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Applications of Qualitative Theory of Ordinary Differential Equations in Fluid Dynamics

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Submitted by

Abdulkarim Mohamed Hasan Khalil

B.Sc.in Mathematics, University of Bahrain,
Kingdom of Bahrain, 2002

Supervised by

Prof . Nail Suleiman Khabeev

Department of Mathematics
University of Bahrain

KINGDOM OF BAHRAIN

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Thesis Discussion Committee Decision

Student Name: <u>Abdulkarim / Mohamed Hasan</u>		Student ID: <u>19971517</u>
Department: <u>Mathematics</u>		College: <u>Science</u>
Program: <u>MSC. Mathematics</u>		
Thesis Title: <u>Applications of Qualitative Theory of ordinary differential Equations in fluid dynamics</u>		
Supervisor Name: <u>Prof. Nail Suleiman</u>		Student GPA: <u>3.96</u>

The Discussion Committee met on 16 Thursday 10/2014 (day and date) 10/2014 Hall no. S50-05 (hall/room no.) 05. Upon discussing the student whose details are stated above in public, the discussion committee recommends the following:

The student awarded grade A- In the thesis subject Math 599

- ☒ Without making any changes to the thesis
☐ After making the modifications attached with this decision

Discussion Committee Members Approval:

- | Name | Signature |
|--|--------------------|
| 1. <u>Prof. Nail Suleiman</u> (Advisor) | <u>[Signature]</u> |
| 2. <u>Prof. AL-Hammedi</u> (Internal Examiner) | <u>[Signature]</u> |
| 3. <u>Prof. Haydar Alwan</u> (External Examiner) | <u>[Signature]</u> |



ABSTRACT

APPLICATIONS OF QUALITATIVE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS IN FLUID DYNAMICS

In the present thesis the qualitative theory of differential equations is used along with topological considerations to discuss problems in fluid dynamics. Special attention is given to the qualitative aspects of dynamics of bubble in a viscous incompressible liquid, in particular to the collapsing of empty spherical bubble and oscillations of gas bubble, The theory relies much on the determining the critical radius of bubble, so that the bubble has a different behavior when its radius less than that critical value and when its greater that critical value.

TO MY FAMILY
AND
MY FRIENDS

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Chapter 1

INTRODUCTION

1.1 Background and history

The need for more complete understanding of the hydromechanical problem of cavitation and the gas bubble phenomena of underwater explosions has encouraged more and more detailed investigations into the pulsations of underwater gas bubbles.

The bubble as a physical object has been observed and studied for centuries. Newton observed the formation of an air bubble while pressing liquid between two lenses, but he did not realize at the time from where the bubble originates. Euler, in 1754, was the first to predict that a liquid moving at high velocity could be subject to a tensile stress which might result in what is today termed cavitation.

Berthelot (1850) carried out an experiment for observing cavitation in a liquid known now as the “Berthelot tube method”.

Later Reynolds (1878), demonstrated experimentally cavitation by heating today referred to as boiling. Parallel to these physical experiments various engineering designs encountered the phenomenon of cavitation and the damage that it causes to propellers.

The term “cavitation” describing the appearance of bubbles in liquids when the

pressure falls to a sufficiently low value was coined by William (Froude 1895).

The study of oscillations of vapour gas bubbles in a liquid is of considerable practical interest specifically in regard to the questions of possible use of bubble screens for dumping shock waves and the use of acoustic disturbances for intensification of technological processes.

Lord Rayleigh (1917) was the first who formulated mathematical model describing collapse of the empty spherical cavity in the liquid. He solved the problem of the collapsing spherical cavity in an incompressible liquid. He made the same assumption that the liquid is incompressible and flow around the cavity is spherically symmetric with only a radial component. After this study, several studies made the assumptions in investigating the the collapse of empty bubble and oscillations of a gas bubble as being the same as Rayleighs model: (Plesset 1949); (Lauterborn 1976); (Plesset and Prosperetti 1977).

To formulate a bubble model, various parameters of the liquid surrounding the bubble and of the gas contents inside the bubble need to be specified. For a spherical bubble the dependent variables can be condensed to just one, the bubble radius r_1 whose variation with time t has determined. beside r_1 further density and viscosity of the liquid ρ and η should be included to the list of basic parameters.

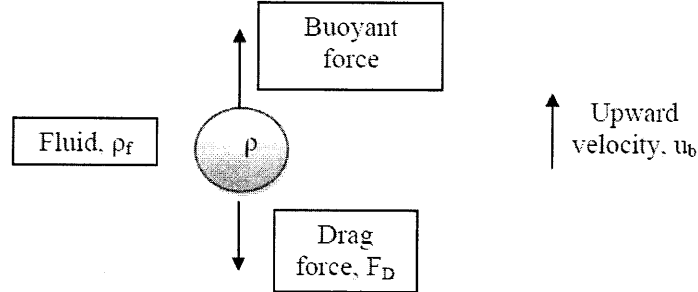


Figure 1.1: Single bubble rising due to density differences under gravity

1.2 Statement of Purpose

A single bubble in a liquid is a remarkable object. In many cases it attains a spherical shape due to surface tension and floats around in the liquid driven by the various forces that it is susceptible to, notably pressure forces of all kinds from static to acoustic, from buoyancy to drag. The single bubble is the building block and starting point for describing more complex bubble configurations.

The present thesis is devoted to investigation of the character of bubble collapse in the viscous liquid.

We assume that the bubble is spherical and empty. The radial oscillations of gas bubble in a liquid will be also investigated. In both cases the study of the problems will be done on the basis of theory of differential equations. Numerical calculations will be used for illustration of analytical results.

The following steps will be done in the process of study this problem:

1. To study qualitative theory of differential equations since it is not simple subject.

2. To use methods of qualitative theory of differential equations for the problem of spherical bubble collapse in the viscous liquid.

3. To calculate numerically the critical value of Reynolds number. The bubbles, which are smaller than a critical size, are filled slowly in an infinitely long time. The filling of large bubble takes place rapidly with an unlimited accumulation of energy during collapse. A quantitative formula should be obtained for critical radius of the bubble.

4. In the second part of the thesis the dynamics of a gas bubble in a viscous incompressible liquid will be studied on the base of qualitative theory of differential equations.

It is generally agreed that intense oscillations of cavity bubbles without collapse, constitute one of the main reasons for the cavity erosion of materials. When the dimension of cavity bubble reaches a certain limiting value, strong pressure pulses may occur in the surrounding liquid, which can cause erosion by local cycle loads (Numachi 1958, p.125); (Sutton 1957).

The aim of the present study is to derive a qualitative formula defining the critical bubble size separating two different regimes of radial bubble motion: oscillatory and monotonically aperiodic. Analysing Rayleigh equation for bubble radial motion the

isolated singularities will be found.

1.3 Justification of the problem

Since collapse and intense oscillations of gas bubbles are the main reasons for the cavity erosion of materials it is very important to study this process from practical point of view. It is important to study the effect of many parameters acting in the complex to the intensity of erosion of materials.

For this purposes fundamental mathematical models describing the process should be created. On the base of this model analytical and numerical investigation should be done using advanced mathematical theory and numerical methods.

1.4 Theoretical framework

The problem of the spherical bubble collapse will be considered under the following assumptions:

a spherical symmetry of the process and pressure uniformity within the bubble. In the course of bubble oscillations the homobaricity prevails when the size of bubble is much less than the length of a sound wave in the gas. On the other hand, when the bubble radius changes monotonically (collapse) the homobaricity condition take place when the velocity of the bubble surface is much less than the sound speed in the gas. This statement of the problem is valid for a wide range of bubble sizes, since the time of pressure equalization is very small.

1.4.1 Methods and procedures

The methods of qualitative theory of differential equations will be used together with advanced numerical methods to calculate critical value of Reynolds number which separates different types of bubble collapse.

The figures illustrating the collapse and oscillations of bubbles for different values of basic parameters will be presented.

The aim of the present study is to derive a qualitative formula defining the critical bubble size separating two different regimes of radial bubble motion, first in case of empty spherical bubble which we will discuss it in chapter 4, and secondly in case of

gas bubble which we will discuss it in chapter 5.

In chapter 4, we will see that for different Reynold number R the solution may belong to different families: in the case of the node they correspond to an unlimited increase of the velocity $\xi \sim \alpha^{-3/2}$ where ξ is the velocity and α is the radius of spherical bubble; in the case of the saddle point they correspond to a decelerated motion of the bubble with $\xi \sim \alpha$. as a result of which its filling occurs only after an infinite time. In the intermediate case, which corresponds to the dividing line **OA**, the filling occurs after a finite time, and the velocity near the center grows as $\xi \sim \alpha^{-1}$.

The Reynolds number corresponding to the dividing line is the critical Reynolds number. It separates two fundamentally different classes of solution. Its critical value $R = R_*$ will be determine in this chapter.

In chapter 5, the two different regime are oscillatory and monotonically aperiodic. Analysing Rayleigh equation for bubble radial motion the isolated singularities will be found.

Analysing roots of characteristic equation the character of singularity should be determined.

a. If the roots of the characteristic equation are real, distinct and same sign, consequently the singularity is a node. All integral curves in the neighbourhood of this singularity enter it at the same angle and motion will be monotonic.

b. If the roots of the characteristic equation are complex conjugate, the singular

point will be focus. In this case each integral curve will be contained between two logarithmic spirals. The gas bubble will oscillate infinitely long time with diminishing amplitude.

The distinct difference between the process of collapsing of bubble in viscous liquid and the oscillations of gas bubble in liquid will be studied in details. The role of surface tension should be investigated for both cases. The obtained results will be used for practical problem of analysing of accidents at nuclear power stations.

1.4.2 Proposed chapter outline

The thesis will contain five chapters:

Chapter one (Introduction)

It includes:

- a. Background and history of the research.
- b. Statement of Purpose.
- c. Justification of the problem. d. Theoretical framework and Research

Overview.

Chapter two (Review of Literature)

In this chapter I will present a background comprehensive review of the literature in addition to that covered in the introduction.

Chapter three (The Methodology of this thesis)

In this chapter I will present the principle of the qualitative theory of differential equations and I call it "Some elements of the qualitative theory of differential equations" which represent our research main tools methods.

Chapter four (The collapse of bubbles in a viscous liquid)

In this chapter I will present the first applications of qualitative theory of differential equations in fluid dynamics, which is the problem of collapse of empty bubble in viscous liquid, also through out this chapter I use the numerical methods and the powerful of Mathematica 7 program to find the value of critical bubble radius.

Chapter five (On the dynamics of a gas bubble in a viscous incompressible liquid)

In this chapter I will present the investigation of dynamics of gas bubble in viscous incompressible liquid. This will include analytical study of the problem by using powerful methods of qualitative theory of differential equations and advanced numerical analysis.

Chapter six (The Results)

Chapter seven (Discussion)

In this chapter I will present the summary and conclusions.

Chapter 2

LITERATURE REVIEW

The first analysis of a problem in cavitation and bubble dynamics was made by Rayleigh (1917), who solved the problem of the collapse of an empty cavity in large mass of liquid. Rayleigh also considered in this same paper the problem of a gas-filled cavity under the assumption that the gas undergoes isothermal compression.

His interest in these problems presumably arose from concern with cavitation and cavitation damage. With neglect of surface tension and liquid viscosity and with the assumption of liquid incompressibility, Rayleigh showed from the momentum equation that the bubble boundary $r_1(t)$ obeyed the relation

$$r_1 \frac{d^2 r_1}{dt^2} + \frac{3}{2} \left(\frac{dr_1}{dt} \right)^2 = \frac{p(r_1) - p_0}{\rho} \quad (2.1)$$

where ρ is the liquid density, p_0 is the pressure in the liquid at a large distance from the bubble, and $p(r_1) = p_1$ is the pressure in the liquid at the bubble boundary. For this Rayleigh problem, $p(r_1)$ is also the pressure within the bubble. A more refined treatment was successfully completed by Herring (1941).

In the presence of viscosity, Poritsky (1952) found by numerical integration that the time of collapsing is infinite for some case.

Shu (1952) found that the time of collapsing is infinite if the parameter $4/R$ is

less than square root of 6 and is finite otherwise, where R is the Reynold number.

Whereas Zababakhin (1960, p.1129), found that the critical parameter is 8.4 .

While Rayleigh neglected surface tension and liquid viscosity and kept the pressure p_0 constant, his dynamical equation (2.1) is easily extended to include these effects by Plesset (1949) to describe the experimental observations cavitation bubble growth and collapse in a liquid flow, his equation was

$$r_1 \frac{d^2 r_1}{dt^2} + \frac{3}{2} \left(\frac{dr_1}{dt} \right)^2 + \frac{4\eta}{\rho r_1} \frac{dr_1}{dt} + \frac{2\sigma}{\rho r_1} + \frac{p_0 - p}{\rho} = 0 \quad (2.2)$$

here η denote the viscosity σ denotes the surface tension of the liquid, and p denotes the pressure of gas within the bubble. Equation (2.2) is called generalized Rayleigh-Plesset equation for bubble dynamics. The first application of Rayleigh-Plesset (1949) was on the problem of traveling cavitation bubbles.

Sometime later Gilmores (1952) took a different approach and postulated equations to describe the growth or collapse of a spherical bubble in a viscous compressible liquid.

The small amplitude (linearized) oscillations of a permanent gas bubble in a liquid were first considered by Minnaert (1933), who showed that a substantial contribution to the sound emitted by running water comes from the free radial pulsations of entrained air bubbles. More recently the same type of problem has been considered in greater detail by several investigators including Devin (1959), Plesset and Hsieh (1960), Frolov (1968), Chapman and Plesset (1971), and Prosperetti (1976), Plesset

and Prosperetti (1977).

Papers on bubble physics and dynamics after then are countless. It is difficult to cover the topic completely in a single volume: most are oriented to one of the many aspects. Soo (1967) considers the fluid dynamics of multiphase systems with an accent on solid particles. Clift et al. (1978) give a good summary on bubbles, drops and particles motion in flows and the mass and heat transfers from and to particles.

Trevena (1987) emphasizes the experimental work on the creating tension in liquids, cavitation and erosion. Young (1989), who is a specialist in sonoluminescence, focuses on acoustic cavitation. The basic theory of light scattering by small particles, including air bubbles, is given by Van de Hulst (1957), while Jerlov (1976) considers the same topic in the context of oceanography. The influence of surfactants on bubble characteristics can be better understood by considering not only the physics, but also the chemistry of the bubble interface and aqueous solution Rosen (1978); Adamson (1982).

Although the papers on bubble are countless Brennen (1995) try to provides the most complete coverage of bubble dynamics, with rigorous mathematical derivations of all dynamical equations. And recently Kedrinskii (2005) provide another complete coverage on bubble dynamics.

In Kedrinskii (2005) book he presented the research results of underwater ex-

plosions. For example the shock-wave transformation in bubbly liquids, shock-wave amplification due to collision and focusing, and the formation of bubble detonation waves in reactive bubbly liquids.

As we see some authors study the problem under the assumption that there is no gas inside the bubble, and other authors consider the problem when there is gas inside the bubbles.

These assumptions will lead to different behaviour of the bubble, in the first case the bubble will collapse after finite time when the radius of bubble is smaller than some critical radius, whereas in the second case the bubble will oscillated, and there is some critical radius for such oscillation.

The main difficulties involved in theoretical investigations of the Rayleigh-Plesset equation are that the solutions can be obtained only numerically and the bubble wall velocity increases to infinity as the bubble collapses. Thus, computer simulations of the bubble motion take a great deal of time and may lead to significant errors in the numerically calculated solutions, especially when the bubble achieves supersonic speeds. Unfortunately, the analytically described bubble dynamics was obtained only for the collapse in inviscid liquids.

In the present thesis, I will present a way to avoid the above difficulties for viscous liquids, I will implement the theory of qualitative theory of differential equations and numerical methods to calculate the critical parameter of bubble in both cases.

Chapter 3

SOME ELEMENTS OF THE QUALITATIVE THEORY OF DIFFERENTIAL EQUATIONS

3.1 Introduction

Jules Henri Poincare (1854 - 1912) was a French mathematician, theoretical physicist, engineer, and a philosopher of science. He is often described as a polymath, and in mathematics as The Last Universalist, since he excelled in all fields of the discipline as it existed during his lifetime.

After defending his doctoral thesis on the study of singular points of the system of differential equations, Poincare wrote a series of memoirs under the title "On curves defined by differential equations" (1881-1882). In these articles, he built a new branch of mathematics, called "qualitative theory of differential equations. Poincare showed that even if the differential equation can not be solved in terms of known functions, yet from the very form of the equation, a wealth of information about the properties and behavior of the solutions can be found. In particular, Poincare investigated the nature of the trajectories of the integral curves in the plane, gave a classification of singular points (saddle, focus, center, node), introduced the concept of a limit cycle and the loop index, and showed that the number of limit cycles is al-

ways finite, except for some special cases. Poincare also developed a general theory of integral invariants and solutions of the variational equations. For the finite-difference equations, he created a new direction the asymptotic analysis of the solutions. He applied all these achievements to study practical problems of mathematical physics and celestial mechanics, and the methods used were the basis of its topological works.

3.2 Phase-Plane Analysis

The systems with which this chapter is concerned are those of the form

$$\dot{x} = X(x, y), \quad \dot{y} = Y(x, y) \tag{3.1}$$

satisfied by the functions $x = x(t)$, $y = y(t)$, where X and Y are continuous real-valued functions.

They are often referred to as *plane autonomous systems* of ordinary differential equations. Here, plane indicates that just two functions x, y , dependent on the variable t , are involved and autonomous that neither of the functions X, Y contains this variable explicitly.

It is not in general possible to find an analytic solution to an arbitrary differential equation and, even when it is, the solution is often expressed in a form (such as a series or integral, or implicitly) which does not allow easy discussion of its more important properties. This chapter provides an introduction to a qualitative study in which one can locate properties of a differential equation without having to find a solution.

3.3 Phase-Plane and stability

When the real plane $\mathbb{R}^2$, with axes x and y , is used to plot solutions $x = x(t), y = y(t)$ of (3.1), it will be referred to as the phase-plane. The curve traced out by $(x(t), y(t))$ as t varies (usually over some finite or infinite interval) is called a phase-path or trajectory. A diagram depicting the phase-paths is called a phase-diagram. Whenever $X \neq 0$, (3.1) gives rise to

$$\frac{dy}{dx} = \frac{Y(x, y)}{X(x, y)} \quad (3.2)$$

The existence and uniqueness theorem of ordinary differential equations tells us that, when Y/X is sufficiently well-behaved and subject to an appropriate boundary condition, (3.2) has a unique solution. So, save for singular points, we should expect one and only one phase-path through each point in the phase-plane.

Constant solutions $x(t) = x_0, y(t) = y_0$ of (3.1) correspond to (singular) points (x_0, y_0) satisfying

$$X(x_0, y_0) = Y(x_0, y_0) = 0. \quad (3.3)$$

such points are called critical points, or equilibrium points.

The nature of the phase-paths near a critical point can often give important information about the solutions of the system (3.1). Intuitively, if near a critical point (a, b) the phase-paths $(x, y) = (x(t), y(t))$ have all their points close to (a, b) , for t greater than some t_0 in the domain of (x, y) , then the point (a, b) is called stable. We

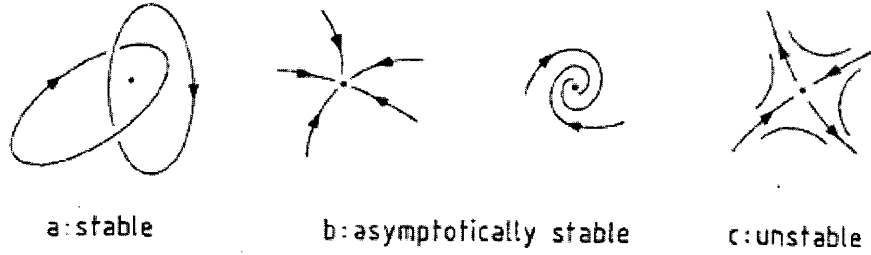


Figure 3.1: Stable and unstable steady solutions.

may formalise this idea as follows. For $(a, b) \in \mathbb{R}^2$ and a solution $(x, y) = (x(t), y(t))$ of (3.1), let

$$F(x, y, a, b, t) := \sqrt{(x(t) - a)^2 + (y(t) - b)^2}$$

Definition 1 A critical point (a, b) is stable if and only if given $\varepsilon > 0$, there exist $\delta > 0$ and a real t_0 such that for any solution $(x, y) = (x(t), y(t))$ of (3.1) for which $F(x, y, a, b, t_0) < \delta$,

$$F(x, y, a, b, t) < \varepsilon$$

for every t in the domain of (x, y) which is greater than t_0 . A critical point is unstable if and only if it is not stable.

3.4 Linear theory

In this section, we will analyse in some detail the phase-diagrams representing the linear system

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy \quad (3.4)$$

where a, b, c, d are real constants. we will search for solutions of (3.4) in the form

$$x = Ae^{\lambda t}, \quad y = Be^{\lambda t} \quad (3.5)$$

where A, B are (possibly complex) constants. Substituting (3.5) in (3.4), we find that

$$(a - \lambda)A + bB = 0, \quad cA + (d - \lambda)B = 0 \quad (3.6)$$

For not both of A, B to be zero, it is necessary that

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0,$$

that is, that

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0 \quad (3.7)$$

the remaining analysis turns on the nature of the roots λ_1, λ_2 of this *characteristic equation*.

(i) λ_1, λ_2 real, unequal and of the same sign

Using equations (3.6), we see that the root $\lambda = \lambda_1$ of the characteristic equation (3.7)

gives rise to a (fixed) ratio $A : B$.

We suppose this ratio is $1 : C_1$, with C_1 non-zero, giving the (real) solution

$$x(t) = K_1 e^{\lambda_1 t}, \quad y(t) = K_1 C_1 e^{\lambda_1 t}$$

where K_1 is an arbitrary real constant.

If the ratio $A : B$ is $1 : C_2$ when $\lambda = \lambda_2$, there is similarly the (real) solution

$$x(t) = K_2 e^{\lambda_2 t}, \quad y(t) = K_2 C_2 e^{\lambda_2 t}$$

where K_2 is an arbitrary real constant. The general solution is the sum of these two solutions, namely,

$$\begin{cases} x(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} \\ y(t) = K_1 C_1 e^{\lambda_1 t} + K_2 C_2 e^{\lambda_2 t} \end{cases} \quad (3.8)$$

where K_1, K_2 are arbitrary real constants. (Note that the existence and uniqueness theorem of ordinary differential equations would allow us to determine, on any interval, the constants K_1, K_2 by using initial conditions for x and y .) When $\dot{x} \neq 0$, the phase-paths satisfy

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\lambda_1 K_1 C_1 \lambda_1 e^{\lambda_1 t} + \lambda_2 K_2 C_2 \lambda_2 e^{\lambda_2 t}}{\lambda_1 K_1 \lambda_1 e^{\lambda_1 t} + \lambda_2 K_2 \lambda_2 e^{\lambda_2 t}} \quad (3.9)$$

We now suppose that λ_1, λ_2 are both negative and, without loss of generality, that $\lambda_1 > \lambda_2$. When $K_1 \neq 0, K_2 = 0$, we have the phase-path $y = C_1 x$; and when $K_1 = 0, K_2 \neq 0$, we have the phase-path $y = C_2 x$. Further, for each phase-path,

$$\frac{dy}{dx} \rightarrow C_1 \quad \text{as } t \rightarrow \infty, \quad \frac{dy}{dx} \rightarrow C_2 \quad \text{as } t \rightarrow -\infty$$

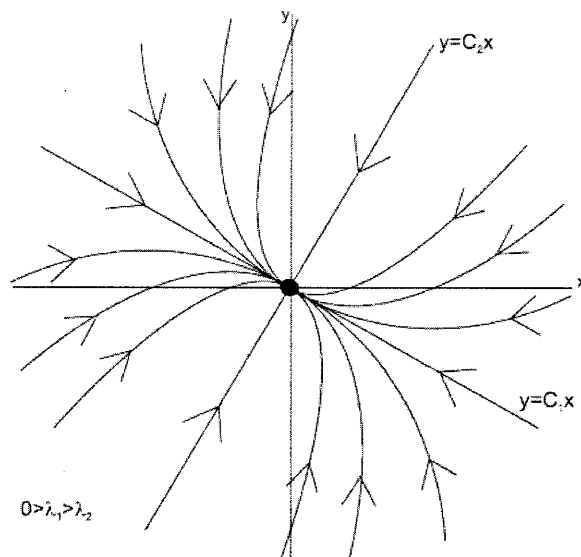


Figure 3.2: The critical point is a node.

The directions of the phase-paths and the behaviour of solutions at zero and infinity follow directly from consideration of $x, y, \dot{x}, \dot{y}$.

We are now in a position to draw the phase-diagram. The origin is a stable critical point. When λ_1, λ_2 are both positive, the phase-diagram is similar, but the arrows point away from the origin and the origin is an unstable critical point. In both cases, this type of critical point is called a *nodal point*, or *node*.

(ii) $\lambda_1 = \lambda_2 \neq 0$

When b, c are not both zero, the real general solution of the system in parametric form is

$$x(t) = (K_1 + K_2 t)e^{\lambda_1 t}$$

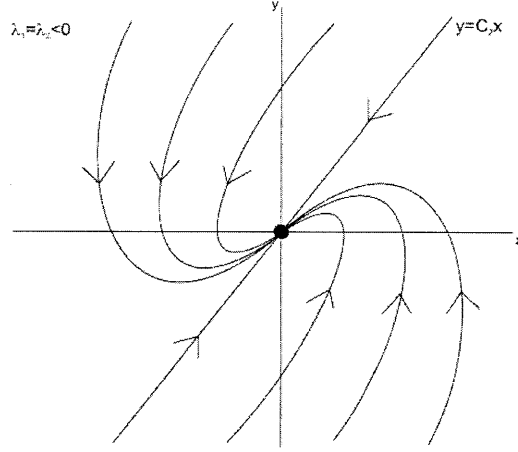


Figure 3.3: The critical point is inflected node.

$$y(t) = (K_1 C_1 + K_2 C_2 t) e^{\lambda_1 t}$$

where C_1, C_2, K_1, K_2 are real constants (with the same provisos concerning C_1, C_2 as in case (i)). When λ_1 is negative, the phase-diagram can be drawn as follows. It can be considered as the limiting case of the diagram in (i) as $\lambda_2 \rightarrow \lambda_1$. The origin is a stable critical point. When λ_1 is positive, the origin is unstable and the arrows in the phase-diagram point away from the origin. This type of critical point is called an *inflected nodal point*, or *inflected node*.

(iii) λ_1, λ_2 real and of opposite sign

Without loss of generality, assume $\lambda_1 > 0 > \lambda_2$. The equations (3.8), (3.9) of (i) apply. In this case $y = C_1 x$ and $y = C_2 x$ are the only phase-paths through the origin, the first being described from the origin to infinity, the second from infinity to the

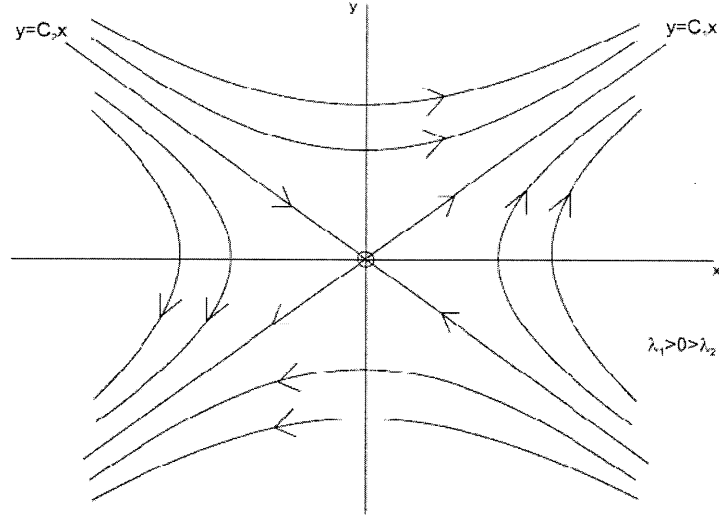


Figure 3.4: The critical point is saddle point.

origin, as t increases. The phase-diagram may be drawn as above origin is an unstable critical point, and is a saddle-point.

(iv) $\lambda_1 = \bar{\lambda}_2 = \beta + i\alpha$, where β, α are both non-zero real numbers

The real general solution may be written

$$x = Ke^{\beta t} \sin(\alpha t + \delta)$$

$$y = KCe^{\beta t} \sin(\alpha t + \epsilon)$$

where K, C, δ, ϵ , are real constants. The periodicity of the sine function indicates that the phase-paths are spirals, approaching the origin as t increases if β is negative, but approaching infinity if β is positive. The phase-diagram is therefore as figure 3.5. The origin is thus stable if and only if β is negative. This type of critical point is called a

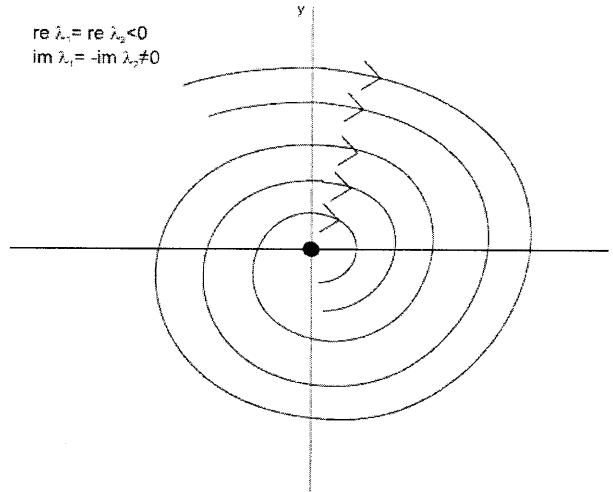


Figure 3.5: The critical point is focus or spiral point.

spiral-point, or *focus*.

(v) $\lambda_1 = \bar{\lambda}_2 = i\alpha$, where α is a non-zero real number Now the system has the real general solution

$$x = K \sin(\alpha t + \delta)$$

$$y = KC \sin(\alpha t + \epsilon)$$

where K, C, δ, ϵ are real constants. The phase-paths are ellipses and the phase-diagram is shown in figure 3.6. The origin is necessarily stable and, the critical point is called a *centre*, or *vortex*.

(vi) $\lambda_1 \neq 0, \lambda_2 = 0$

The phase-paths form a set of parallel lines.

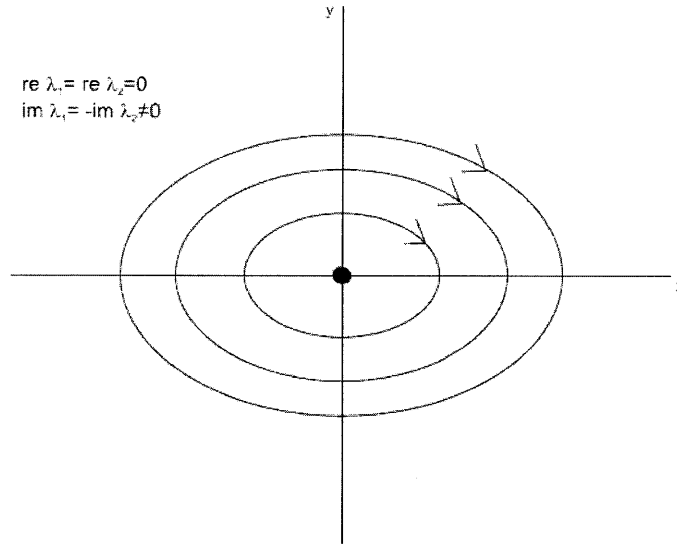


Figure 3.6: The critical point is center or vertex.

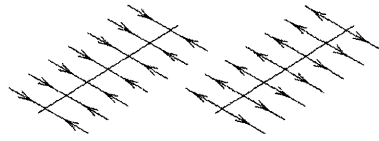


Figure 3.7: The phase-paths when case of $\lambda_1 \neq 0, \lambda_2 = 0$

(vii) $\lambda_1 = \lambda_2 = 0$

(a) **Case 1** When the coefficient matrix $\mathbf{A}$ of (3.6) is diagonal, then $\mathbf{A} = \mathbf{0}$

and every point of the plane is a critical point.

(b) **Case 2** If coefficient matrix $\mathbf{A}$ of (3.6) is not diagonal, then from the orbits are on straight lines.

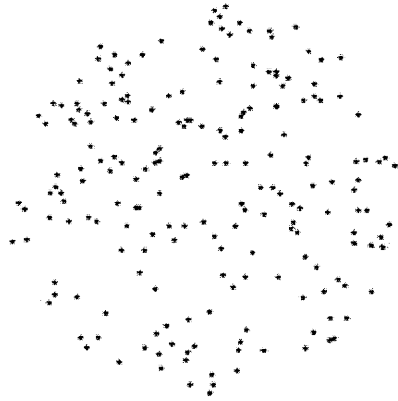


Figure 3.8: $\lambda_1 = \lambda_2 = 0$ and the coefficient matrix $\mathbf{A}$ is diagonal

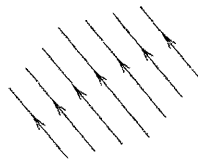


Figure 3.9: $\lambda_1 = \lambda_2 = 0$ and the coefficient matrix $\mathbf{A}$ is not diagonal

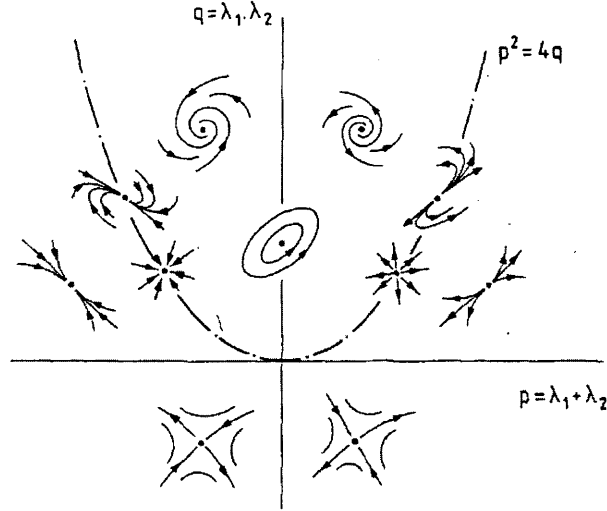


Figure 3.10: General classification of a linear system in $\mathbb{R}^2$.

Summary

By referring to equation (3.7), if we let $p = a + d = \lambda_1 + \lambda_2$, and $q = (ad - bc) = \lambda_1 \lambda_2$, then equation (3.7) can be written as

$$\left(\lambda - \frac{1}{2}p\right)^2 - \left(\frac{1}{2}p\right)^2 + q = 0$$

Then the several different cases of the singular point are:

- (i) $q < 0$: the eigenvalues are real with different signs. The singular point is a saddle point and is unstable.
- (ii) $0 < q < \left(\frac{1}{2}p\right)^2$: the eigenvalues are real, unequal and of the same sign. The singular point is a stable node (sink) if $p < 0$ and an unstable node (source) if $p > 0$.
- (iii) $q > \left(\frac{1}{2}p\right)^2$ ($p \neq 0$): the eigenvalues are conjugate complex with non zero real

part. The phase-paths shows a focus or spiral near the singular point. The singular point is a stable focus if $p < 0$ and an unstable focus if $p > 0$.

(iv) $q > 0, p = 0$: the eigenvalues are purely imaginary. The trajectories form closed curves surrounding the origin and the singular point is called a center.

(v) $q = (\frac{1}{2}p)^2$ ($p \neq 0$): the eigenvalues are real and equal. The nature of the phase-paths depends on the Jordan form of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

if it has the Jordan form

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix},$$

the trajectories form a star shaped node, which is stable for $p < 0$ and unstable if $p > 0$. If it has the Jordan form

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

the phase-paths is called an inflected node; all trajectories tend to the origin in the same direction and are parallel at infinity.

(vi) $q = 0, p \neq 0$: one of the eigenvalues is zero; the phase-paths consists of a family of parallel paths and a line of singular points, which is stable for $p < 0$ and unstable if $p > 0$.

3.5 Linearisation

We can surmise from the foregoing, if the nature of the critical points of the plane autonomous system (3.1) is known, one is in a good position to attempt to draw a phase-diagram. One rather useful method of discussing the critical points is by approximating X and Y by linear functions in a neighbourhood of each such point and then using the linear theory. The approximation is achieved by using Taylor expansions. By first making a translation of axes, we may suppose that the critical point under discussion is at the origin. Supposing that X and Y are twice continuously differentiable, the Taylor expansions of X , Y at the origin may be written

$$\begin{cases} X(x, y) = ax + by + M(x, y) \\ Y(x, y) = cx + dy + N(x, y) \end{cases} \quad (3.10)$$

where

$$a = X_x(0, 0), \quad b = X_y(0, 0), \quad c = Y_x(0, 0), \quad d = Y_y(0, 0)$$

and

$$M(x, y) = O(r^2), \quad N(x, y) = O(r^2),$$

as $r = \sqrt{x^2 + y^2} \rightarrow \infty$.

Our hope is then that the linear approximation (3.4) will have the same type of critical point as the original system. However these, 'linearized' phase-paths are not necessarily equivalent with those near the singular point of the original non-linear

system. The relation between both phase-paths is given by the theorem of Hartman-Grobman.

Hartman-Grobman's Theorem

If the equation (3.7) has no eigenvalues with zero real part, then the family of trajectories near a singular point of a nonlinear system and those of the locally linearized system have the same topological structure; which means that in a neighbourhood of the singular point there exists a homeomorphic mapping which maps trajectories of the non-linear system into trajectories of the linear system.

When the linear system (3.4) has no eigenvalues with zero real part, the singular point is called hyperbolic or nondegenerate.

The theorem of Hartman-Grobman implies that local linearization near singular points will be an effective method when analysing phase-paths of non-linear systems. The isolated singular points for a linear system are saddles, nodes, foci and centers. Since for nodes, foci and saddle points the real parts of the roots of (3.7) are not equal zero holds, Hartman-Grobman's theorem implies that adding sufficiently smooth nonlinearities does not change the phase-paths near these types of singular points.

Centers in non-linear systems satisfy the real parts of the roots of (3.7) are equal zero, and their existence cannot be shown by linearization. As an illustration consider the nonlinear system

$$\dot{x} = y, \quad \dot{y} = -x + \varepsilon x^2 y$$

with eigenvalues $\lambda_1, \lambda_2 = \pm i$. Unless $\varepsilon = 0$, the singular point $(0,0)$ is not a center as in the linearized system, but a nonlinear focus, stable if $\varepsilon < 0$.

In general, if the linear approximation has a node, saddle-point or spiral, then so does the original system.

Chapter 4

THE COLLAPSE OF BUBBLES IN A VISCOUS LIQUID

4.1 Introduction

In this chapter we will concern to the filling of an empty spherical bubble in a viscous liquid. The existence of two different types of motion was discovered: bubbles, which are smaller than a critical size, are filled slowly in an infinitely long time; the filling of large bubbles takes place rapidly with an unlimited accumulation of energy during collapse.

A quantitative formula is obtained for the critical radius of the bubble.

Assume that for some reason a bubble has formed inside the liquid, which in the future can again be filled under the action of the surrounding pressure. The problem of the filling of a spherical bubble in an inviscid incompressible liquid was studied for the first time in Rayleigh (1917). He found, in particular, that the velocity of motion its surface toward the center grow indefinitely as $r^{-3/2}$, as the bubble completes the process of collapsing, i.e. an unlimited accumulation of energy takes place. This phenomenon is assumed to be a possible for the rapid wear in screw propellers and turbines which operate under cavitations conditions. The collapse of bubbles at a metal surface can damage it severely.

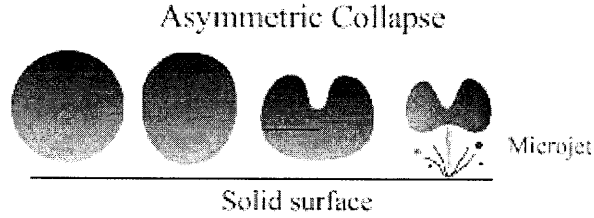


Figure 4.1: Metal surface damage due to collapse of bubble.

The review of the papers devoted to the collapse of bubble was done in Bogoyavlenskiy (1999); Frolov (1968, p.1118); Kou Yamada et al. (2006); Lauterborn et al. (2010); Leighton (1994); M.Al-Mannai & N.Khabeev (2012); Nigmatulin, Khabeev & Nagiev (1981); T.Kurz et al. (2007). Now let us study the Rayleigh problem for the viscous liquid. Such a statement of the problem corresponds more nearly to the actual conditions, although it still does not give an accurate description of the phenomenon, since it does not take into account the compressibility of the liquid, the inevitable presence of its vapor in the bubble and the possible instability of the bubble.

4.2 Applying the qualitative theory of differential equations to find the critical radius of empty spherical bubble

Assume that a spherical bubble of radius a was formed in a liquid of density ρ , pressure p_0 (far from bubble) and viscosity η . There is no pressure inside the bubble and the initial velocity is zero. The motion of the bubble will be spherically symmetric and can be described by the Navier-Stokes equations of the form

$$\frac{\partial u}{\partial r} + \frac{2u}{r} = 0, \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0 \quad (4.1)$$

Here $u(r, t)$ is the velocity, $p(r, t)$ is the pressure. The viscosity does not enter equation (4.1) because in the general equation it occurs in the term

$$\eta(\text{grad div } u - \text{curl curl } u) \quad (4.2)$$

Which in our case is identically equal to zero. This is so because $\text{div } u = 0$, since the liquid is incompressible and $\text{curl } u = 0$ because of spherical symmetry.

The normal stress σ_{rr} at the free surface of the bubble is zero (boundary with a vacuum), and since

$$\sigma_{rr} = -p + 2\eta \frac{\partial u}{\partial r} \quad (4.3)$$

Hence we conclude that

$$p_1 = 2\eta \left(\frac{\partial u}{\partial r} \right)_1 \quad (4.4)$$

i.e. the viscosity has entered the boundary condition. Here and in the following, the

index 1 denotes a value at the boundary. The second boundary condition is

$$p = p_0 \quad \text{for} \quad r = \infty \quad (4.5)$$

Integrate both sides of the first equation of (4.1) with respect to r

$$\int_{r_1}^r \frac{\partial u}{u} = \int_{r_1}^r \frac{-2\partial r}{r}.$$

Where r_1 is a functions of t .

This implies that

$$\ln u(r, t) - \ln u(r_1, t) = -2(\ln r - \ln r_1)$$

Note that $u(r_1, t)$ is a function of t , since r_1 is a functions of t .

Hence $u(r_1, t)$ represent the velocity at the boundary of the bubble, and so we can denoted by u_1 as we mention before.

After simplifying the previous relations we obtain

$$u(r, t) = \frac{u_1 r_1^2}{r^2} \quad (4.6)$$

Where $u_1 r_1^2$ is a function of t .

Now substitute (4.6) into the second equation of (4.1)

$$\frac{\partial}{\partial t} \left(\frac{u_1 r_1^2}{r^2} \right) + \left(\frac{u_1 r_1^2}{r^2} \right) \frac{\partial}{\partial r} \left(\frac{u_1 r_1^2}{r^2} \right) + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0$$

Which implies

$$\frac{1}{r^2} (2r_1 u_1^2 + r_1^2 \frac{du_1}{dt}) + \frac{1}{r^5} (-2r_1^4 u_1^2) + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0$$

Then integrate with respect to r from r_1 to ∞ , taking into account the boundary conditions on p

$$(2r_1 u_1^2 + r_1^2 \frac{du_1}{dt}) \int_{r_1}^{\infty} \frac{1}{r^2} dr + (-2r_1^4 u_1^2) \int_{r_1}^{\infty} \frac{1}{r^5} dr + \frac{1}{\rho} \int_{r_1}^{\infty} \frac{\partial p}{\partial r} dr = 0$$

Which gives

$$r_1 \frac{du_1}{dt} + \frac{3}{2} u_1^2 + \frac{p_0}{\rho} - \frac{p_1}{\rho} = 0 \quad (4.7)$$

Divide both sides by $r_1 u_1$ and then use the boundary condition (4.4)

$$\frac{1}{u_1} \frac{du_1}{dt} + \frac{3}{2} \frac{u_1}{r_1} + \frac{p_0}{\rho r_1 u_1} - \frac{2\eta}{\rho r_1 u_1} \left(\frac{\partial u}{\partial r} \right)_1 = 0 \quad (4.8)$$

By chain rule

$$\frac{du_1}{dt} = \frac{du_1}{dr_1} \frac{dr_1}{dt} = u_1 \frac{du_1}{dr_1} \implies \frac{du_1}{dr_1} = \frac{1}{u_1} \frac{du_1}{dt}$$

So (4.8) can be written as

$$\frac{du_1}{dr_1} + \frac{3}{2} \frac{u_1}{r_1} + \frac{p_0}{\rho r_1 u_1} - \frac{2\eta}{\rho r_1 u_1} \left(\frac{\partial u}{\partial r} \right)_1 = 0 \quad (4.9)$$

Now by using (4.6) we can find $\left(\frac{\partial u}{\partial r} \right)_1$ and then substitute to the previous equation,

then we will get

$$\frac{du_1}{dr_1} + \frac{3}{2} \frac{u_1}{r_1} + \frac{p_0}{\rho r_1 u_1} + \frac{4\nu}{r_1^2} = 0 \quad \left(\nu = \frac{\eta}{\rho} \right) \quad (4.10)$$

Here ν is the kinematic viscosity.

Now let us introduce new equation corresponds to (4.10) with a new variables so that the special constants p_0 , ρ and ν are disappeared, and so we can study the

family of solutions for any values of such constants. (such new variables is called non-dimensional variables)

To do that we multiply both sides of (4.10) by $\frac{\rho\nu}{p_0}$

$$\frac{\rho\nu}{p_0} \frac{du_1}{dr_1} + \frac{3}{2} \frac{\rho\nu}{p_0} \frac{u_1}{r_1} + \frac{\nu}{r_1 u_1} + \frac{\rho}{p_0} \frac{4\nu^2}{r_1^2} = 0 \quad (4.11)$$

Equation (4.11) can be written as

$$\frac{d\left(\sqrt{\frac{\rho}{p_0}} u_1\right)}{d\left(\frac{1}{\nu}\sqrt{\frac{p_0}{\rho}} r_1\right)} + \frac{3}{2} \frac{\left(\sqrt{\frac{\rho}{p_0}} u_1\right)}{\left(\frac{1}{\nu}\sqrt{\frac{p_0}{\rho}} r_1\right)} + \frac{1}{\left(\sqrt{\frac{\rho}{p_0}} u_1\right) \left(\frac{1}{\nu}\sqrt{\frac{p_0}{\rho}} r_1\right)} + \frac{4}{\left(\frac{1}{\nu}\sqrt{\frac{p_0}{\rho}} r_1\right)^2} = 0 \quad (4.12)$$

Where d is the derivative operator.

And then the non-dimensional variables are

$$\xi = \frac{u_1}{\sqrt{p_0/\rho}}, \quad \alpha = \frac{1}{\nu} \sqrt{\frac{p_0}{\rho}} r_1 \quad (4.13)$$

Let the initial radius of the bubble a , so we can introduce the Reynolds number to be

$$R = \frac{a}{\nu} \sqrt{\frac{p_0}{\rho}} \quad (4.14)$$

Hence equation (4.12) becomes

$$\frac{d\xi}{d\alpha} + \frac{3}{2} \frac{\xi}{\alpha} + \frac{1}{\xi\alpha} + \frac{4}{\alpha^2} = 0 \quad (4.15)$$

Since the initial velocity is zero ($u_1(a) = 0$), then the initial condition is of form $\xi(R) = 0$.

As we see, our problem under study has a family of solution which depends on the number R .

Before we continue the discussion our case study, let us first study some special cases (in case of absent of some assumptions of our case study), to let us understand the full picture of bubble collapsing.

We originally assume that the is a spherical bubble formed in a liquid of density ρ , pressure p_0 (far from bubble) and viscosity η , and there is no pressure inside the bubble and the initial velocity is zero.

Some special cases

(Case1) If the liquid is an ideal liquid (non viscous liquid)

This means that($\nu = 0, \eta = 0$), and then from equations (4.4) and (4.14) $p_1 = 0$ and $R = \infty$.

Now equation (4.10) can be written as

$$\frac{du_1}{dr_1} + \frac{3}{2} \frac{u_1}{r_1} + \frac{p_0}{\rho r_1 u_1} = 0 \quad (4.16)$$

Multiply both sides of (4.16) by $2r_1^3 u_1$

$$2r_1^3 u_1 \frac{du_1}{dr_1} + 3r_1^2 u_1^2 + 2 \frac{p_0}{\rho} r_1^2 = 0$$

Or it can be written as

$$r_1^3 \frac{d}{dr_1} (u_1^2) + \frac{d}{dr_1} (r_1^3) u_1^2 + 2 \frac{p_0}{\rho} r_1^2 = 0$$

That is

$$\frac{d}{dr_1}(r_1^3 u_1^2) = -2 \frac{p_0}{\rho} r_1^2 \quad (4.17)$$

Integrate equation (4.17) with respect to r_1 , keeping in mind the initial condition

$$u_1(a) = 0$$

$$r_1^3 u_1^2 = \frac{-2p_0}{3\rho} (r_1^3 - a^3)$$

Hence

$$u_1 = \pm \sqrt{\frac{2p_0}{3\rho} \left(\frac{a^3}{r_1^3} - 1 \right)} \quad (4.18)$$

In case of bubble collapse, which is our aim of study

$$u_1 = - \sqrt{\frac{2p_0}{3\rho} \left(\frac{a^3}{r_1^3} - 1 \right)} \quad (4.19)$$

And as $r_1 \rightarrow 0$, then $u_1 \sim r^{-3/2}$, and $u_1 \rightarrow -\infty$ which is in fact the Rayleigh (1917) argument.

Let us try to find the collapsing time in this case (in case of ideal liquid).

We have

$$\frac{dr_1}{dt} = - \sqrt{\frac{2p_0}{3\rho} \left(\frac{a^3}{r_1^3} - 1 \right)} \quad (4.20)$$

Equation (4.21) is separable, then we can sperate the variables and integrate

$$\int_{r_1=a}^{r_1=0} \frac{dr_1}{- \sqrt{\frac{2p_0}{3\rho} \left(\frac{a^3}{r_1^3} - 1 \right)}} = \int_{t=0}^{t=t_{TC}} dt$$

Where t_{TC} is the collapsing time.

By making change of variable, let

$$x = \frac{r_1^3}{a^3} \quad (4.21)$$

After substituting the new variable x and change the lower and upper limit of the integral, and this will yield

$$\int_0^1 \frac{a \, dx}{3x^{2/3} \sqrt{\frac{2p_0}{3\rho} \left(\frac{1}{x} - 1\right)}} = t_{TC} \quad (4.22)$$

Simplifying the last integral of equation (4.22)

$$t_{TC} = \int_0^1 \frac{a}{\sqrt{\frac{6p_0}{\rho}}} x^{-1/6} (1-x)^{-1/2} dx \quad (4.23)$$

Evaluate the integral in (4.23)

$$t_{TC} = \frac{a}{\sqrt{\frac{6p_0}{\rho}}} B\left(\frac{5}{6}, \frac{1}{2}\right) = \frac{a}{\sqrt{\frac{p_0}{\rho}}} \frac{\Gamma(\frac{5}{6})\Gamma(\frac{1}{2})}{\sqrt{6}\Gamma(\frac{5}{6} + \frac{1}{2})} \quad (4.24)$$

Where B and Γ are the Beta and the Gamma functions respectively.

Then

$$t_{TC} = 0.915 \frac{a}{\sqrt{\frac{p_0}{\rho}}} \quad (4.25)$$

In fact, we can extend the previous work, and find the general relation between the time and bubble radius.

By (4.21), (4.23),

$$t = \int_{\frac{3}{\sqrt[3]{a^3 x}}}^1 \frac{a}{\sqrt{\frac{6p_0}{\rho}}} X^{-1/6} (1-X)^{-1/2} dX \quad (4.26)$$

Integrate (4.26)

$$t = \frac{a}{\sqrt[3]{6 \frac{p_0}{\rho}}} \left[6/5 X^{5/6} {}_2F_1(1/2, 5/6; \frac{11}{6}; X) \right]_{\sqrt[3]{a^3 x}}^1 \quad (4.27)$$

where ${}_2F_1$ is the hypergeometric function.

Substitute the lower limit and upper limit of the integral, and this will yield

$$t = \frac{a}{\sqrt[3]{6 \frac{p_0}{\rho}}} \left[\frac{6}{5} {}_2F_1(1/2, 5/6; \frac{11}{6}; 1) - \frac{6}{5} (a^3 x)^{5/18} {}_2F_1(1/2, 5/6; \frac{11}{6}; \sqrt[3]{a^3 x}) \right] \quad (4.28)$$

Simplify (4.28)

$$t = \frac{a}{\sqrt[3]{\frac{p_0}{\rho}}} \left[0.915 - \frac{\sqrt{6}}{5} (a^3 x)^{5/18} {}_2F_1(1/2, 5/6; \frac{11}{6}; \sqrt[3]{a^3 x}) \right] \quad (4.29)$$

Hence the relation between the time and the bubble radius is

$$t = \frac{a}{\sqrt[3]{\frac{p_0}{\rho}}} \left[0.915 - \frac{\sqrt{6}}{5} (r_1)^{5/6} {}_2F_1(1/2, 5/6; \frac{11}{6}; r_1) \right] \quad (4.30)$$

Equation (4.30) form an analytic solution of (4.20).

Also we can find the relation between the radial velocity of the bubble and the time.

From (4.18)

$$r_1 = \frac{\sqrt[3]{2p_0} a}{\sqrt[3]{3\rho u_1^2 + 2p_0}} \quad (4.31)$$

Hence the explicit relation between the time and the velocity will be

$$t = \frac{a}{\sqrt[3]{\frac{p_0}{\rho}}} \left[0.915 - \frac{\sqrt{6}}{5} \left(\frac{2p_0 a^3}{3\rho u_1^2 + 2p_0} \right)^{5/18} {}_2F_1(1/2, 5/6; \frac{11}{6}; \frac{2p_0 a}{3\rho u_1^2 + 2p_0}) \right] \quad (4.32)$$

We can summarize case (1) if $R = \infty$ ($\nu = 0, \eta = 0$), then the solution of the differential equation coincides with Rayleigh (1917) solution. For this family, the velocity near the center grows as $\xi \sim a^{-3/2}$.

(Case2) If the liquid is a viscous liquid with $p_0 = 0, u_1(a) = 0$

Let us study the problem in case of absence of an external pressure in a viscous liquid.

Substitute equation (4.4) into (4.7)

$$r_1 \frac{du_1}{dt} + \frac{3}{2}u_1^2 - \frac{2\eta}{\rho} \left(\frac{\partial u}{\partial r} \right)_1 = 0 \quad (4.33)$$

Differentiate (4.6) with respect to r_1 and substitute it into (4.33), and then simply the result

$$r_1 \frac{du_1}{dt} + \frac{3}{2}u_1^2 + \frac{4\nu u_1}{r_1} = 0 \quad (4.34)$$

Multiply equation (4.34) by $r_1^{1/2}$

$$r_1^{3/2} \frac{du_1}{dt} + \frac{3}{2}r_1^{1/2}u_1^2 = -\frac{4\nu u_1}{r_1^{1/2}} \quad (4.35)$$

Or it can be written as

$$r_1^{3/2} \frac{d^2 r_1}{dt^2} + \frac{3}{2}r_1^{1/2} \left(\frac{dr_1}{dt} \right)^2 = -\frac{4\nu \frac{dr_1}{dt}}{r_1^{1/2}} \quad (4.36)$$

Hence equation (4.36) can be written as

$$\frac{d}{dt} (r_1^{3/2} \dot{r}_1) = -\frac{4\nu \frac{dr_1}{dt}}{r_1^{1/2}} \quad (4.37)$$

Integrate equation (4.37) with respect to t ,

$$r_1^{3/2} \dot{r}_1 - a^{3/2} \dot{r}_1(0) = -8\nu(r_1^{1/2} - a^{1/2}) \quad (4.38)$$

Given that the initial velocity equal zero, then

$$r_1^{3/2} \frac{dr_1}{dt} = -8\nu(r_1^{1/2} - a^{1/2}) \quad (4.39)$$

Divided both sides by $r_1^{3/2}$

$$\frac{dr_1}{dt} = -8\nu \frac{r_1^{1/2} - a^{1/2}}{r_1^{3/2}} \quad (4.40)$$

Then (4.40) is separable, so we can split the variables and integrate

$$\int_a^{r_1} \frac{r_1^{3/2}}{\sqrt{r_1} - \sqrt{a}} dr_1 = -8\nu \int_0^t dT \quad (4.41)$$

By integrating both sides,

$$\lim_{a \rightarrow w} \left[1/2 r_1^2 + 2/3 \sqrt{a} r_1^{3/2} + a r_1 + 2 a^{3/2} \sqrt{r_1} + 2 a^2 \ln(-\sqrt{r_1} + \sqrt{a}) \right]_w^{r_1} = -8\nu t$$

Hence $t = \infty$ for any radius r_1 .

So we deduce that the bubble whenever it form at any radius, take infinite long time to collapse.

Remark Equation (4.40) has no singular point.

(Case3) If the liquid is a viscous liquid with $p_0 = 0$, $u_1(a) \neq 0$

Now let us study the problem in case of absence of an external pressure and the initial velocity $\dot{r}_1(0) \neq 0$.

By using case (2) we have

$$r_1^{3/2}\dot{r}_1 - a^{3/2}\dot{r}_1(0) = -8\nu(r_1^{1/2} - a^{1/2}) \quad (4.42)$$

This integral allows an analytical estimate of the condition of unbounded cumulation (asymptotic closing of the cavity in an infinite long time)

Divided both sides of (4.42) by $\nu a^{1/2}$

$$\frac{r_1^{3/2}}{\nu a^{1/2}}\dot{r}_1 - \frac{a\dot{r}_1(0)}{\nu} = -8\left(\frac{r_1^{1/2}}{a^{1/2}} - 1\right) \quad (4.43)$$

We introduce the Reynolds number R_* by (where we consider the collapse rate is negative)

$$R_* = \frac{a|\dot{r}_1(0)|}{\nu} \quad (4.44)$$

The formula of cumulated energy is given by

$$E = r_1^2 \left(\frac{dr_1}{dt}\right)^2 \quad (4.45)$$

Assuming $\dot{r}_1 = 0$, we obtain the condition of termination of the cumulation process by viscosity at a certain radius $r_1 = r_{1*}$ is:

$$R_* = -8\left(\frac{r_{1*}^{1/2}}{a^{1/2}} - 1\right) \quad (4.46)$$

Or

$$\sqrt{\frac{r_{1*}}{a}} = 1 - \frac{R_*}{8} \quad (4.47)$$

Which implies that the critical Reynolds number is 8.

This implies that for $R_* < 8$ the cavity stops collapsing before reaching the center, for

critical value $R_* = 8$ it is filled over infinite time, and for greater values it collapses with unbound velocity.

Hence the critical radius in this case is

$$r_{1*} = 8 \frac{\nu}{|\dot{r}_1(0)|} \quad (4.48)$$

In the above discussion, we find the critical Reynold number by using a formula of cumulated energy, instead let us applies qualitative theory of differential equation to ensure that $R_* = 8$.

Now by using (4.44), equation (4.43) can be written as

$$\dot{r}_1 = \frac{-8 \frac{r_1^{1/2}}{a^{1/2}} + 8 - R_*}{\frac{r_1^{3/2}}{\nu a^{1/2}}} \quad (4.49)$$

Then the isolated singularities point(s) of ordinary differential equation (4.49) appears when the numerator and denominator are equals to zero at the same time, and this happens only at one isolated singularity $(t_*, 0)$ when $8 - R_* = 0$, or $R_* = 8$.

Now as we did in case (2) let us try to find an analytic solution of the differential equation (4.34).

First we will introduce dimensionless variables and constant

$$\tilde{r}_1 = \frac{r_1}{a}, \quad \tilde{t} = \frac{tU_0}{a}, \quad \tilde{\eta} = \frac{\eta}{a\rho U_0} \quad (4.50)$$

Here $\tilde{r}_1, \tilde{t}$ and $\tilde{\eta}$ are the dimensionless bubble radius, time, and viscosity, respectively,

and $u_1(0) = -U_0$

Then

$$r_1 \frac{du_1}{dt} = U_0^2 \tilde{r}_1 \frac{d\tilde{u}_1}{d\tilde{t}}, \quad \frac{3}{2} u_1^2 = \frac{3}{2} U_0^2 \tilde{u}_1^2, \quad \frac{4\eta u_1}{\rho r_1} = \frac{4U_0^2 \tilde{\eta} \tilde{u}_1}{\tilde{r}_1} \quad (4.51)$$

Substitute (4.51) into (4.34)

$$U_0^2 \tilde{r}_1 \frac{d\tilde{u}_1}{d\tilde{t}} + \frac{3}{2} U_0^2 \tilde{u}_1^2 + \frac{4U_0^2 \tilde{\eta} \tilde{u}_1}{\tilde{r}_1} = 0 \quad (4.52)$$

Divided both sides of (4.52) by U_0^2

$$\tilde{r}_1 \frac{d\tilde{u}_1}{d\tilde{t}} + \frac{3}{2} \tilde{u}_1^2 + \frac{4\tilde{\eta} \tilde{u}_1}{\tilde{r}_1} = 0 \quad (4.53)$$

$$\tilde{r}_1(\tilde{t} = 0) = \frac{r_1(\tilde{t} = 0)}{a} = \frac{r_1(\frac{tU_0}{a} = 0)}{a} = \frac{r_1(t = 0)}{a} = \frac{a}{a} = 1 \quad (4.54)$$

$$\begin{aligned} \frac{d\tilde{r}_1}{d\tilde{t}}(\tilde{t} = 0) &= \frac{dt}{d\tilde{t}} \frac{1}{dt} \left(\frac{r_1(\frac{tU_0}{a} = 0)}{a} \right) = \frac{a}{U_0} \frac{\frac{dr_1}{dt}(\frac{tU_0}{a} = 0)}{a} = \\ &= \frac{1}{U_0} \frac{dr_1}{dt}(t = 0) = \frac{1}{U_0}(-U_0) = -1 \end{aligned} \quad (4.55)$$

Using (4.42) and the initial condition (4.54) and (4.55) we can evaluate the first integral of (4.53)

$$\tilde{r}_1^{3/2} \dot{\tilde{r}_1} + 1 = -8\tilde{\eta}(\tilde{r}_1^{1/2} - 1) \quad (4.56)$$

Or

$$\dot{\tilde{r}_1} = \frac{-8\tilde{\eta} \tilde{r}_1^{1/2} + 8\tilde{\eta} - 1}{\tilde{r}_1^{3/2}} \quad (4.57)$$

Integrate

$$\int_1^{\tilde{r}_1} \frac{\tilde{r}_1^{3/2}}{-8\tilde{\eta} \sqrt{\tilde{r}_1} + 8\tilde{\eta} - 1} d\tilde{r}_1 = \int_0^{\tilde{t}} d\tilde{T} \quad (4.58)$$

we will yields,

$$\begin{aligned}
& \frac{-1}{49152\tilde{\eta}^5}(-25600\tilde{\eta}^4 + 6656\tilde{\eta}^3 - 672\tilde{\eta}^2 + 24\tilde{\eta} + 3072\tilde{r}_1^2\tilde{\eta}^4 + \\
& 4096\tilde{r}_1^{3/2}\tilde{\eta}^4 - 512\tilde{r}_1^{3/2}\tilde{\eta}^3 + 6144\tilde{r}_1\tilde{\eta}^4 - 1536\tilde{r}_1\tilde{\eta}^3 + 96\tilde{r}_1\tilde{\eta}^2 \\
& + 12288\sqrt{\tilde{r}_1}\tilde{\eta}^4 - 4608\sqrt{\tilde{r}_1}\tilde{\eta}^3 + 576\sqrt{\tilde{r}_1}\tilde{\eta}^2 - 24\tilde{\eta}\sqrt{\tilde{r}_1} + \\
& 3\ln(8\tilde{\eta}\sqrt{\tilde{r}_1} - 8\tilde{\eta} + 1) + 12288\ln(8\tilde{\eta}\sqrt{\tilde{r}_1} - 8\tilde{\eta} + 1)\tilde{\eta}^4 - \\
& 6144\ln(8\tilde{\eta}\sqrt{\tilde{r}_1} - 8\tilde{\eta} + 1)\tilde{\eta}^3 + 1152\ln(8\tilde{\eta}\sqrt{\tilde{r}_1} - 8\tilde{\eta} + 1)\tilde{\eta}^2 \\
& - 96\ln(8\tilde{\eta}\sqrt{\tilde{r}_1} - 8\tilde{\eta} + 1)\tilde{\eta}) = \tilde{t}
\end{aligned} \tag{4.59}$$

From equation (4.59) we can find $\tilde{t}_{TC}$, by letting $\tilde{r}_1 \rightarrow 0$.

$$\begin{aligned}
& \frac{-1}{49152\tilde{\eta}^5}(-25600\tilde{\eta}^4 + 6656\tilde{\eta}^3 - 672\tilde{\eta}^2 + 24\tilde{\eta} + 3\ln(-8\tilde{\eta} + 1) + \\
& 12288\ln(-8\tilde{\eta} + 1)\tilde{\eta}^4 - 6144\ln(-8\tilde{\eta} + 1)\tilde{\eta}^3 + \\
& 1152\ln(-8\tilde{\eta} + 1)\tilde{\eta}^2 - 96\ln(-8\tilde{\eta} + 1)\tilde{\eta}) = \tilde{t}_{TC}
\end{aligned} \tag{4.60}$$

Now from (4.57), we can deduce that the condition of singular point is $1 - 8\tilde{\eta}_* = 0$,
or $\tilde{\eta}_* = \frac{1}{8} = 0.125$.

We see from the figure (4.3) that the collapsing times for $\tilde{\eta} < 0.125$ varies from 0.4 ($\tilde{\eta} = 0$) to 0.5 ($\tilde{\eta} = 0.125$).

That is if the viscosity ≈ 0 the collapse time will be

$$t_{TC} = 0.4 \frac{a}{U_0}$$

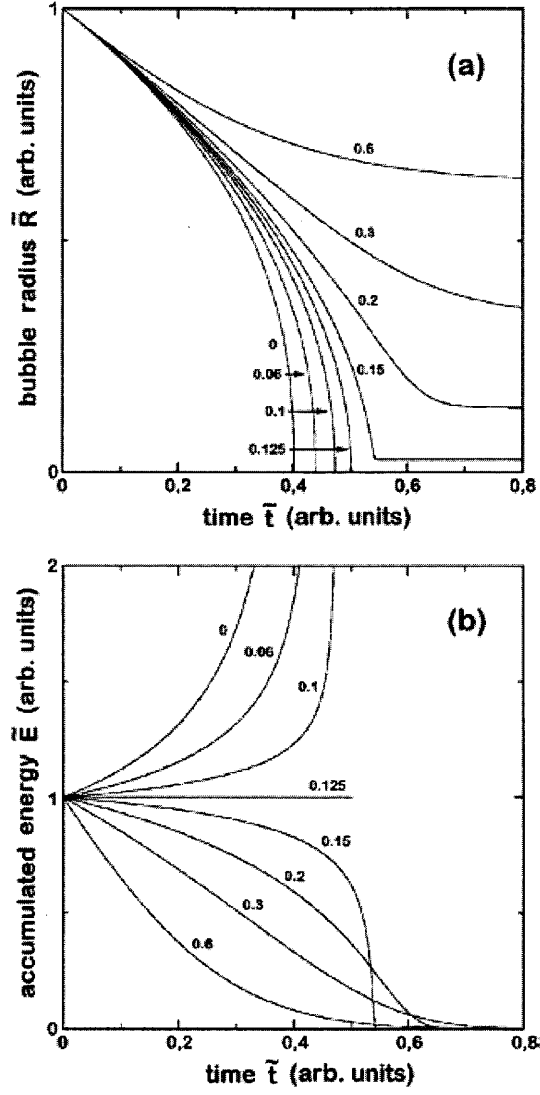


Figure 4.2: (a) Bubble radius $\tilde{R}$ as a function of time $\tilde{t}$, (b) accumulated energy $\tilde{E}$ as a function of time $\tilde{t}$. Values of parameter $\tilde{\eta}$ are shown at curves.

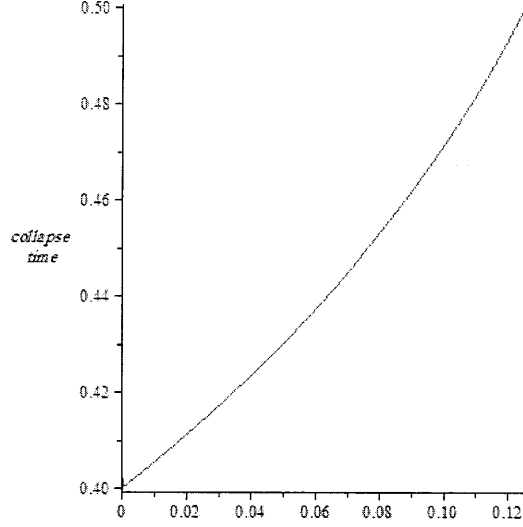


Figure 4.3: The relation between nondimensional parameter $\tilde{\eta}$ and the collapse time t_{TC} for $\tilde{\eta} < 0.125$.

and in case of $0 < \tilde{\eta} < 0.125$, the collapse time will be

$$t_{TC} = 0.5 \frac{a}{U_0}$$

Remark The critical Reynold number is $R_* = \frac{1}{\tilde{\eta}_*} = 8$, which agree with our previous result .

Hence for $R_* > 8$, or $\tilde{\eta}_* < \frac{1}{8}$, the accumulated energy increases to infinity, that is, the velocity increase to infinity and the bubble collapse takes place.

And for $R_* < 8$, or $\tilde{\eta}_* > \frac{1}{8}$ take infinite long time to collapse. For $R_* = 8$ the bubble collapse in finite time (see figure 4.2).

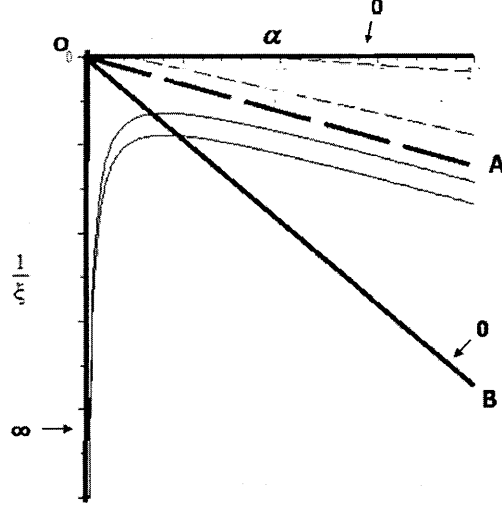


Figure 4.4: The origin point has multiple singularity.

Let us back now to our case study. Let us study the behaviour of the velocity near the center in the in the case of nonzero viscosity. But as we discuss before the velocity ξ in some cases unbounded. So instead we will study ξ^{-1} .

$$\frac{d\xi^{-1}}{d\alpha} = \frac{d\xi^{-1}}{d\xi} \frac{d\xi}{d\alpha} = -\xi^{-2} \frac{d\xi}{d\alpha} \quad (4.61)$$

Or

$$-\xi^2 \frac{d\xi^{-1}}{d\alpha} = \frac{d\xi}{d\alpha} \quad (4.62)$$

Substitute (4.62) into (4.15)

$$-\xi^2 \frac{d\xi^{-1}}{d\alpha} + \frac{3}{2} \frac{\xi}{\alpha} + \frac{1}{\xi\alpha} + \frac{4}{\alpha^2} = 0 \quad (4.63)$$

Simplify (4.63)

$$\frac{d\xi^{-1}}{d\alpha} = \frac{3\alpha\xi^{-1} + 2\alpha\xi^{-3} + 8\xi^{-2}}{2\alpha^2} \quad (4.64)$$

The numerator and denominator of (4.64) are polynomials in α , ξ^{-1} so they are continuously differentiable, Then the differential equation (4.64) has isolated singularities point(s) if any when the numerator and denominator are equals to zero at the same time.

This is happens only when $\alpha = 0$, $\xi^{-1} = 0$.

Hence the point $(\alpha, \xi^{-1}) = (0, 0)$ is a multiple singularity for equation (4.64), as shown in figure 4.4.

We represent equation (4.64) in the form

$$\frac{d\xi^{-1}}{d\alpha} = \frac{\xi^{-2}}{\alpha} \left(\frac{3}{2}\xi + \xi^{-1} + \frac{4}{\alpha} \right) \quad (4.65)$$

The loci of the zeros and infinities are shown in Figure 4.4 by means of heavy lines and identified by 0 and ∞ , whereas the integral curves are denoted by the light lines. the dot-dash line **OA** is the only integral curve which emanates from the origin with a finite slope. Its slopes x is determined from (4.65) by substitution of the solution $\xi^{-1} = x\alpha$ near $\alpha = 0$.

We have $\frac{d\xi^{-1}}{d\alpha} = x$, then substitute into equation (4.65)

$$x = \frac{x^2\alpha^2}{\alpha} \left(\frac{3}{2} \frac{1}{x\alpha} + x\alpha + \frac{4}{\alpha} \right)$$

After simplifying the previous equation, it will becomes

$$-1 = 2x^2\alpha^2 + 8x$$

Now x is a finite slope, then as $\alpha \rightarrow 0$, $2x^2\alpha^2 \rightarrow 0$

Hence

$$-1 = 8x$$

And then

$$x = -1/8$$

Next for ∞ locus we can see from (4.65) it happens only as

$$\alpha \rightarrow 0^+, \quad \xi^{-1} \rightarrow -\infty$$

Then

$$\frac{d\xi^{-1}}{d\alpha} \rightarrow -\infty$$

Now to find the loci of zeros, we set equation (4.65) equals zero.

$$\frac{\xi^{-2}}{\alpha} \left(\frac{3}{2}\xi + \xi^{-1} + \frac{4}{\alpha} \right) = 0$$

Then either

$$\xi^{-2} = 0 \tag{4.66}$$

Or

$$\frac{3}{2}\xi + \xi^{-1} + \frac{4}{\alpha} = 0 \tag{4.67}$$

case 1 if $\xi^{-2} = 0$.

Then the horizontal axis $\xi^{-1} = 0$ form the first locus of zeros. And then if we

substitute $\xi^{-1} = 0$ in (4.67), we will get $\frac{3}{2}\xi + \frac{4}{\alpha} = 0$, or

$$\xi^{-1} = \frac{-3}{8}\alpha \quad (4.68)$$

Which form the second locus of zeros which corresponds to the line **OB**.

case 2 if $\frac{3}{2}\xi + \xi^{-1} + \frac{4}{\alpha} = 0$, and $\xi^{-1} \neq 0$.

Then by quadratic formula

$$\xi = \frac{\frac{-8}{3\alpha} \pm \sqrt{\frac{64}{9\alpha^2} - \frac{8}{3}}}{2} \quad (4.69)$$

Or

$$\xi^{-1} = \frac{2}{\frac{-8}{3\alpha} \pm \sqrt{\frac{64}{9\alpha^2} - \frac{8}{3}}} \quad (4.70)$$

If we analyze (4.70), we will see that it is asymptotic to (4.68) as $\alpha \approx 0$, and is limited for just $\alpha < \frac{3}{8}$.

Hence we neglect this case, because it is almost the same as (4.68), but it is limited.

In fact this zero locus intersect the integral curves at the same points as (4.68) but for limited value of α .

The locus **OB** of the zeros passes by lower integral curves of the of the separatrix.

The integral curves above **OA** enter a node, where $\xi \sim \alpha^{-3/2}$ Rayleigh (1917).

Below **OA** they represent a saddle where

$$\xi \rightarrow (-1/4)\alpha \quad \text{as} \quad \alpha \rightarrow 0. \quad (4.71)$$

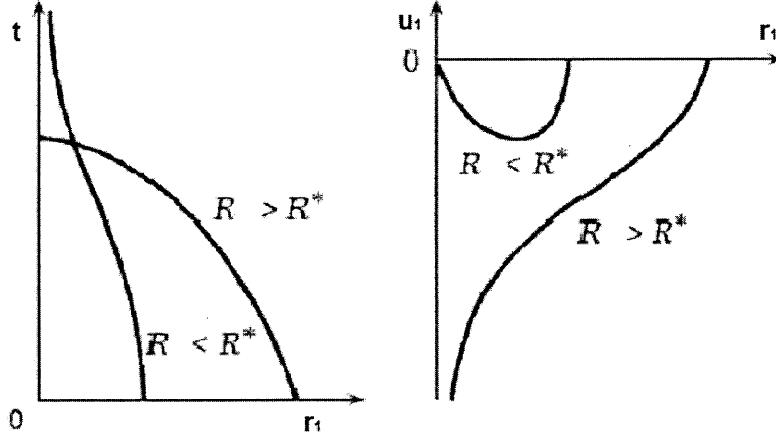


Figure 4.5: Physical and phase (u_1, r_1) planes of bubble collapse in a viscous liquid

Or (4.71) can be written in terms of u_1 and r_1 by using (4.13) and the definition of kinematic viscosity ν as follows

$$u_1 \rightarrow -\frac{r_1 p_0}{4\eta} \quad \text{as} \quad r_1 \rightarrow 0 \quad (4.72)$$

The curve which corresponds to the solution is that which leads towards point corresponding to the initial condition, i.e. $\alpha = R$, $\xi = 0$.

Generally speaking, for different number R the solution may belong to different families: in the case of the node they correspond to an unlimited increase of the velocity $\xi \sim \alpha^{-3/2}$; in the case of the saddle point they correspond to a decelerated motion of the bubble with $\xi \sim \alpha$. as a result of which its filling occurs only after an infinite time. In the intermediate case, which corresponds to the dividing line **OA**,

the filling occurs after a finite time, and the velocity near the center grows as $\xi \sim \alpha^{-1}$.

The Reynolds number corresponding to the dividing line is the critical Reynolds number. It separates two fundamentally different classes of solution. Its critical value $R = R_*$ can be determined by construction the separatrix from $\alpha = 0$ (near that point its asymptotic representation is known $\xi^{-1} = \frac{-1}{8}\alpha$) to $\xi = 0$, where $\alpha = R$. So we can say that to find the critical value $R = R_*$ we have to discuss the approximate solution of equation (4.15) under the following conditions

$$\xi(0^+) = -8/R_* \quad \text{and} \quad \xi(R_*) = 0 \quad (4.73)$$

and then we solve the resultant algebraic equation to find R_* .

First let us divide the interval $[0, R_*]$ into 6 equally parts, then the stepsize h and x_i will be

$$h = \frac{R_* - 0}{6} \quad \text{and} \quad x_i = a + hi = 0 + \frac{R_*}{6}i = \frac{R_*}{6}i$$

Let the approximate solution at each point x_i is w_i , and since the exact solution at $\alpha = R_*$ equal 0, then we may say that $w_6 = 0$, also since the exact solution near $\alpha = 0$ is $\frac{-8}{R_*}$, then we may assume $w_1 = \frac{-8}{R_*}$.

Now we will use Euler formula that approximate the solution of first order differential equations

$$w_{i+1} = w_i + hf(x_i, w_i) \quad (4.74)$$

Which implies

$$w_{i+1} = w_i - \frac{R_*}{6} \left[\frac{3}{2} \frac{w_i}{\frac{R_*}{6}i} + \frac{1}{\frac{R_*}{6}iw_i} + \frac{4}{(\frac{R_*}{6})^2i^2} \right]$$

After simplifying we will get the following recurrence relation

$$w_1 = \frac{-8}{R_*} \quad \text{and} \quad w_{i+1} = \left(1 - \frac{3}{2i}\right)w_i - \frac{1}{iw_i} - \frac{24}{R_*i^2} \quad \text{for } i = 1, 2, \dots, 5 \quad (4.75)$$

for $i = 1$

$$w_2 = \frac{R_*^2 - 160}{8R_*} \quad (4.76)$$

for $i = 2$

$$w_3 = \frac{56320 - 640R_*^2 + R_*^4}{32R_*(R_*^2 - 160)} \quad (4.77)$$

for $i = 3$

$$w_4 = \frac{14129561600 - 349962240R_*^2 + 2631680R_*^4 - 6400R_*^6 + 3R_*^8}{192R_*(-160 + R_*^2)(56320 - 640R_*^2 + R_*^4)} \quad (4.78)$$

for $i = 4$

$$w_5 = 5(258315550977949696000 - 13573574124109824000R_*^2 + 280754560371261440R_*^4$$

$$- 2961480613888000R_*^6 + 17112405901312R_*^8 - 53694693376R_*^{10} +$$

$$84398080R_*^{12} - 54528R_*^{14} + 9R_*^{16})/$$

$$(1536R_*(-160 + R_*^2)(56320 - 640R_*^2 + R_*^4)(14129561600 - 349962240R_*^2 +$$

$$2631680R_*^4 - 6400R_*^6 + 3R_*^8)) \quad (4.79)$$

for $i = 5$

$$\begin{aligned} w_6 = & (14102114401080159549242470894938357760000000 - \\ & 1533879876486520927880035686741442560000000 R_*^2 + \\ & 73870123755538965454135394306936012800000 R_*^4 - \\ & 2092552525212194899214781510582272000000 R_*^6 + \\ & 38997204504941602278910984124891136000 R_*^8 - \\ & 505920221596023921529367040098304000 R_*^{10} + \\ & 4709982692097829133791174171033600 R_*^{12} - \\ & 31939027941518913602161370726400 R_*^{14} + \\ & 158367504008248833308926935040 R_*^{16} - \\ & 571008513317780062543544320 R_*^{18} + \\ & 1474778866026053495685120 R_*^{20} - \\ & 2658147401872669933568 R_*^{22} + \\ & 3209558592222920704 R_*^{24} - \\ & 2437714665799680 R_*^{26} + \\ & 1053223575552 R_*^{28} - \\ & 216221184 R_*^{30} + \end{aligned}$$

$$\begin{aligned}
& 14175R_*^{32})/ \\
& (76800R_*(-160+R_*^2)(56320-640R_*^2+R_*^4)(14129561600-349962240R_*^2+2631680R_*^4- \\
& 6400R_*^6+3R_*^8)(258315550977949696000-13573574124109824000R_*^2+ \\
& 280754560371261440R_*^4-2961480613888000R_*^6+17112405901312R_*^8- \\
& 53694693376R_*^{10}+84398080R_*^{12}-54528R_*^{14}+9R_*^{16})) \quad (4.80)
\end{aligned}$$

Set $w_6 = 0$, then the numerator of (4.80) is equals zero, and this numerator is a polynomial of order 32, so it has at most 32 real roots.

Now by solving the equation using Mathematica (7) then one of the outputs will be

$$R_* = 8.397630465392549 \approx 8.4 \quad \text{where} \quad R = \frac{a}{\nu} \sqrt{\frac{p_0}{\rho}}$$

The other roots will be rejected because if we again do the same work by dividing the intervals $[0, R_*]$ into 5 parts or 7 parts for example the other roots disappeared and the root 8.4 appears always, the second reason to accept the root 8.4 is that it is near the critical Reynold (8)that we found in the special case (2).

Hence we are sure that that the critical Reynold number is 8.4 in our case study .

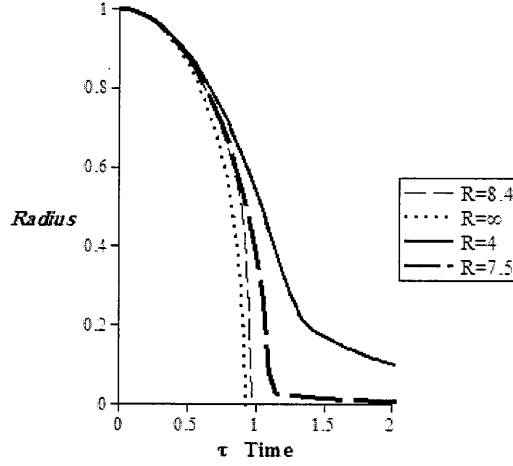


Figure 4.6: Effect of change the Reynold numbers.

We can see from previous work that for $R < 8.4$ the filling of the bubble takes place slowly in an infinitely long time. The accumulation of energy is completely dissipated by the viscosity.

For $R > 8.4$ the velocity near the isolated singularity point grows indefinitely as in the Rayleigh problem (without viscosity), i.e. as constant times $r_1^{-3/2}$, but with a smaller value of constant. In the interim case for $R_* = 8.4$ the bubble is filling in finite time, and the velocity of the bubble wall grows indefinitely, but more weakly, as r_1^{-1} .

The schemes of all three cases of motion are shown in Figure 4.6 which presents the calculated radius vs time curves for the collapse of a bubble in liquid with different viscosity (different values of parameter R). Figure shows that when $\nu=0$ ($R=\text{infinity}$)

the time required for total collapse of the bubble of the bubble from $r_1 = a$ to $r_1 = 0$ is given by the formula $t = 0.915 \frac{a}{\sqrt{\frac{p_0}{\rho}}}$.

Also it is shown that when $R < 8.4$ the time for collapse is infinite. Calculations confirm that if R is big enough, the time of collapse is limited, and if R is small the time of the bubble life is unlimited.

For given p_0 , ρ and ν one can speak about a critical radius a_* of the bubble

$$a_* = 8.4\nu\sqrt{\frac{\rho}{p_0}} \quad (4.81)$$

For $a < a_*$ the energy accumulation is completely dissipated by the viscosity. The velocity at the end of the filling of small bubbles

$$u_1 = -\frac{r_1 p_0}{4\eta} \quad (4.82)$$

does not depend on the density and on their initial radius a . In practice, the critical radius a_* is very small, i.e. the viscosity dissipates the accumulation only in bubbles of very small size; for instance, for $p_0 = 1$ atm in water ($\nu = 0.01, \rho = 1$) $a_* = 0.8$ micron, and in glycerin ($\nu = 6.8, \rho = 0.8$) $a_* = 0.5mm$.

In this chapter we discussed the collapse of empty bubble, the case of gas bubble is more complicated because the gas pressure slows down the bubble wall motion and hence the bubble will not collapse it is oscillated. In the next chapter we will discuss the gas bubble using the qualitative theory of differential equation.

Chapter 5

ON THE DYNAMICS OF A GAS BUBBLE IN A VISCOUS INCOMPRESSIBLE LIQUID

5.1 Introduction

It is generally agreed that intense oscillations of cavity bubbles without collapse, constitute one of the main reasons for the cavity erosion of materials. When the dimension of a cavity bubble reaches a certain limiting value, strong pressure pulses may occur in the surrounding liquid, which can cause erosion by local cyclic loads Numachi (1958); Sutton (1957).

Oscillations of cavity bubbles in a viscous liquid, exhibit a number of distinctive features caused by the viscosity. Poritsky (1952) and Zababakhin (1960) noted the fundamental influence of viscosity while investigating the behavior of a spherical cavity in a viscous, incompressible liquid. The existence of two different types of motion was discovered: bubbles which are smaller than a critical size, are filled slowly in an infinitely long time; the filling of large bubbles takes place rapidly with an unlimited accumulation of energy during collapse.

There are two types of cavitations, Inertial cavitation and Non-inertial cavitation. Inertial cavitation is the process where a void or bubble in a liquid rapidly collapses,



Figure 5.1: Cavitation damage to a Francis turbine.

producing a shock wave. An Example of shock wave is shown in figure 5.2. Inertial cavitation occurs in nature in the strikes of mantis shrimps and pistol shrimps, as well as in the vascular tissues of plants. In man-made objects, it can occur in control valves, pumps, propellers and impellers. Non-inertial cavitation is the process in which a bubble in a fluid is forced to oscillate in size or shape due to some form of energy input, figure 5.3 shows an example of bubble oscillations. such as an acoustic field. Such cavitation is often employed in ultrasonic cleaning baths and can also be observed in pumps, propellers, etc. Bakanov (1961) use dimensional analysis to derive a qualitative formula defining the critical bubble size D_* separating the inertial and

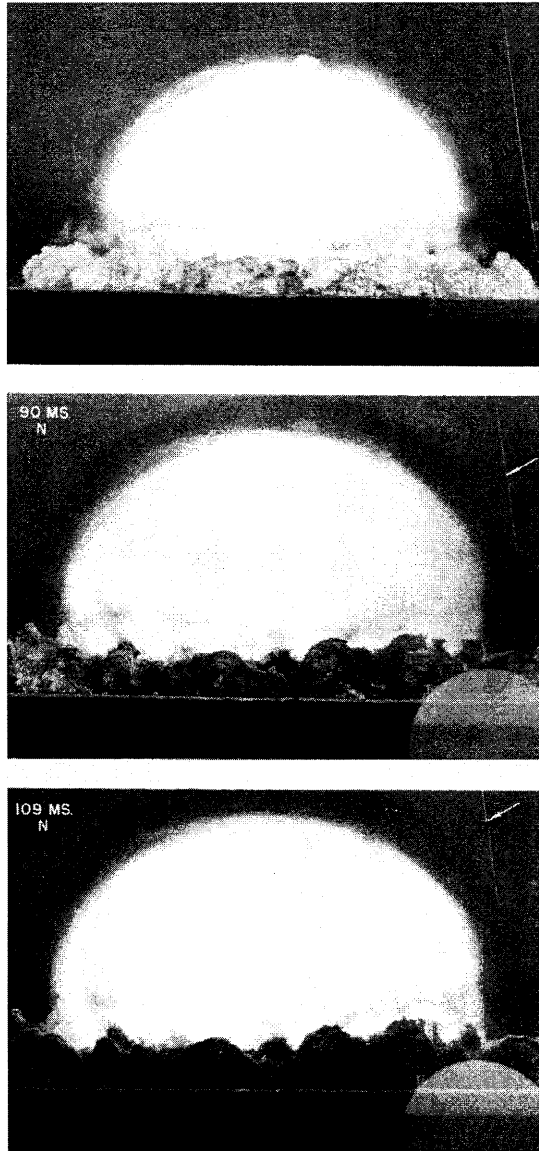


Figure 5.2: Shock wave propagating into a stationary medium, ahead of the fireball of an explosion. The shock is made visible by the shadow effect.

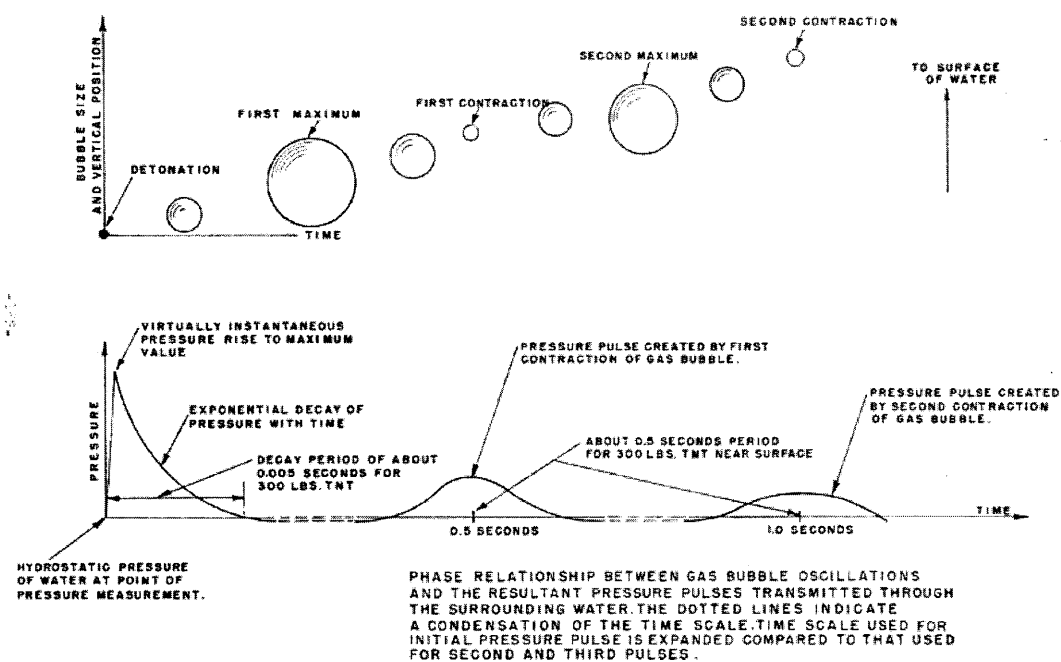


Figure 5.3: Bubble oscillation.

inertial less mode of expansion of a gaseous sphere in a viscous liquid

$$D_* = \left(\frac{\eta\tau_*}{\rho}\right)^{1/2} \quad (5.1)$$

Where η and ρ are the dynamic viscosity and density of the liquid, respectively, and τ_* is the characteristic time of the process, determined experimentally.

In this chapter we find that, when the bubble in a viscous incompressible liquid is filled with large gas, then two modes of motion exist, depending on the initial radius of bubble, oscillatory or monotonically a periodic.

Below we derive a formula for critical diameter of the gas bubble.

Let us suppose that a spherical gas bubble is situated in an infinite, viscous, incompressible liquid. We assume that the pressure and density of the gas are uniform throughout the bubble. This of course is true, provided that the velocity of the boundary of the gaseous sphere is much smaller than the velocity of sound in the gas at a given temperature. Viscosity of gas is assumed to be negligible. The following nonlinear second order differential equation describes the variation in the radius of the bubble

$$r_1 \frac{d^2 r_1}{dt^2} + \frac{3}{2} \left(\frac{dr_1}{dt}\right)^2 + \frac{4\eta}{\rho r_1} \frac{dr_1}{dt} + \frac{2\sigma}{\rho r_1} + \frac{p_0 - p}{\rho} = 0 \quad (5.2)$$

And the initial conditions are

$$r_1 = R_0, \quad \frac{dr_1}{dt} = 0, \text{ where } t = 0 \quad (5.3)$$

Here $r_1 = r_1(t)$ is the radius of the bubble, σ denotes the surface tension of the liquid,

p_0 denotes constant pressure of the liquid away from the bubble and p denotes the pressure of gas within the bubble.

Assume that the process of expansion and contraction of gas within the bubble is adiabatic, we can derive the following relation between p and r_1 at any instant as follows:

From thermodynamics, for an ideal gas,

$$P_0 V_0^\gamma = P V^\gamma \quad (5.4)$$

Where P is the pressure and V is the volume of the gas and the subscript 0 refers to some initial state, where γ is a constant (the specific heat ratio or some authors call it adiabatic index).

Now from (5.4) we deduce the relation between the volume of the bubble v and the pressure within it p

$$p_0 v_0^\gamma = p v^\gamma \quad (5.5)$$

Where again the subscript 0 refers to some initial state. Since the volume of a sphere is proportional to its cube radius, then

$$\frac{v_0}{v} = \left(\frac{R_0}{r_1}\right)^3 \quad (5.6)$$

Which implies

$$\left(\frac{v_0}{v}\right)^\gamma = \left(\frac{R_0}{r_1}\right)^{3\gamma} \quad (5.7)$$

Then from (5.5)

$$\frac{\dot{p}}{p_0} = \left(\frac{R_0}{r_1}\right)^{3\gamma} \quad (5.8)$$

Or

$$\dot{p} = p_0 \left(\frac{R_0}{r_1}\right)^{3\gamma} \quad (5.9)$$

In the case of adiabatic oscillations of a bubble in water, we have $\gamma = \frac{4}{3}$ Sirotiuk (1967). We also assume that there is no diffusion of gas through the boundary of the bubble.

The process of obtaining the general solution of (5.2) is fairly difficult even with various simplifying assumptions. Some conclusions concerning the appearance of cavitation described by this equation can, however, be drawn by perusing some general properties of the integral curves of (5.2) on the phase plane. This method gives a measure of success in the qualitative investigations of non-linear differential equations.

5.2 Study of the oscillation of gas bubble using the qualitative theory of differential equations

Let us study the differential equation (5.2) first we introduce the following dimensionless variables:

$$u = \frac{r_1}{R_0}, \quad \tau = \frac{t}{R_0} \left(\frac{\rho'}{\rho} \right)^{\frac{1}{2}} \quad (5.10)$$

Then

$$\frac{du}{d\tau} = \frac{du}{dt} \frac{dt}{d\tau} = \frac{1}{R_0} \frac{dr_1}{dt} R_0 \left(\frac{\rho}{\rho'} \right)^{1/2} = \frac{dr_1}{dt} \left(\frac{\rho}{\rho'} \right)^{1/2} \quad (5.11)$$

Hence

$$\frac{dr_1}{dt} = \left(\frac{\rho'}{\rho} \right)^{1/2} \frac{du}{d\tau} \quad (5.12)$$

Differentiate equation (5.11) again with respect to τ

$$\frac{d^2u}{d\tau^2} = \frac{d}{d\tau} \left[\frac{dr_1}{dt} \left(\frac{\rho}{\rho'} \right)^{1/2} \right] = \frac{dt}{d\tau} \frac{d}{dt} \left[\frac{dr_1}{dt} \left(\frac{\rho}{\rho'} \right)^{1/2} \right] = R_0 \left(\frac{\rho}{\rho'} \right)^{1/2} \left(\frac{\rho}{\rho'} \right)^{1/2} \frac{d^2r_1}{dt^2}$$

This implies

$$\frac{d^2r_1}{dt^2} = \frac{1}{R_0} \left(\frac{\rho'}{\rho} \right) \frac{d^2u}{d\tau^2}$$

Multiply both sides by r_1 , this will yield

$$r_1 \frac{d^2r_1}{dt^2} = \frac{r_1}{R_0} \left(\frac{\rho'}{\rho} \right) \frac{d^2u}{d\tau^2} = \left(\frac{\rho'}{\rho} \right) u u'' \quad (5.13)$$

Also by using equation (5.12), we find that

$$\frac{3}{2} \left(\frac{dr_1}{dt} \right)^2 = \frac{3}{2} \left(\frac{\rho'}{\rho} \right) (u')^2 \quad (5.14)$$

Then from equations (5.10),(5.12), (5.13) and (5.14), Equation (5.2) becomes

$$\left(\frac{p_0}{\rho}\right)uu'' + \frac{3}{2}\left(\frac{p_0}{\rho}\right)(u')^2 + \frac{4\eta}{\rho R_0 u}\left(\frac{p_0}{\rho}\right)^{1/2}u' + \frac{2\sigma}{\rho R_0 u} + \frac{p_0 - p}{\rho} = 0$$

Multiply both sides of the last equation by $\left(\frac{\rho}{p_0}\right)$

$$uu'' + \frac{3}{2}(u')^2 + \frac{4\eta}{R_0\sqrt{\rho p_0}}\frac{u'}{u} + \frac{2\sigma}{R_0 p_0}\frac{1}{u} + \frac{p_0}{p_0} - \frac{p}{p_0} = 0 \quad (5.15)$$

Now, since the adiabatic index of water is 4/3 then

$$p = p_0\left(\frac{R_0}{r_1}\right)^4 = p_0 u^{-4} \quad (5.16)$$

Substitute equation (5.16) into equation (5.15)

$$uu'' + \frac{3}{2}u'^2 + \frac{4\eta}{R_0\sqrt{\rho p_0}}\frac{u'}{u} + \frac{2\sigma}{R_0 p_0}\frac{1}{u} + \frac{p_0}{p_0} - u^{-4} = 0 \quad (5.17)$$

Assume that

$$u' = \frac{du}{d\tau} = y(u) \quad , \quad a = \frac{4\eta}{R_0\sqrt{\rho p_0}} \quad , \quad b = \frac{2\sigma}{R_0 p_0} \quad , \quad c = \frac{p_0}{p_0} \quad (5.18)$$

Hence

$$\frac{dy}{du} = \frac{dy}{d\tau} \frac{d\tau}{du} = u'' \frac{1}{y}$$

Which implies that

$$y \frac{dy}{du} = u'' \quad (5.19)$$

Substitute equations (5.18) and (5.19) into equation (5.17)

$$uy \frac{dy}{du} + \frac{3}{2}y^2 + a \frac{y}{u} + b \frac{1}{u} + c - u^{-4} = 0 \quad (5.20)$$

We can reduce (5.20) to the following first order differential equation:

$$\frac{dy}{du} = \frac{1 - cu^4 - bu^3 - ayu^3 - 1.5y^2u^4}{yu^5} \quad (5.21)$$

The numerator and denominator of equation (5.21) are continuously differentiable, then the isolated singularities point(s) of ordinary differential equation (5.21) if any appears when the numerator and denominator are equals to zero at the same time.

Now if we set the denominator equals zero, then either $y = 0$ or $u = 0$.

Case1: If $u = 0$.

Then in this case the numerator equals 1, and hence at $u = 0$ no isolated singular point.

Case2: If $y = 0$.

Then in this case the numerator equals $1 - cu^4 - bu^3$.

To find the isolated singular point(s) if any we must equalize the numerator by zero.

Define the function f by

$$f(u) = 1 - bu^3 - cu^4$$

Where the variable u is nonnegative and b and c are positive constants.

Differentiate the previous function with respect to u .

$$f'(u) = -3bu^2 - 4cu^3 < 0$$

For all u of the domain of f .

Hence f is decreasing function on its domain.

Also we note that f is continuous, $f(0) = 1$ and $\lim_{u \rightarrow \infty} f(u) = -\infty$.

Hence the function f has a unique value u_* such that $f(u_*) = 0$.

i.e. the first order differential equation has one isolated singularity

$$y = 0 \quad , \quad u = u_* \tag{5.22}$$

Where u_* is the only positive root of

$$1 - bu^3 - cu^4 = 0 \tag{5.23}$$

We note that u_* can be < 1 , or > 1 .

Now since it is difficult to discuss the type of isolated singular point for a nonlinear ordinary differential equation (5.21), so we will use linear approximation, noting that the linear approximation is applicable since the numerator and denominator of equation (5.21) are twice continuously differentiable.

The approximation is achieved by using Taylor expansions. By first making a translation of axes, we may suppose that the singular point under discussion is at the origin.

Our hope is then that the linear approximation will have the same type of singular point as the original system.

In general, if the linear approximation has a node, saddle-point or spiral, then so does the original system.

Hence if

$$\frac{dy}{du} = \frac{f(u, y)}{g(u, y)} \quad (5.24)$$

Then by expanding the numerator and denominator of (5.24) around $(u_*, 0)$

$$\frac{dy}{du} = \frac{f(u_*, 0) + \frac{\partial f}{\partial u}|_{(u_*, 0)}(u - u_*) + \frac{\partial f}{\partial y}|_{(u_*, 0)}(y - 0) + \dots}{g(u_*, 0) + \frac{\partial g}{\partial u}|_{(u_*, 0)}(u - u_*) + \frac{\partial g}{\partial y}|_{(u_*, 0)}(y - 0) + \dots} \quad (5.25)$$

Let $x = u - u_*$, then by chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \frac{dx}{du} = \frac{dy}{dx} (1) = \frac{dy}{dx} \quad (5.26)$$

Then by using equations (5.25), (5.26), the ordinary differential equation (5.21) can be written near the singularity as

$$\frac{dy}{dx} = \frac{-(4cu_*^3 + 3bu_*^2)x - ayu_*^3 + P_2(x, y)}{yu_*^5 + Q_2(x, y)} \quad (5.27)$$

Where $P_2(x, y)$ and $Q_2(x, y)$ are functions which are second order infinitesimals when x, y are first order infinitesimals.

For simplicity, we will divide the numerator and the denominator of right hand side of (5.27) by u_*^2 , and then equation (5.28) becomes

$$\frac{dy}{dx} = \frac{-(4cu_* + 3b)x - ayu_* + P_2'(x, y)}{yu_*^3 + Q_2'(x, y)} \quad (5.28)$$

Where again $P_2'(x, y)$ and $Q_2'(x, y)$ are functions which are second order infinitesimals.

Now for any linear differential equation of the form

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by} \quad (5.29)$$

has *characteristic polynomial* $\chi(\lambda)$ is given by the formula

$$\chi(\lambda) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix}$$

So the *characteristic equation* of (5.28) is given by

$$\chi(\lambda) = \begin{vmatrix} 0 - \lambda & u_*^3 \\ -(4cu_* + 3b) & -au_* - \lambda \end{vmatrix} = 0$$

Or

$$\lambda^2 + au_*\lambda + (3b + 4cu_*)u_*^3 = 0 \quad (5.30)$$

To determine the character of the singularity, we use the discriminant of (5.30).

The discriminant of (5.30) is equal to

$$d = a^2u_*^2 - 4(3b + 4cu_*)u_*^3 \quad (5.31)$$

Investigating the roots of (5.30) we find that two basically distinct possibilities may exist.

They are:

1) roots the characteristic equation are real, distinct and of the same sign ($d > 0$), consequently the singularity is a node. All integral curves in the neighborhood of this singularity enter it at the same angle (see figure 3.2).

To determine the slope of common tangent, we will use equations (3.6), (3.8), and

(3.11), and they give us the formula of the slope which is

$$m = \frac{\lambda_1 - a}{b}$$

Where λ_1 is one of the roots of (5.30).

By using the quadratic formula we find that

$$\lambda_1 = \frac{-au_* - [a^2u_*^2 - 4u_*^3(3b + 4cu_*)]^{1/2}}{2} = -u_* \frac{a + [a^2 - 4u_*(3b + 4cu_*)]^{1/2}}{2}$$

And hence the slope of the common tangent is

$$m = \frac{-u_* \frac{a + [a^2 - 4u_*(3b + 4cu_*)]^{1/2}}{2} - 0}{u_*^3} \quad (5.32)$$

After simplifying (5.32) the slope of common tangent is given by

$$m = -\frac{a + [a^2 - 4u_*(3b + 4cu_*)]^{1/2}}{2u_*^2} \quad (5.33)$$

And then we deduce the equation of common tangent which given by

$$y + kx = 0, \quad k = \frac{a + [a^2 - 4u_*(3b + 4cu_*)]^{1/2}}{2u_*^2} \quad (5.34)$$

As $x \rightarrow 0$, the velocity of the boundary of the bubble tends also to zero in a linear manner.

Important note The case where the two roots of (5.30) are real, distinct and with different sign (in such case the singular point is called saddle) never occur since the constant term of (5.30) is positive.

2) roots of the characteristic equation are complex conjugate. The singularity point is a focus point, each integral curve is contained between two logarithmic spirals and approaches the point $y = 0, u = u_*$, circumscribing it an infinite number of times. The gas bubble oscillates infinitely long with diminishing amplitude.

When $d = 0$, we have the intermediate case when the characteristic equation (5.30) has a multiple root and singularity is an inflected node or called a degenerate node (see figure 3.6).

All integral curves enter the singularity and are tangent at this point to

$$y + \frac{ax}{2u_*^2} = 0 \quad (5.35)$$

The critical bubble radius separating two distinct modes of behavior, is obtained by solving the following pairs of equations

$$d = 0, \quad 1 - bu_*^3 - cu_*^4 = 0 \quad (5.36)$$

If the initial radius is larger than the critical one, we have a focus, if it is smaller we have a node.

In the case of a perfect fluid ($\eta = 0$), then a in equation (5.21) is also zero.

In this case equation (5.21) can be written as

$$yu^5 dy + (1.5y^2u^4 + cu^4 + bu^3 - 1)du = 0 \quad (5.37)$$

let $M(u, y) = yu^5$ and $N(x, y) = 1.5y^2u^4 + cu^4 + bu^3 - 1$

$$\frac{\partial M}{\partial u} = 5yu^4, \quad \frac{\partial N}{\partial y} = 3yu^4$$

Since

$$\frac{\partial M}{\partial u} \neq \frac{\partial N}{\partial y}$$

So equation (5.37) is not exact, so let us try to find an integrating factor

$$\begin{aligned}\mu(u) &= \exp\left(\int \frac{\frac{\partial N}{\partial y} - \frac{\partial M}{\partial u}}{M} du\right) \\ \mu(u) &= \exp\left(\int \frac{3yu^4 - 5yu^4}{yu^5} du\right) = \exp\left(\int \frac{-2}{u} du\right) = u^{-2}\end{aligned}$$

Equation (5.37) will be exact if we multiply it by previous integrating factor

$$yu^3 dy + (1.5y^2u^2 + cu^2 + bu - u^{-2})du = 0 \quad (5.38)$$

So

$$F(u, y) = \int (yu^3) dy = \frac{y^2}{2}u^3 + g(u)$$

Where $g(u)$ is arbitrary function of u

Now

$$\frac{\partial F}{\partial u} = N(u, y)$$

$$1.5y^2u^2 + g'(u) = 1.5y^2u^2 + cu^2 + bu - u^{-2}$$

And hence

$$g(u) = c\frac{u^3}{3} + b\frac{u^2}{2} + u^{-1} + k_2$$

Where k_1 is constant

And so the solution of (5.38) will be

$$\frac{y^2}{2}u^3 + c\frac{u^3}{3} + b\frac{u^2}{2} + u^{-1} + k_1 = 0$$

Multiply both sides by $2u$

$$y^2 u^4 + c \frac{2u^4}{3} + bu^3 + 2 + k_1 u = 0 \quad (5.39)$$

Or

$$y^2 u^4 + \frac{p_0}{p'_0} \frac{2u^4}{3} + \frac{2\sigma}{R_0 p'_0} u^3 + 2 + k_1 u = 0 \quad (5.40)$$

Multiply both sides by $\frac{p'_0}{\rho} R_0^4$

$$\frac{p'_0}{\rho} y^2 u^4 R_0^4 + \frac{p_0}{\rho} \frac{2u^4}{3} R_0^4 + \frac{2\sigma}{R_0 \rho} u^3 R_0^4 + 2 \frac{p'_0}{\rho} R_0^4 + c_1 r_1 = 0 \quad (5.41)$$

Where c_1 is constant

Simplify equation (5.41)

$$\frac{p'_0}{\rho} y^2 r_1^4 + \frac{p_0}{\rho} \frac{2r_1^4}{3} + \frac{2\sigma}{\rho} r_1^3 + 2 \frac{p'_0}{\rho} R_0^4 + c_1 r_1 = 0 \quad (5.42)$$

Now by (5.12) and (5.18)

$$y^2 = \left(\frac{dr_1}{dt} \right)^2 \left(\frac{\rho}{p'_0} \right) \quad (5.43)$$

By saying

$$\frac{dr_1}{dt} = Y(t) \quad (5.44)$$

Substitute (5.43) and (5.44) into (5.42)

$$Y^2 r_1^4 + \frac{p_0}{\rho} \frac{2r_1^4}{3} + \frac{2\sigma}{\rho} r_1^3 + 2 \frac{p'_0}{\rho} R_0^4 + c_1 r_1 = 0 \quad (5.45)$$

Now let us approximate the integral (5.39) by Taylor series near the singularity and

neglecting the third and the higher order infinitesimals, the formula is given by

$$F(u + u_*, y + 0) \approx F(u_*, 0) + u \frac{\partial F}{\partial u}(u_*, 0) + y \frac{\partial F}{\partial y}(u_*, 0) + \frac{1}{2!} [u^2 \frac{\partial^2 F}{\partial u^2}(u_*, 0) +$$

$$2uy \frac{\partial^2 F}{\partial u \partial y}(u_*, 0) + y^2 \frac{\partial^2 F}{\partial y^2}(u_*, 0)]$$

Now let us compute each term separately

$$F(u_*, 0) = c \frac{2u_*^4}{3} + bu_*^3 + 2 + k_1 u_* = c_2 \quad (5.46)$$

Where c_2 is a constant

$$u \frac{\partial F}{\partial u}(u_*, 0) = u \left(c \frac{8u_*^2}{3} + 3bu_*^2 + k_1 \right) = c_3 u \quad (5.47)$$

Where c_3 is a constant

$$y \frac{\partial F}{\partial y}(u_*, 0) = y \times 0 = 0 \quad (5.48)$$

$$u^2 \frac{\partial^2 F}{\partial u^2}(u_*, 0) = u^2 (8cu_*^2 + 6bu_*) \quad (5.49)$$

$$2uy \frac{\partial^2 F}{\partial u \partial y}(u_*, 0) = 2uy \times 0 = 0 \quad (5.50)$$

$$y^2 \frac{\partial^2 F}{\partial y^2}(u_*, 0) = y^2 (2u_*^4) \quad (5.51)$$

Hence

$$F(u_* + u, 0 + y) = c_3 + c_2 u + \frac{1}{2!} [u^2 (8cu_*^2 + 6bu_*) + y^2 (2u_*^4)] \quad (5.52)$$

Hence we obtain an equation of the ellipse

$$y^2 u_*^4 + u^2 (3bu_* + 4cu_*^2) + c_2 u + c_3 = 0, (c_2, c_3 = \text{constant}) \quad (5.53)$$

with the singularity situated at its center. Here the gas bubble performs a steady oscillation.

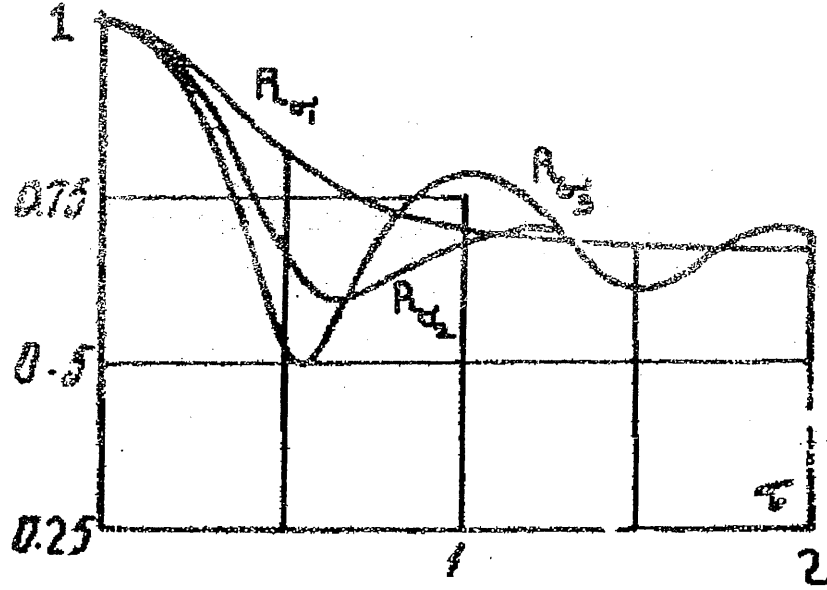


Figure 5.4: Bubble oscillations in different values of initial radius

It should be noted that there is a distinct difference between the process of collapsing of bubble in a viscous liquid and that of expansion/contraction of a gaseous bubble discussed above. In the first case the surface tension has qualitative effect on the collapse of cavity (during the process remaining finite for any value of the viscosity coefficient Poritsky (1952)), while in the latter case two distinct modes of behavior are observed even in the absence of surface tension, the only variable being the critical radius of the bubble.

In particular, putting $b = 0$ in (5.36), and then solving the following equations simultaneously

$$a^2 - 16cu_*^2 = 0 \quad (5.54)$$

$$1 - cu_*^4 = 0 \quad (5.55)$$

Now by using equation (5.10), we can reduce equation (5.54) to

$$R_* = \frac{\eta}{\sqrt{\rho p_0}} \quad (5.56)$$

Again by using equations (5.10) and (5.55), we can reduce equation (5.56) to

$$R_{0*} = \left(\frac{p_0}{p'_0}\right)^{1/4} R_* \quad (5.57)$$

Or

$$R_{0*} = \left(\frac{p_0}{p'_0}\right)^{1/4} \frac{\eta}{\sqrt{\rho p_0}} = \frac{\eta}{\sqrt{\rho}(p_0 p'_0)^{0.25}} \quad (5.58)$$

The actual value of the critical radius is fairly small, for example when $p_0 = 1$ atm and $p'_0 = 0.2$ atm, we have for water $R_{0*} = 0.178 \times 10^{-3}$ mm and for glycerine, $R_{0*} = 0.24$ mm.

Figure (5.4) shows the results of a direct computation, with $b = 0$, for the values $R_{01} = 0.15 \times 10^{-3}$ mm $< R_{0*}$, $R_{03} = 1 \times 10^{-3}$ mm $> R_{0*}$ and $R_{02} = 0.4 \times 10^{-2}$ mm $> R_{0*}$.

Chapter 6

THE RESULTS

In this thesis, we implement the qualitative theory of differential equations to conduct theoretical investigation of the dynamics of cavitation bubbles using the Navier-Stokes equations, and gas bubble using Rayleigh-Plesset equations.

Chapter 4, was the first application of qualitative theory of differential equations, in this chapter we concerned about collapse of bubbles in a viscous liquid, in this chapter we proved that the empty bubbles, which are smaller than a critical size, are filled slowly in an infinitely long time. The filling of large bubble takes place rapidly with an unlimited accumulation of energy during collapse. A quantitative formula was obtained for critical value of the bubble. This critical value was shown to be equals 8 (nondimensional) in case of absence of far pressure $p_0 = 0$, and if $p_0 \neq 0$ it was shown to be 8.4.

Also we discussed some special cases, to understand the full picture bubble collapsing.

The summary of the main results of the special cases are:

(In case of non viscous liquid)

- 1) We derived relation between the radius and the time during collapsing of empty

bubble which is

$$t = \frac{a}{\sqrt{\frac{p_0}{\rho}}} \left[0.915 - \frac{\sqrt{6}}{5} (r_1)^{5/6} {}_2F_1(1/2, 5/6; \frac{11}{6}; r_1) \right]$$

2) We derived formula for collapsing time which was

$$t_{TC} = 0.915 \frac{a}{\sqrt{\frac{p_0}{\rho}}}$$

3) We derived relation between the velocity and the radius during collapsing of empty bubble which was

$$u_1 = -\sqrt{\frac{2p_0}{3\rho} \left(\frac{a^3}{r_1^3} - 1 \right)}$$

(In case of viscous liquid with $p_0 = 0$)

1) We proved that the critical Reynold number is 8.

2) We derived relation between the time and the radius during collapsing (equation 4.60)

3) We derived formula of the collapsing time (equation 4.61)

4) We proved that the collapsing time lies on $[0.4 \frac{a}{U_0}, 0.5 \frac{a}{U_0}]$

In chapter 5, we discussed the gas bubble, and we obtained a qualitative formula defining the critical bubble size separating two different regimes of radial bubble motion: oscillatory and monotonically aperiodic, and this critical radius is equals

$$R_{0*} = \frac{\eta}{\sqrt{\rho}(p_0 p_0')^{0.25}}, \text{ where } \eta, \rho, p_0 \text{ and } p_0' \text{ are the dynamic viscosity, density of liquid,}$$

pressure of liquid away from the bubble and initial pressure of gas within the bubble respectively.

Chapter 7

DISCUSSIONS

The main idea of the present thesis is to demonstrate that the qualitative theory of differential equations, when applied to problems in fluid dynamics, will contribute to the understanding of qualitative aspects of fluid flows, in particular to the collapsing of empty spherical bubble and oscillations of gas bubble.

It is obvious that insight into the qualitative structure of flow fields is of great importance and appears as an ultimate aim of flow research. For, qualitative insight fashions our knowledge; it serves as a good guide for further quantitative investigations. Moreover, qualitative information can become very useful, especially when it is applied in close correspondence with numerical methods, in order to interpret the numerical results. A qualitative analysis may be crucial for the investigation of the flow in the neighbourhood of singularities where a numerical method is not reliable anymore due to discretisation errors being unacceptable.

Up till now, familiar research methods frequently based on rigorous analyses, careful numerical procedures and sophisticated experimental techniques have increased considerably our qualitative knowledge of flows, albeit that the information is often obtained indirectly by a process of a careful but cumbersome examination of quanti-

tative data.

In the past decade, new methods are under development that yield the qualitative information more directly.

In the present thesis we concerned on the singularities of the differential equation that generated from the bubble dynamics by using qualitative theory of differential equations.

These methods, make use of the knowledge available in the qualitative theory of differential equations to describe the motion of the bubbles inside the liquid, which give us the the types of motions before and after specific parameter.

I recommend any researcher who's research related to any types of physical dynamic system to use qualitative theory of differential equations, to have a full description of the phenomena of the dynamic objects.

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الملخص

تهتم هذه الأطروحة بتوظيف نظرية المعادلات التفاضلية النوعية جنباً إلى جنب مع الاعتبارات التبولوجية لمعالجة بعض المواضيع والمشكلات في ديناميكا السوائل.

تركز الاهتمام في هذه الأطروحة لحركة الفقاعة في سائل لزج غير قابل للانضغاط ، ولا سيما انهيار الفقاعات الكروية الفارغة ، وتذبذب الفقاعات الكروية الغازية.

وقد دُرست الجوانب النوعية لسلوك حلول المعادلة التفاضلية التي توصف حركة الفقاعة دون معرفة الحل الفعلي للمعادلة وخصوصاً حول النقاط الانفرادية المعزولة.

و تم التوصل إلى نصف القطر الحرج في الحالتين عندما تكون الفقاعة الكروية فارغة أو غازية.

ومن ثم دُرِس سلوك الفقاعة عندما يكون نصف قطرها أكبر من نصف القطر الحرج وعندما يكون أصغر من نصف القطر الحرج.

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كلية العلوم

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إعداد

عبدالكريم محمد حسن خليل

بكالوريوس في الرياضيات، جامعة البحرين، كلية العلوم، ٢٠٠٢م

إشراف

أ. د. نايل سليمان خابيف

أستاذ الرياضيات

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