -summability dominated subset determined by (x_n^*) which contains E .
$E \oplus Z$	The algebraic sum of two subspaces E and Z .
$\ f\ _{\varepsilon(p)}$	The norm of p-Pettis integrable functions .
$\mathfrak{A}(X, Y)$	Space of finite rank operators from X into Y .
$\ \cdot\ _{\vee}$	The injective norm .
(I, μ)	The unit interval with the Lebesgue measure .
$I_p(X, Y)$	Space of p-integrable operators from X into Y .
I_X	The identity operator on X .
J_p	The canonical (inclusion) map from $C(K)$ into $L^p(K, \mu)$.

J_Y	The canonical embedding of Y into Y^* .
$\mathbf{K}$	Scalar field of either the complex numbers or the real numbers.
$K(X, Y)$	Space of compact operators from X into Y .
$\ker(T)$	The kernel of an operator T .
$L(X, Y)$	Space of bounded linear operators from X into Y .
$L^p(I, X)$	Space of p -Bochner's integrable functions from (I, μ) into X .
$\mathcal{L}^p(I, X)$	Space of p -Pettis integrable functions from (I, μ) into X .
$L^p(I, X)$	The completion of $\mathcal{L}^p(I, X)$.
$L^p(\Omega, \mu)$	Space of p -integrable functions on a measure space (Ω, μ) .
$L^p(I)$	The space $L^p(I, \mu)$.
$L^\infty(\Omega, \mu)$	Space of essentially bounded functions on a measure space (Ω, μ) .
$L^\infty(I)$	The space $L^\infty(I, \mu)$.
$\ell^p(X)$	Space of absolutely p -summable sequences in X .
$\ell^p\langle X \rangle$	Space of weakly p -summable sequences in X .
ℓ^p	The space $\ell^p(\mathbf{K})$.
ℓ_n^p	The n -dimensional Banach space with the p -norm.
ℓ^∞	Space of bounded sequences in $\mathbf{K}$.
$M(K)$	Space of regular Borel measures on K .
$\mathbf{N}$	Set of natural numbers.
$N_p(X, Y)$	Space of p -nuclear operators from X into Y .
$\ \cdot\ _{n(p)}$	The p -nuclear norm.
$\Pi_p(X, Y)$	Space of p -summing operators from X into Y .
$\ \cdot\ _p$	The p -norm of $L^p(\Omega, \mu)$.
$\ \cdot\ _{\pi(p)}$	The p -summing norm.
$\ \cdot\ _\wedge$	The projective norm.

$\mathbf{R}$	The field of real numbers .
$\mathfrak{R}$	An arbitrary operator ideal .
$S(I, X)$	Space of simple functions from (I, μ) into X .
$\ \cdot \ _{\sigma(2)}$	The norm of Hilbert-Schmidt operators .
$W(X, Y)$	Space of weakly compact operators from X into Y .
X^*	The dual space of a Banach space X .
$X \otimes Y$	The tensor product of X with Y .
$X \overset{\vee}{\otimes} Y$	The injective tensor product .
$X \overset{\wedge}{\otimes} Y$	The projective tensor product .
$x \otimes y$	An atom in $X \otimes Y$.
x^*	A linear functional on X .
$\langle x, x^* \rangle$	The evaluation map .
$\ (x_n)\ _{\alpha(p)}$	The norm in $\ell^p(X)$.
$\ (x_n)\ _{\varepsilon(p)}$	The norm in $\ell^p\langle X \rangle$.
1_E	The characterestic function of a set E .
$[x]$	$\text{span}\{x\}$.
$\ z^*\ _{\alpha d}$	The norm of $X^* \otimes Y^*$ as a subspace of $(X \otimes Y, \ \cdot \ _{\alpha})^*$.

CHAPTER ONE

TENSOR PRODUCTS OF BANACH SPACES

Introduction: In this chapter we introduce the basic material that we will use in our work . The chapter is divided into four sections . Section 1 deals with the basic properties of the tensor product of Banach spaces . In section 2 we give the definition of the injective tensor product, basic results , and a basic example . In section 3 we present some function and sequence spaces . Some basic properties and relationships of such spaces are discussed . In section 4 we state the definition of the projective tensor product and some well-known facts .

1.1. Tensor Products :

Throughout this section , let X and Y be Banach spaces and let X^* and Y^* denote their respective duals .

For $x \in X$ and $y \in Y$ define

$$x \otimes y : X^* \rightarrow Y ,$$
$$(x \otimes y)(x^*) := \langle x, x^* \rangle y .$$

Clearly $x \otimes y$ is a bounded linear operator from X^* into Y . Let

$$K := \{ x \otimes y : x \in X \text{ and } y \in Y \} \subseteq L(X^*, Y) .$$

The span of K in $L(X^*, Y)$ is a subspace of $L(X^*, Y)$. We shall write $X \otimes Y$ for the span of K . $X \otimes Y$ is called the *tensor product* of X and Y .

Thus $X \otimes Y$ is just a linear subspace of $L(X^*, Y)$ which contains finite rank operators. We shall call $x \otimes y$ an *atom* in $X \otimes Y$.

Now we list some elementary properties of $X \otimes Y$:

$$(T_1) \quad \lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y) \text{ for all scalars } \lambda.$$

$$(T_2) \quad (x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y.$$

$$(T_3) \quad x \otimes 0 = 0 \otimes y \equiv 0, \text{ the zero operator.}$$

$$(T_4) \quad \text{Each } z \text{ in } X \otimes Y \text{ can be written in the form}$$

$$z = \sum_{i=1}^n \lambda_i a_i \otimes b_i \text{ with } \|a_i\| = \|b_i\| = 1.$$

$$\text{Indeed : If } z = \sum_{i=1}^n x_i \otimes y_i \text{ then}$$

$$z = \sum_{i=1}^n \|x_i\| \left(\frac{x_i}{\|x_i\|} \right) \otimes \|y_i\| \left(\frac{y_i}{\|y_i\|} \right) = \sum_{i=1}^n \lambda_i a_i \otimes b_i$$

$$\text{where } \lambda_i = \|x_i\| \cdot \|y_i\|, \quad a_i = \frac{x_i}{\|x_i\|}, \text{ and } b_i = \frac{y_i}{\|y_i\|}.$$

Theorem 1.1.1 : $\mathfrak{I}(X, Y) = X^* \otimes Y$.

Proof : Let $A \in \mathfrak{I}(X, Y)$ and let $\{b_1, \dots, b_n\}$ be a basis of the range of A . For each $i = 1, \dots, n$ define $y_i^* \in (A(X))^*$ by

$$y_i^*(b_i) := 1 \text{ and } y_i^*(b_j) := 0 \text{ if } i \neq j.$$

Set $x_i^* := y_i^* \circ A$ for $i = 1, \dots, n$.

We claim that $A = \sum_{i=1}^n x_i^* \otimes b_i$. To do this , let $x \in X$. Then we have

$$Ax = \sum_{j=1}^n \alpha_j b_j \quad \text{with} \quad \alpha_1, \dots, \alpha_n \text{ are scalars .}$$

Now

$$\begin{aligned} \left(\sum_{i=1}^n x_i^* \otimes b_i \right) (x) &= \sum_{i=1}^n \langle x, x_i^* \rangle b_i \\ &= \sum_{i=1}^n \langle x, y_i^* \circ A \rangle b_i \\ &= \sum_{i=1}^n y_i^* (Ax) b_i \\ &= \sum_{i=1}^n \left[y_i^* \left(\sum_{j=1}^n \alpha_j b_j \right) \right] b_i \\ &= \sum_{i=1}^n \sum_{j=1}^n \alpha_j y_i^* (b_j) b_i \\ &= \sum_{i=1}^n \alpha_i b_i \\ &= Ax \quad . \end{aligned}$$

This concludes the claim .

The converse is obviously true . □

The following consequence is immediate and the proof is omitted.

Corollary 1.1.2 : *If X is reflexive , then $X \otimes Y = \mathfrak{F}(X^*, Y)$.*

Theorem 1.1.3 : Every $z := \sum_{i=1}^n x_i \otimes y_i$ in $X \otimes Y$ is equivalent to either $0 \otimes 0$ or has a representation $z = \sum_{i=1}^n a_i \otimes b_i$ where $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are linearly independent sets .

Proof : Assume that $x_1, \dots, x_n$ are dependent . With no loss of generality , we may assume $x_n = \sum_{i=1}^{n-1} \alpha_i x_i$. Then

$$z = \sum_{i=1}^{n-1} x_i \otimes y_i + \left(\sum_{i=1}^{n-1} \alpha_i x_i \right) \otimes y_n .$$

Applying (T_2) then (T_1) we obtain

$$\left(\sum_{i=1}^{n-1} \alpha_i x_i \right) \otimes y_n = \sum_{i=1}^{n-1} x_i \otimes \alpha_i y_n .$$

Hence it follows from (T_2) that $z = \sum_{i=1}^{n-1} x_i \otimes (y_i + \alpha_i y_n)$. Consequently ,

$$z = \sum_{i=1}^{n-1} a_i \otimes y_i \quad \text{with } a_i = x_i \quad \text{and } b_i = y_i + \alpha_i y_n .$$

We may repeat this process until we arrive at either a representation in which $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are linearly independent or a representation of the form $x \otimes 0$ or $0 \otimes y$. In either case it is equivalent to $0 \otimes 0$. This ends the proof . $\square$

Theorem 1.1.4 : Let $\{y_1, \dots, y_m\}$ be a linearly independent set in Y . If $\{x_1, \dots, x_m\}$ is any set of elements in X such that

$$z = \sum_{i=1}^m x_i \otimes y_i = 0 \quad \text{in } X \otimes Y, \text{ then } x_i = 0 \text{ for } i = 1, \dots, m.$$

Proof : Since for any $x^* \in X^*$,

$$\left(\sum_{i=1}^m x_i \otimes y_i \right) (x^*) = \sum_{i=1}^m \langle x_i, x^* \rangle y_i = 0$$

and since $y_1, \dots, y_m$ are linearly independent, we have

$$\langle x_i, x^* \rangle = 0 \quad \text{for all } x^* \in X^* \text{ and } i = 1, \dots, m.$$

Hence by the Hahn Banach Theorem $x_i = 0$ for $i = 1, \dots, m$. □

Definition 1.1.5 : A norm $\| \cdot \|$ on $X \otimes Y$ is called a **cross norm** if

$$\|x \otimes y\| = \|x\| \|y\| \text{ for all } x \in X \text{ and } y \in Y.$$

Consider the tensor product $X^* \otimes Y^*$. We may regard each $z \in X^* \otimes Y^*$ as a linear form on $X \otimes Y$ in the following sense :

For $x^* \in X^*$ and $y^* \in Y^*$,

$$(x^* \otimes y^*)(x \otimes y) := \langle (x \otimes y)(x^*), y^* \rangle = \langle x, x^* \rangle \langle y, y^* \rangle .$$

Clearly $x^* \otimes y^*$ is linear . Furthermore if $\| \cdot \|_\alpha$ is a cross norm on $X \otimes Y$ with X and Y are finite dimensional , then for any $x \in X$ and $y \in Y$,

$$\begin{aligned} |\langle x, x^* \rangle \langle y, y^* \rangle| &= |\langle x, x^* \rangle| |\langle y, y^* \rangle| \\ &\leq \|x^*\| \|y^*\| \|x\| \|y\| \\ &\leq \|x^*\| \|y^*\| \|x \otimes y\|_\alpha . \end{aligned}$$

Thus $x^* \otimes y^* \in (X \otimes Y, \| \cdot \|_\alpha)^*$.

For $z^* := \sum_{i=1}^n x_i^* \otimes y_i^*$ in $X^* \otimes Y^*$ we define

$$\|z^*\|_{\alpha d} := \sup \left\{ \left| \left(\sum_{i=1}^n x_i^* \otimes y_i^* \right)(z) \right| : \|z\|_\alpha = 1 \text{ and } z \in X \otimes Y \right\} .$$

This proves :

Proposition 1.1.6 : Let $\| \cdot \|_\alpha$ be a cross norm on $X \otimes Y$. If X and Y are finite dimensional spaces , then $X^* \otimes Y^*$ is a linear subspace of $(X \otimes Y, \| \cdot \|_\alpha)^*$.

From now on , $\| \cdot \|_{\alpha d}$ is the norm of $X^* \otimes Y^*$ as a subspace of $(X \otimes Y, \| \cdot \|_\alpha)^*$.

Definition 1.1.7 : A norm $\| \cdot \|$ on $X \otimes Y$ is said to be a *reasonable cross norm* if

$$(R_1) \quad \|x \otimes y\| = \|x\| \|y\| \quad \text{for all } x \in X, y \in Y, \text{ and}$$

$$(R_2) \quad \|x^* \otimes y^*\|_d = \|x^*\| \|y^*\| \quad \text{for all } x^* \in X^* \text{ and } y^* \in Y^*.$$

Condition (R_2) means that $x^* \otimes y^*$ is a bounded linear functional on $(X \otimes Y, \|\cdot\|)$ and has norm equal to $\|x^*\| \|y^*\|$.

Theorem 1.1.8 : A norm $\|\cdot\|$ on $X \otimes Y$ is a reasonable cross norm if and only if

$$(i) \quad \|x \otimes y\| \leq \|x\| \|y\| \quad \text{for all } x \in X, y \in Y, \text{ and}$$

$$(ii) \quad \|x^* \otimes y^*\|_d \leq \|x^*\| \|y^*\| \quad \text{for all } x^* \in X^* \text{ and } y^* \in Y^*.$$

Proof : If $\|\cdot\|$ is a reasonable cross norm, then (i) and (ii) are satisfied trivially.

Conversely, let (i) and (ii) be satisfied. For $x \in X, y \in Y, x^* \in X^*$ and $y^* \in Y^*$ one has

$$\begin{aligned} \left| \langle x, x^* \rangle \langle y, y^* \rangle \right| &= \left| (x^* \otimes y^*)(x \otimes y) \right| \\ &\leq \|x^* \otimes y^*\|_d \|x \otimes y\| \end{aligned} \quad (*)$$

It follows from condition (i) that

$$\left| \langle x, x^* \rangle \langle y, y^* \rangle \right| \leq \|x^* \otimes y^*\|_d \|x\| \|y\|.$$

Taking supremum over $\|x\| = \|y\| = 1$ we obtain

$$\|x^*\| \|y^*\| \leq \|x^* \otimes y^*\|_d.$$

Similarly by using condition (ii) in (*) , then taking supremum over

$$\|x^*\| = \|y^*\| = 1 \text{ we get } \|x\| \|y\| \leq \|x \otimes y\| .$$

Hence $\| \cdot \|$ is a reasonable cross norm . $\square$

Theorem 1.1.9 : *If $\| \cdot \|$ is a reasonable cross norm on $X \otimes Y$, then $\| \cdot \|_d$ is a reasonable cross norm on $X^* \otimes Y^*$.*

Proof : Since $\| \cdot \|$ is a reasonable cross norm , $\| \cdot \|_d$ is a cross norm on $X^* \otimes Y^*$. Hence

$$\|x^* \otimes y^*\|_d \leq \|x^*\| \|y^*\| \text{ for all } x^* \in X^* \text{ and } y^* \in Y^* .$$

In order to show that $\| \cdot \|_d$ is a reasonable cross norm , take $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$. By Theorem 1.1.8 we need only to show that

$$\|x^{**} \otimes y^{**}\|_{dd} \leq \|x^{**}\| \|y^{**}\| .$$

By Goldstine Theorem [36,p 31] there exist nets (x_β) and (y_γ) in X and Y respectively such that x^{**} and y^{**} are the weak* limits of (x_β) and (y_γ) respectively . Further

$$\|x_\beta\| \leq \|x^{**}\| \text{ and } \|y_\gamma\| \leq \|y^{**}\| \text{ for all } \beta \text{ and } \gamma . \quad (*)$$

Let $f := \sum_{i=1}^n x_i^* \otimes y_i^*$ in $X^* \otimes Y^*$ be arbitrary . Since f is a continuous

linear functional on $X \otimes Y$, using (*) we get :

$$\begin{aligned}
\left| f(x_\beta \otimes y_\gamma) \right| &\leq \|f\|_d \|x_\beta \otimes y_\gamma\| \\
&= \|f\|_d \|x_\beta\| \|y_\gamma\| \quad \text{for all } \beta, \gamma. \\
&\leq \|f\|_d \|x^{**}\| \|y^{**}\|.
\end{aligned}$$

Thus

$$\left| \sum_{i=1}^n \langle x_\beta, x_i^* \rangle \langle y_\gamma, y_i^* \rangle \right| \leq \|f\|_d \|x^{**}\| \|y^{**}\|.$$

By taking the limit over β and γ we obtain

$$\left| \sum_{i=1}^n \langle x^{**}, x_i^* \rangle \langle y^{**}, y_i^* \rangle \right| \leq \|f\|_d \|x^{**}\| \|y^{**}\|.$$

This means that

$$\left| (x^{**} \otimes y^{**})(f) \right| \leq \|x^{**}\| \|y^{**}\| \|f\|_d.$$

Hence $\|x^{**} \otimes y^{**}\| \leq \|x^{**}\| \|y^{**}\|$. This completes the proof. $\square$

Theorem 1.1.10 : Suppose that $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are two norms on $X \otimes Y$ with $\|\cdot\|_\alpha \geq \|\cdot\|_\beta$. Then $\|\cdot\|_{\alpha d} \leq \|\cdot\|_{\beta d}$ on $X^* \otimes Y^*$.

Proof : Let $f \in X^* \otimes Y^*$. Since $\|\cdot\|_\alpha \geq \|\cdot\|_\beta$ on $X \otimes Y$, it follows that

$$\left\{ f(z) : z \in X \otimes Y, \|z\|_\alpha \leq 1 \right\} \subseteq \left\{ f(z) : z \in X \otimes Y, \|z\|_\beta \leq 1 \right\}.$$

Consequently, $\|f\|_{\alpha d} \leq \|f\|_{\beta d}$ which concludes the result. $\square$

1.2. The Injective Tensor Product :

There are many cross norms that one can put on $X \otimes Y$. One of the most important of these norms, is the so-called, the *injective* norm which is defined as follows :

For $z := \sum_{i=1}^n x_i \otimes y_i$ in $X \otimes Y$:

$$\|z\|_{\vee} := \sup \left\{ \left| \sum_{i=1}^n \langle x_i, x^* \rangle \langle y_i, y^* \rangle \right| : x^* \in B_1(X^*) \text{ and } y^* \in B_1(Y^*) \right\}.$$

So $\|\cdot\|_{\vee}$ is just the operator norm on $L(X^*, Y)$.

Theorem 1.2.1 : *The norm $\|\cdot\|_{\vee}$ is a reasonable cross norm on $X \otimes Y$.*

Proof : let $x \in X$ and $y \in Y$. Then

$$\begin{aligned} \|x \otimes y\|_{\vee} &= \sup \left\{ \left| \langle x, x^* \rangle \langle y, y^* \rangle \right| : x^* \in B_1(X^*) \text{ and } y^* \in B_1(Y^*) \right\} \\ &= \sup \left\{ \left| \langle x, x^* \rangle \right| : x^* \in B_1(X^*) \right\} \sup \left\{ \left| \langle y, y^* \rangle \right| : y^* \in B_1(Y^*) \right\} \\ &= \|x\| \|y\|. \end{aligned}$$

Hence

$$\|x \otimes y\|_{\vee} = \|x\| \|y\| \text{ for all } x \in X \text{ and } y \in Y. \text{ Now let}$$

$x^* \otimes y^* \in X^* \otimes Y^*$ and $z := \sum_{i=1}^n x_i \otimes y_i$ be an element in $X \otimes Y$.

Then it follows from

$$(x^* \otimes y^*)(z) = \sum_{i=1}^n \langle x_i, x^* \rangle \langle y_i, y^* \rangle$$

that

$$\begin{aligned} |(x^* \otimes y^*)(z)| &= \|x^*\| \|y^*\| \left| \sum_{i=1}^n \left\langle x_i, \frac{x^*}{\|x^*\|} \right\rangle \left\langle y_i, \frac{y^*}{\|y^*\|} \right\rangle \right| \\ &\leq \|x^*\| \|y^*\| \|z\|_\vee \end{aligned}$$

Therefore by Theorem 1.1.8 , $\|\cdot\|_\vee$ is a reasonable cross norm . $\square$

Theorem 1.2.2 : *The norm $\|\cdot\|_\vee$ is the smallest reasonable cross norm on $X \otimes Y$. More precisely , if $\|\cdot\|_\alpha$ is any reasonable cross norm on $X \otimes Y$, then*

$$\|z\|_\vee \leq \|z\|_\alpha \text{ for all } z \in X \otimes Y .$$

Proof : Let $\|\cdot\|_\alpha$ be any reasonable cross norm on $X \otimes Y$ and let

$z := \sum_{i=1}^n x_i \otimes y_i$ be any element in $X \otimes Y$. Then for $x^* \otimes y^* \in X^* \otimes Y^*$

we have

$$\begin{aligned} |(x^* \otimes y^*)(z)| &\leq \|x^* \otimes y^*\|_{\alpha d} \|z\|_\alpha \\ &= \|x^*\| \|y^*\| \|z\|_\alpha . \end{aligned}$$

Thus it follows from

$$\left| \sum_{i=1}^n \langle x_i, x^* \rangle \langle y_i, y^* \rangle \right| \leq \|x^*\| \|y^*\| \|z\|_\alpha$$

that

$$\left| \sum_{i=1}^n \left\langle x_i, \frac{x^*}{\|x^*\|} \right\rangle \left\langle y_i, \frac{y^*}{\|y^*\|} \right\rangle \right| \leq \|z\|_\alpha \quad \text{for all } x^* \in X^* \text{ and } y^* \in Y^* .$$

Thus $\|z\|_\vee \leq \|z\|_\alpha$ $\square$

Theorem 1.2.3 : *A cross norm $\|\cdot\|_\alpha$ on $X \otimes Y$ is reasonable if and only if $\|\cdot\|_\alpha \geq \|\cdot\|_\vee$.*

Proof : Suppose that $\|\cdot\|_\alpha \geq \|\cdot\|_\vee$. Then by Theorem 1.1.10 we have $\|\cdot\|_{\alpha d} \leq \|\cdot\|_{\vee d}$. So

$$\begin{aligned} \|x^* \otimes y^*\|_{\alpha d} &\leq \|x^* \otimes y^*\|_{\vee d} \\ &= \|x^*\| \|y^*\| \end{aligned}$$

for all $x^* \in X^*$ and $y^* \in Y^*$. Since $\|\cdot\|_\alpha$ is a reasonable cross norm on

$X \otimes Y$, $X^* \otimes Y^* \subseteq (X \otimes Y, \|\cdot\|_\alpha)^*$. So for $x^* \otimes y^* \in X^* \otimes Y^*$ we have

$$\begin{aligned} |\langle x, x^* \rangle \langle y, y^* \rangle| &= |(x^* \otimes y^*)(x \otimes y)| \\ &\leq \|x^* \otimes y^*\|_{\alpha d} \|x \otimes y\|_\alpha \\ &= \|x^* \otimes y^*\|_{\alpha d} \|x\| \|y\| . \end{aligned}$$

Taking the supremum on both sides over $\|x\| = \|y\| = 1$ we obtain

$$\|x^*\| \|y^*\| \leq \|x^* \otimes y^*\|_{\alpha d}.$$

Combining the above results we see $\|\cdot\|_{\alpha d}$ is a cross norm on $X^* \otimes Y^*$.

Conversely, if $\|\cdot\|_{\alpha d}$ is a cross norm on $X^* \otimes Y^*$, then $\|\cdot\|_{\alpha}$ is a reasonable cross norm on $X \otimes Y$. Thus by Theorem 1.2.2 we have

$$\|\cdot\|_{\alpha} \geq \|\cdot\|_{\vee}. \quad \square$$

The space $X \otimes Y$ with the injective norm need not be complete. We let $\overset{\vee}{X \otimes Y}$ denote the completion of $X \otimes Y$. In fact, $\overset{\vee}{X \otimes Y}$ is the closure of $X \otimes Y$ in $L(X^*, Y)$. So $(\overset{\vee}{X \otimes Y}, \|\cdot\|_{\vee})$ is a closed subspace of $L(X^*, Y)$.

Definition 1.2.4 : The Banach space $(\overset{\vee}{X \otimes Y}, \|\cdot\|_{\vee})$ is called the *completed injective tensor product* of X with Y .

One can easily prove the following

Proposition 1.2.4 : The Banach space $(\overset{\vee}{X \otimes Y}, \|\cdot\|_{\vee})$ is isometrically isomorphic to $(\overset{\vee}{Y \otimes X}, \|\cdot\|_{\vee})$.

Example 1.2.5 : $C(K, Y)$:

Let K be a compact Hausdorff space and let Y be a Banach space .
 Let $C(K, Y)$ denote the Banach space of all continuous functions f from K into Y equipped with the norm

$$\|f\|_{\infty} := \sup_{t \in K} \|f(t)\| .$$

To simplify the notation , we write $C(K)$ for $C(K, \mathbb{K})$. We refer to [13] for more details on $C(K, Y)$.

Theorem 1.2.6 : *For any compact Hausdorff space K and any Banach space Y , $C(K, Y)$ is isometrically isomorphic to $C(K) \overset{\vee}{\otimes} Y$.*

Proof : See [20, p. 9]

Corollary 1.2.7 : *Let E and K be compact Hausdorff spaces . Then $C(E \times K)$ is isometrically isomorphic to $C(E) \overset{\vee}{\otimes} C(K)$.*

Proof : See [20, p. 10] .

1.3. Some Function and Sequence Spaces :

Throughout this section let X be a Banach space and let μ denote the Lebesgue measure on the unit interval I .

(i) The Space $L^p(I, X)$:

Definition 1.3.1 : Let $L^p(I, X)$ be the space (of equivalence classes) of p -Bochner integrable functions from (I, μ) into X with $1 \leq p < \infty$.

To simplify the notation we write $L^p(I)$ for $L^p(I, \mathbf{K})$.

For $f \in L^p(I, X)$ we define

$$\|f\|_{\alpha(p)} := \left\{ \int_I \|f(t)\|^p d\mu(t) \right\}^{\frac{1}{p}} .$$

Theorem 1.3.2 : $(L^p(I, X), \|\cdot\|_{\alpha(p)})$ is a Banach space .

Proof : See [9]

We refer to [9] for the basic properties of vector valued p -integrable functions .

(ii) The space $L^p(I, X)$:

Definition 1.3.3 : A strongly measurable function $f : I \rightarrow X$ is called *p -Pettis integrable* , $1 \leq p < \infty$, if

$$\int_I \left| \langle f(t), x^* \rangle \right|^p d\mu(t) < \infty \quad \text{for all } x^* \in X^* .$$

We let $\mathcal{L}^p\langle I, X \rangle$ denote the space (of equivalence classes) of all p -Pettis integrable functions . It is well-known [14] that $\mathcal{L}^p\langle I, X \rangle$ is a normed space when equipped with

$$\|f\|_{\mathcal{E}(p)} := \sup \left\{ \left(\int_I |\langle f(t), x^* \rangle|^p d\mu(t) \right)^{\frac{1}{p}} : x^* \in B_1(X^*) \right\} .$$

It is also known , [9] , that $\mathcal{L}^p\langle I, X \rangle$ is not complete . So let $L^p\langle I, X \rangle$ denote its completion .

For a compact Hausdorff space K , we shall remark that if (K, μ) is a finite measure space, then one can define $L^p\langle K, X \rangle$ in a similar way .

We refer to [9] for the basic properties of the p -Pettis integrable functions.

Let $S(I, X)$ denote the space of all simple functions from I into X .

Lemma 1.3.4 : $S(I, X)$ is a dense subspace of $L^p\langle I, X \rangle$.

Proof : Let $f \in \mathcal{L}^p\langle I, X \rangle$. Since f is strongly measurable , choose a simple function $S \in S(I, X)$ such that

$$\|f(t) - S(t)\| \leq \varepsilon \quad \text{for } \mu\text{-a.e } t \in I .$$

Hence $\|f - S\|_{\mathcal{E}(p)} \leq \varepsilon$. Indeed : For $x^* \in B_1(X^*)$, we have

$$\int_I |\langle f(t) - S(t), x^* \rangle|^p d\mu(t) \leq \int_I \|f(t) - S(t)\|^p d\mu(t) \leq \varepsilon^p . \quad \square$$

For $p=1$, it was proved [9, p. 223] that $L^1\langle I, X \rangle$ is isometrically isomorphic to $L^1(I) \overset{\vee}{\otimes} X$. For $1 < p < \infty$ we have :

Theorem 1.3.5 : $L^p\langle I, X \rangle$ is isometrically isomorphic to $L^p(I) \overset{\vee}{\otimes} X$.

Proof : Consider the map

$$\phi : L^p\langle I, X \rangle \rightarrow L^p(I) \overset{\vee}{\otimes} X ,$$

$$\phi\left(\sum_{i=1}^n 1_{E_i} x_i\right) := \sum_{i=1}^n 1_{E_i} \otimes x_i$$

for every simple function $S := \sum_{i=1}^n 1_{E_i} x_i$ where each 1_{E_i} , with $i=1, \dots, n$ is

the characteristic function of E_i such that $\bigcup_{i=1}^n E_i = I$. Then it follows from

[10, p. 286] and the Hahn-Banach Theorem that

$$\begin{aligned} \|S\|_{\mathcal{E}(p)} &= \sup \left\{ \left(\int_I |\langle S(t), x^* \rangle|^p d\mu(t) \right)^{\frac{1}{p}} : x^* \in B_1(X^*) \right\} \\ &= \sup \left\{ \int_I \theta(t) \langle S(t), x^* \rangle d\mu(t) : x^* \in B_1(X^*) \text{ and } \theta \in B_1(L^{p^*}) \right\} \\ &= \sup \left\{ \int_I \theta(t) \sum_{i=1}^n \langle x_i, x^* \rangle 1_{E_i}(t) d\mu(t) : x^* \in B_1(X^*) \text{ and } \theta \in B_1(L^{p^*}) \right\} \end{aligned}$$

$$\begin{aligned}
&= \sup \left\{ \sum_{i=1}^n \langle x_i, x^* \rangle \int_I \theta(t) 1_{E_i}(t) d\mu(t) : x^* \in B_1(X^*) \text{ and } \theta \in B_1(L^{p^*}) \right\} \\
&= \sup \left\{ \sum_{i=1}^n \langle x_i, x^* \rangle \langle 1_{E_i}, \theta \rangle : x^* \in B_1(X^*) \text{ and } \theta \in B_1(L^{p^*}) \right\} \\
&= \|\phi(S)\|_{\vee} .
\end{aligned}$$

Now it follows from Lemma 1.3.4 that the simple functions are dense in $L^p(I, X)$. Moreover, the space $S(I) \otimes X$ is dense in $L^p(I) \otimes X$, where $S(I)$ is the space of simple functions on I . Indeed :

Let $f := \sum_{i=1}^n f_i \otimes x_i$ be in $L^p(I) \otimes X$. Since $S(I)$ is dense in $L^p(I)$,

there is a sequence (s_{im}) in $L^p(I)$ such that $\lim_{m \rightarrow \infty} \|f_i - s_{im}\|_p = 0$.

Now set

$$\hat{s}_m := \sum_{i=1}^n s_{im} \otimes x_i .$$

Clearly, $(\hat{s}_m)$ is a sequence of simple functions in $S(I) \otimes X$. Moreover

$$\begin{aligned}
\lim_{m \rightarrow \infty} \|f - \hat{s}_m\|_{\vee} &= \lim_{m \rightarrow \infty} \left\| \sum_{i=1}^n (f_i - s_{im}) \otimes x_i \right\|_{\vee} \\
&\leq \lim_{m \rightarrow \infty} \sum_{i=1}^n \|f_i - s_{im}\|_p \cdot \|x_i\| = 0 .
\end{aligned}$$

Further, one can easily show that finite rank elements in $L^p(I) \overset{\vee}{\otimes} X$ are contained in the range of ϕ . Therefore ϕ is an isometry between a

dense subspace of $L^p\langle I, X \rangle$ onto a dense subspace of $L^p(I) \overset{\vee}{\otimes} X$. Thus

$$L^p(I) \overset{\vee}{\otimes} X = \phi(L^p\langle I, X \rangle) . \quad \square$$

(iii) The Space $\ell^p(X)$:

Definition 1.3.6 : An X -valued sequence (x_n) is said to be *absolutely p -summable*, with $1 \leq p < \infty$, if $(\|x_n\|) \in \ell^p$.

The space of all such sequences is denoted by $\ell^p(X)$. Clearly $\ell^p(X)$ is a linear space with the pointwise linear operations.

For $(x_n) \in \ell^p(X)$ we define

$$\|(x_n)\|_{\alpha(p)} := \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}} .$$

The proof of the following theorem consists of routine verification of steps and will be omitted.

Theorem 1.3.7 : $(\ell^p(X), \|\cdot\|_{\alpha(p)})$ is a Banach space.

(iv) The Space $\ell^p\langle X \rangle$:

Definition 1.3.8 : An X -valued sequence (x_n) is said to be *weakly p -summable*, with $1 \leq p < \infty$, if $\left(\langle x_n, x^* \rangle \right) \in \ell^p$ for all $x^* \in X^*$.

The space of all such sequences is denoted by $\ell^p\langle X \rangle$. Obviously $\ell^p\langle X \rangle$ is a linear space with the pointwise linear operations.

For $(x_n) \in \ell^p\langle X \rangle$ we define

$$\|(x_n)\|_{\varepsilon(p)} := \sup \left\{ \left(\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p \right)^{\frac{1}{p}} : x^* \in B_1(X^*) \right\}.$$

We have to mention that $\ell^p\langle X \rangle$ have been studied by Cohen, [5] and Apiola, [2] .

Theorem 1.3.9 : $\| \cdot \|_{\varepsilon(p)}$ is a norm on $\ell^p\langle X \rangle$.

Proof : We need only to show that the $\| \cdot \|_{\varepsilon(p)}$ is finite . Indeed : Let $(x_n) \in \ell^p\langle X \rangle$ be fixed . For each $k \in \mathbb{N}$ define

$$T_k : \ell^{p^*} \rightarrow X ,$$

$$T_k(\alpha_n) := \sum_{n=1}^k \alpha_n x_n .$$

Clearly that T_k is a bounded linear operator for all k . Let $x^* \in X^*$. Then

$$\begin{aligned} |\langle T_k(\alpha_n), x^* \rangle| &\leq \sum_{n=1}^k |\alpha_n| |\langle x_n, x^* \rangle| \\ &\leq \|(\alpha_n)\|_{\alpha(p^*)} \left(\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p \right)^{\frac{1}{p}} \end{aligned}$$

for all k . Hence for all $(\alpha_n) \in \ell^{p^*}$ and $x^* \in X^*$ we have

$$\sup_k \left| \langle T_k(\alpha_n), x^* \rangle \right| < \infty.$$

It follows from the Uniform Boundedness Principle [33, p. 190] that

$$\sup_k \|T_k\| < M < \infty.$$

Now let $x^* \in B_1(X^*)$. Since $(\langle x_n, x^* \rangle) \in \ell^p$, there is $(\beta_n) \in \ell^{p^*}$ with

$$\|(\beta_n)\|_{\ell^{p^*}} = 1 \text{ such that } \left(\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p \right)^{\frac{1}{p}} = \sum_{n=1}^{\infty} \beta_n \langle x_n, x^* \rangle. \text{ Thus}$$

$$\begin{aligned} \left| \langle T_k(\beta_n), x^* \rangle \right| &= \left| \sum_{n=1}^k \beta_n \langle x_n, x^* \rangle \right| \\ &= \left(\sum_{n=1}^k |\langle x_n, x^* \rangle|^p \right)^{\frac{1}{p}} \\ &\leq \|T_k\| < M \text{ for all } k. \end{aligned}$$

Hence $\left(\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p \right)^{\frac{1}{p}} \leq M$ for all $x^* \in B_1(X^*)$ which shows that

$\|(x_n)\|_{\ell^p(X)}$ is finite. This ends the proof. $\square$

Theorem 1.3.10 : $(\ell^p(X), \|\cdot\|_{\ell^p(X)})$ is a Banach space.

Proof : Let (x_m) be a Cauchy sequence in $\ell^p(X)$, where

$$x_m := (x_1^m, x_2^m, \dots) \quad .$$

Then for every $\varepsilon > 0$, there is an N_0 such that $\|x_m - x_n\|_{\varepsilon(P)} < \varepsilon$ for all $m, n \geq N_0$. Hence it follows from

$$\sup \left\{ \left(\sum_{j=1}^{\infty} |\langle x_j^m - x_j^n, x^* \rangle|^p \right)^{\frac{1}{p}} : x^* \in B_1(X^*) \right\} < \varepsilon \text{ for all } m, n > N_0$$

that for every $j \in \mathbb{N}$

$$|\langle x_j^m - x_j^n, x^* \rangle| < \varepsilon \text{ for all } x^* \in B_1(X^*) \text{ and for } m, n > N_0 .$$

By the Hahn-Banach Theorem, we obtain

$$\|x_j^m - x_j^n\| < \varepsilon \text{ for all } m, n \geq N_0 .$$

Thus for each $j \in \mathbb{N}$, $(x_j^m)_m$ is a Cauchy sequence in X . Since X is complete, $z_j := \lim_m x_j^m \in X$ exists for all $j \in \mathbb{N}$.

On the other hand, for $x^* \in X^*$, $\left(\left(\langle x_j^m, x^* \rangle \right)_{j=1}^{\infty} \right)_m$ is a Cauchy sequence in ℓ^p which must converge to some $(\beta_j) \in \ell^p$. Since, for each $j \in \mathbb{N}$, $x_j^m \xrightarrow{m} z_j$, we have $\langle x_j^m, x^* \rangle \xrightarrow{m} \langle z_j, x^* \rangle$. Hence for all $j \in \mathbb{N}$, $\beta_j = \langle z_j, x^* \rangle$ which implies that $(z_j) \in \ell^p(X)$.

It follows from $\|x_m - x_n\|_{\varepsilon(P)} < \varepsilon$ for all $m, n \geq N_0$ that for all $x^* \in B_1(X^*)$ we have

$$\left(\sum_{j=1}^{\infty} \left| \langle x_j^m - x_j^n, x^* \rangle \right|^p \right)^{\frac{1}{p}} \leq \varepsilon .$$

Letting $n \rightarrow \infty$ we obtain

$$\left(\sum_{j=1}^{\infty} \left| \langle x_j^m - z_j, x^* \rangle \right|^p \right)^{\frac{1}{p}} \leq \varepsilon .$$

Thus $\|(x_j^m) - (z_j)\|_{\varepsilon(p)} \leq \varepsilon$ for all $m \geq N_0$. $\square$

Proposition 1.3.11 : $\ell^p(X) \subseteq \ell^p\langle X \rangle$ and $\|\cdot\|_{\varepsilon(p)} \leq \|\cdot\|_{\alpha(p)}$.

Proof : Let $(x_n) \in \ell^p(X)$ and let $x^* \in X^*$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \langle x_n, x^* \rangle \right|^p &= \|x^*\|^p \sum_{n=1}^{\infty} \left| \left\langle x_n, \frac{x^*}{\|x^*\|} \right\rangle \right|^p \\ &\leq \|x^*\|^p \sum_{n=1}^{\infty} \sup \left\{ \left| \langle x_n, z^* \rangle \right|^p : z^* \in B_1(X^*) \right\} \\ &= \|x^*\|^p \sum_{n=1}^{\infty} \|x_n\|^p . \end{aligned}$$

Hence $(x_n) \in \ell^p\langle X \rangle$. Moreover , if $x^* \in B_1(X^*)$, then

$$\|(x_n)\|_{\varepsilon(p)} \leq \|(x_n)\|_{\alpha(p)} . \quad \square$$

Theorem 1.3.12 : *Let $T \in L(X, Y)$.Then*

$$\Lambda: \ell^p \langle X \rangle \rightarrow \ell^p \langle Y \rangle ,$$

$$\Lambda(x_i) := (Tx_i)$$

is a bounded linear operator . Moreover ,

$$\|\Lambda(x_i)\|_{\varepsilon(p)} \leq \|T\| \| (x_i) \|_{\varepsilon(p)} \quad \text{for } (x_i) \in \ell^p \langle X \rangle .$$

Proof :

$$\begin{aligned} \|\Lambda(x_i)\|_{\varepsilon(p)} &= \sup \left\{ \left(\sum_{i=1}^{\infty} |\langle Tx_i, y^* \rangle|^p \right)^{\frac{1}{p}} : y^* \in B_1(Y^*) \right\} \\ &= \sup \left\{ \left(\sum_{i=1}^{\infty} |\langle x_i, T^* y^* \rangle|^p \right)^{\frac{1}{p}} : y^* \in B_1(Y^*) \right\} \\ &= \|T^*\| \sup \left\{ \left(\sum_{i=1}^{\infty} \left| \left\langle x_i, \frac{T^* y^*}{\|T^*\|} \right\rangle \right|^p \right)^{\frac{1}{p}} : y^* \in B_1(Y^*) \right\} \\ &\leq \|T\| \sup \left\{ \left(\sum_{i=1}^{\infty} |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} : x^* \in B_1(X^*) \right\} \end{aligned}$$

This shows that $\|\Lambda(x_i)\|_{\varepsilon(p)} \leq \|T\| \|(x_i)\|_{\varepsilon(p)}$. $\square$

Example 1.3.13 : Every orthonormal sequence (x_n) in a Hilbert space H is weakly 2-summable and $\|(x_n)\|_{\varepsilon(2)} = 1$. Indeed :

From Bessel's inequality we have

$$\sum_{n=1}^{\infty} |\langle x_n, y \rangle|^2 \leq \|y\|^2 \quad \text{for all } y \in H.$$

This shows that

$$\|(x_n)\|_{\varepsilon(2)} \leq 1.$$

Since for fixed m , $\|x_m\| = 1$, it follows from

$$\sum_{n=1}^{\infty} |\langle x_n, x_m \rangle|^2 = 1 \quad \text{that} \quad \|(x_n)\|_{\varepsilon(2)} = 1. \quad \square$$

Proposition 1.3.14 : Let (e_n) be the standard basis of ℓ^2 and let $\frac{1}{p} + \frac{1}{p^*} = 1$ with $1 \leq p < \infty$. Then

$$(e_n) \in \ell^p \left\langle \ell^{p^*} \right\rangle \quad \text{and} \quad \|(e_n)\|_{\varepsilon(p)} = 1.$$

Proof : We will prove the statement for $p = 1$ only, because for $1 < p < \infty$, the proof is elementary. So let $x^* \in B_1\left((\ell^\infty)^*\right)$ and let $x|_{\ell_k^\infty}^*$ denote the restriction of x^* on ℓ_k^∞ . Then

$$\sum_{n=1}^k |\langle e_n, x^* \rangle| = \sum_{n=1}^k \left| \left\langle e_n, x|_{\ell_k^\infty}^* \right\rangle \right| \leq \left\| x|_{\ell_k^\infty}^* \right\|_{\alpha(1)} \leq 1 \quad \text{for all } k$$

because $\ell_k^\infty = \text{span}\{e_1, \dots, e_k\}$ in ℓ^∞ and $x|_{\ell_k^\infty}^* \in \ell_k^1$. This shows

$\|(e_n)\|_{\varepsilon(1)} \leq 1$. The equality holds by choosing $x^* := (e_n)$ in $B_1((\ell^\infty)^*)$.

This completes the proof. $\square$

Theorem 1.3.15 :

(a) If $1 < p < \infty$, then $\ell^p\langle X \rangle$ is isometrically isomorphic to

$$L(\ell^{p^*}, X).$$

(b) $\ell^1\langle X \rangle$ is isometrically isomorphic to $L(c_0, X)$.

Proof : (a) Let $(x_n) \in \ell^p\langle X \rangle$. For $(\alpha_n) \in \ell^{p^*}$ we have

$$\begin{aligned} \left\| \sum_{n=k}^m \alpha_n x_n \right\| &\leq \sup \left\{ \sum_{n=k}^m |\alpha_n| \cdot |\langle x_n, x^* \rangle| : x^* \in B_1(X^*) \right\} \\ &\leq \left(\sum_{n=k}^m |\alpha_n|^{p^*} \right)^{\frac{1}{p^*}} \| (x_n) \|_{\varepsilon(p)} \end{aligned} \quad (1)$$

for all $k, m \in \mathbb{N}$. Hence $\sum_{n=1}^{\infty} \alpha_n x_n$ exists in X because its partial sum is

a Cauchy sequence in a Banach space. Thus by putting

$$T(\alpha_n) := \sum_{n=1}^{\infty} \alpha_n x_n,$$

we obtain a continuous linear operator from ℓ^{p^*} into X . Clearly that

$$\|T\| \leq \|(x_n)\|_{\varepsilon(p)} .$$

Conversely , we show that every operator T in $L(\ell^p, X)$ can be obtained in this way . By Proposition 1.3.14 (e_i) , the natural basis of ℓ^2 , is weakly p -summable sequence in ℓ^p and $\|(e_i)\|_{\varepsilon(p)} = 1$. Set

$$(x_i) := (Te_i) .$$

Now it follows from Theorem 1.3.12 that

$$(x_i) \in \ell^p \langle X \rangle \text{ with } \|(x_i)\|_{\varepsilon(p)} \leq \|T\| .$$

So if $(\alpha_n) \in \ell^p$ then $(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n e_n$ and consequently

$$T(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n Te_n = \sum_{n=1}^{\infty} \alpha_n x_n .$$

Moreover $\|T\| = \|(x_n)\|_{\varepsilon(p)} .$

(b) The proof is the same . □

As a consequence of the Hahn-Banach Theorem , we have

$$\|x\| = \sup \left\{ \left| \langle x, x^* \rangle \right| : x^* \in B_1(X^*) \right\} \text{ for all } x \in X .$$

In some cases , $\|x\|$ can be obtained by taking the right hand supremum only over a certain subset E of $B_1(X^*)$. This observation leads to the following [28, p. 39]

Definition 1.3.16 : A subset E of X^* is said to be *norming* for X if

$$\|x\| = \sup \left\{ \left| \langle x, x^* \rangle \right| : x^* \in E \right\} \text{ for all } x \in X .$$

It follows immediately that $E \subseteq B_1(X^*)$.

Example 1.3.17 : Let E be the set of extreme points in $B_1(X^*)$.

Then it is well-known [33, p. 184] that

$$\|x\| = \sup \left\{ \left| \langle x, x^* \rangle \right| : x^* \in E \right\} \text{ for all } x \in X .$$

Hence E is a norming subset of X^* .

Example 1.3.18 : Let K be a compact Hausdorff space . It follows from the above example that

$$\|f\| = \sup \left\{ \left| \langle f, \mu \rangle \right| : \mu \in E \right\} \text{ for all } f \in C(K) ,$$

where E is the set of extreme points of $B_1(M(K))$. It was proved [6, p. 147] that

$$E = \left\{ \alpha \delta_t : |\alpha| = 1 \text{ and } t \in K \right\} ,$$

where δ_t denotes the Dirac evaluation measure . Thus

$$\begin{aligned} \|f\| &= \sup \left\{ \left| \langle f, \alpha \delta_t \rangle \right| : |\alpha| = 1 \text{ and } t \in K \right\} \\ &= \sup \left\{ |f(t)| : t \in K \right\} . \end{aligned}$$

This shows that $D := \{\delta_t : t \in K\}$ is a norming subset in $M(K)$ for $C(K)$.

Theorem 1.3.19 : Let E be any norming subset of X^* and let

$$(x_n) \in \ell^p \langle X \rangle . \text{ Then } \|(x_n)\|_{\varepsilon(p)} = \sup_{x^* \in E} \left(\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p \right)^{\frac{1}{p}} .$$

Proof : Let $(x_n) \in \ell^p \langle X \rangle$. Following the same lines of the proof of Theorem 1.3.15 by putting

$$T(\alpha_n) := \sum_{n=1}^{\infty} \alpha_n x_n ,$$

we obtain a continuous linear operator from ℓ^{p^*} (or c_0) into X . Since E is a norming subset for X , we can replace $B_1(X^*)$ by E in (1) in the proof of Theorem 1.3.15 to get

$$\|T\| \leq \sup_{x^* \in E} \left(\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p \right)^{\frac{1}{p}} \leq \|(x_n)\|_{\varepsilon(p)} .$$

By Proposition 1.3.14 (e_i) , the natural basis of ℓ^2 , is weakly p -summabl in ℓ^{p^*} with $\|(e_i)\|_{\varepsilon(p)} = 1$.

Now by Theorem 1.3.12 we have

$$(x_i) = (T e_i) \in \ell^p \langle X \rangle \text{ with } \|(x_i)\|_{\varepsilon(p)} \leq \|T\| .$$

This proves that $\|(x_n)\|_{\varepsilon(p)} = \sup_{x^* \in E} \left(\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p \right)^{\frac{1}{p}} . \quad \square$

An immediate result from the above theorem is the following :

Corollary 1.3.20 : Let $x_1, \dots, x_n \in X$. Then

$$(1) \quad \left\| (x_i)_{i=1}^n \right\|_{\varepsilon(p)} = \sup \left\{ \left\| \sum_{i=1}^n \alpha_i x_i \right\| : \sum_{i=1}^n |\alpha_i|^{p^*} \leq 1 \right\} \text{ for } 1 < p < \infty .$$

$$(2) \quad \left\| (x_i)_{i=1}^n \right\|_{\varepsilon(1)} = \sup \left\{ \left\| \sum_{i=1}^n \alpha_i x_i \right\| : |\alpha_i| \leq 1 \text{ and } i = 1, \dots, n \right\} .$$

Proposition 1.3.21 : Let K be any compact Hausdorff space and

$$\text{let } f_1, \dots, f_n \in C(K). \text{ Then } \left\| (f_i)_{i=1}^n \right\|_{\varepsilon(p)} = \sup_{t \in K} \left(\sum_{i=1}^n |f_i(t)|^p \right)^{\frac{1}{p}} .$$

Proof : Let D be the set of all Dirac evaluation measures on $C(K)$. By Example 1.3.18, D is a norming subset of $M(K)$ for $C(K)$. Hence by Theorem 1.3.19 we have

$$\left\| (f_i)_{i=1}^n \right\|_{\varepsilon(p)} = \sup_{\delta_t \in D} \left(\sum_{i=1}^n |\langle f_i, \delta_t \rangle|^p \right)^{\frac{1}{p}} = \sup_{t \in K} \left(\sum_{i=1}^n |f_i(t)|^p \right)^{\frac{1}{p}} . \quad \square$$

Theorem 1.3.22 : Let (I, μ) be the unit interval with the Lebesgue measure. Let $f_1, \dots, f_n \in L^\infty(I, \mu)$. Then

$$\left\| (f_i)_{i=1}^n \right\|_{\varepsilon(p)} \geq \text{ess-sup}_{t \in I} \left(\sum_{i=1}^n |f_i(t)|^p \right)^{\frac{1}{p}} .$$

Proof : Let $\Lambda := \left\| (f_i)_{i=1}^n \right\|_{\varepsilon(p)}$. Then $\Lambda \geq \left(\sum_{i=1}^n |\langle f_i, v \rangle|^p \right)^{\frac{1}{p}}$ for all

functionals $v \in B_1 \left((L^\infty(I))^* \right)$. It follows from [9, p. 49] that

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_{t_0}^{t_0+h} f_i(t) d\mu(t) = f_i(t_0) \quad \text{for a.e } t_0 \in I .$$

Now , for each $j \in \mathbb{N}$, define

$$v_j := \begin{cases} j d\mu(t) & \text{on } \left[t_0, t_0 + \frac{1}{j} \right] \\ 0 & \text{elsewhere} . \end{cases}$$

Now $|\langle f_i, v_j \rangle| \leq \int_0^1 |f_i(t)| dv_j \leq \int_{t_0}^{t_0 + \frac{1}{j}} j |f_i(t)| d\mu(t) \leq \|f_i\|_\infty$ implies that

$\|v_j\| \leq 1$ for all j . Thus

$$\Lambda \geq \left(\sum_{i=1}^n |\langle f_i, v_j \rangle|^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^n \left| j \int_{t_0}^{t_0 + \frac{1}{j}} f_i(t) d\mu(t) \right|^p \right)^{\frac{1}{p}} \quad \text{for all } j .$$

By letting $j \rightarrow \infty$, we conclude

$$\Lambda \geq \left(\sum_{i=1}^n |f_i(t_0)|^p \right)^{\frac{1}{p}} \quad \text{for a.e } t_0 \in I .$$

This shows that $\left\| (f_i)_{i=1}^n \right\|_{\varepsilon(p)} \geq \operatorname{ess-sup}_{t \in I} \left(\sum_{i=1}^n |f_i(t)|^p \right)^{\frac{1}{p}}$. □

1.4. The Projective Tensor Product :

Let X and Y be Banach spaces . In this section we discuss the second norm on $X \otimes Y$ that we are interested in .

For $z \in X \otimes Y$ we define

$$\|z\|_{\wedge} := \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : x_i \in X, y_i \in Y, \text{ and } z = \sum_{i=1}^n x_i \otimes y_i \right\} .$$

Theorem 1.4.1 : $\|\cdot\|_{\wedge}$ is a norm on $X \otimes Y$.

Proof : We will only prove that if $z \neq 0$, then $\|z\|_{\wedge} \neq 0$. So let

$$z := \sum_{i=1}^n x_i \otimes y_i \neq 0 \quad \text{in } X \otimes Y .$$

As an element of $L(X^*, Y)$, $\|z\| \neq 0$. Thus there exists $x_0^* \in X^*$ such that $\|x_0^*\| = 1$ and $\|z(x_0^*)\| > 0$.

Hence

$$\|z(x_0^*)\| = \left\| \sum_{i=1}^n \langle x_i, x_0^* \rangle y_i \right\| \leq \sum_{i=1}^n |\langle x_i, x_0^* \rangle| \|y_i\| \leq \sum_{i=1}^n \|x_i\| \|y_i\| .$$

Since this holds for all representation of z , we have $\|z\|_{\wedge} > 0$. $\square$

Theorem 1.4.2 : The projective norm , $\|\cdot\|_{\wedge}$, is a reasonable cross norm on $X \otimes Y$.

Proof : Let $x \in X$ and $y \in Y$. It follows from the definition of $\|\cdot\|_{\wedge}$ that

$$\|x \otimes y\|_{\wedge} \leq \|x\| \|y\|.$$

Now for $x^* \in X^*$ and $y^* \in Y^*$ we have

$$(x^* \otimes y^*)(z) = \sum_{i=1}^n \langle x_i, x^* \rangle \langle y_i, y^* \rangle \text{ for any } z := \sum_{i=1}^n x_i \otimes y_i \text{ in the}$$

space $X \otimes Y$. Further

$$|(x^* \otimes y^*)(z)| \leq \sum_{i=1}^n |\langle x_i, x^* \rangle| \cdot |\langle y_i, y^* \rangle| \leq \|x^*\| \|y^*\| \sum_{i=1}^n \|x_i\| \|y_i\|.$$

Since this holds for all representation of z , we obtain

$$|(x^* \otimes y^*)(z)| \leq \|x^*\| \|y^*\| \|z\|_{\wedge}.$$

Thus $\|x^* \otimes y^*\|_d \leq \|x^*\| \|y^*\|.$

By Theorem 1.1.8 the result follows. $\square$

Theorem 1.4.3 : *The projective norm, $\|\cdot\|_{\wedge}$, is the largest reasonable cross norm on $X \otimes Y$.*

Proof : Let $\|\cdot\|_{\alpha}$ be any reasonable cross norm on $X \otimes Y$ and let

$$z := \sum_{i=1}^n x_i \otimes y_i \text{ in } X \otimes Y. \text{ Then}$$

$$\|z\|_{\alpha} = \left\| \sum_{i=1}^n x_i \otimes y_i \right\|_{\alpha} \leq \sum_{i=1}^n \|x_i \otimes y_i\|_{\alpha} = \sum_{i=1}^n \|x_i\| \|y_i\|.$$

Since this holds for all representations of z , we have $\|z\|_\alpha \leq \|z\|_\wedge$. $\square$

The space $X \otimes Y$ with the projective norm need not be complete. We let $\hat{X \otimes Y}$ denote the completion of $(X \otimes Y, \|\cdot\|_\wedge)$.

Definition 1.4.4 : The Banach space $(\hat{X \otimes Y}, \|\cdot\|_\wedge)$ is called the *completed projective tensor product* of X and Y .

Theorem 1.4.5 : If z in $\hat{X \otimes Y}$ and $\varepsilon > 0$, then there exist $x_i \in X$ and $y_i \in Y$ such that

- (1) $z = \sum_{i=1}^{\infty} x_i \otimes y_i$;
- (2) $\|x_i\| = \|y_i\| \rightarrow 0$; and
- (3) $\|z\|_\wedge \leq \sum_{i=1}^{\infty} \|x_i\| \|y_i\| \leq \|z\|_\wedge + \varepsilon$.

Proof : Since $X \otimes Y$ is dense in $\hat{X \otimes Y}$, choose (z_n) in $X \otimes Y$ such that $\|z - z_n\|_\wedge < \frac{\varepsilon}{2^{n+2}}$ for $n = 1, 2, \dots$.

From the definition of $\|\cdot\|_\wedge$, we can write

$$z_1 = \sum_{i=1}^{k(1)} x_i \otimes y_i \quad \text{where} \quad \sum_{i=1}^{k(1)} \|x_i\| \|y_i\| \leq \|z_1\|_\wedge + \frac{\varepsilon}{4} .$$

Since

$$\|z_1 - z\|_{\wedge} < \frac{\varepsilon}{8} \quad \text{and} \quad \left| \|z_1\|_{\wedge} - \|z\|_{\wedge} \right| \leq \|z_1 - z\|_{\wedge} < \frac{\varepsilon}{4} ,$$

we have $\|z_1\|_{\wedge} < \|z\|_{\wedge} + \frac{\varepsilon}{4}$. Hence

$$z_1 = \sum_{i=1}^{k(1)} x_i \otimes y_i \quad \text{with} \quad \sum_{i=1}^{k(1)} \|x_i\| \|y_i\| < \|z\|_{\wedge} + \frac{\varepsilon}{2} .$$

Clearly we can assume $\|x_i\| = \|y_i\|$ for all i . Now for each $n \in \mathbb{N}$,

$$\|z_{n+1} - z_n\|_{\wedge} \leq \|z_{n+1} - z\|_{\wedge} + \|z - z_n\|_{\wedge} < \frac{\varepsilon}{2^{n+3}} + \frac{\varepsilon}{2^{n+2}} < \frac{\varepsilon}{2^{n+1}} .$$

Thus we can represent $z_{n+1} - z_n$ as follows :

$$z_{n+1} - z_n = \sum_{i=k(n)+1}^{k(n+1)} x_i \otimes y_i \quad \text{with} \quad \sum_{i=k(n)+1}^{k(n+1)} \|x_i\| \|y_i\| < \frac{\varepsilon}{2^{n+1}} .$$

We claim that $\sum_{i=1}^{\infty} x_i \otimes y_i$ is absolutely convergent in $\hat{X \otimes Y}$. Indeed :

$$\begin{aligned} \sum_{i=1}^{\infty} \|x_i \otimes y_i\|_{\wedge} &= \sum_{i=1}^{\infty} \|x_i\| \|y_i\| \\ &= \sum_{i=1}^{k(1)} \|x_i\| \|y_i\| + \sum_{n=1}^{\infty} \sum_{i=k(n)+1}^{k(n+1)} \|x_i\| \|y_i\| \\ &< \|z\|_{\wedge} + \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} \\ &= \|z\|_{\wedge} + \varepsilon . \end{aligned}$$

Moreover , it follows from

$$\begin{aligned} \left\| \left(\sum_{i=1}^{k(n)} x_i \otimes y_i \right) - z \right\|_{\wedge} &= \|z_1 + (z_2 - z_1) + \dots + (z_n - z_{n-1}) - z\|_{\wedge} \\ &= \|z_n - z\|_{\wedge} \end{aligned}$$

that the series converges to z in $\hat{X \otimes Y}$. Hence we have

$$\|z\|_{\wedge} \leq \sum_{i=1}^{\infty} \|x_i \otimes y_i\|_{\wedge} = \sum_{i=1}^{\infty} \|x_i\| \|y_i\| \leq \|z\|_{\wedge} + \varepsilon .$$

Since $\|x_i\| \cdot \|y_i\| \rightarrow 0$ and $\|x_i\| = \|y_i\|$ we must have $\|x_i\| = \|y_i\| \rightarrow 0$. $\square$

Theorem 1.4.6 : *Let (E, μ) be any measure space . Then $L^1(E, X)$ is isometrically isomorphic to $L^1(E) \hat{\otimes} X$.*

Proof : See [20, p. 11] .

Theorem 1.4.7 : $\left(\hat{X \otimes Y} \right)^*$ is isometrically isomorphic to $L(X, Y^*)$.

Proof : For any $A \in \left(\hat{X \otimes Y} \right)^*$, define

$$F_A : X \rightarrow Y^*,$$

$$\langle F_A x, y \rangle := A(x \otimes y) \quad \text{for } x \in X \text{ and } y \in Y .$$

Clearly F_A is a linear map . Since

$$|\langle F_A x, y \rangle| = |A(x \otimes y)| \leq \|A\| \|x \otimes y\|_{\wedge} = \|A\| \|x\| \|y\|$$

is true for all $x \in B_1(X)$ and $y \in B_1(Y)$, it follows that

$$\|F_A\| = \sup \left\{ |\langle F_A x, y \rangle| : x \in B_1(X) \text{ and } y \in B_1(Y) \right\} \leq \|A\| . \quad (*)$$

Let $z := \sum_{i=1}^n x_i \otimes y_i$ in $X \otimes Y$. Since

$$\begin{aligned} |A(z)| &= \left| \sum_{i=1}^n A(x_i \otimes y_i) \right| \\ &= \left| \sum_{i=1}^n \langle F_A x_i, y_i \rangle \right| \\ &\leq \|F_A\| \sum_{i=1}^n \|x_i\| \|y_i\| \end{aligned}$$

is true for all representations of z , it follows that $|A(z)| \leq \|F_A\| \|z\|_{\wedge}$ which implies $\|A\| \leq \|F_A\|$. (**)

It follows from (*) and (**) that $\|F_A\| = \|A\|$.

In order to see that the mapping is onto, let $F \in L(X, Y^*)$. Define

$$\begin{aligned} A: X \hat{\otimes} Y &\rightarrow \mathbf{K}, \\ A\left(\sum_{i=1}^n x_i \otimes y_i\right) &:= \sum_{i=1}^n \langle F x_i, y_i \rangle. \end{aligned}$$

Then we claim that A is a well-defined linear functional. Indeed :

$$\text{Let } z := \sum_{i=1}^n x_i \otimes y_i = \sum_{i=1}^m \hat{x}_i \otimes \hat{y}_i \text{ in } X \hat{\otimes} Y.$$

We need to show that

$$\sum_{i=1}^n \langle F x_i, y_i \rangle = \sum_{i=1}^m \langle F \hat{x}_i, \hat{y}_i \rangle$$

or equivalently whenever

$$z := \sum_{i=1}^n x_i \otimes y_i = 0 \otimes 0 \quad \text{then} \quad \sum_{i=1}^n \langle F x_i, y_i \rangle = 0 \quad .$$

If $y_1, \dots, y_n$ are linearly independent, by definition we have

$$z(x^*) = \sum_{i=1}^n \langle x_i, x^* \rangle y_i \quad \text{for all } x^* \in X^* \quad .$$

Hence $\langle x_i, x^* \rangle = 0$ for all $x^* \in X^*$ implies that $x_i = 0$ for all i .

$$\text{Thus} \quad \sum_{i=1}^n \langle F x_i, y_i \rangle = 0 \quad .$$

Now if $y_1, \dots, y_n$ are linearly dependent, then without loss of generality ,

$$y_n = \sum_{i=1}^{n-1} \lambda_i y_i \quad .$$

As in the proof of Theorem 1.1.3 we obtain

$$\sum_{i=1}^n x_i \otimes y_i = \sum_{i=1}^{n-1} (x_i + \lambda_i x_n) \otimes y_i = 0 \otimes 0 \quad .$$

Moreover

$$\begin{aligned} \sum_{i=1}^n \langle F x_i, y_i \rangle &= \sum_{i=1}^{n-1} \langle F x_i, y_i \rangle + \langle F x_n, y_n \rangle \\ &= \sum_{i=1}^{n-1} \langle F x_i, y_i \rangle + \sum_{i=1}^{n-1} \langle F x_n, \lambda_i y_i \rangle \\ &= \sum_{i=1}^{n-1} \langle F(x_i + \lambda_i x_n), y_i \rangle \quad . \end{aligned}$$

If $y_1, \dots, y_n$ are linearly independent, then as above we conclude that

$$x_i + \lambda_i x_n = 0 \quad \text{for } i = 1, \dots, n \quad .$$

Hence $\sum_{i=1}^n \langle F x_i, y_i \rangle = 0$.

By continuing the above argument , we either establish $\sum_{i=1}^n \langle F x_i, y_i \rangle = 0$

or get $z = x \otimes y = 0 \otimes 0$. If the second case occurs then

$$\sum_{i=1}^n \langle F x_i, y_i \rangle = \langle F x, y \rangle = 0$$

because either $x = 0$ or $y = 0$. This concludes the claim .

Since for any

$$z := \sum_{i=1}^n x_i \otimes y_i \text{ in } X \hat{\otimes} Y,$$

$$\begin{aligned} \left| A \left(\sum_{i=1}^n x_i \otimes y_i \right) \right| &= \left| \sum_{i=1}^n \langle F x_i, y_i \rangle \right| \\ &\leq \|F\| \sum_{i=1}^n \|x_i\| \|y_i\| \end{aligned}$$

is true for all representations of z , we have

$$A \in \left(X \hat{\otimes} Y \right)^* \text{ and } \|A\| \leq \|F\| .$$

Finally

$$F_A = F .$$

This completes the proof .

□

A nice consequence of Theorem 1.4.7 is :

Proposition 1.4.8 : *For any $z \in \hat{X \otimes Y}$,*

$$\|z\|_{\wedge} = \sup \left\{ |\langle z, u \rangle| : u \in L(X, Y^*) \text{ and } \|u\| \leq 1 \right\} .$$

Proof : The proof follows from the Hahn-Banach Theorem and Theorem 1.4.7 . $\square$

CHAPTER TWO

OPERATOR IDEALS

Introduction : In this chapter we will introduce several important classes of operator ideals , namely the ideals of , compact , weakly compact , p-nuclear , completely continuous , and absolutely p-summing operators . We discuss in details absolutely p-summing operators . Several important characterizations , which are basic to this theory , are proved . We present the basic properties and some examples . Later we introduce the $\mathcal{L}_\lambda^p$ -spaces and show that the p-summing operator ideal is not small . We show the connection between p-summing operators and various classes of linear operators . We discuss the duality of the p-summing operators in the finite dimensional case . We also present an application to the theory of Hilbert spaces .

2.1. Basic Concepts and Definitions :

Definition 2.1.1 : Suppose that , for every pair of Banach spaces X and Y , we are given a subset $\mathfrak{R}(X, Y)$ of $L(X, Y)$. The class

$$\mathfrak{R} := \bigcup_{X, Y} \mathfrak{R}(X, Y)$$

is said to be an *operator ideal* if the following are satisfied :

- (O₁) $x^* \otimes y \in \mathfrak{R}(X, Y)$ for all $x^* \in X^*$ and $y \in Y$.
(O₂) $S + T \in \mathfrak{R}(X, Y)$ for $S, T \in \mathfrak{R}(X, Y)$.
(O₃) $BTA \in \mathfrak{R}(Z, W)$ for all $T \in \mathfrak{R}(X, Y)$, $A \in L(Z, X)$,
 $B \in L(Y, W)$, and for all Banach spaces X, Y, Z, W .

$\mathfrak{R}(X, Y)$ is called a **component** of the operator ideal $\mathfrak{R}$.

Two operator ideals $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are said to be **equal** , if they coincide componentwise .

Also we shall write $\mathfrak{R}_1 \subseteq \mathfrak{R}_2$ to mean that $\mathfrak{R}_1(X, Y) \subseteq \mathfrak{R}_2(X, Y)$ holds for every pair X, Y of Banach spaces . The adjectives “**largest**” and “**smallest**” used below in connection with ideals always refer to this order relation .

Proposition 2.1.2 : *Let $\mathfrak{R}$ be an operator ideal . Then all components $\mathfrak{R}(X, Y)$ are linear subspaces of $L(X, Y)$.*

Proof : By (O₂) it remains to show that $T \in \mathfrak{R}(X, Y)$ and $\lambda \in \mathbf{K}$ imply $\lambda T \in \mathfrak{R}(X, Y)$. So let I_X and I_Y be the identity operators of X and Y respectively . By (O₃) it follows $\lambda T = (\lambda I_Y) T I_X \in \mathfrak{R}(X, Y)$. $\square$

Definition 2.1.3 : A function $\|\cdot\|_\alpha$ which assigns to every operator $T \in \mathfrak{R}$ a non-negative number $\|T\|_\alpha$ is called an **ideal-norm** on the operator ideal $\mathfrak{R}$ if , for arbitrary Banach spaces X, Y, Z, W , it has the following properties :

$$(N_1) \quad \|x^* \otimes y\|_\alpha = \|x^*\| \|y\| \quad \text{for all } x^* \in X^* \text{ and } y \in Y .$$

$$(N_2) \quad \|S + T\|_\alpha \leq \|S\|_\alpha + \|T\|_\alpha \quad \text{for all } S, T \in \mathfrak{R}(X, Y) .$$

$$(N_3) \quad \|BTA\|_\alpha \leq \|B\| \|T\|_\alpha \|A\| \quad \text{for all } T \in \mathfrak{R}(X, Y) ,$$

$$A \in L(Z, X) , \text{ and } B \in L(Y, W) .$$

Correspondingly , the couple $(\mathfrak{R}, \|\cdot\|_\alpha)$ is called a ***normed operator ideal*** ,
or briefly a ***normed ideal*** .

Definition 2.1.4 : A normed operator ideal $(\mathfrak{R}, \|\cdot\|_\alpha)$ is a ***Banach operator ideal*** if , for every pair X , Y of Banach spaces ,
 $(\mathfrak{R}(X, Y), \|\cdot\|_\alpha)$ is a Banach space .

Proposition 2.1.5 : Let $(\mathfrak{R}, \|\cdot\|_\alpha)$ be a normed operator ideal . For
every component $\mathfrak{R}(X, Y)$, we have

$$\|T\| \leq \|T\|_\alpha \quad \text{for all } T \in \mathfrak{R}(X, Y) .$$

Proof : See [13, p. 423] .

Proposition 2.1.6 : Let $(\mathfrak{R}, \|\cdot\|_\alpha)$ be a normed operator ideal . Then
 $(\mathfrak{R}(X, Y), \|\cdot\|_\alpha)$ is a normed space for all Banach spaces X and Y .

Proof : If $\|T\|_\alpha = 0$, then it follows from Proposition 2.1.5 that
 $T = 0$. Furthermore , if $T \neq 0$, then by (1.1 - T₃) and (N₁) we have

$$\|T\|_{\alpha} = \|0 \otimes 0\|_{\alpha} = \|0\| \cdot \|0\| = 0 .$$

Now let λ be any scalar and let $T \in \mathfrak{R}(X, Y)$. Moreover let I_X and I_Y be the identity operators of X and Y respectively. Hence by (N_3) we have

$$\|\lambda T\|_{\alpha} = \|(\lambda I_Y) T I_X\|_{\alpha} \leq \|(\lambda I_Y)\| \|T\|_{\alpha} \|I_X\| = |\lambda| \|T\|_{\alpha} .$$

Similarly we have

$$\|T\|_{\alpha} = \left\| \frac{1}{\lambda} (\lambda T) \right\|_{\alpha} \leq \frac{1}{|\lambda|} \|\lambda T\|_{\alpha} .$$

Thus $\|\lambda T\|_{\alpha} = |\lambda| \|T\|_{\alpha}$.

Finally, by (N_2) , the triangular inequality of $\|\cdot\|_{\alpha}$ is satisfied and therefore the function $\|\cdot\|_{\alpha}$ indeed is a norm on $\mathfrak{R}(X, Y)$. $\square$

Definition 2.1.7 : An operator ideal $\mathfrak{R}$ is said to be *small* if, whenever $L(X, Y) = \mathfrak{R}(X, Y)$ then either X or Y is finite dimensional.

$\mathfrak{R}$ is said to be *closed*, if all components $\mathfrak{R}(X, Y)$ are closed linear subspaces of $L(X, Y)$.

That is, $\mathfrak{R}$ is a Banach operator ideal with the ordinary operator norm.

Definition 2.1.8 : An operator $J \in L(X, Y)$ is said to be a *metric injection* if it is an isometric into map.

Definition 2.1.9 : A Banach operator ideal $(\mathfrak{R}, \|\cdot\|_{\alpha})$ is said to be *injective* if, for arbitrary Banach spaces X, Y, Z , given any metric injection $J \in L(Y, Z)$,

$T \in L(X, Y)$ and $JT \in \mathfrak{R}(X, Z)$ imply $T \in \mathfrak{R}(X, Y)$.

This means that the answer to the question of whether or not an operator $T \in L(X, Y)$ belongs to $\mathfrak{R}$ is independent on the particular choice of the range space for T .

2.2. Examples of Operator Ideals :

Throughout this section , let X and Y be arbitrary Banach spaces unless the contrary is explicitly stated .

(I) Bounded Linear Operators :

Let $L := \bigcup_{X, Y} L(X, Y)$. One can easily prove the :

Proposition 2.2.1 : *L is the largest operator ideal . Moreover L is a Banach operator ideal when equipped with the operator norm .*

(II) Finite Rank Operators :

Let $\mathfrak{F} := \bigcup_{X, Y} \mathfrak{F}(X, Y)$.

Proposition 2.2.2 : *$\mathfrak{F}$ is the smallest operator ideal .*

Proof : One can easily verify that $\mathfrak{F}$ satisfies the ideal axioms . We need only to show that $\mathfrak{F}$ is the smallest operator ideal . To do this let $\mathfrak{R}$ be arbitrary operator ideal . It follows from (O_1) and Proposition 2.1.2 that

$\mathfrak{I}(X, Y) = X^* \otimes Y \subseteq \mathfrak{R}(X, Y)$ which shows that $\mathfrak{I} \subseteq \mathfrak{R}$. $\square$

(III) Compact Operators :

Definition 2.2.3 : An operator $T \in L(X, Y)$ is said to be *compact* if it maps the closed unit ball of X into a relatively compact subset of Y . The set of compact operators is denoted by $K(X, Y)$. Further let

$$K := \bigcup_{X, Y} K(X, Y) .$$

Theorem 2.2.4 : $K(X, Y)$ is a closed linear subspace of $L(X, Y)$ in the operator norm .

Proof : See [32, p. 388] .

Theorem 2.2.5 : K is a closed operator ideal .

Proof : See [10, p. 486] and [13, p. 368] .

We refer to [15] and [32] for more on $K(X, Y)$.

(IV) Weakly Compact Operators :

Definition 2.2.6 : An operator $T \in L(X, Y)$ is said to be *weakly compact* if T maps the closed unit ball of X into a relatively weakly compact subset of Y .

We denote the set of all weakly compact operators from X into Y by $W(X, Y)$. Further set $W := \bigcup_{X, Y} W(X, Y)$.

Theorem 2.2.7 : W is a closed operator ideal.

Proof : See [32] and [10].

Theorem 2.2.8 : If either X or Y is reflexive, then

$$W(X, Y) = L(X, Y).$$

Proof : See [10, p. 483].

We refer to [32] and [10] for the basic properties of $W(X, Y)$.

(V) p -Nuclear Operators :

Definition 2.2.9 : Let $1 \leq p < \infty$. An operator $T \in L(X, Y)$ is called *p -nuclear* if T can be written in the form

$$T = \sum_{n=1}^{\infty} x_n^* \otimes y_n,$$

where $(x_n^*) \in \ell^p(X^*)$ and $(y_n) \in \ell^p \langle Y \rangle$.

Each such representation of T is called *p -nuclear representation*. The set of all p -nuclear operators from X into Y is denoted by $N_p(X, Y)$ and we let $N_p := \bigcup_{X, Y} N_p(X, Y)$.

For $T \in N_p(X, Y)$ we define

$$\|T\|_{n(p)} := \inf \left\{ \|(x_n^*)\|_{\alpha(p)} \cdot \|(y_n)\|_{\varepsilon(p^*)} \right\},$$

where the infimum is taken over all p -nuclear representations of T .

Theorem 2.2.10 : $(N_p, \|\cdot\|_{n(p)})$ is a Banach operator ideal.

Proof : See [13, p. 435] and [35, p. 40].

Proposition 2.2.11 : Let $1 \leq p \leq q < \infty$. Then $N_p \subseteq N_q$ and

$$\|\cdot\|_{n(q)} \leq \|\cdot\|_{n(p)} \text{ on } N_p.$$

Proof : See [35, p. 41].

Theorem 2.2.12 : Let $1 \leq p < \infty$ and let $T \in L(X, Y)$. Then the following are equivalent :

- (1) $T \in N_p(X, Y)$.
- (2) There exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ A \downarrow & & \uparrow B \\ \ell^\infty & \xrightarrow{D} & \ell^p \end{array}$$

such that $D \in L(\ell^\infty, \ell^p)$ is an operator of the form $D(\alpha_i) = (\sigma_i \alpha_i)$ with $(\sigma_i) \in \ell^p$, $1 \leq p < \infty$, $B \in L(\ell^p, Y)$ and $A \in L(X, \ell^\infty)$. In this case

$$\|T\|_{n(p)} = \inf \left\{ \|A\| \|(\sigma_i)\| \|B\| \right\},$$

where the infimum is taken over all possible factorizations .

Proof : See [13, p. 437] .

(VI) p -Integral Operators :

Definition 2.2.13 : Let $1 \leq p < \infty$. An operator $T \in L(X, Y)$ is called *p -integral* , if there exists a compact Hausdorff space K and a probability measure μ on K and operators $A \in L(X, C(K))$ and $B \in L(L^p(K, \mu), Y^{**})$ such that

$$J_Y T = B J_p A ,$$

Where $J_p \in L(C(K), L^p(K, \mu))$ is the natural inclusion map and $J_Y \in L(Y, Y^{**})$ is the natural embedding .

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \xrightarrow{J_Y} & Y^{**} \\ \downarrow A & & & & \uparrow B \\ C(K) & \xrightarrow{J_p} & L^p(K, \mu) & & \end{array} \quad (*)$$

The set of all p -integral operators from X into Y is denoted by $I_p(X, Y)$. For $T \in I_p(X, Y)$, we define

$$\|T\|_{i(p)} := \inf \left\{ \|B\| \cdot \|A\| \right\},$$

where the infimum is taken over all factorizations $(*)$.

Theorem 2.2.14 : Let $1 \leq p < \infty$ and let $I_p := \bigcup_{X,Y} I_p(X,Y)$. Then $(I_p, \|\cdot\|_{i(p)})$ is a normed operator ideal .

Proof : See [35, p. 42] .

(VII) Completely Continuous Operators :

Definition 2.2.15 : An operator $T \in L(X,Y)$ is called *completely continuous* , if every weakly convergence sequence (x_n) in X is mapped into a norm convergent sequence (Tx_n) in Y .

The set of all completely continuous operators from X into Y is denoted by $\mathcal{C}(X,Y)$. Further we let

$$\mathcal{C} := \bigcup_{X,Y} \mathcal{C}(X,Y) .$$

Proposition 2.2.16 : $\mathcal{C}$ is an operator ideal .

Proof : See [27, p. 47] .

(IX) Absolutely p -Summing Operators :

Definition 2.2.17 : Let $1 \leq p < \infty$. An operator $T \in L(X, Y)$ is said to be *absolutely p -summing* , or briefly *p -summing* , if there exists a constant $\lambda \geq 0$ such that

$$\left(\sum_{i=1}^n \|T x_i\|^p \right)^{\frac{1}{p}} \leq \lambda \sup \left\{ \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} : x^* \in B_1(X^*) \right\} \quad (**)$$

for every finite family $\{x_1, \dots, x_n\} \subseteq X$.

The set of all absolutely p -summing operators from X into Y is denoted by $\Pi_p(X, Y)$. In the special case $p=1$ we simply say absolutely summing operators . Further let

$$\Pi_p := \bigcup_{X, Y} \Pi_p(X, Y) \text{ .}$$

For $T \in \Pi_p(X, Y)$, we define

$$\|T\|_{\pi(p)} := \inf \{ \lambda \} ,$$

the infimum being taken over all $\lambda \geq 0$ for which (**) holds .

Thus if $T \in \Pi_p(X, Y)$, from (**) we have :

$$\left(\sum_{i=1}^n \|T x_i\|^p \right)^{\frac{1}{p}} \leq \|T\|_{\pi(p)} \sup \left\{ \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} : x^* \in B_1(X^*) \right\} .$$

We will study $\Pi_p(X, Y)$ in some details .

2.3. Algebraic Properties of p-Summing Operators :

Throughout this section let X and Y be arbitrary Banach spaces and let $1 \leq p < \infty$.

Lemma 2.3.1 : *For every $T \in \Pi_p(X, Y)$, we have $\|T\| \leq \|T\|_{\pi(p)}$.*

Proof : Let $T \in \Pi_p(X, Y)$. By definition we have

$$\left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} \leq \|T\|_{\pi(p)} \sup \left\{ \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} : x^* \in B_1(X^*) \right\}$$

for $x_1, \dots, x_n$ in X . Taking $n = 1$, we get

$$\begin{aligned} \|Tx\| &\leq \|T\|_{\pi(p)} \sup \left\{ |\langle x, x^* \rangle| : x^* \in B_1(X^*) \right\} \\ &= \|T\|_{\pi(p)} \|x\| \quad \text{for all } x \in X . \end{aligned}$$

This proves that $\|T\| \leq \|T\|_{\pi(p)}$. $\square$

Lemma 2.3.2 : $(\Pi_p(X, Y), \|\cdot\|_{\pi(p)})$ is a normed linear space .

Proof : Let $S, T \in \Pi_p(X, Y)$. By Minkowski's inequality we have

$$\begin{aligned} \left(\sum_{i=1}^n \|(S+T)(x_i)\|^p \right)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^n \|Sx_i\|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} \\ &\leq [\|S\|_{\pi(p)} + \|T\|_{\pi(p)}] \sup_{x^* \in B_1(X^*)} \left\{ \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} \right\} \end{aligned}$$

for $x_1, \dots, x_n \in X$.

Hence

$$S + T \in \Pi_p(X, Y) \quad \text{and} \quad \|S + T\|_{\pi(p)} \leq \|S\|_{\pi(p)} + \|T\|_{\pi(p)} .$$

Further let $\alpha \in \mathbf{K}$ and $T \in \Pi_p(X, Y)$. Then for $x_1, \dots, x_n \in X$ we have

$$\begin{aligned} \left(\sum_{i=1}^n \|\alpha T x_i\|^p \right)^{\frac{1}{p}} &= |\alpha| \left(\sum_{i=1}^n \|T x_i\|^p \right)^{\frac{1}{p}} \\ &\leq |\alpha| \|T\|_{\pi(p)} \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} . \end{aligned}$$

This shows $\|\alpha T\|_{\pi(p)} \leq |\alpha| \|T\|_{\pi(p)}$.

Thus $\Pi_p(X, Y)$ is a linear subspace of $L(X, Y)$. To complete the proof , we may assume that $\alpha \neq 0$. Then by the same argument above we obtain

$$\|T\|_{\pi(p)} \leq \frac{1}{|\alpha|} \|\alpha T\|_{\pi(p)} .$$

Consequently

$$\|\alpha T\|_{\pi(p)} = |\alpha| \|T\|_{\pi(p)} .$$

Clearly , if $T = 0$ then $\|T\|_{\pi(p)} = 0$. Moreover , whenever $\|T\|_{\pi(p)} = 0$ it follows from Lemma 2.3.1 that $T = 0$ and hence $\|\cdot\|_{\pi(p)}$ is indeed a norm on $\Pi_p(X, Y)$. $\square$

Theorem 2.3.3 : $(\Pi_p(X, Y), \|\cdot\|_{\pi(p)})$ is a Banach space .

Proof : Let (T_n) be a Cauchy sequence in $\Pi_p(X, Y)$. By Lemma 2.3.1 , (T_n) is also a Cauchy sequence in $L(X, Y)$. Since $L(X, Y)$ is

complete , (T_n) has a limit $T \in L(X, Y)$ with respect to the operator norm . Given $\varepsilon > 0$ we choose N_0 so that

$$\|T_m - T_n\|_{\pi(p)} \leq \varepsilon \quad \text{for all } m, n \geq N_0 .$$

This means that

$$\left(\sum_{i=1}^k \|(T_m - T_n)(x_i)\|^p \right)^{\frac{1}{p}} \leq \varepsilon \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^k |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}}$$

for $x_1, \dots, x_k \in X$. Letting $m \rightarrow \infty$, we get for all $n \geq N_0$

$$\left(\sum_{i=1}^k \|(T - T_n)(x_i)\|^p \right)^{\frac{1}{p}} \leq \varepsilon \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^k |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} .$$

Thus

$$T - T_n \in \Pi_p(X, Y) \quad \text{and} \quad \|T - T_n\|_{\pi(p)} \leq \varepsilon \quad \text{for all } n \geq N_0 .$$

Since $\Pi_p(X, Y)$ is a linear space ,

$$T = (T - T_n) + T_n \in \Pi_p(X, Y) \quad \text{for all } n \geq N_0 .$$

Thus T is also the limit of (T_n) with respect to the norm of $\Pi_p(X, Y)$.

This ends the proof . $\square$

Theorem 2.3.4 : $(\Pi_p, \|\cdot\|_{\pi(p)})$ is an injective Banach operator ideal .

Proof : Since , by Theorem 2.3.3 , all components of Π_p are Banach spaces with $\|\cdot\|_{\pi(p)}$, it remains to show that Π_p is an injective

normed operator ideal .

So let $x^* \in X^*$ and $y \in Y$ be arbitrary and let $x_1, \dots, x_n \in X$. Then

$$\begin{aligned} \left(\sum_{i=1}^n \left\| (x^* \otimes y)(x_i) \right\|^p \right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n \left\| \langle x_i, x^* \rangle y \right\|^p \right)^{\frac{1}{p}} \\ &= \|x^*\| \|y\| \left(\sum_{i=1}^n \left| \left\langle x_i, \frac{x^*}{\|x^*\|} \right\rangle \right|^p \right)^{\frac{1}{p}} \\ &\leq \|x^*\| \|y\| \sup_{z^* \in B_1(X^*)} \left(\sum_{i=1}^n \left| \langle x_i, z^* \rangle \right|^p \right)^{\frac{1}{p}} . \end{aligned}$$

Hence

$$x^* \otimes y \in \Pi_p(X, Y) \text{ with } \|x^* \otimes y\|_{\pi(p)} \leq \|x^*\| \|y\| .$$

On the other hand , by Lemma 2.3.1 and Theorem 1.2.1 , it follows that

$$\|x^*\| \|y\| = \|x^* \otimes y\| \leq \|x^* \otimes y\|_{\pi(p)} .$$

Consequently $\|x^* \otimes y\|_{\pi(p)} = \|x^*\| \|y\|$.

Now let X, Y, W , and Z be arbitrary Banach spaces . Further , let $T \in \Pi_p(X, Y)$, $A \in L(W, X)$, and $B \in L(Y, Z)$. Then

$$\begin{aligned} \left(\sum_{i=1}^n \left\| (BTA)(w_i) \right\|^p \right)^{\frac{1}{p}} &\leq \|B\| \left(\sum_{i=1}^n \left\| T(A w_i) \right\|^p \right)^{\frac{1}{p}} \\ &\leq \|B\| \|T\|_{\pi(p)} \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^n \left| \langle A w_i, x^* \rangle \right|^p \right)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
&\leq \|B\| \|T\|_{\pi(p)} \|A^*\| \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^n \left| \left\langle w_i, \frac{A^* x^*}{\|A^*\|} \right\rangle \right|^p \right)^{\frac{1}{p}} \\
&\leq \|B\| \|T\|_{\pi(p)} \|A\| \sup_{w^* \in B_1(W^*)} \left(\sum_{i=1}^n |\langle w_i, w^* \rangle|^p \right)^{\frac{1}{p}}
\end{aligned}$$

for $w_1, \dots, w_n \in W$. Thus

$$BTA \in \Pi_p(W, Z) \text{ and } \|BTA\|_{\pi(p)} \leq \|B\| \|T\|_{\pi(p)} \|A\|.$$

Hence $(\Pi_p, \|\cdot\|_{\pi(p)})$ is a normed operator ideal.

For injectivity, let $T \in L(X, Y)$ and let $J \in L(Y, Z)$ be any metric injection such that $JT \in \Pi_p(X, Z)$ where Z is arbitrary Banach space.

Then for $x_1, \dots, x_n \in X$ we have

$$\begin{aligned}
\left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n \|J(Tx_i)\|^p \right)^{\frac{1}{p}} \\
&\leq \|JT\|_{\pi(p)} \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

This proves that $T \in \Pi_p(X, Y)$ which means that $(\Pi_p, \|\cdot\|_{\pi(p)})$ is an injective Banach operator ideal. $\square$

Theorem 2.3.5 : Let $1 \leq p \leq q < \infty$. Then $\Pi_p \subseteq \Pi_q$ and

$$\|T\|_{\pi(q)} \leq \|T\|_{\pi(p)} \text{ for all } T \in \Pi_p.$$

Proof : Choose $r > 0$ so that $\frac{1}{r} + \frac{1}{q} = \frac{1}{p}$, i.e., $q = p + \frac{pq}{r}$. Let

$T \in \Pi_p(X, Y)$ and let $x_1, \dots, x_n \in X$. Set

$$\alpha_i := \|T x_i\|_r^{\frac{q}{r}}, \text{ for } i = 1, \dots, n.$$

Since $T \in \Pi_p(X, Y)$, we have

$$\begin{aligned} \left(\sum_{i=1}^n \|T x_i\|^q \right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n \|T(\alpha_i x_i)\|^p \right)^{\frac{1}{p}} \\ &\leq \|T\|_{\pi(p)} \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^n |\langle \alpha_i x_i, x^* \rangle|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Now using the generalized Hölder's inequality we obtain

$$\left(\sum_{i=1}^n \|T x_i\|^q \right)^{\frac{1}{p}} \leq \|T\|_{\pi(p)} \left(\sum_{i=1}^n |\alpha_i|^r \right)^{\frac{1}{r}} \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^q \right)^{\frac{1}{q}}.$$

Thus

$$\begin{aligned} \left(\sum_{i=1}^n \|T x_i\|^q \right)^{\frac{1}{p} - \frac{1}{r}} &= \left(\sum_{i=1}^n \|T x_i\|^q \right)^{\frac{1}{q}} \\ &\leq \|T\|_{\pi(p)} \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^q \right)^{\frac{1}{q}}. \end{aligned}$$

This proves that

$$T \in \Pi_q(X, Y) \text{ and } \|T\|_{\pi(q)} \leq \|T\|_{\pi(p)} \text{ for } T \in \Pi_p(X, Y). \quad \square$$

2.4. Characterizations of p-Summing Operators :

Throughout this section , let X and Y be arbitrary Banach spaces and let $1 \leq p < \infty$.

The term “ absolutely p-summing “ is derived from the fact that such operators take weakly p-summable sequences into absolutely p-summable ones . More precisely we have the following criterion :

Theorem 2.4.1 : *An operator $T \in L(X, Y)$ is absolutely p-summing if and only if it maps weakly p-summable sequences in X into absolutely p-summable sequences in Y .*

Proof : Let $T \in \Pi_p(X, Y)$ and let $(x_n) \in \ell^p(X)$. Then

$$\left(\sum_{i=1}^k \|Tx_i\|^p \right)^{\frac{1}{p}} \leq \|T\|_{\pi(p)} \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^k |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} \text{ for all } k \in \mathbb{N} .$$

Passing to the limit as $k \rightarrow \infty$ yields

$$\|(Tx_i)\|_{\alpha(p)} \leq \|T\|_{\pi(p)} \cdot \|(x_i)\|_{\varepsilon(p)} \quad (*)$$

Conversely , Suppose T maps weakly p-summable sequences in X into absolutely p-summable sequences in Y . Define

$$\hat{T}: \ell^p(X) \rightarrow \ell^p(Y) ,$$

$$\hat{T}(x_n) := (Tx_n) .$$

Clearly $\hat{T}$ is linear . We claim that $\hat{T}$ is continuous . To do this let (x_k) be a sequence in $\ell^p(X)$, where

$$x_k := (x_1^k, x_2^k, \dots, x_n^k, \dots) \text{ for all } n \in \mathbb{N}.$$

Further, assume that (x_k) converges to a limit $x := (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n, \dots)$ in $\ell^p\langle X \rangle$ and $\hat{T}(x_k)$ converges to a limit (y_n) in $\ell^p(Y)$. We need to show that $\hat{T}x = y$.

By proposition 1.3.11 we have for all $k \in \mathbb{N}$

$$\begin{aligned} \|y - \hat{T}x\|_{\varepsilon(p)} &\leq \|y - \hat{T}x_k\|_{\varepsilon(p)} + \|\hat{T}x_k - \hat{T}x\|_{\varepsilon(p)} \\ &\leq \|y - \hat{T}x_k\|_{\alpha(p)} + \|(Tx_n^k) - (T\hat{x}_n)\|_{\varepsilon(p)} \\ &\leq \|y - \hat{T}x_k\|_{\alpha(p)} + \sup_{y^* \in B_1(Y^*)} \left(\sum_{n=1}^{\infty} |\langle T(x_n^k - \hat{x}_n), y^* \rangle|^p \right)^{\frac{1}{p}} \\ &\leq \|y - \hat{T}x_k\|_{\alpha(p)} + \|T^*\| \cdot \sup_{y^* \in B_1(Y^*)} \left(\sum_{n=1}^{\infty} \left| \left\langle x_n^k - \hat{x}_n, \frac{T^*y^*}{\|T^*\|} \right\rangle \right|^p \right)^{\frac{1}{p}} \\ &\leq \|y - \hat{T}x_k\|_{\alpha(p)} + \|T\| \cdot \|x_k - x\|_{\varepsilon(p)}. \end{aligned}$$

Letting $k \rightarrow \infty$, we have

$$\|y - \hat{T}x\|_{\varepsilon(p)} \leq 0 \text{ implies } y = \hat{T}x.$$

Thus by the Closed Graph Theorem [15, p. 292] $\hat{T}$ must be continuous.

Hence

$$\left(\sum_{i=1}^{\infty} \|Tx_i\|^p \right)^{\frac{1}{p}} \leq \|\hat{T}\| \cdot \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^{\infty} |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} \quad (**)$$

for all $(x_i) \in \ell^p\langle X \rangle$. Therefore T is absolutely p -summing. $\square$

We can reformulate Theorem 2.4.1 in the following strong way:

Theorem 2.4.2 *An operator $T \in L(X, Y)$ is absolutely p -summing if and only if*

$$\hat{T}(x_n) := (Tx_n)$$

defines a continuous linear operator from $\ell^p(X)$ into $\ell^p(Y)$.

$$\text{In this case, } \|T\|_{\pi(p)} = \|\hat{T}\|.$$

Theorem 2.4.3 : *An operator $T \in L(X, Y)$ is absolutely p -summing if and only if*

$$\tilde{T}(f)(t) := T(f(t))$$

defines a continuous linear operator from $L^p(I, X)$ into $L^p(I, Y)$.

In this case,

$$\|T\|_{\pi(p)} = \|\tilde{T}\|.$$

Proof : Let $T \in \Pi_p(X, Y)$. Then obviously $\tilde{T}$ is a well-defined linear operator on the subspace $S(I, X)$. So for every simple function

$$s := \sum_{i=1}^n 1_{E_i} x_i \text{ where } 1_{E_i} \text{'s are the characteristic functions of the}$$

measurable subsets E_i 's and $\bigcup_{i=1}^n E_i = I$ we have

$$\begin{aligned}
\|\tilde{T}(s)\|_{\alpha(p)}^p &= \int_I \|T(s(t))\|^p d\mu(t) \\
&= \int_I \left\| T\left(\sum_{i=1}^n 1_{E_i}(t) x_i\right) \right\|^p d\mu(t) \\
&= \int_I \left\| \sum_{i=1}^n 1_{E_i}(t) T x_i \right\|^p d\mu(t) \quad .
\end{aligned}$$

By disjointness of E_i 's and linearity of the integral we have

$$\begin{aligned}
\|\tilde{T}(s)\|_{\alpha(p)}^p &= \sum_{i=1}^n \int_{E_i} \|T x_i\|^p d\mu(t) \\
&= \sum_{i=1}^n \|T x_i\|^p \mu(E_i) \\
&= \sum_{i=1}^n \left\| T\left([\mu(E_i)]^{\frac{1}{p}} x_i\right) \right\|^p \quad .
\end{aligned}$$

Since $T \in \Pi_p(X, Y)$ we have

$$\begin{aligned}
\|\tilde{T}(s)\|_{\alpha(p)}^p &\leq \|T\|_{\pi(p)}^p \sup_{x^* \in B_1(X^*)} \sum_{i=1}^n \left| \langle x_i [\mu(E_i)]^{\frac{1}{p}}, x^* \rangle \right|^p \\
&= \|T\|_{\pi(p)}^p \sup_{x^* \in B_1(X^*)} \sum_{i=1}^n \mu(E_i) \cdot \left| \langle x_i, x^* \rangle \right|^p
\end{aligned}$$

It follows from

$$\int_I \left| \langle s(t), x^* \rangle \right|^p d\mu(t) = \int_I \left| \sum_{i=1}^n 1_{E_i}(t) \langle x_i, x^* \rangle \right|^p d\mu(t)$$

$$\begin{aligned}
&= \sum_{j=1}^n \int_{E_j} \left| \sum_{i=1}^n 1_{E_i}(t) \langle x_i, x^* \rangle \right|^p d\mu(t) \\
&= \sum_{j=1}^n \int_{E_j} \left| 1_{E_j}(t) \langle x_j, x^* \rangle \right|^p d\mu(t) \\
&= \sum_{j=1}^n \mu(E_j) \left| \langle x_j, x^* \rangle \right|^p
\end{aligned}$$

that

$$\begin{aligned}
\|\tilde{T}(s)\|_{\alpha(p)} &\leq \|T\|_{\pi(p)} \sup_{x^* \in B_1(X^*)} \left(\int_I \left| \langle s(t), x^* \rangle \right|^p d\mu(t) \right)^{\frac{1}{p}} \\
&= \|T\|_{\pi(p)} \cdot \|s\|_{\varepsilon(p)} .
\end{aligned} \tag{*}$$

Thus $\tilde{T}$ is a continuous linear operator from the simple functions $S(I, X)$ in $L^p(I, X)$ into the simple functions $S(I, Y)$ in $L^p(I, Y)$. By Lemma 1.3.4 and by [15, p. 100], $\tilde{T}$ has a continuous extension

$$\tilde{T}_0: L^p(I, X) \rightarrow L^p(I, Y) .$$

Now let $f \in L^p(I, X)$. By Lemma 1.3.4, there is a sequence (s_n) of simple functions such that

$$\lim_n \|s_n - f\|_{\varepsilon(p)} = 0 \quad \text{and} \quad f(t) = \lim_n s_n(t) \quad \mu - \text{a.e.} .$$

Therefore we have

$$\tilde{T}_0(f)(t) = \left(\lim_n \tilde{T}_0(s_n) \right)(t) = \lim_n \tilde{T}(s_n)(t) = \lim_n T(s_n(t))$$

and for $\mu - \text{a.e. } t \in I$,

$$\tilde{T}(f)(t) = T(f(t)) = T\left(\lim_n s_n(t)\right) = \lim_n T(s_n(t)) .$$

Hence as a member of $L^p(I, Y)$, it follows

$$\tilde{T}(f) = \tilde{T}_o(f)$$

and consequently $\tilde{T}$ is well-defined on $L^p(I, X)$.

Moreover , by continuity of $\tilde{T}$, we have from (*)

$$\|\tilde{T}(f)\|_{\alpha(p)} \leq \|T\|_{\pi(p)} \|f\|_{\varepsilon(p)} \text{ for } f \in L^p(I, X)$$

and therefore $\|\tilde{T}\| \leq \|T\|_{\pi(p)}$.

Conversely , let $x_1, \dots, x_n \in X$. Consider

$$f: I \rightarrow X ,$$

$$f := \sum_{i=1}^n 1_{E_i} x_i ,$$

with $E_i \cap E_j = \emptyset$ for $i \neq j$ and $\mu(E_i) = \mu(E_j) = \beta$ for $i, j = 1, \dots, n$. Then

by the assumption we have

$$\begin{aligned} \|\tilde{T}(f)\|_{\alpha(p)} &= \left(\int_I \|\tilde{T}(f)(t)\|^p d\mu(t) \right)^{\frac{1}{p}} \\ &= \left(\int_I \|T(f(t))\|^p d\mu(t) \right)^{\frac{1}{p}} \\ &\leq \|\tilde{T}\| \sup_{x^* \in B_1(X^*)} \left(\int_I |\langle f(t), x^* \rangle|^p d\mu(t) \right)^{\frac{1}{p}} \end{aligned}$$

$$\leq \|\tilde{T}\| \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^n \int_{E_i} |\langle f(t), x^* \rangle|^p d\mu(t) \right)^{\frac{1}{p}}.$$

But $f(t) = x_i$ on E_i and $\beta = \mu(E_i) = \mu(E_j)$ for $i, j = 1, \dots, n$. Hence

$$\begin{aligned} \|\tilde{T}(f)\|_{\alpha(p)} &= \left(\int_I \|\tilde{T}(f)(t)\|^p d\mu(t) \right)^{\frac{1}{p}} \\ &= \left(\int_I \left\| \sum_{j=1}^n 1_{E_j}(t) T x_j \right\|^p d\mu(t) \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^n \int_{E_i} \left\| \sum_{j=1}^n 1_{E_j}(t) T x_j \right\|^p d\mu(t) \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^n \int_{E_i} \|T x_i\|^p d\mu(t) \right)^{\frac{1}{p}} \\ &= \beta^{\frac{1}{p}} \left(\sum_{i=1}^n \|T x_i\|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Similarly we have

$$\left(\int_I |\langle f(t), x^* \rangle|^p d\mu(t) \right)^{\frac{1}{p}} = \beta^{\frac{1}{p}} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}}.$$

Consequently we obtain

$$\left(\sum_{i=1}^n \|T x_i\|^p \right)^{\frac{1}{p}} \leq \|\tilde{T}\| \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} .$$

This proves that

$$T \in \Pi_p(X, Y) \quad \text{and} \quad \|T\|_{\pi(p)} = \|\tilde{T}\| \quad . \quad \square$$

Now we follow a common habit and call ***probability measure*** on a compact Hausdorff space K every $\mu \in M(K)$ such that μ is positive and $\|\mu\| = \mu(K) = 1$.

A fundamental result linking measure theory to the theory of absolutely p -summing operators is the following :

Theorem 2.4.4 (Pietsch Domination Theorem) :

An operator $T \in L(X, Y)$ is absolutely p -summing if and only if there are a regular Borel probability measure μ on $B_1(X^*)$ and a constant $\lambda \geq 0$ such that

$$\|T x\| \leq \lambda \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \quad \text{for all } x \in X . \quad (*)$$

In this case ,

$$\|T\|_{\pi(p)} = \inf \{ \lambda : \lambda \text{ satisfies } (*) \} .$$

Proof : Let $T \in L(X, Y)$ for which $(*)$ is satisfied . Then for $x_1, \dots, x_n \in X$ we have

$$\begin{aligned}
\sum_{i=1}^n \|T x_i\|^p &\leq \lambda^p \sum_{i=1}^n \int_{B_1(X^*)} |\langle x_i, x^* \rangle|^p d\mu(x^*) \\
&\leq \lambda^p \int_{B_1(X^*)} \sum_{i=1}^n |\langle x_i, x^* \rangle|^p d\mu(x^*) \\
&\leq \lambda^p \int_{B_1(X^*)} \sup_{x^* \in B_1(X^*)} \sum_{i=1}^n |\langle x_i, x^* \rangle|^p d\mu(x^*) \\
&= \lambda^p \cdot \sup_{x^* \in B_1(X^*)} \sum_{i=1}^n |\langle x_i, x^* \rangle|^p .
\end{aligned}$$

This proves

$$T \in \Pi_p(X, Y) \text{ and } \|T\|_{\pi(p)} \leq \lambda \text{ for all } \lambda \text{ satisfying } (*) .$$

Thus

$$\|T\|_{\pi(p)} \leq \inf \{ \lambda : \lambda \text{ satisfies } (*) \} . \quad (**)$$

Conversely , Let $T \in \Pi_p(X, Y)$. Then for $x_1, \dots, x_n \in X$ we define

$$f_{x_1, \dots, x_n} : B_1(X^*) \rightarrow \mathbb{R} ,$$

$$f_{x_1, \dots, x_n}(x^*) := \|T\|_{\pi(p)}^p \cdot \sum_{i=1}^n |\langle x_i, x^* \rangle|^p - \sum_{i=1}^n \|T x_i\|^p .$$

Now each $f_{x_1, \dots, x_n}$ is continuous with respect to the weak*-toplogy of $B_1(X^*)$. Moreover ,

$$T \in \Pi_p(X, Y) \quad \text{implies} \quad \sup_{x^* \in B_1(X^*)} f_{x_1, \dots, x_n}(x^*) \geq 0 \quad .$$

From this and from compactness of $B_1(X^*)$, [6, p. 130] , one can deduce that $f_{x_1, \dots, x_n}$ is somewhere nonnegative .Let

$$P := \left\{ f_{x_1, \dots, x_n} : x_1, \dots, x_n \in X \right\} \quad .$$

It can be shown easily that P is a convex subset of $C(B_1(X^*))$.

Furthermore P is disjoint from the convex subset

$$A := \left\{ f \in C(B_1(X^*)) : f(x^*) < 0 \text{ for each } x^* \in B_1(X^*) \right\} \quad .$$

Now A is clearly open in $C(B_1(X^*))$.

By the Separation Theorem [23, p. 65] , there are $\alpha \in \mathbb{R}$ and $\mu \in M(B_1(X^*))$ such that

$$\langle g, \mu \rangle < \alpha \leq \langle f_{x_1, \dots, x_n}, \mu \rangle \quad \text{for all } g \in A \text{ and } x_1, \dots, x_n \in X \quad .$$

By Riesz Representation Theorem [6, p. 75] we have

$$\int_{B_1(X^*)} g(x^*) d\mu(x^*) < \alpha \leq \int_{B_1(X^*)} f_{x_1, \dots, x_n}(x^*) d\mu(x^*)$$

for all $g \in A$ and $x_1, \dots, x_n \in X$.

Because $0 \in P$, we have $\alpha \leq 0$. Since A contains the constant function given by the negative numbers we get $\alpha = 0$ and $\mu(B_1(X^*)) > 0$. Further

we may assume that $\|\mu\| = 1$. From the above right hand inequality we obtain

$$\int_{B_1(X^*)} f_x(x^*) d\mu(x^*) \geq 0 \quad \text{for all } x \in X.$$

But this just says that

$$\|T_x\| \leq \lambda \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \quad \text{for all } x \in X.$$

Finally combining this result with (**) we have

$$\|T\|_{\pi(p)} = \inf \{ \lambda : \lambda \text{ satisfies } (*) \}. \quad \square$$

Suppose K_0 is a compact subset of $B_1(X^*)$ in the weak*-topology such that

$$\sup_{x^* \in K_0} \sum_{i=1}^n |\langle x_i, x^* \rangle|^p = \sup_{x^* \in B_1(X^*)} \sum_{i=1}^n |\langle x_i, x^* \rangle|^p \quad (***)$$

for every finite family $\{x_1, \dots, x_n\} \subseteq X$.

Then we can replace $B_1(X^*)$ by K_0 in the preceding theorem. For example, let K_0 be any norming subset of X^* which is compact in the weak*-topology. By Theorem 1.3.19, (***) is satisfied and thus we may restate the Pietsch Domination Theorem in the following more general way :

Theorem 2.4.5 (Pietsch Domination Theorem) :

Let K_0 be any norming subset of X^* which is compact in the weak*-topology . An operator $T \in L(X, Y)$ is absolutely p -summing if and only if there are a regular Borel probability measure μ on K_0 and a constant $\lambda \geq 0$ such that

$$\|Tx\| \leq \lambda \left(\int_{K_0} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \text{ for all } x \in X . \quad (*)$$

In this case ,

$$\|T\|_{\pi(p)} = \inf \{ \lambda : \lambda \text{ satisfies } (*) \} .$$

Now we state the Pietsch Domination Theorem in a special but rather illustrative situation .

Proposition 2.4.6 : Let K be any compact Hausdorff space . An operator $T \in L(C(K), Y)$ is absolutely p -summing if and only if there are a regular Borel probability measure μ on K and a constant $\lambda \geq 0$ such that

$$\|Tf\| \leq \lambda \left(\int_K |f(x)|^p d\mu(x) \right)^{\frac{1}{p}} \text{ for all } f \in C(K) .$$

Proof : Using Proposition 1.3.21 , we see that an operator $T \in L(C(K), Y)$ is absolutely p -summing if and only if there is a constant $\lambda \geq 0$ such that

$$\left(\sum_{i=1}^n \|T f_i\|^p \right)^{\frac{1}{p}} \leq \lambda \sup_{t \in K} \left(\sum_{i=1}^n |f_i(t)|^p \right)^{\frac{1}{p}} .$$

Now the proof is completely along the lines of the proof of the Pietsch Domination Theorem . $\square$

We shall remark that one can use the Pietsch Domination Theorem to prove Theorem 2.3.5 directly as in the following :

Corollary 2.4.7 : *Let $1 \leq p \leq q < \infty$. Then $\Pi_p \subseteq \Pi_q$ and*

$$\|T\|_{\pi(q)} \leq \|T\|_{\pi(p)} \text{ for all } T \in \Pi_p .$$

Proof : Let $T \in \Pi_p(X, Y)$. Then there is a regular Borel probability measure μ defined on $B_1(X^*)$ for which

$$\|Tx\| \leq \|T\|_{\pi(p)} \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \text{ for all } x \in X .$$

Choose r so that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. By using the generalized Holder's inequality we have

$$\|Tx\| \leq \|T\|_{\pi(p)} \left(\int_{B_1(X^*)} 1^r d\mu(x^*) \right)^{\frac{1}{r}} \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^q d\mu(x^*) \right)^{\frac{1}{q}}$$

$$= \|T\|_{\pi(p)} \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^q d\mu(x^*) \right)^{\frac{1}{q}} \quad \text{for all } x \in X .$$

Hence by the Pietsch Domination Theorem we see that

$$T \in \Pi_q \quad \text{and} \quad \|T\|_{\pi(q)} \leq \|T\|_{\pi(p)} . \quad \square$$

Lemma 2.4.8 : *Let K be any compact Hausdorff space and let $\mu \in M(K)$ be positive . Then the canonical map*

$$J_p : C(K) \rightarrow L^p(K, \mu) ,$$

is absolutely p -summing .

In this case ,

$$\|J_p\|_{\pi(p)} = \|J_p\| = \|\mu\|^{\frac{1}{p}} .$$

Proof : Let δ_x be the point measure associated with $x \in X$. Since $\|\delta_x\| = 1$, we have for $f_1, \dots, f_n \in C(K)$,

$$\begin{aligned} \sum_{i=1}^n \|J_p f_i\|_p^p &= \sum_{i=1}^n \|f_i\|_p^p \\ &= \sum_{i=1}^n \int_K |f_i(x)|^p d\mu(x) \\ &= \int_K \sum_{i=1}^n |\langle f_i, \delta_x \rangle|^p d\mu(x) \\ &\leq \mu(K) \sup_{\gamma \in B_1(M(K))} \sum_{i=1}^n |\langle f_i, \gamma \rangle|^p . \end{aligned}$$

Thus

$$J_p \in \Pi_p(C(K), L^p(K, \mu)) \text{ and } \|J_p\|_{\pi(p)} \leq \mu(K)^{\frac{1}{p}} = \|\mu\|^{\frac{1}{p}}. \quad (*)$$

Furthermore ,

$$\|\mu\|^{\frac{1}{p}} = \left(\int_K 1 \, d\mu(x) \right)^{\frac{1}{p}} \leq \sup_{f \in B_1(C(K))} \left(\int_K |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} = \|J_p\|.$$

By Lemma 2.3.1 and (*) we obtain $\|J_p\|_{\pi(p)} = \|J_p\| = \|\mu\|^{\frac{1}{p}}$. □

From now on , J_p denotes the canonical map from $C(K)$ into $L^p(K, \mu)$.

Theorem 2.4.9 (Factorization Theorem) :

An operator $T \in L(X, Y)$ is absolutely p -summing if and only if there exists a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \xrightarrow{J_Y} & \ell^\infty(B_1(Y^*)) \\ \downarrow A & & & & \uparrow B \\ C(K) & \xrightarrow{J_p} & L^p(K, \mu) & & \end{array}$$

for some compact Hausdorff space K , a probability measure $\mu \in M(K)$,

$A \in L(X, C(K))$, and $B \in L(L^p(K, \mu), \ell^\infty(B_1(Y^*)))$.

In this case ,

A can be chosen to be isometric and B to satisfy $\|T\|_{\pi(p)} = \|B\|$.

Proof : Let T be represented as described . That is

$$J_Y T = B J_p A$$

It follows from Lemma 2.4.8 and the ideal properties that $J_Y T$ is absolutely p -summing . Since Π_p is injective and J_Y is a metric injection , T must be absolutely p -summing .

Conversely , let $T \in \Pi_p(X, Y)$. By the Pietsch Domination Theorem , there is a probability measure $\mu \in B_1(X^*)$ such that

$$\|Tx\| \leq \|T\|_{\pi(p)} \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \quad \text{for all } x \in X .$$

By setting $Ax := f_x$ where $f_x(x^*) := \langle x, x^* \rangle$, we obtain an operator

$A \in L(X, C(B_1(X^*)))$ with $\|A\| = 1$. Indeed : Let $x \in X$. Then

$$\|Ax\|_{\infty} = \|f_x\|_{\infty} = \sup \left\{ |\langle x, x^* \rangle| : x^* \in B_1(X^*) \right\} = \|x\| .$$

Clearly

$$J_p A \in \Pi_p(X, L^p(K, \mu)) , \text{ where } K := B_1(X^*) .$$

Set $X^p := J_p A(X) = \{ f_x : x \in X \}$ equipped with the $L^p(K, \mu)$ - topology .

Then

$$B_o(J_p A x) := T x$$

defines a continuous linear operator from X^p into Y . Indeed :

$$\begin{aligned}\|B_0(J_p A x)\| &= \|T x\| \leq \|T\|_{\pi(p)} \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \\ &= \|T\|_{\pi(p)} \|J_p A x\|_p .\end{aligned}$$

Thus

$$B_0 \in L(X^p, Y) \quad \text{and} \quad \|B_0\| \leq \|T\|_{\pi(p)} , \quad (*)$$

and it follows that $J_Y B_0 \in L(X^p, \ell^\infty(B_1(X^*)))$.

On the other hand, since $\|A\| = \|J_p\|_{\pi(p)} = 1$, we have

$$\|T\|_{\pi(p)} \leq \|B_0\| \|J_p\|_{\pi(p)} \|A\| = \|B_0\| . \quad (**)$$

By [13, p. 132], $J_Y B_0$ has a continuous extension

$$B \in L(L^p(K, \mu), \ell^\infty(B_1(X^*)))$$

such that

$$J_Y T = B J_p A \quad \text{and} \quad \|J_Y B_0\| = \|B\| . \quad (***)$$

But J_Y is an isometry, hence $\|J_Y B_0\| = \|B_0\|$.

Combining the above results we obtain

$$J_Y T = B J_p A \quad \text{and} \quad \|T\|_{\pi(p)} = \|B\| . \quad \square$$

Theorem 2.4.10 : *Let K be any compact Hausdorff space . An operator $T \in L(C(K), Y)$ is absolutely p -summing if and only if there is a commutative diagram*

$$\begin{array}{ccc}
C(K) & \xrightarrow{T} & Y \\
& \searrow J_p & \nearrow B \\
& L^p(K, \mu) &
\end{array}$$

for some probability measure $\mu \in M(K)$ and $B \in L(L^p(K, \mu), Y)$.

In this case ,

$$\|T\|_{\pi(p)} = \inf \{ \|B\| \}$$

where the infimum is taken over all possible factorizations .

Proof : If $T = BJ_p$, then it follows from Lemma 2.4.8 that

$$J_p \in \Pi_p(C(K), L^p(K, \mu)) \quad \text{and} \quad \|J_p\|_{\pi(p)} = 1 \quad .$$

Hence by the ideal properties we obtain

$$T = BJ_p \in \Pi_p(C(K), Y) \quad \text{and} \quad \|T\|_{\pi(p)} \leq \|B\| \quad .$$

This proves the sufficiency of the above condition .

Conversely , let $T \in \Pi_p(C(K), Y)$. Then by Theorem 2.4.6 there is a regular Borel probability measure μ on K such that

$$\|Tf\| \leq \|T\|_{\pi(p)} \left(\int_K |f(t)|^p d\mu(t) \right)^{\frac{1}{p}} \quad \text{for all } f \in C(K) \quad .$$

Let $X^p := J_p(C(K))$ equipped with the $L^p(K, \mu)$ -topology . Hence

$T \in L(X^p, Y)$. Since the space of continuous functions on K is dense in $L^p(K, \mu)$ [1, p. 209] , it follows from [15] that T extends to an operator $B \in L(L^p(K, \mu), Y)$ with $\|T\| = \|B\|$. Hence we obtain

$$BJ_p = T \quad \text{and} \quad \|B\| \leq \|T\|_{\pi(p)} .$$

This ends the proof . $\square$

If $p = 2$, then the criterion given by the Factorization Theorem 2.4.9 can be improved . We see this in the following :

Theorem 2.4.11 : *An operator $T \in L(X, Y)$ is absolutely 2-summing if and only if there is a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ A \downarrow & & \uparrow B \\ C(K) & \xrightarrow{J_2} & L^2(K, \mu) \end{array}$$

for some compact Hausdorff space K , a probability measure $\mu \in M(K)$,

$A \in L(X, C(K))$, $B \in L(L^2(K, \mu), Y)$.

In this case ,

$$\|T\|_{\pi(2)} = \inf \{ \|B\| \|A\| \}$$

where the infimum is taken over all possible factorizations .

Proof : Let $T \in \Pi_2(X, Y)$. In view of the proof of Theorem 2.4.9 with the same notations , there is a continuous linear operator B_0 from X^2 into Y . Obviously B_0 has a continuous extension from $\overline{X^2}$ into Y , say , B_1 , with $\|B_1\| = \|T\|_{\pi(2)}$ and $T = B_1 J_2 A$.

Since $L^2(K, \mu)$ is a Hilbert space, it follows from [13, p. 244] that there is an orthogonal projection P from $L^2(K, \mu)$ onto $\overline{X^2}$. Now define

$$B: L^2(K, \mu) \rightarrow Y,$$

$$B(z) := B_1 P(z)$$

Clearly B is linear and since $\|P\| = 1$, we have $\|B\| = \|B_1 P\| \leq \|B_1\|$.

On the other hand, let z be any element in $\overline{X^2}$. Then

$$\|B_1 z\| = \|Bz\| \leq \|B\| \|z\| \text{ implies } \|B_1\| \leq \|B\|.$$

Hence we have

$$\|B\| = \|T\|_{\pi(2)} \text{ and } T = B J_2 A. \quad (*)$$

Conversely, if T can be represented as described. Then by Lemma 2.4.8 and by the ideal properties, $T \in \Pi_2(X, Y)$. Moreover,

$$\|T\|_{\pi(2)} = \|B J_2 A\|_{\pi(2)} \leq \|B\| \|A\|$$

for all possible factorizations of this form. Hence

$$\|T\|_{\pi(2)} \leq \inf \{ \|B\| \|A\| \}.$$

The equality holds by using $(*)$. □

Theorem 2.4.12 : For $T \in L(X, Y)$ the following are equivalent :

- (i) $T \in \Pi_1(X, Y)$
- (ii) $TS \in \Pi_1(c_o, Y)$ for every $S \in L(c_o, X)$.

In this case,

$$\|T\|_{\pi(1)} = \sup \{ \|TS\|_{\pi(1)} : S \in L(c_o, X), \|S\| \leq 1 \}.$$

Proof : (ii) $\Rightarrow$ (i) Suppose that the condition (ii) is satisfied . Let $(x_n) \in \ell^1\langle X \rangle$. Define

$$S: c_0 \rightarrow X ,$$

$$S(a_n) := \sum_{n=1}^{\infty} a_n x_n .$$

Then $S(a_n)$ converges in X . Indeed : For all $k, m \in \mathbb{N}$ we have

$$\begin{aligned} \left\| \sum_{n=m}^{k+m} a_n x_n \right\| &= \sup_{x^* \in B_1(X^*)} \left| \left\langle \sum_{n=m}^{k+m} a_n x_n, x^* \right\rangle \right| \\ &\leq \sup_{x^* \in B_1(X^*)} \sum_{n=m}^{k+m} |a_n| \left| \langle x_n, x^* \rangle \right| \\ &\leq \|(x_n)\|_{\varepsilon(1)} \sup_{n=m, \dots, k+m} |\alpha_n| . \end{aligned}$$

Since (α_n) is a null sequence, the partial sum of $S(a_n)$ is a Cauchy sequence in the Banach space X . Hence $S(a_n) \in X$.

Furthermore , S is clearly linear and it follows from above that

$$\|S(a_n)\| \leq \|(x_n)\|_{\varepsilon(1)} \|(a_n)\|_{\infty} \quad \text{for all } (a_n) \in c_0 .$$

This proves that $S \in L(c_0, X)$ and $\|S\| \leq \|(x_n)\|_{\varepsilon(1)}$.

Let (e_n) be the natural basis of ℓ^2 . Then , by Proposition 1.3.14 , $(e_n) \in \ell^1\langle c_0 \rangle$ with $\|(e_n)\|_{\varepsilon(1)} = 1$. Also $Se_n = x_n$ for each $n \in \mathbb{N}$.

Since $TS \in \Pi_1(c_o, Y)$, it follows from Theorem 2.4.2 that

$$\|T(x_n)\|_{\alpha(1)} = \sum_{n=1}^{\infty} \|Tx_n\| = \sum_{n=1}^{\infty} \|TS e_n\| \leq \|(x_n)\| \left\| \frac{TS}{\|S\|} \right\|_{\pi(1)}.$$

Hence by Theorem 2.4.2 also, T is absolutely summing with

$$\|T\|_{\pi(1)} \leq \sup \left\{ \|TS\|_{\pi(1)} : S \in L(c_o, X), \|S\| \leq 1 \right\}.$$

(i) $\Rightarrow$ (ii) This follows directly from the ideal properties of Π_1 . $\square$

Theorem 2.4.13 : Let $1 < p < \infty$. For $T \in L(X, Y)$, the following are equivalent :

$$(i) \quad T \in \Pi_p(X, Y).$$

$$(ii) \quad TR \in \Pi_p(\ell^{p^*}, Y) \text{ for all } R \in L(\ell^{p^*}, X) \text{ with } \frac{1}{p} + \frac{1}{p^*} = 1.$$

In this case

$$\|T\|_{\pi(p)} = \sup \left\{ \|TR\|_{\pi(p)} : R \in L(\ell^{p^*}, X), \|R\| \leq 1 \right\}.$$

Proof : (ii) $\Rightarrow$ (i) : It follows from Theorem 1.3.15 that given (x_i) in $\ell^p(X)$ then

$$R: \ell^{p^*} \rightarrow X \text{ defined by } R(\alpha_i) := \sum_{i=1}^{\infty} \alpha_i x_i$$

is a continuous linear operator with $\|R\| = \|(x_i)\|_{\varepsilon(p)}$.

If (e_i) denotes the natural basis of ℓ^2 , then $\|(e_i)\|_{\varepsilon(p)} = 1$, by Proposition

1.3.14. Moreover, we have $Re_i = x_i$ for all $i \in \mathbb{N}$. Set

$$\|T\|_o := \sup \left\{ \|TR\|_{\pi(p)} : R \in L(\ell^{p^*}, X), \|R\| \leq 1 \right\}.$$

Since $TR \in \Pi_p(\ell^{p^*}, Y)$, it follows from Theorem 2.4.2 that

$$\begin{aligned} \|T(x_i)\|_{\alpha(p)} &= \left(\sum_{i=1}^{\infty} \|Tx_i\|^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^{\infty} \|TR e_i\|^p \right)^{\frac{1}{p}} \\ &\leq \|TR\|_{\pi(p)} \|e_i\|_{\varepsilon(p)} \\ &= \|R\| \left\| \frac{TR}{\|R\|} \right\|_{\pi(p)} \\ &\leq \|T\|_o \|x_i\|_{\varepsilon(p)}. \end{aligned}$$

By Theorem 2.4.2 again we have

$$T \in \Pi_p(X, Y) \quad \text{and} \quad \|T\|_{\pi(p)} \leq \|T\|_o.$$

(i) $\Rightarrow$ (ii) : Let $T \in \Pi_p(X, Y)$. It follows from the ideal properties that

$$TR \in \Pi_p(\ell^{p^*}, Y) \quad \text{and} \quad \|TR\|_{\pi(p)} \leq \|T\|_{\pi(p)} \|R\|$$

for all $R \in L(\ell^{p^*}, X)$. Consequently, $\|T\|_o \leq \|T\|_{\pi(p)}$.

This completes the proof. $\square$

Theorem 2.4.14 : Let H and K be Hilbert spaces. For $T \in L(H, K)$

the following are equivalent :

$$(i) \quad T \in \Pi_p(H, K).$$

$$(ii) \quad \|T\|_o := \sup \left\{ \|ST\|_{\pi(p)} : S \in \mathfrak{I}(K) \text{ and } \|S\| \leq 1 \right\} < \infty.$$

In this case, $\|T\|_{\pi(p)} = \|T\|_o$.

Proof : (i) $\Rightarrow$ (ii) : Let $T \in \Pi_p(H, K)$. By the ideal properties we have

$$\|ST\|_{\pi(p)} \leq \|S\| \|T\|_{\pi(p)} \leq \|T\|_{\pi(p)} \quad \text{for all } S \in \mathfrak{I}(K) \text{ with } \|S\| \leq 1 .$$

Hence $\|T\|_o \leq \|T\|_{\pi(p)} .$

(ii) $\Rightarrow$ (i) : Let $x_1, \dots, x_n \in H$ and let P be the orthogonal projection of K onto the closure of the span of $Tx_1, \dots, Tx_n$. Since P is a finite rank operator , $P \in \Pi_p(H, K)$ and hence , by the ideal properties , $PT \in \Pi_p(H, K)$. Now

$$\begin{aligned} \left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n \|PTx_i\|^p \right)^{\frac{1}{p}} \\ &\leq \|PT\|_{\pi(p)} \sup_{x^* \in B_1(H)} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} . \end{aligned}$$

Since $\|P\| = 1$, it follows from

$$\left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} \leq \|T\|_o \sup_{x^* \in B_1(H)} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} .$$

that

$$T \in \Pi_p(H, K) \quad \text{and} \quad \|T\|_{\pi(p)} \leq \|T\|_o .$$

combining the above results we obtain

$$\|T\|_{\pi(p)} = \|T\|_o .$$

This ends the proof . $\square$

2.5. Examples of p-Summing Operators :

Throughout this section , let X and Y be arbitrary Banach spaces and let $1 \leq p < \infty$.

(1) p-Integral Operators :

Theorem 2.5.1 : $I_p(X, Y) \subseteq \Pi_p(X, Y)$.

Proof : Let $T \in I_p(X, Y)$. By Definition 2.2.13 , there exists a compact Hausdorff space K and a probability measure μ on K and operators $A \in L(X, C(K))$ and $B \in L(L^p(K, \mu), Y^{**})$ such that

$$J_Y T = B J_p A ,$$

where $J_p \in L(C(K), L^p(K, \mu))$ is the natural inclusion map and $J_Y \in L(Y, Y^{**})$ denotes the natural embedding . It follows from Lemma 2.4.8 that J_p is absolutely p-summing . By the ideal properties , $J_Y T \in \Pi_p(X, Y^{**})$. Since J_Y is a metric injection and since , by Theorem 2.3.4 , Π_p is injective , T must be absolutely p-summing . Hence

$$I_p(X, Y) \subseteq \Pi_p(X, Y) .$$

This completes the proof .

□

(2) Multiplicative and Diagonal Operators :

Definition 2.5.2 : Let (I, μ) be the unit interval with the Lebesgue measure. An operator $T \in L(L^\infty(I, \mu), L^p(I, \mu))$ is said to be **multiplicative**, if there exists a function $g \in L^p(I, \mu)$ such that

$$T(f) = f \cdot g \quad \text{for all } f \in L^\infty(I, \mu) .$$

Theorem 2.5.3 : Let $T \in L(L^\infty(I, \mu), L^p(I, \mu))$ be a multiplicative operator . Then T is absolutely p -summing .

Proof : Since T is a multiplicative operator , there is a function $g \in L^p(I, \mu)$ such that $T(f) = f \cdot g$ for all $f \in L^\infty(I, \mu)$. Further , for $f_1, \dots, f_n \in L^\infty(I, \mu)$, we have

$$\begin{aligned} \left(\sum_{i=1}^n \|T f_i\|_p^p \right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n \int_0^1 |g(t) f_i(t)|^p d\mu(t) \right)^{\frac{1}{p}} \\ &= \left(\int_0^1 \sum_{i=1}^n |g(t) f_i(t)|^p d\mu(t) \right)^{\frac{1}{p}} \\ &\leq \|g\|_p \text{ess-sup} \left\{ \left(\sum_{i=1}^n |f_i(t)|^p \right)^{\frac{1}{p}} : t \in I \right\} . \end{aligned}$$

It follows from Theorem 1.3.22 that

$$\left(\sum_{i=1}^n \|T f_i\|_p^p \right)^{\frac{1}{p}} \leq \|g\|_p \cdot \sup_{\gamma \in B_1 \left((L^\infty(I, \mu))^* \right)} \left(\sum_{i=1}^n |\langle f_i, \gamma \rangle|^p \right)^{\frac{1}{p}}.$$

Thus $T \in \Pi_p(X, Y)$ and $\|T\|_{\pi(p)} \leq \|g\|_p$. $\square$

Corollary 2.5.4 : *The embedding operator $J: L^\infty(I, \mu) \rightarrow L^p(I, \mu)$ is absolutely p -summing .*

Proof : Since $J(f) = f$ for all $f \in L^\infty(I, \mu)$, J is a multiplicative operator with $g(t) = 1$ for all $t \in I$. Applying the above theorem , the result follows . $\square$

Definition 2.5.5 : An operator $D \in L(\ell^\infty, \ell^p)$ is called a **diagonal** operator if , there is a sequence $(\alpha_n) \in \ell^p$ such that

$$D(\beta_n) = (\alpha_n \beta_n) \quad \text{for all } (\beta_n) \in \ell^\infty.$$

To prove the theorem for diagonal operators we need the following :

Proposition 2.5.6 : *Let $T \in L(X, Y)$. If there exists a sequence (x_n^*) in $\ell^p(X^*)$ such that*

$$\|T x\| \leq \left(\sum_{n=1}^{\infty} |\langle x, x_n^* \rangle|^p \right)^{\frac{1}{p}} \quad \text{for all } x \in X,$$

then

$$T \in \Pi_p(X, Y) \text{ and } \|T\|_{\pi(p)} \leq \left\| (x_n^*) \right\|_{\alpha(p)} .$$

Proof : For $x_1, \dots, x_m \in X$ we have

$$\begin{aligned} \left(\sum_{i=1}^m \|Tx_i\|^p \right)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^m \sum_{n=1}^{\infty} |\langle x_i, x_n^* \rangle|^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=1}^{\infty} \sum_{i=1}^m |\langle x_i, x_n^* \rangle|^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=1}^{\infty} \|x_n^*\|^p \cdot \sum_{i=1}^m \left| \left\langle x_i, \frac{x_n^*}{\|x_n^*\|} \right\rangle \right|^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=1}^{\infty} \|x_n^*\|^p \right)^{\frac{1}{p}} \cdot \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^m |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} . \end{aligned}$$

This proves $T \in \Pi_p(X, Y)$ and $\|T\|_{\pi(p)} \leq \left\| (x_n^*) \right\|_{\alpha(p)} . \quad \square$

Theorem 2.5.7 : Let $D \in L(\ell^\infty, \ell^p)$ be a diagonal operator . Then D is absolutely p -summing .

Proof : Since D is a diagonal operator , there exists a sequence $(\alpha_n) \in \ell^p$ such that $D(\beta_n) = (\alpha_n \beta_n)$ for all $(\beta_n) \in \ell^\infty$. Now if (e_n) denotes the natural basis of ℓ^p , we have

$$\|D(\beta_n)\|_{\alpha(p)} = \left(\sum_{n=1}^{\infty} |\alpha_n \beta_n|^p \right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |\langle (\beta_k), \alpha_n e_n \rangle|^p \right)^{\frac{1}{p}} .$$

Since $(\alpha_n e_n) \in \ell^p \left((\ell^\infty)^* \right)$, it follows from Proposition 2.5.6 that

$$D \in \Pi_p(\ell^\infty, \ell^p) \quad \text{and} \quad \|D\|_{\pi(p)} \leq \|(\alpha_n)\|_{\alpha(p)} . \quad \square$$

(3) p -Nuclear Operator :

Theorem 2.5.8 : $N_p(X, Y) \subseteq \Pi_p(X, Y)$.

Proof : Let $T \in N_p(X, Y)$. By Theorem 2.2.12 , there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ A \downarrow & & \uparrow B \\ \ell^\infty & \xrightarrow{D} & \ell^p \end{array}$$

where D is a diagonal operator . Since , by Theorem 2.5.7 , D is absolutely p -summing , it follows from the ideal properties that T is also absolutely p -summing . $\square$

(4) p -Decomposable Operators :

Definition 2.5.9 : Let (I, μ) be the unit interval with the Lebesgue measure . An operator $A \in L(X, L^p(I, \mu))$ is called *p -decomposable* , if

there exists a map $f:I \rightarrow X^*$, such that the following conditions are satisfied :

(i) The function $\langle x, f(t) \rangle$ is μ - measurable and

$$(Ax)(t) = \langle x, f(t) \rangle \quad \mu - a.e \text{ for all } x \in X .$$

(ii) There exists $g \in L^p(I, \mu)$ such that $\|f(t)\| \leq g(t) \quad \mu - a.e .$

Theorem 2.5.10 : Let $A \in L(X, L^p(I, \mu))$ be p -decomposable . Then A is absolutely p -summing .

Proof : Let $x_1, \dots, x_n \in X$. Then , by Definition 2.5.9 , we have

$$\begin{aligned} \sum_{i=1}^n \|Ax_i\|_p^p &= \sum_{i=1}^n \int_0^1 |\langle x_i, f(t) \rangle|^p d\mu(t) \\ &= \int_0^1 \sum_{i=1}^n |\langle x_i, f(t) \rangle|^p d\mu(t) \\ &= \int_0^1 \|f(t)\|^p \sum_{i=1}^n \left| \left\langle x_i, \frac{f(t)}{\|f(t)\|} \right\rangle \right|^p d\mu(t) \\ &\leq \left(\int_0^1 |g(t)|^p d\mu(t) \right) \cdot \sup_{x^* \in B_1(X^*)} \sum_{i=1}^n |\langle x_i, x^* \rangle|^p . \end{aligned}$$

Hence

$$A \in \Pi_p(X, L^p(I, \mu)) \text{ and } \|A\|_{\pi(p)} \leq \|g\|_p .$$

This completes the proof. $\square$

(5) Hilbert-Schmidt Operators :

Lemma 2.5.11 : *Let H and K be Hilbert spaces with orthonormal bases (e_i) and (b_i) respectively . Then for every operator $T \in L(H, K)$ and every orthonormal basis (f_i) of H we have*

$$\sum_{i=1}^{\infty} \|Tf_i\|^2 = \sum_{i=1}^{\infty} \|Te_i\|^2 = \sum_{j=1}^{\infty} \|T^*b_j\|^2$$

where the sums may or may not be finite .

Proof : Using Parseval's equation , we obtain

$$\begin{aligned} \sum_{i=1}^{\infty} \|Te_i\|^2 &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle Te_i, b_j \rangle|^2 \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle e_i, T^*b_j \rangle|^2 \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\langle e_i, T^*b_j \rangle|^2 \\ &= \sum_{j=1}^{\infty} \|T^*b_j\|^2 . \end{aligned}$$

Similarly , for any orthonormal basis (f_i) of H , we get

$$\sum_{i=1}^{\infty} \|Tf_i\|^2 = \sum_{i=1}^{\infty} \|T^*b_j\|^2 .$$

This proves

$$\sum_{i=1}^{\infty} \|Tf_i\|^2 = \sum_{i=1}^{\infty} \|Te_i\|^2 = \sum_{j=1}^{\infty} \|T^*b_j\|^2 . \quad \square$$

Definition 2.5.12 : Let H and K be Hilbert spaces . An operator $T \in L(H, K)$ is said to be ***Hilbert-Schmidt*** if $\left(\|T e_i\|^2 \right)$ is summable for some orthonormal basis (e_i) of H .

The set of Hilbert-Schmidt operators is denoted by $C_2(H, K)$. Further let

$$C_2 := \bigcup_{H, K} C_2(H, K) .$$

The preceding lemma implies that , for $T \in C_2(H, K)$, the expression

$$\|T\|_{\sigma(2)} := \left(\sum_{i=1}^n \|T e_i\|^2 \right)^{\frac{1}{2}}$$

does not depend on the special choice of (e_i) .

Theorem 2.5.13 : C_2 is a Banach operator ideal on the class of Hilbert spaces . Furthermore , $\| \cdot \|_{\sigma(2)}$ can be generated by the inner product

$$\langle S, T \rangle := \sum_{i=1}^{\infty} \langle S e_i, T e_i \rangle \text{ for } S, T \in C_2(H, K) .$$

Proof : See [27, p. 214] .

We refer to [30] and [27] for more properties on Hilbert-Schmidt operators .

Theorem 2.5.14 : *Let H and K be Hilbert spaces . An operator $T \in L(H, K)$ is absolutely 2-summing if and only if it is Hilbert-Schmidt .*

In this case ,

$$\|T\|_{\pi(2)} = \|T\|_{\sigma(2)} \quad \text{for } T \in C_2(H, K).$$

Proof : Let $T \in \Pi_2(H, K)$ and let (e_i) be any orthonormal basis of H .

Then for each $n \in \mathbf{N}$, we have

$$\left(\sum_{i=1}^n \|Te_i\|^2 \right)^{\frac{1}{2}} \leq \|T\|_{\pi(2)} \sup_{z \in B_1(H)} \left(\sum_{i=1}^n |\langle e_i, z \rangle|^2 \right)^{\frac{1}{2}}.$$

It follows from Parseval's equation that

$$\left(\sum_{i=1}^n \|Te_i\|^2 \right)^{\frac{1}{2}} \leq \|T\|_{\pi(2)} \quad \text{for all } n \in \mathbf{N}.$$

Therefore $T \in C_2(H, K)$ and $\|T\|_{\sigma(2)} \leq \|T\|_{\pi(2)}.$

Conversely , let $T \in C_2(H, K)$ and let $x_1, \dots, x_n \in H$. Moreover , let (b_j) be any orthonormal basis of K . Using Parseval's equation , we now obtain :

$$\begin{aligned} \left(\sum_{i=1}^n \|Tx_i\|^2 \right)^{\frac{1}{2}} &= \left(\sum_{i=1}^n \sum_{j=1}^{\infty} |\langle Tx_i, b_j \rangle|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{j=1}^{\infty} \sum_{i=1}^n |\langle x_i, T^* b_j \rangle|^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{j=1}^{\infty} \|T^* b_j\|^2 \sum_{i=1}^n \left| \left\langle x_i, \frac{T^* b_j}{\|T^* b_j\|} \right\rangle \right|^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{j=1}^{\infty} \|T^* b_j\|^2 \right)^{\frac{1}{2}} \sup_{z \in B_1(H)} \left(\sum_{i=1}^n |\langle x_i, z \rangle|^2 \right)^{\frac{1}{2}} .
\end{aligned}$$

Consequently ,

$$T \in \Pi_2(H, K) \quad \text{and} \quad \|T\|_{\pi(2)} \leq \left(\sum_{j=1}^{\infty} \|T^* b_j\|^2 \right)^{\frac{1}{2}}$$

It follows from Lemma 2.5.11 that $\|T\|_{\pi(2)} \leq \|T\|_{\sigma(2)}$. □

Corollary 2.5.15 : *Let H be a Hilbert space of dimension n and let I_H denote the identity operator . Then $\|I_H\|_{\pi(2)} = \sqrt{n}$.*

Proof : Let $e_1, \dots, e_n$ be an orthonormal basis of H . By Theorem 2.5.14 we have

$$\|I_H\|_{\pi(2)} = \left(\sum_{i=1}^n \|I_H e_i\|^2 \right)^{\frac{1}{2}} = \sqrt{n} . \quad \square$$

(6) An operator Which is not 1-Summing :

Proposition 2.5.16 : *The injection $J: \ell^2 \rightarrow c_0$ is not absolutely summing .*

Proof : Let (e_i) denote the standard basis of ℓ^2 . Then the sequence $\left(\frac{e_i}{i}\right)$ is weakly summable . Indeed : For $x := (x_n) \in \ell^2$, it follows from

$$\left\langle x, \frac{e_i}{i} \right\rangle = \left\langle (x_n), \frac{e_i}{i} \right\rangle = \frac{x_i}{i}$$

that

$$\sum_{i=1}^{\infty} \left| \left\langle x, \frac{e_i}{i} \right\rangle \right| = \sum_{i=1}^{\infty} \frac{|x_i|}{i} \leq \left(\sum_{i=1}^{\infty} \frac{1}{i^2} \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}} < \infty .$$

Now

$$\sum_{i=1}^{\infty} \left\| J\left(\frac{e_i}{i}\right) \right\|_{\infty} = \sum_{i=1}^{\infty} \left\| \frac{e_i}{i} \right\|_{\infty} = \sum_{i=1}^{\infty} \frac{1}{i} = \infty .$$

Consequently , by Theorem 2.4.1 , J is not absolutely summing . $\square$

We shall remark that the above proposition is also true if c_0 is replaced by ℓ^{∞} .

2.6. Smallness of p-Summing Operators :

Throughout this section , let $1 \leq p < \infty$ and $1 \leq \lambda < \infty$.

Definition 2.6.1 : A Banach space X is said to be an $\mathcal{L}_{\lambda}^p$ -space if for every finite dimensional subspace B of X there exists a finite dimensional subspace E of X containing B such that

$$d(E, \ell_n^p) := \inf \left\{ \|T\| \|T^{-1}\| : T \in L(E, \ell_n^p) \text{ is invertible} \right\} \leq \lambda$$

where $n := \dim E$. The space X is called an $\mathcal{L}^p$ -space if it is an $\mathcal{L}^p_\lambda$ -space for some $\lambda \geq 1$.

Thus we are considering a class of Banach spaces whose finite dimensional structure is the same as the structure of spaces ℓ^p_n .

Proposition 2.6.2 : *For a Banach space X , the following are equivalent :*

- (1) X is an $\mathcal{L}^p_\lambda$ -space .
- (2) There is a net $(E_\alpha)_{\alpha \in \Lambda}$ of finite dimensional subspaces of X , directed by the inclusion, such that $\bigcup_{\alpha} E_\alpha$ is dense in X , and for all α

$$d(E_\alpha, \ell^p_n) \leq \lambda \quad \text{where} \quad n := \dim E_\alpha .$$

Proof : See [19, p. 168] .

Example 2.6.3 : Let (Ω, μ) be a measure space . The space $L^p(\Omega, \mu)$ is an $\mathcal{L}^p_1$ -space . Indeed : Let Λ be the set formed by all the finite sets $\alpha := \{A_1, \dots, A_n\}$, with any $n \in \mathbb{N}$, of disjoint measurable subsets of Ω of finite measures such that $\bigcup_{i=1}^n A_i = \Omega$.

Moreover, for any $\alpha \in \Lambda$, let $E_\alpha := \text{span} \{1_{A_1}, \dots, 1_{A_n}\}$ in $L^p(\Omega, \mu)$, where 1_{A_i} is the characteristic function of A_i .

Now let $\alpha \in \Lambda$ and let $n := \dim E_\alpha$. With no loss of generality, we may assume that $\mu(A_i) \neq 0$ for $i = 1, \dots, n$. Define

$$J: E_\alpha \rightarrow \ell_n^p,$$

$$J\left(\sum_{i=1}^n \beta_i 1_{A_i}\right) := (\beta_1 \mu(A_1)^{\frac{1}{p}}, \dots, \beta_n \mu(A_n)^{\frac{1}{p}}).$$

Clearly J is a well-defined linear operator which is onto. Further, let

$$z := \left(\sum_{i=1}^n \beta_i 1_{A_i}\right) \in E_\alpha.$$

Then

$$\begin{aligned} \|z\|_p &= \left(\int_{\Omega} \left| \sum_{i=1}^n \beta_i 1_{A_i}(t) \right|^p d\mu(t) \right)^{\frac{1}{p}} \\ &= \left(\sum_{j=1}^n \int_{A_j} \left| \sum_{i=1}^n \beta_i 1_{A_i}(t) \right|^p d\mu(t) \right)^{\frac{1}{p}} \\ &= \left(\sum_{j=1}^n |\beta_j|^p \mu(A_j) \right)^{\frac{1}{p}} = \|Jz\|_{\alpha(p)}. \end{aligned}$$

Thus E_α is isometrically isomorphic to ℓ_n^p and therefore $d(E_\alpha, \ell_n^p) \leq 1$.

Since the simple functions are dense in $L^p(\Omega, \mu)$, we have

$L^p(\Omega, \mu) = \overline{\bigcup_{\alpha} E_\alpha}$. Now in view of Proposition 2.6.2, the result is

concluded.

Now we state the following fact which is known as the **Grothendieck's** inequality .

Proposition 2.6.4 : *Let $(a_{ij})_{i,j=1}^n$ be a real-valued matrix and let M be a positive number such that*

$$\left| \sum_{i=1}^n \sum_{j=1}^n a_{ij} t_i s_j \right| \leq M$$

for every real $(t_i)_{i=1}^n$ and $(s_j)_{j=1}^n$ satisfying $|t_i| \leq 1$ and $|s_j| \leq 1$. Then

for arbitrary vectors $(x_i)_{i=1}^n$ in an inner product space H ,

$$\sum_{j=1}^n \left\| \sum_{i=1}^n a_{ij} x_i \right\| \leq K_G \cdot M \cdot \max_i \|x_i\|$$

where K_G is the Grothendieck universal constant .

Proof : See [22, p. 280] .

Theorem 2.6.5 : *Let X be an $\mathcal{L}^1$ -space and let H be a Hilbert space . Then $\Pi_1(X, H) = L(X, H)$.*

Proof : Let λ be such that X is an $\mathcal{L}_{\lambda}^1$ -space and let $(x_i) \in \ell^1\langle X \rangle$ with $\|(x_i)\|_{\varepsilon(1)} \leq 1$. Then , for each $n \in \mathbf{N}$, we have

$$\sum_{i=1}^n \left| \langle x_i, x^* \rangle \right| \leq \|x^*\| \quad \text{for all } x^* \in X^* . \quad (1)$$

By definition 2.6.1 , there are a finite dimensional subspace E of X containing $\{x_1, \dots, x_n\}$ and an invertible operator

$$S: \ell_m^1 \rightarrow E \quad \text{with } m = \dim E, \|S\| = 1, \|S^{-1}\| \leq \lambda.$$

Suppose $y_1, \dots, y_n \in \ell_m^1$ are chosen so that $y_i = S^{-1} x_i$ for $i = 1, \dots, n$.

If (e_i) denotes the standard basis of ℓ_m^1 , then

$$y_i = \sum_{j=1}^m a_{ij} e_j \quad \text{for } i = 1, \dots, n.$$

Let $y^* := (s_1, \dots, s_m) \in \ell_m^\infty = (\ell_m^1)^*$ with $\|y^*\| \leq 1$ and take any real numbers $t_1, \dots, t_n$ such that $|t_i| \leq 1$ for $i = 1, \dots, n$. Then

$$\begin{aligned} \left| \sum_{i=1}^n \sum_{j=1}^m a_{ij} t_i s_j \right| &\leq \sum_{i=1}^n |t_i| \left| \sum_{j=1}^m a_{ij} s_j \right| \\ &\leq \sum_{i=1}^n \left| \sum_{j=1}^m a_{ij} s_j \right| \\ &= \sum_{i=1}^n |\langle y_i, y^* \rangle| \\ &= \sum_{i=1}^n |\langle S^{-1} x_i, y^* \rangle| \\ &= \sum_{i=1}^n \left| \langle x_i, (S^{-1})^* y^* \rangle \right|. \end{aligned} \tag{2}$$

Since $(S^{-1})^*: \ell_m^\infty \rightarrow E^*$, by the Hahn-Banach Theorem $(S^{-1})^* y^*$ can be extended continuously to a linear functional $x_0^* \in X^*$ such that

$$\|x_o^*\| = \|(S^{-1})^* y^*\|.$$

Thus from (2) and (1) respectively we have :

$$\left| \sum_{i=1}^n \sum_{j=1}^m a_{ij} t_i s_j \right| \leq \sum_{i=1}^n |\langle x_i, x_o^* \rangle| \leq \|(S^{-1})^* y^*\| \leq \lambda.$$

Now let $T \in L(X, H)$, set $u_i := T x_i$ for $i = 1, \dots, n$. Then

$$u_i = T S y_i = \sum_{j=1}^m a_{ij} T S e_j \quad \text{for } i = 1, \dots, n$$

and

$$\sum_{i=1}^n \|u_i\| = \sum_{i=1}^n \left\| \sum_{j=1}^m a_{ij} T S e_j \right\|.$$

Applying Proposition 2.6.4 we obtain :

$$\sum_{i=1}^n \|T x_i\| \leq K_G \lambda \max_j \|T S e_j\| \leq K_G \lambda \|T\|.$$

Since the above inequality holds for all n , we get

$$\sum_{i=1}^{\infty} \|T x_i\| \leq K_G \lambda \|T\| \quad \text{for all } (x_i) \in B_1(\ell^1\langle X \rangle).$$

Now let $(x_i) \in \ell^1\langle X \rangle$. By the above argument we obtain

$$\sum_{i=1}^{\infty} \|T x_i\| \leq K_G \lambda \|T\| \|(x_i)\|_{\varepsilon(1)}.$$

Hence by Theorem 2.4.2 we obtain

$$T \in \Pi_1(X, H) \quad \text{and} \quad \|T\|_{\pi(1)} \leq K_G \lambda \|T\|. \quad \square$$

Proposition 2.6.6 : *Let X be an $\mathcal{L}^1$ -space and let H be a Hilbert space . Then*

$$\Pi_p(X, H) = \Pi_q(X, H) = L(X, H) \text{ for all } 1 \leq p, q < \infty .$$

Proof : Without loss of generality , we may assume $1 \leq p \leq q < \infty$.

Combining Theorem 2.3.5 and Theorem 2.6.5 we obtain

$$L(X, H) = \Pi_1(X, H) \subseteq \Pi_p(X, H) \subseteq \Pi_q(X, H) \subseteq L(X, H)$$

and the result is concluded . $\square$

Theorem 2.6.7 : *Let X be an $\mathcal{L}^1$ -space and let Y be any Banach space . Then*

$$\Pi_1(X, Y) = \Pi_r(X, Y) \text{ for all } 1 \leq r \leq 2 .$$

Proof : Let $T \in \Pi_2(X, Y)$. By Theorem 2.4.11 ,

$$T = BJ_2 A \text{ where } B \in L(L^2(K, \mu), Y) \text{ and } A \in L(X, C(K))$$

for some compact Hausdorff space K . Since $J_2 A \in L(X, L^2(K, \mu))$, by Proposition 2.6.6 $J_2 A \in \Pi_1(X, L^2(K, \mu))$. Therefore $T \in \Pi_1(X, Y)$.

Now for $1 \leq r \leq 2$,

$$\Pi_1(X, Y) \subseteq \Pi_r(X, Y) \subseteq \Pi_2(X, Y) \subseteq \Pi_1(X, Y)$$

implies that $\Pi_1(X, Y) = \Pi_r(X, Y)$ for all $1 \leq r \leq 2$. $\square$

Now we are in a position to state the main theorem of this section :

Theorem 2.6.8 : Π_p is not a small operator ideal .

Proof : Let (Ω, μ) be a measure space . Then in view of Proposition 2.6.6 we have

$$\Pi_p\left(L^1(\Omega, \mu), L^2(\Omega, \mu)\right) = L(L^1(\Omega, \mu), L^2(\Omega, \mu)) .$$

Since both $L^1(\Omega, \mu)$ and $L^2(\Omega, \mu)$ are not finite dimensional spaces , it follows that Π_p is not small . $\square$

With the aid of the theory of tensor products , the above theorem can be proved in a simpler way .That is , it is enough to show that $\Pi_1(\ell^1, \ell^2) = L(\ell^1, \ell^2)$. So for completeness , we include the proof here .

Theorem 2.6.9 : $\Pi_1(\ell^1, \ell^2) = L(\ell^1, \ell^2)$.

Proof : By Theorem 1.4.7 and Theorem 1.4.6 respectively , we have

$$L(\ell^1, \ell^2) = \left(\ell^1 \hat{\otimes} \ell^2 \right)^* = \left(\ell^1(\ell^2) \right)^* = \ell^\infty(\ell^2) ,$$

where the equalities here mean isometrically isomorphic . So if $T \in L(\ell^1, \ell^2)$, there is an $f \in \ell^\infty(\ell^2)$ such that $T = T_f$. Now let $a_1, \dots, a_n \in \ell^1$. Then

$$\sum_{i=1}^n \|T a_i\|_{\alpha(2)} = \sum_{i=1}^n \left\| \sum_{j=1}^{\infty} a_{ij} f_j \right\|_{\alpha(2)} .$$

Note that the matrix $a:=(a_{ij})$ is a bounded linear operator from ℓ_n^∞ into ℓ_n^1 . Hence for $(t_i) \in B_1(\ell_n^\infty)$, by the Hahn-Banach Theorem, it follows from

$$\|a(t_i)\|_{\alpha(1)} = \sup_{(s_j) \in B_1(\ell_n^\infty)} \left| \sum_{i=1}^n \sum_{j=1}^{\infty} a_{ij} t_i s_j \right|$$

that

$$\left| \sum_{i=1}^n \sum_{j=1}^{\infty} a_{ij} t_i s_j \right| \leq \|a(t_i)\|_{\alpha(1)} \leq \|a\| = M .$$

So by the Grothendieck's inequality, the result follows.

2.7. Further Properties :

Throughout this section, let X and Y be arbitrary Banach spaces unless explicitly stated and let $1 \leq p < \infty$.

Theorem 2.7.1 : $\Pi_p(X, Y) \subseteq W(X, Y)$.

Proof : Let $T \in \Pi_p(X, Y)$. By Theorem 2.3.5, we may assume that $p > 1$. From the Pietsch Domination Theorem, one can easily show that the following diagram commutes :

$$\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
A \downarrow & & \uparrow B \\
C(B_1(X^*)) & \xrightarrow{J_p} & Z
\end{array}$$

where A is the canonical isometry from X into $C(B_1(X^*))$ and

$$Z := \overline{J_p A(X)} \quad \text{in} \quad L^p(C(B_1(X^*)), \mu)$$

for some probability measure μ . Since Z is reflexive, it follows from Theorem 2.2.8 that $J_p \in W(C(B_1(X^*)), Z)$. By the ideal properties of W , T must be weakly compact. $\square$

Theorem 2.7.2 : $\Pi_p(X, Y) \subseteq \mathcal{C}(X, Y)$.

Proof : Let $x_n \rightarrow x$ weakly in X . The sequence $(x_n - x)$ is norm bounded. It follows from

$$\langle x_n - x, x^* \rangle \rightarrow 0 \quad \text{in } \mathbf{K} \quad \text{for all } x^* \in X^*$$

that $(x_n - x)$ converges pointwise on $B_1(X^*)$. Moreover, by the Pietsch Domination Theorem, there is a probability measure μ on $B_1(X^*)$ such that

$$\|T(x_n - x)\|^p \leq \|T\|_{\pi(p)}^p \int_{B_1(X^*)} |\langle x_n - x, x^* \rangle|^p d\mu(x^*) \quad \text{for all } n.$$

Thus by the Dominated Convergence Theorem , the right hand side of the above inequality tends to 0 as $n \rightarrow \infty$. This shows that

$$Tx_n \rightarrow Tx \text{ in } Y \text{ which implies that } T \in \mathcal{C}(X, Y) . \quad \square$$

Theorem 2.7.3 : *Let $T \in \Pi_p(X, Y)$. Then T transforms weakly compact sets into norm compact sets .*

Proof : Let M be a weakly compact set in X and let (Tx_n) be a sequence in $T(M)$. Since (x_n) is a sequence in the weakly compact set M , it has a subsequence (x_{n_k}) such that $x_{n_k} \rightarrow x$ weakly in $M \subseteq X$. By Theorem 2.7.2 , $Tx_{n_k} \rightarrow Tx$ in $T(M) \subseteq Y$. Hence $T(M)$ is a norm (sequentially) compact set in Y . $\square$

Theorem 2.7.4 : *If X is reflexive , then $\Pi_p(X, Y) \subseteq K(X, Y)$.*

Proof : Since X is reflexive , by [6, p. 132] , $B_1(X)$ is weakly compact . Therefore , by Theorem 2.7.3 , $\overline{T(B_1(X))}$ in Y is compact which shows that T itself is compact . $\square$

Proposition 2.7.5 : *The identity operator on a Banach space X is absolutely p -summing if and only if X is finite dimensional .*

Proof : Let I_X be the identity operator on X . If I_X is absolutely p -summing , then by Theorem 2.7.1 $B_1(X)$ must be weakly compact . Hence , by [6,p 132] , X must be reflexive . Applying Proposition 2.7.4 we see that I_X must be compact and hence X must be finite dimensional.

The converse is obviously true . $\square$

Theorem 2.7.6 : Let Z be a Banach space and let $1 \leq p, q, r < \infty$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $T \in \Pi_p(X, Y)$ and $S \in \Pi_q(Y, Z)$, then

$$ST \in \Pi_r(X, Z) \quad \text{and} \quad \|ST\|_{\pi(r)} \leq \|S\|_{\pi(q)} \|T\|_{\pi(p)} .$$

Proof : It follows from the Pietsch Domination Theorem that there is a regular Borel probability measure μ on $B_1(X^*)$ such that

$$\|Tx\| \leq \|T\|_{\pi(p)} \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \quad \text{for all } x \in X .$$

For each $x \in X$, one has the evaluation map

$$f_x : B_1(X^*) \rightarrow \mathbf{K} ,$$

$$f_x(x^*) := \langle x, x^* \rangle .$$

Now let $X^p := \{f_x : x \in X\}$. One can easily check that X^p is a subspace of $L^p(B_1(X^*), \mu)$.

Further , let $y^* \in Y^*$. Define :

$$\phi_{y^*} : X^p \rightarrow \mathbf{K} ,$$

$$\langle f_x, \phi_{y^*} \rangle := \langle T x, y^* \rangle .$$

Obviously ϕ_{y^*} is linear . Moreover , ϕ_{y^*} is continuous also . Indeed :

$$\begin{aligned} \left| \langle f_x, \phi_{y^*} \rangle \right| &\leq \|T x\| \|y^*\| \\ &\leq \|T\|_{\pi(p)} \left(\int_{B_1(X^*)} |f_x(x^*)|^p d\mu(x^*) \right)^{\frac{1}{p}} \|y^*\| \\ &\leq \|T\|_{\pi(p)} \|f_x\| \|y^*\| \quad \text{for all } x \in X . \end{aligned}$$

By the Hahn-Banach Theorem , ϕ_{y^*} extends to a linear functional $\hat{\phi}_{y^*}$ on $L^p(B_1(X^*), \mu)$. Moreover , by [10, p. 286] , there is ψ_{y^*} in the space $L^{p^*}(B_1(X^*), \mu)$, where $\frac{1}{p} + \frac{1}{p^*} = 1$, such that

$$\|\psi_{y^*}\|_{p^*} = \|\phi_{y^*}\| \leq \|T\|_{\pi(p)} \|y^*\|$$

and

$$\langle T x, y^* \rangle = \int_{B_1(X^*)} f_x(x^*) \psi_{y^*}(x^*) d\mu(x^*) .$$

Now from $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ we have $\frac{p^*}{q} + \frac{p^*}{r^*} = 1$. So

$$\begin{aligned}
|\langle Tx, y^* \rangle| &\leq \int_{B_1(X^*)} |\langle x, x^* \rangle| |\psi_{y^*}(x^*)| d\mu(x^*) \\
&\leq \int_{B_1(X^*)} \left(|\langle x, x^* \rangle|^{\frac{r}{p}} \right) \left(|\langle x, x^* \rangle|^{\frac{r}{q}} |\psi_{y^*}(x^*)|^{\frac{p^*}{q}} |\psi_{y^*}(x^*)|^{\frac{p^*}{r^*}} \right) d\mu(x^*).
\end{aligned}$$

Applying Holder's inequality we obtain

$$|\langle Tx, y^* \rangle| \leq \lambda \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^r d\mu(x^*) \right)^{\frac{1}{p}}$$

where

$$\lambda := \left(\int_{B_1(X^*)} \left(|\langle x, x^* \rangle|^{\frac{p^* r}{q}} |\psi_{y^*}(x^*)|^{\frac{p^* p^*}{q}} \right) \left(|\psi_{y^*}(x^*)|^{\frac{p^* p^*}{r^*}} \right) d\mu(x^*) \right)^{\frac{1}{p^*}}.$$

Again by Holder's inequality, with $\frac{p^*}{q} + \frac{p^*}{r^*} = 1$, we have

$$\lambda \leq \int_{B_1(X^*)} \left(|\langle x, x^* \rangle|^r |\psi_{y^*}(x^*)|^{p^*} d\mu(x^*) \right)^{\frac{1}{q}} \left(\int_{B_1(X^*)} |\psi_{y^*}(x^*)|^{p^*} d\mu(x^*) \right)^{\frac{1}{r^*}}.$$

Hence

$$|\langle Tx, y^* \rangle| \leq \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^r d\mu(x^*) \right)^{\frac{1}{p}} \int_{B_1(X^*)} \left(|\langle x, x^* \rangle|^r |\psi_{y^*}(x^*)|^{p^*} d\mu(x^*) \right)^{\frac{1}{q}} \left(\int_{B_1(X^*)} |\psi_{y^*}(x^*)|^{p^*} d\mu(x^*) \right)^{\frac{1}{r^*}}.$$

Now let $x_1, \dots, x_n \in X$. Set

$$\hat{x}_i := \left(\int_{B_1(X^*)} |\langle x_i, x^* \rangle|^r d\mu(x^*) \right)^{-\frac{1}{p}} \cdot x_i \quad \text{for } i=1, \dots, n.$$

Thus

$$\left| \langle T \hat{x}_i, y^* \rangle \right|^q \leq \int_{B_1(X^*)} \left(\left| \langle x_i, x^* \rangle \right|^r \left| \psi_{y^*}(x^*) \right|^p d\mu(x^*) \right) \left(\int_{B_1(X^*)} \left| \psi_{y^*}(x^*) \right|^p d\mu(x^*) \right)^{\frac{q}{r^*}}.$$

Therefore

$$\begin{aligned} \left(\sum_{i=1}^n \left| \langle T \hat{x}_i, y^* \rangle \right|^q \right)^{\frac{1}{q}} &\leq \left(\sum_{i=1}^n \int_{B_1(X^*)} \left| \langle x_i, x^* \rangle \right|^r \left| \psi_{y^*}(x^*) \right|^p d\mu(x^*) \right)^{\frac{1}{q}} \left\| \psi_{y^*} \right\|_{p^*}^{\frac{p}{r^*}} \\ &\leq \sup_{B_1(X^*)} \left(\sum_{i=1}^n \left| \langle x_i, x^* \rangle \right|^r \right)^{\frac{1}{q}} \left\| \psi_{y^*} \right\|_{p^*}^{\frac{p}{r^*} + \frac{p}{q}} \\ &= \sup_{B_1(X^*)} \left(\sum_{i=1}^n \left| \langle x_i, x^* \rangle \right|^r \right)^{\frac{1}{q}} \left\| \psi_{y^*} \right\|_{p^*} \\ &\leq \sup_{B_1(X^*)} \left(\sum_{i=1}^n \left| \langle x_i, x^* \rangle \right|^r \right)^{\frac{1}{q}} \|T\|_{\pi(p)} \|y^*\|. \end{aligned}$$

Since S is absolutely q -summing, we have

$$\begin{aligned} \left(\sum_{i=1}^n \|S(T \hat{x}_i)\|^q \right)^{\frac{1}{q}} &\leq \|S\|_{\pi(q)} \sup_{y^* \in B_1(Y^*)} \left(\sum_{i=1}^n \left| \langle T \hat{x}_i, y^* \rangle \right|^q \right)^{\frac{1}{q}} \\ &\leq \|S\|_{\pi(q)} \|T\|_{\pi(p)} \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^n \left| \langle x_i, x^* \rangle \right|^r \right)^{\frac{1}{q}}. \end{aligned}$$

Now it follows from

$$\left(\sum_{i=1}^n \|S T x_i\|^r \right)^{\frac{1}{r}} = \left(\sum_{i=1}^n \|S T \hat{x}_i\|^r \left\{ \int_{B_1(X^*)} \left| \langle x_i, x^* \rangle \right|^r d\mu(x^*) \right\}^{\frac{r}{p}} \right)^{\frac{1}{r}}$$

and from Holder's inequality that

$$\begin{aligned}
\left(\sum_{i=1}^n \|ST_{x_i}\|^r \right)^{\frac{1}{r}} &\leq \left(\sum_{i=1}^n \|ST_{\hat{x}_i}\|^q \right)^{\frac{1}{q}} \left(\sum_{i=1}^n \left\{ \int_{B_1(X^*)} |\langle x_i, x^* \rangle|^r d\mu(x^*) \right\} \right)^{\frac{1}{p}} \\
&\leq \|S\|_{\pi(q)} \cdot \|T\|_{\pi(p)} \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^r \right)^{\frac{1}{r}} .
\end{aligned}$$

This proves

$$ST \in \Pi_r(X, Z) \quad \text{and} \quad \|ST\|_{\pi(r)} \leq \|S\|_{\pi(q)} \|T\|_{\pi(p)} \quad . \quad \square$$

We pass now to a duality problem for absolutely p -summing operators. We shall discuss this problem only in the finite dimensional setting . To do this we need the following motivation :

Let X be a finite dimensional space . If $T \in L(X)$, then

$$T = \sum_{i=1}^n x_i^* \otimes x_i \quad \text{where} \quad x_i^* \in X^* \quad \text{and} \quad x_i \in X \quad .$$

The trace of T is defined as $\text{tr}(T) := \sum_{i=1}^n \langle x_i, x_i^* \rangle$.

It is elementary and well-known that this definition does not depend on any representation of T .

Theorem 2.7.8 : *Let X and Y be finite dimensional Banach spaces and let $\frac{1}{p} + \frac{1}{p^*} = 1$. Then $\left(\Pi_p(X, Y), \|\cdot\|_{\pi(p)} \right)^*$ is isometrically isomorphic to $\left(N_{p^*}(Y, X), \|\cdot\|_{n(p^*)} \right)$.*

Proof : Let $T \in \Pi_p(X, Y)$. Define

$$J_T : N_p^*(Y, X) \rightarrow \mathbf{K},$$

$$J_T(S) := \text{tr}(TS) .$$

Clearly J_T is well-defined . Further , let $S = \sum_{i=1}^n y_i^* \otimes x_i$ in $N_p^*(Y, X)$.

Since the estimate

$$\begin{aligned} |\text{tr}(TS)| &\leq \left| \sum_{i=1}^n \langle T x_i, y_i^* \rangle \right| \\ &\leq \left(\sum_{i=1}^n \|T x_i\|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|y_i^*\|^{p^*} \right)^{\frac{1}{p^*}} \\ &\leq \|T\|_{\pi(p)} \left(\sum_{i=1}^n \|y_i^*\|^{p^*} \right)^{\frac{1}{p^*}} \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} \end{aligned}$$

holds for every representation of S we get $|J_T(S)| \leq \|T\|_{\pi(p)} \|S\|_{n(p^*)}$

which implies that $\|J_T\| \leq \|T\|_{\pi(p)}$.

On the other hand let $\varepsilon > 0$. Let $x_1, \dots, x_n \in X$ be such that

$$\sup_{x^* \in B_1(X^*)} \sum_{i=1}^n |\langle x_i, x^* \rangle|^p \leq 1 \quad \text{and} \quad \left(\sum_{i=1}^n \|T x_i\|^p \right)^{\frac{1}{p}} \geq (1 - \varepsilon) \|T\|_{\pi(p)} .$$

Pick $y_i^* \in Y^*$ with $\|y_i^*\| = 1$ and $\langle T x_i, y_i^* \rangle = \|T x_i\|$. Let us also fix numbers $\alpha_1, \dots, \alpha_n$ such that

$$\left(\sum_{i=1}^n |\alpha_i|^{p^*} \right)^{\frac{1}{p^*}} = 1 \quad \text{and} \quad \sum_{i=1}^n \alpha_i \|T x_i\| = \left(\sum_{i=1}^n \|T x_i\|^p \right)^{\frac{1}{p}} .$$

Define $S \in L(Y, X)$ by $S := \sum_{i=1}^n \alpha_i y_i^* \otimes x_i$. It follows from

$$\|S\|_{n(p^*)} \leq \left(\sum_{i=1}^n \|\alpha_i y_i^*\|^{p^*} \right)^{\frac{1}{p^*}} \sup_{x^* \in B_1(X^*)} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} \leq 1 .$$

and from

$$|\operatorname{tr}(TS)| = \left| \sum_{i=1}^n \alpha_i \langle T x_i, y_i^* \rangle \right| = \left(\sum_{i=1}^n \|T x_i\|^p \right)^{\frac{1}{p}}$$

$$\text{that } (1 - \varepsilon) \|T\|_{\pi(p)} \leq \left(\sum_{i=1}^n \|T x_i\|^p \right)^{\frac{1}{p}} \leq \|J_T\| .$$

Since $\varepsilon > 0$ is arbitrary, we have $\|T\|_{\pi(p)} \leq \|J_T\|$. Thus the mapping

$$\Pi_p(X, Y) \rightarrow \left(N_{p^*}(Y, X) \right)^* : T \rightarrow J_T$$

is an isometry between finite dimensional Banach spaces and hence it is

onto as well. This shows that $\left(N_{p^*}(Y, X), \|\cdot\|_{n(p^*)} \right)^*$ is isometrically

isomorphic to $\left(\Pi_p(X, Y), \|\cdot\|_{\pi(p)} \right)$. Since $N_{p^*}(Y, X)$ is a finite

dimensional Banach space which must be reflexive, the result follows.

This ends the proof. $\square$

2.8. An Application of p-Summing Operators :

Theorem 2.8.1 : *A Banach space X is a Hilbert space if and only if for all Banach spaces Y and for all absolutely 2-summing operators T mapping X into Y , the adjoint operator T^* is absolutely 2-summing and $\|T^*\|_{\pi(2)} \leq \|T\|_{\pi(2)}$.*

Proof : Let X be a Hilbert space and let Y be any Banach space . Further , let $T \in \Pi_2(X, Y)$. By Theorem 2.4.11 we have the following commutative diagram :

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ A \downarrow & & \uparrow B \\ C(K) & \xrightarrow{J_2} & L^2(K, \mu) \end{array}$$

with $\|A\| = 1$ and $\|B\| = \|T\|_{\pi(2)}$. Now the operator $J_2 A: X \rightarrow L^2(K, \mu)$ is an absolutely 2-summing operator between the Hilbert spaces X and $L^2(K, \mu)$. Hence by Theorem 2.5.14 we have $J_2 A \in C_2(X, L^2(K, \mu))$. Furthermore , since the Hilbert-Schmidt operators are closed under the adjointness , it follows that

$$(J_2 A)^* \in (C_2(L^2(K, \mu), X)) = \Pi_2((L^2(K, \mu), X))$$

with

$$\|(J_2 A)^*\|_{\pi(2)} = \|(J_2 A)^*\|_{\sigma(2)} = \|J_2 A\|_{\sigma(2)} = \|J_2 A\|_{\pi(2)} .$$

It follows from $T^* = (J_2 A)^* B^*$ and from the ideal properties of Π_2 that

T^* is absolutely 2-summing and

$$\|T^*\|_{\pi(2)} \leq \|(J_2 A)^*\|_{\pi(2)} \|B^*\| \leq \|J_2 A\|_{\pi(2)} \|B\| \leq \|J_2\|_{\pi(2)} \|A\| \|B\|.$$

But $\|J_2\|_{\pi(2)} = 1$, and $\|A\| = 1$, and $\|B\| = \|T\|_{\pi(2)}$. This shows that

$$\|T^*\|_{\pi(2)} \leq \|T\|_{\pi(2)}.$$

To prove the converse, we shall show that X^* is a Hilbert space by showing the norm of X^* satisfies the parallelogram law. Let, as usual, ℓ_2^2 be the two dimensional Hilbert space and let $\{e_1, e_2\}$ be the natural basis of ℓ_2^2 . Further let $x_1^*, x_2^* \in X^*$. Consider the linear operator

$$T: X \rightarrow \ell_2^2,$$

$$Tx := \sum_{i=1}^2 \langle x, x_i^* \rangle e_i.$$

Since T is finite rank operator, it is absolutely 2-summing. Also since

$$\|Tx\|_{\alpha(2)} = \left(\sum_{i=1}^2 |\langle x, x_i^* \rangle|^2 \right)^{\frac{1}{2}} \text{ for all } x \in X,$$

it follows from Theorem 2.5.6 that

$$\|T\|_{\pi(2)} \leq \left(\|x_1^*\|^2 + \|x_2^*\|^2 \right)^{\frac{1}{2}}. \quad (1)$$

Since, by the assumption, T^* is absolutely 2-summing such that

$$\|T^*\|_{\pi(2)} \leq \|T\|_{\pi(2)}$$

we must have

$$\begin{aligned} \left(\|T^* e_1\|^2 + \|T^* e_2\|^2 \right)^{\frac{1}{2}} &\leq \|T^*\|_{\pi(2)} \sup_{x^* \in B_1(\ell_2^2)} \left(\sum_{i=1}^2 |\langle e_i, x^* \rangle|^2 \right)^{\frac{1}{2}} \\ &\leq \|T\|_{\pi(2)} . \end{aligned}$$

But $T^* e_i = x_i^*$ for $i=1,2$. Hence

$$\left(\|x_1^*\|^2 + \|x_2^*\|^2 \right)^{\frac{1}{2}} \leq \|T\|_{\pi(2)} . \quad (2)$$

Combining the two results (1) and (2) we obtain

$$\|T\|_{\pi(2)} = \left(\|x_1^*\|^2 + \|x_2^*\|^2 \right)^{\frac{1}{2}} . \quad (3)$$

Now let

$$w_1^* := \frac{x_1^* + x_2^*}{\sqrt{2}} , \quad w_2^* := \frac{x_1^* - x_2^*}{\sqrt{2}} ,$$

$$f_1 := \frac{e_1 + e_2}{\sqrt{2}} , \text{ and } f_2 := \frac{e_1 - e_2}{\sqrt{2}} .$$

Clearly that $\{f_1, f_2\}$ is also a basis for ℓ_2^2 . Therefore by the same

argument above we get

$$\|T\|_{\pi(2)} = \left(\|w_1^*\|^2 + \|w_2^*\|^2 \right)^{\frac{1}{2}} = \left(\frac{\|x_1^* + x_2^*\|^2}{2} + \frac{\|x_1^* - x_2^*\|^2}{2} \right)^{\frac{1}{2}} . \quad (4)$$

Combining (3) and (4) we obtain

$$\left\| x_1^* + x_2^* \right\|^2 + \left\| x_1^* - x_2^* \right\|^2 = 2 \left(\left\| x_1^* \right\|^2 + \left\| x_2^* \right\|^2 \right) \quad .$$

This proves X^* is a Hilbert space . It follows immediately [6, p. 132]

that X is a Hilbert space . This completes the proof . $\square$

CHAPTER THREE

UNIFORMLY DOMINATED SETS

Introduction : In this chapter, we create a further study, which is an original work, on certain types of subsets of absolutely p -summing operators. The chapter is divided into three sections. Section 1 presents the definition of uniformly dominated sets and discusses some basic facts. In section 2, we study what is called summability, sequentially, totally sequentially, finitely, and finitely pointwise dominated sets of absolutely p -summing operators. Some basic properties and relationships of such sets are also discussed. In section 3, we present our fundamental theorems and some basic results.

3.1. Basic Concepts and Definitions :

Throughtout this section , let X and Y be arbitrary Banach spaces and let $1 \leq p < \infty$.

Definition 3.1.2 : A subset E of $\Pi_p(X, Y)$ is called λ -*uniformly dominated* if there are a constant $\lambda > 0$ and a regular probability Borel measure μ on $B_1(X^*)$ such that

$$\|Tx\| \leq \lambda \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \quad \text{for all } x \in X \text{ and } T \in E .$$

We shall write $E(p, \lambda, \mu)$ to denote the largest λ -uniformly dominated subset determined by μ which contains E .

Theorem 3.1.2 : $E(p, \lambda, \mu)$ is a closed convex subset of $\Pi_p(X, Y)$.

Proof : One can prove easily that $E(p, \lambda, \mu)$ is a convex subset of $\Pi_p(X, Y)$.

Now to show that $E(p, \lambda, \mu)$ is a closed subset of $\Pi_p(X, Y)$, let (T_n) be a sequence in $E(p, \lambda, \mu)$ such that $T_n \rightarrow T$ in $\Pi_p(X, Y)$. It follows from Lemma 2.3.1 that $T_n \rightarrow T$ in $L(X, Y)$ which implies that

$$\|T_n x\| \rightarrow \|T x\| \text{ for all } x \in X.$$

Since (T_n) is a sequence in $E(p, \lambda, \mu)$, we have, for all $n \in \mathbb{N}$,

$$\|T_n x\| \leq \lambda \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \text{ for all } x \in X.$$

Letting $n \rightarrow \infty$, we obtain

$$\|T x\| \leq \lambda \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \text{ for all } x \in X.$$

Hence $T \in E(p, \lambda, \mu)$. This ends the proof. $\square$

Example 3.1.3 : Let $(x_n^*) \in \ell^p(X^*)$ and let $\lambda > 0$. Further let

$$M := \left\{ T \in L(X, Y) : \|Tx\| \leq \lambda \left(\sum_{n=1}^{\infty} \frac{1}{2^n} |\langle x, x_n^* \rangle|^p \right)^{\frac{1}{p}} \text{ for all } x \in X \right\} .$$

Then M is a β -uniformly dominated subset of $\Pi_p(X, Y)$ for some β .

Indeed : Let $T \in M$. Then

$$\|Tx\| \leq \lambda \left(\sum_{n=1}^{\infty} \frac{1}{2^n} |\langle x, x_n^* \rangle|^p \right)^{\frac{1}{p}} \leq \lambda \left(\sum_{n=1}^{\infty} |\langle x, x_n^* \rangle|^p \right)^{\frac{1}{p}} \text{ for all } x \in X .$$

Hence , by Proposition 2.5.6 , $E \subseteq \Pi_p(X, Y)$.

Now for each $n \in \mathbb{N}$, set $z_n^* := \frac{x_n^*}{\|(x_n^*)\|_{\alpha(p)}}$. Then $\|z_n^*\| \leq 1$.

Moreover , let $\delta_{z_n^*}$ be the point mass measure at z_n^* . Define

$$\mu := \sum_{n=1}^{\infty} \frac{\delta_{z_n^*}}{2^n} .$$

Clearly μ is a regular probability measure on $B_1(X^*)$. For each $n \in \mathbb{N}$,

set $E_n := \{z_n^*\}$ and let $E_0 := B_1(X^*) \setminus \{z_n^* : n \in \mathbb{N}\}$. Since the E_n 's are

pairwise disjoint subsets , we have

$$\begin{aligned} \int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) &= \sum_{n=0}^{\infty} \int_{E_n} |\langle x, x^* \rangle|^p d\mu(x^*) \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle x, z_n^* \rangle|^p \end{aligned}$$

$$= \frac{1}{\|(x_n^*)\|_{\alpha(p)}} \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle x, x_n^* \rangle|^p$$

This shows that

$$\|Tx\| \leq \lambda \|(x_n^*)\|_{\alpha(p)} \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}}$$

for all $x \in X$ and $T \in E$. By setting $\beta := \lambda \|(x_n^*)\|_{\alpha(p)}$, we see that M is a β -uniformly dominated subset of $\Pi_p(X, Y)$.

Theorem 3.1.4 : *Let E be a λ -uniformly dominated subset of $\Pi_p(X, Y)$. If $x \in X$, then there is a $Q \in E(p, \lambda, \mu)$ such that*

$$\|Tx\| \leq \|Qx\| \quad \text{for all } T \in E.$$

Proof : Let $x \in X$ be fixed. Then for all $T \in E$, we have

$$\|Tx\| \leq \lambda \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}}.$$

It follows from the Hahn-Banach Theorem and Theorem 1 in [10, p. 286] that there is a $g \in L^p(B_1(X^*), \mu)$ with $\|g\|_{p^*} = 1$ such that

$$\|Tx\| \leq \lambda \int_{B_1(X^*)} \langle x, x^* \rangle g(x^*) d\mu(x^*) \quad \text{for all } T \in E.$$

Now let $e \in Y$ with $\|e\| = 1$ be fixed. Define a bounded linear operator

$Q: X \rightarrow Y$ by

$$Qz := \lambda \left\{ \int_{B_1(X^*)} \langle z, x^* \rangle g(x^*) d\mu(x^*) \right\} \cdot e .$$

Hence

$$\|Qx\| = \lambda \int_{B_1(X^*)} \langle x, x^* \rangle g(x^*) d\mu(x^*) .$$

Thus

$$\|Tx\| \leq \|Qx\| \quad \text{for all } T \in E .$$

Now since $\|g\|_{p^*} = 1$ then using Holder's inequality we obtain

$$\|Qz\| = \lambda \left| \int_{B_1(X^*)} \langle z, x^* \rangle g(x^*) d\mu(x^*) \right| \leq \lambda \left(\int_{B_1(X^*)} |\langle z, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}}$$

for all $z \in X$. By the Pietsch Domination Theorem 2.4.4 , $Q \in \Pi_p(X, Y)$ and consequently $Q \in E(p, \lambda, \mu)$. This ends the proof . $\square$

3.2. Further Types of Dominated Sets :

Definition 3.2.1 : A subset E of $\Pi_p(X, Y)$ is called λ -*summability dominated* , if there are a constant $\lambda > 0$ and a sequence $(x_n^*) \in \ell^p(X^*)$ such that

$$\|Tx\| \leq \lambda \left(\sum_{n=1}^{\infty} |\langle x, x_n^* \rangle|^p \right)^{\frac{1}{p}} \quad \text{for all } x \in X \text{ and all } T \in E .$$

We shall write $E\{p, \lambda, (x_n^*)\}$ to denote the largest λ -summability dominated subset of $\Pi_p(X, Y)$ determined by (x_n^*) which contains E .

Theorem 3.2.2: $E\{p, \lambda, (x_n^*)\}$ is a closed convex subset of $\Pi_p(X, Y)$.

Proof : We shall prove that $E\{p, \lambda, (x_n^*)\}$ is closed in $\Pi_p(X, Y)$ only .

So let (T_m) be a sequence in $E\{p, \lambda, (x_n^*)\}$ such that $T_m \rightarrow T$ in $\Pi_p(X, Y)$. It follows from Lemma 2.3.1 that

$$\|T_m x\| \rightarrow \|Tx\| \quad \text{for all } x \in X .$$

Since (T_m) is a sequence in $E\{p, \lambda, (x_n^*)\}$, we have for all $m \in \mathbb{N}$,

$$\|T_m x\| \leq \lambda \left(\sum_{n=1}^{\infty} |\langle x, x_n^* \rangle|^p \right)^{\frac{1}{p}} \quad \text{for all } x \in X .$$

Letting $m \rightarrow \infty$, we obtain

$$\|Tx\| \leq \lambda \left(\sum_{n=1}^{\infty} |\langle x, x_n^* \rangle|^p \right)^{\frac{1}{p}} \quad \text{for all } x \in X .$$

By Proposition 2.5.6 , $T \in \Pi_p(X, Y)$ and hence $T \in E\{p, \lambda, (x_n^*)\}$. This proves that $E\{p, \lambda, (x_n^*)\}$ is closed in $\Pi_p(X, Y)$. $\square$

Example 3.2.3 : Let $(x_m^*) \in \ell^p(X^*)$. Then one can easily verify that

$$M := \left\{ T := \lambda \sum_{m=1}^{\infty} x_m^* \otimes y_m : (y_m) \in B_1(\ell^{p^*}(Y)) \right\}$$

is λ -summability dominated subset of $\Pi_p(X, Y)$.

Theorem 3.2.4: Let E be a λ -summability dominated subset of $\Pi_p(X, Y)$. If $x \in X$, then there is a $Q \in E\{p, \lambda, (x_n^*)\}$ such that

$$\|Tx\| \leq \|Qx\| \text{ for all } T \in E.$$

Proof : Let $x \in X$ be fixed and $T \in E$. From the Hahn-Banach Theorem there exists a $(t_n) \in B_1(\ell^{p^*})$ such that

$$\|Tx\| \leq \lambda \left(\sum_{n=1}^{\infty} |\langle x, x_n^* \rangle|^p \right)^{\frac{1}{p}} = \lambda \sum_{n=1}^{\infty} \langle x, x_n^* \rangle t_n.$$

Now define

$$Q: X \rightarrow Y,$$

$$Qz := \lambda \left(\sum_{n=1}^{\infty} \langle z, x_n^* \rangle t_n \right) \cdot e,$$

where $e \in Y$ is fixed with $\|e\| = 1$. Clearly Q is a continuous linear operator with

$$\|Tx\| \leq \|Qx\| \text{ for all } T \in E.$$

Moreover $Q \in E\{p, \lambda, (x_n^*)\}$. Indeed :

$$\|Qz\| = \lambda \left| \sum_{n=1}^{\infty} \langle z, x_n^* \rangle t_n \right| \leq \lambda \left(\sum_{n=1}^{\infty} |\langle z, x_n^* \rangle|^p \right)^{\frac{1}{p}} \text{ for all } z \in X .$$

By Proposition 2.5.6 , $Q \in E\{p, \lambda, (x_n^*)\}$. $\square$

To introduce another type of dominated sets we need the following

Definition 3.2.5 : Let E be a subset of $\Pi_p(X, Y)$. A sequence (Q_n) in E is said to be a **strong sequence for** E , if there is a $Q \in E$ such that $Q_n \rightarrow Q$ strongly . (Q_n) is called a **partial strong sequence for** E , if there are a $Q \in E$ and a subsequence (Q_{n_k}) such that $Q_{n_k} \rightarrow Q$ strongly .

Clearly every strong sequence for E is a partial strong sequence for E but the converse is not generally true .

Definition 3.2.6 : A subset E of $\Pi_p(X, Y)$ is called **λ -sequentially dominated** , (with $\lambda > 0$), if for each sequence (x_n) in X , there is a partial strong sequence (Q_n) for E such that

$$\|Tx_n\| \leq \|Q_n x_n\| \text{ and } \|Q_n\|_{\pi(p)} \leq \lambda \quad (*)$$

for all $T \in E$ and all $n \in \mathbb{N}$.

E is said to be **λ -totally sequentially dominated** , with $\lambda > 0$, if for each sequence (x_n) in X , there is a strong sequence (Q_n) for E such that $(*)$ holds .

Example 3.2.7 : Let K be a compact Hausdorff space . It is known [10, p. 163] that $(M(K), \leq)$ with its usual order relation is a complete lattice . So let Z be a subset of positive measures in $B_1(M(K))$ which is bounded above with respect to the ordered relation . Since $M(K)$ is a complete lattice , Z has a least upper bound , say μ_o . Then the set

$$E := \{ \lambda \mu \otimes y : \mu \in Z \cup \{ \mu_o \} \text{ and } y \in B_1(Y) \}$$

is a λ -totally sequentially dominated subset of $\Pi_p(C(K), Y)$. Indeed :

Let (f_n) be a sequence in $C(K)$ and let $e \in Y$ with $\|e\| = 1$ be fixed . Set

$$Q_n := \lambda \mu_o \otimes e \quad \text{for all } n \in \mathbb{N} .$$

Now let $T := \lambda \mu \otimes y$ for some $\mu \in Z$ and $y \in B_1(Y)$. Then for all $n \in \mathbb{N}$,

$$\|T f_n\| \leq \lambda |\langle f_n, \mu \rangle| \leq \lambda |\langle f_n, \mu_o \rangle| = \|Q_n f_n\| \quad \text{and} \quad \|Q_n\|_{\pi(p)} \leq \lambda .$$

Moreover , (Q_n) is a strong sequence for E . This shows that E is λ -totally sequentially dominated .

We shall remark that the set E in the above example is a λ -uniformly as well as a λ -summability dominated set .

Theorem 3.2.8 : Let $1 < p < \infty$ and let E be a subset of $\Pi_p(X, Y)$.

If $E = E(p, \lambda, \mu)$ or $E = \{p, \lambda, (x_n^*)\}$ with $\|(x_n^*)\|_{\alpha(p)} = 1$, then E is

λ -sequentially dominated.

Proof : Let (x_n) be any sequence in X and $E = E(p, \lambda, \mu)$. By Theorem 3.1.4, for each $n \in \mathbb{N}$, there exists a $Q_n \in E(p, \lambda, \mu)$ such that

$$\|Tx_n\| \leq \|Q_n x_n\| \quad \text{for all } T \in E \text{ and all } n \in \mathbb{N}.$$

Further, Q_n can be chosen as

$$Q_n x := \lambda \left(\int_{B_1(X^*)} \langle x, x^* \rangle g_n(x^*) d\mu(x^*) \right) \cdot e \quad (*)$$

where

$$g_n \in B_1 \left(L^p \left(B_1(X^*), \mu \right) \right) \text{ and } e \in Y \text{ is fixed with } \|e\| = 1.$$

It remains to show that (Q_n) is a partial strong sequence for E .

Now $(*)$ can be written as

$$Q_n x = \lambda \langle f_x, g_n \rangle \cdot e \quad \text{where } f_x(x^*) := \langle x, x^* \rangle.$$

Since the closed unit ball of $L^p \left(B_1(X^*), \mu \right)$ is weakly compact and since (g_n) is a sequence in that unit ball, there is a subsequence (g_{n_k}) which converges weakly to a limit g with $\|g\|_{p^*} \leq 1$. That is

$$\langle g_{n_k}, f \rangle \rightarrow \langle g, f \rangle \quad \text{for all } f \in L^p \left(B_1(X^*), \mu \right). \quad (**)$$

Thus from $(**)$, we get that for any $x \in X$ the following holds :

$$\lim_{k \rightarrow \infty} \int_{B_1(X^*)} \langle x, x^* \rangle g_{n_k}(x^*) d\mu(x) = \int_{B_1(X^*)} \langle x, x^* \rangle g(x^*) d\mu(x).$$

Set

$$Qx := \lambda \left(\int_{B_1(X^*)} \langle x, x^* \rangle g(x^*) d\mu(x^*) \right) \cdot e .$$

Obviously Q defines a continuous linear operator from X into Y with $Q \in E$. Moreover, for fixed $x \in X$, we have

$$\lim_{k \rightarrow \infty} \|Q_{n_k} x - Qx\| = \lim_{k \rightarrow \infty} \left| \lambda \int_{B_1(X^*)} \langle x, x^* \rangle [g_{n_k}(x^*) - g(x^*)] d\mu(x^*) \right| = 0.$$

Hence (Q_{n_k}) converges strongly to Q . This concludes the first case.

Now we consider the case when $E = \{p, \lambda, (x_n^*)\}$. By Theorem 3.2.4, for each $n \in \mathbb{N}$, there exists a $Q_n \in E \setminus \{p, \lambda, (x_n^*)\}$ such that

$$\|Tx_n\| \leq \|Q_n x_n\| \quad \text{for all } T \in E \text{ and all } n \in \mathbb{N}.$$

Further Q_n can be chosen as

$$Q_n x := \lambda \left(\sum_{m=1}^{\infty} \langle x, x_m^* \rangle f_n(m) \right) \cdot e .$$

with $e \in Y$ is fixed with $\|e\| = 1$, and $\|f_n\|_{\alpha(p^*)} = 1$. One easily note that

$$Q_n x = \lambda \left(\left(\langle x, x_m^* \rangle \right), f_n \right) \cdot e .$$

Again since $\|f_n\|_{\alpha(p^*)} = 1$ and $B_1(\ell^{p^*})$ is weakly compact, (f_n) has a

subsequence (f_{n_k}) which converges weakly to some $f \in B_1(\ell^{p^*})$. Set

$$Qx := \lambda \left(\sum_{m=1}^{\infty} \langle x, x_m^* \rangle f(m) \right) \cdot e .$$

Now similar arguments to the first part of the proof ends the proof . $\square$

Definition 3.2.9 : A subset E of $\Pi_p(X, Y)$ is called λ -*finitely pointwise dominated* at $\{x_1, \dots, x_n\} \subseteq X$, if there is a $Q \in E$ such that

$$\|Q\|_{\pi(p)} \leq \lambda \quad \text{and} \quad \|Tx_i\| \leq \|Qx_i\| \quad \text{for all } T \in E \text{ and } i=1, \dots, n .$$

We say E is λ -*finitely pointwise dominated* if it is λ -finitely pointwise dominated at every finite family $\{x_1, \dots, x_n\} \subseteq X$.

Definition 3.2.10 : A subset $A := \{x_\omega : \omega \in \Omega\} \subseteq X$ is called *kernel independent* , if

$$X^* = [x_\omega^*] \oplus \ker(x_\omega) \quad \text{for all } \omega \in \Omega \quad \text{and} \quad \langle x_\omega, x_\beta^* \rangle = 0 \quad \text{for } \omega \neq \beta .$$

Proposition 3.2.11 : In a Hilbert space H , $\{x, y\}$ is kernel independent if and only if $\langle x, y \rangle = 0$.

Proof : Suppose that $\{x, y\}$ is kernel independent . We may assume that both x and y are non zero elements of H .

Now both of these elements can be thought of as linear functionals on H :

$$x(z) := \langle z, x \rangle \quad \text{and} \quad y(z) := \langle z, y \rangle \quad . \quad (*)$$

Since x and y are different from zero , we have $x \notin \ker(x)$ and $y \notin \ker(y)$. Moreover , since $\{x, y\}$ is kernel independent, we can write

$$\begin{aligned} H^* &= H = [x] \oplus \ker(x) \\ &= [y] \oplus \ker(y) . \end{aligned}$$

Hence , in particular , $\langle x, y \rangle = 0$.

Conversely , let $\langle x, y \rangle = 0$. Again we may assume that both of x and y are non zero elements. It follows from (*) and Proposition 1 in [24, p. 57] that

$$\begin{aligned} H &= [x] \oplus \ker(x) \\ &= [y] \oplus \ker(y) . \end{aligned}$$

But $H^* = H$. Consequently $\{x, y\}$ is kernel independent . $\square$

Theorem 3.2.12 : *Let $M := \{x_1, \dots, x_n\} \subseteq X$ be kernel independent.*

If , for each $i = 1, \dots, n$, $x_i \neq 0$, then M is linearly independent .

Proof : Since M is kernel independent , we have

$$X^* = [x_i^*] \oplus \ker(x_i) \quad \text{for } i = 1, \dots, n .$$

We may choose x_i^* so that $\langle x_i^*, x_i \rangle > 0$. Now let

$$\sum_{i=1}^n \alpha_i x_i = 0 \quad \text{where } \alpha_1, \dots, \alpha_n \text{ are scalars .}$$

Since , for any j , $x_j^* \in \ker(x_i)$ with $i \neq j$, we deduce from

$$\left\langle \sum_{i=1}^n \alpha_i x_i, x_j^* \right\rangle = \alpha_j \langle x_j, x_j^* \rangle = 0$$

that $\alpha_j = 0$. This proves that M is a linearly independent set . $\square$

Definition 3.2.13 : A subset E of $\Pi_p(X, Y)$ is called (λ, p) -*finitely dominated* at $\{x_1, \dots, x_n\} \subseteq X$, if there is a $Q \in \Pi_p(X, Y)$ such that

$$\sum_{i=1}^n \|T x_i\|^p \leq \sum_{i=1}^n \|Q x_i\|^p \quad \text{for all } T \in E \quad \text{and} \quad \|Q\|_{\pi(p)} \leq \lambda .$$

We say E is (λ, p) -*finitely dominated*, if it is (λ, p) -finitely dominated at every finite subset $\{x_1, \dots, x_n\} \subseteq X$.

The following result is immediate and the proof is omitted .

Proposition 3.2.14 : If E is λ -finitely pointwise dominated at $Z := \{x_1, \dots, x_n\} \subseteq X$, then E is (λ, p) -finitely dominated at Z as well .

Theorem 3.2.15 : If E is λ -summability dominated, then E is β -uniformly dominated for some β .

Proof : Since E is λ -summability dominated, we have

$$\|T x\| \leq \lambda \left(\sum_{n=1}^{\infty} |\langle x, x_n^* \rangle|^p \right)^{\frac{1}{p}} \quad \text{for all } x \in X \quad \text{and all } T \in E$$

for some $(x_n^*) \in \ell^p(X^*)$.

Now for each $n \in \mathbb{N}$, set $z_n^* := \frac{x_n^*}{\|(x_n^*)\|}$. Then $\|z_n^*\| \leq 1$. Moreover ,

let $\delta_{z_n^*}$ be the point mass measure at z_n^* .

Define

$$\mu := \frac{1}{\|(x_n^*)\|_{\alpha(p)}^p} \sum_{n=1}^{\infty} \|x_n^*\|^p \delta_{z_n^*} .$$

Clearly μ is a regular probability measure on $B_1(X^*)$. For each $n \in \mathbb{N}$, set $E_n := \{z_n^*\}$ and let $E_0 := B_1(X^*) \setminus \{z_n^* : n \in \mathbb{N}\}$. Since the E_n 's are pairwise disjoint subsets, we have

$$\begin{aligned} \int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) &= \sum_{n=0}^{\infty} \int_{E_n} |\langle x, x^* \rangle|^p d\mu(x^*) \\ &= \frac{1}{\|(x_n^*)\|_{\alpha(p)}^p} \sum_{n=1}^{\infty} \|x_n^*\|^p |\langle x, z_n^* \rangle|^p \\ &= \frac{1}{\|(x_n^*)\|_{\alpha(p)}^p} \sum_{n=1}^{\infty} |\langle x, x_n^* \rangle|^p \end{aligned}$$

This shows that

$$\|Tx\| \leq \lambda \|(x_n^*)\|_{\alpha(p)} \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}}$$

for all $x \in X$ and $T \in E$.

By setting $\beta := \lambda \|(x_n^*)\|_{\alpha(p)}$, we see that E is a β -uniformly dominated subset of $\Pi_p(X, Y)$. This completes the proof. $\square$

3.3. Fundamental Theorems and Main Results :

Throughout this section , let X and Y be arbitrary Banach spaces and let $1 \leq p < \infty$ unless explicitly stated .

Proposition 3.3.1 : *Let X be a separable Banach space and let E be a subset of $\Pi_p(X, Y)$. If E is λ -totally sequentially dominated, then E is β -uniformly dominated for some β .*

Proof : Let $\{x_n : n \in \mathbb{N}\}$ be a countable dense subset of X . Since E is λ -totally sequentially dominated , there is a strong sequence (Q_n) for E such that

$$\|Tx_n\| \leq \|Q_n x_n\| \text{ for all } T \in E \text{ and all } n \in \mathbb{N} .$$

Furthermore , let $Q \in E$ be the strong limit of (Q_n) and let $x \in X$ be fixed . There is a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \xrightarrow{k} x$ in X . Hence it follows immediately that

$$\|Tx_{n_k}\| \leq \|Q_{n_k} x_{n_k}\| \text{ for all } T \in E \text{ and all } k \in \mathbb{N} .$$

Letting $k \rightarrow \infty$ we obtain

$$\|Tx\| \leq \|Qx\| \text{ for all } T \in E \text{ and all } x \in X .$$

By the Pietsch Domination Theorem , there is a probability regular Borel measure μ on $B_1(X^*)$ such that

$$\|Tx\| \leq \beta \left(\int_{B_1(X^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \text{ for all } T \in E \text{ and all } x \in X$$

where $\beta := \|Q\|_{\pi(p)}$ and hence the result . $\square$

Theorem 3.3.2 : *Let E be a λ -uniformly dominated subset of $\Pi_p(X, Y)$. Then , for every $n \in \mathbb{N}$, E is $\lambda \cdot n$ -finitely pointwise dominated at every kernel independent set M in X with $\text{card}(M) \leq n$.*

Proof : Let $\{x_1, \dots, x_m\} \subseteq X$ be any kernel independent set with $m \leq n$. Then

$$\|Tx_i\| \leq \lambda \left(\int_{B_1(X^*)} |\langle x_i, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} = \lambda \int_{B_1(X^*)} \langle x_i, x^* \rangle f_i(x^*) d\mu(x^*)$$

for all $T \in E$, where $f_i \in L^p(B_1(X^*), \mu)$ with $\|f_i\|_p^* = 1$ for $i = 1, \dots, m$.

Since $\{x_1, \dots, x_m\}$ is kernel independent set , for each i , we have

$$\int_{B_1(X^*)} \langle x_i, x^* \rangle f_i(x^*) d\mu(x^*) = \int_{B_i} \langle x_i, x^* \rangle f_i(x^*) d\mu(x^*) \text{ where}$$

$$B_i := \{\alpha x_i^* : |\alpha| \leq 1\} \text{ and } X^* = [x_i^*] \oplus \ker(x_i) .$$

Moreover , by setting

$$Qz := \lambda \left(\sum_{i=1}^m \int_{B_i} \langle z, x^* \rangle f_i(x^*) d\mu(x^*) \right) \cdot e ,$$

where $e \in Y$ is fixed with $\|e\| = 1$, we obtain a continuous linear operator from X into Y . Further, for fixed j , we get

$$\begin{aligned}\|Qx_j\| &= \lambda \left| \sum_{i=1}^m \int_{B_i} \langle x_j, x^* \rangle f_i(x^*) d\mu(x^*) \right| \\ &= \lambda \left| \int_{B_j} \langle x_j, x^* \rangle f_j(x^*) d\mu(x^*) \right| \\ &= \lambda \left(\int_{B_j(x^*)} |\langle x_j, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}},\end{aligned}$$

because $x_i^* \in \ker(x_j)$ for $i \neq j$. This proves that

$$\|Tx_i\| \leq \|Qx_i\| \quad \text{for all } T \in E \text{ and } i = 1, \dots, n.$$

It is elementary to verify that $\|Q\|_{\pi(p)} \leq n \cdot \lambda$. This ends the proof. $\square$

Theorem 3.3.3 : *Let E be a λ -uniformly dominated subset of $\Pi_1(X, Y)$. Then E is λ -finitely pointwise dominated at every finite kernel independent set.*

Proof : Let $\{x_1, \dots, x_m\} \subseteq X$ be any kernel independent set. With no loss of generality, we may assume that $x_i \neq 0$ for $i = 1, \dots, m$. Following the same lines as well as the same notations of Theorem 3.3.2 we obtain

$$\|Tx_i\| \leq \|Qx_i\| \quad \text{for all } T \in E \text{ and } i = 1, \dots, m \text{ with}$$

$$Qz := \lambda \left(\sum_{i=1}^m \int_{B_i} \langle z, x^* \rangle f_i(x^*) d\mu(x^*) \right) \cdot e .$$

Furthermore $\|Q\|_{\pi(1)} \leq \lambda$. Indeed :

$$\begin{aligned} \|Qz\| &\leq \lambda \sum_{i=1}^m \int_{B_i} |\langle z, x^* \rangle| |f_i(x^*)| d\mu(x^*) \\ &\leq \lambda \sum_{i=1}^m \int_{B_i} |\langle z, x^* \rangle| d\mu(x^*) . \end{aligned}$$

Since $\{x_1, \dots, x_m\}$ is a kernel independent set of non zero elements , it follows from Theorem 3.2.12 that $B_1, \dots, B_m$ are pairwise disjoint subsets of $B_1(X^*)$. Therefore

$$\|Qz\| \leq \lambda \int_{B_1(X^*)} |\langle z, x^* \rangle| d\mu(x^*) \quad \text{for all } z \in X .$$

This shows , by the Pietsch Domination Theorem , that $\|Q\|_{\pi(1)} \leq \lambda$. $\square$

We shall mention that similar statements to Theorem 3.3.2 and Theorem 3.3.3 also valid when E is (λ, p) -finitely dominated and the proofs exactly the same .

Theorem 3.3.4 : *If E is λ -finitely pointwise dominated , then E is λ -uniformly dominated .*

Proof : Let $Z := B_1\left(M\left(B_1\left(X^*\right)\right)\right)$. For every $K := \{x_1, \dots, x_n\} \subseteq X$,

let Q_K be the associated dominating operator for E .

Define

$$\varphi_K : Z \rightarrow \mathbb{R} ,$$

$$\varphi_K(\mu) := \sum_{i=1}^n \left[\|Q_K x_i\|^p - \lambda^p \int_{B_1(X^*)} |\langle x_i, x^* \rangle|^p d\mu(x^*) \right] .$$

Then it is easy to see that φ_K is a convex weak*-continuous function on Z .

Let F be the set of all φ_K defined as above . Since , for every finite subset K of X , $Q_K \in E$, there exists , for every $\{\varphi_1, \dots, \varphi_m\} \subseteq F$ and

$\alpha_1, \dots, \alpha_m \in [0, \infty)$ with $\sum_{i=1}^m \alpha_i = 1$, $\varphi \in F$ such that

$$\sum_{i=1}^m \alpha_i \varphi_i(\mu) \leq \varphi(\mu) \quad \text{for all } \mu \in Z .$$

This means that F is concave .

Further , $f := \sum_{i=1}^n |\langle x_i, x^* \rangle|^p$ is a continuous function on $B_1(X^*)$

which is compact in the weak*-topology . Consequently there exists an

$$x_0^* \in B_1(X^*) \text{ such that } \|f\|_\infty = \sum_{i=1}^n |\langle x_i, x_0^* \rangle|^p .$$

Then for each finite family $K := \{x_1, \dots, x_n\} \subseteq X$ we have $\varphi_K(\delta_{x_o^*}) \leq 0$.

Thus F satisfies the conditions of Ky Fan's Lemma [27, p. 40]. So there is a $\mu \in Z$ such that

$$\varphi(\mu) \leq 0 \quad \text{for all } \varphi \in F.$$

Now consider $K := \{x\}$ where $x \in X$. Then for $T \in E$ we have

$$\begin{aligned} \|Tx\|^p - \lambda^p \int_{B_1(x^*)} |\langle x, x^* \rangle|^p d\mu(x^*) &\leq \|Q_K x\|^p - \lambda^p \int_{B_1(x^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \\ &= \varphi_K(\mu) \leq 0. \end{aligned}$$

Therefore

$$\|Tx\|^p \leq \lambda^p \int_{B_1(x^*)} |\langle x, x^* \rangle|^p d\mu(x^*) \quad \text{for all } T \in E.$$

This shows that E is λ -uniformly. $\square$

Theorem 3.3.5 : Let E be a closed convex subset of $\Pi_p(c_o, c_o)$.

Then the following are equivalent :

- (i) $E = E(p, \lambda, \mu)$ for some λ and μ .
- (ii) E is λ -finitely pointwise dominated.

Proof : (ii) $\Rightarrow$ (i) : This follows from Theorem 3.3.4.

(i) $\Rightarrow$ (ii) : Let E be λ -uniformly dominated. Then there is a regular Borel probability measure μ on $B_1(\ell^1)$ such that

$$\|Tx\| \leq \lambda \left(\int_{B_1(\ell^1)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \quad \text{for all } x \in X \text{ and all } T \in E.$$

Now let $Z := \{x_1, \dots, x_n\} \subseteq c_0$. For $i=1, \dots, n$, we can choose

$g_i \in B_1(L^{p^*}(B_1(\ell^1), \mu))$ such that

$$\left(\int_{B_1(\ell^1)} |\langle x_i, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} = \int_{B_1(\ell^1)} \langle x_i, x^* \rangle g_i(x^*) d\mu(x^*).$$

Let (e_i) be the natural basis for c_0 . Define

$$g: B_1(\ell^1) \rightarrow c_0,$$

$$g(x^*) := \sum_{i=1}^n g_i(x^*) e_i.$$

Then g is p^* -Pettis integrable and

$$\begin{aligned} \|g\|_{\varepsilon(p^*)} &= \sup_{y^* \in B_1(\ell^1)} \left(\int_{B_1(\ell^1)} |\langle g(x^*), y^* \rangle|^{p^*} d\mu(x^*) \right)^{\frac{1}{p^*}} \\ &= \sup_{y^* \in B_1(\ell^1)} \left(\int_{B_1(\ell^1)} \left| \left\langle \sum_{i=1}^n g_i(x^*) e_i, y^* \right\rangle \right|^{p^*} d\mu(x^*) \right)^{\frac{1}{p^*}} \end{aligned}$$

$$\begin{aligned}
&= \sup_{y^* \in B_1(\ell^1)} \left(\int_{B_1(\ell^1)} \left| \sum_{i=1}^n g_i(x^*) \langle e_i, y^* \rangle \right|^p d\mu(x^*) \right)^{\frac{1}{p}} \\
&\leq \sup_{y^* \in B_1(\ell^1)} \sum_{i=1}^n |\langle e_i, y^* \rangle| \|g_i\|_{p^*} .
\end{aligned}$$

It follows from

$$\sum_{i=1}^n |\langle e_i, y^* \rangle| \leq \|y^*\|_{\alpha(1)} \leq 1$$

that

$$\|g\|_{\varepsilon(p^*)} \leq 1 \cdot \max_{i=1, \dots, n} \|g_i\|_{p^*} \leq 1 .$$

Now define

$$Q: c_0 \rightarrow c_0 ,$$

$$Qx := \lambda \int_{B_1(\ell^1)} \langle x, x^* \rangle g(x^*) d\mu(x^*) .$$

Then

$$\begin{aligned}
\|Qx\| &\leq \lambda \left(\int_{B_1(\ell^1)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \|g\|_{\varepsilon(p^*)} \\
&\leq \lambda \left(\int_{B_1(\ell^1)} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} .
\end{aligned}$$

Hence $Q \in E(p, \lambda, \mu)$.

Further , for any $T \in E(p, \lambda, \mu)$ we have

$$\begin{aligned}
\|T x_i\|_{\infty} &\leq \lambda \left(\int_{B_1(\ell^1)} |\langle x_i, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}} \\
&\leq \lambda \left| \int_{B_1(\ell^1)} \langle x_i, x^* \rangle g_i(x^*) d\mu(x^*) \right| \\
&\leq \lambda \left| \int_{B_1(\ell^1)} \langle x_i, x^* \rangle \langle g(x^*), e_i \rangle d\mu(x^*) \right| \\
&\leq \lambda \sup_{y^* \in B_1(\ell^1)} \left| \int_{B_1(\ell^1)} \langle x_i, x^* \rangle \langle g(x^*), y^* \rangle d\mu(x^*) \right| \\
&= \|Q x_i\|_{\infty} .
\end{aligned}$$

This ends the proof . $\square$

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