

DYNAMICS OF AN ELASTIC CABLE CARRYING A
MOVING MASS PARTICLE

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ABSTRACT

The dynamic behavior of an elastic catenary cable due to a moving mass along its length is investigated. The equations of motions are derived using the Hamilton's principle for general cable supports that include the horizontal and inclined cables with small and large sags and for variable velocity of the moving mass. These equations of motions are in general nonlinear partial differential equations due to the initial curvature of the cable. The solution of these equations is not feasible using a harmonic motion solution as it is the case for the free vibrations of cables. The reason is the presence of the nonhomogenous terms that appear in the equations of motions and because of the presence of the Coriolis terms, due to the moving mass. Therefore, we used the Galerkin procedure with sine function(Fourier representation) and anti-derivative functions of the compactly supported wavelets as trial basis and direct integration methods to integrate the discretized equations of motion. Two models are obtained, namely linear and nonlinear models. The linear model requires less computations and faster in simulation than the nonlinear model. Though, the linear model is suitable only for small sag-to-span ratios, small mass ratios of the moving mass to the mass of the cable and for small mass velocities. The nonlinear model is applicable to general cable configuration and moving mass specifications. Comparisons are given between the Fourier and wavelets representations and between the linear and nonlinear models by means of examples.

CHAPTER I

INTRODUCTION

1.1 Problem Definition And Motivations

The light weight and the flexibility of cables and their ability to withstand high tension make them attractive structural elements in many applications in engineering. For example, long cables are used for suspension bridges like the famous Golden Gate bridge at San Francisco, California. They are also used in roof suspension to add distinct architectural shapes to many museums, auditoriums and arenas. Cables are often involved in ocean engineering to support attached masses such as sensors, hydrophones, cameras and other electrical instruments. Cables are used in ski lifts and tramways as a means of transportation. Stationary cables that are capable of carrying rocket propelled trolleys have been used by the Sandia Laboratories, Albuquerque, to simulate aircraft flight for evaluation of airborne equipment and to precisely target items for impact testing[29]¹. A similar idea of aircraft flight simulation shall be conducted at the White Sands Missile Range. A Kevlar cable of 4572m in length shall be supported between two mountain tops and a Helicopter shall be used to travel along the cable length for flight testings.

The linear analysis of the free vibrations of inextensible cables or chains is formulated by Routh[32] and asymptotic solutions are given by Saxon and Cahn[35] and Goodey[16]. The inextensible cable analysis failed to converge to the string case as the limit of the sag approaches zero. The first elastic cable analysis is given in 1974 by Irvine and Caughey[21] for cables with small sags. Although the analysis is restricted to cables of small sag-to-span ratios, the results are suitable for many engineering applications, such as power lines and marine cables. The natural frequencies and

¹Corresponds to numbered reference in the bibliography.

mode shapes are the important results at which resonant could occur and failure to cables due to fatigue could happen. The nonlinear analysis of elastic cables is also important to predict the galloping vibrations at which cables could break because of large displacements at low frequencies[17, 24]. In reality static loading on cables are not necessarily absent. Cables could have masses hanging on them[36, 30, 5], steady wind pressure[31] or complicated loads[56]. The determination of natural frequencies and mode shapes in each case require special attention. Traveling bare strings or cables under steady state motions are also considered for free vibrations when the static shape is altered by the horizontal traverse velocity of the string or cable. In this case as in the other cases of free vibrations, the natural frequencies and mode shapes are required. What is interesting about this case is the frequency coalescence and mode reversion which means that the higher modes with many nodes will revert to the lower modes with fewer nodes[38, 47]. If the traveling strings carry attached masses, the wave scattering is important. Waves associated in this case are in the form of reflected by and transmitted through the mass[50]. Forcing can be introduced in the above cases in many ways. However, the forced vibrations of fixed cables that carry moving loads is the research interest of many people[40, 12, 55, 14, 34, 29, 42]. Most of the work in this case is restricted to stretched strings which are assumed to produce no dynamic tension and the interest is in finding the wave propagation, kink angle and response curves. Main assumption of this case is that the cables are assumed inextensible and the dynamic strain is negligible. A finite element analysis studied by Wu and Chen[52], included the effect of elasticity, but admitting horizontal small sag cables only. The detailed literature review can be found in the next section.

In this thesis, we aim to study a wider case of elastic cable carrying a moving mass. We will allow horizontal and inclined cables with arbitrary sag to span ratios. The moving mass could be small or large with constant velocity, constant acceleration

or variable velocity and variable acceleration. The dynamic equations will be derived from Hamilton's principle which are in general nonlinear. These equations will be solved using the Galerkin procedure with Fourier and wavelets analysis. Linear and nonlinear solutions will be provided. The dynamic configuration will be measured from the catenary equilibrium of the cable. The location of the mass, the response of the cable and the cable tension will be shown graphically for different cases. Discussion on the convergence of the method and a comparison between the classical Galerkin procedure and the wavelet-Galerkin method will be given. The formulation of the moving boundary problem will be given, from which, many important aspects can be deduced, such as cable static shape and cable free vibrations.

1.2 Literature Review

Routh[32] gave a partial solution for the vibration of an inelastic chain supported between two points on the same level. The special case he considered was a chain hanging in the form of cycloid, which he solved exactly. Saxon and Cahn[35] obtained an asymptotic solution of the natural frequencies of an inextensible chain vibrating in-plane. The weight of the cable was uniformly distributed and supported at the same level from its two ends. The equilibrium configuration of the chain was a catenary shape which make the results applicable for small and large ratios of sag to span. Approximate expressions for odd and even modes were given. The same problem was investigated by Goodey[16]. He found equivalent expressions for the mode shapes. Unlike Saxon and Cahn[35], who used a fourth order differential equation in the tangential displacement, Goodey[16] used a second order differential equation in the angular displacement. This equation was first derived by Routh[32] for a nonuniform chain. Goodey also used the asymptotic procedure that was followed by Saxon and Cahn[35]. Smith and Thompson[39] investigated the vibration of an inextensible

cable which is supported at different levels. They used power series expansion to approximate the displacements solution. They used the linear extrapolation procedure to find the roots of the resulting characteristic equation. Their solution is valid only for ratios of sag to span not greater than 0.76. As an example, they showed that their results are in agreement with the results obtained by Saxon and Cahn[35] for the case of same level supports. Gale and Smith[15] extended the work of Smith and Thompson[39] to include the out-of-plane modes and for sag to span ratios up to 1.09.

Irvine and Caughey[21] developed a linear theory of the free vibration of a uniform suspended cable that is extensible and has same level supports. The sag to span ratio is limited to 1 : 8 or less, so that a parabolic approximation is used for the static configuration of the cable. They found expressions of the natural frequencies for the out-of-plane and the in-plane motions. For the case of the symmetric in-plane motion they found that the natural frequencies are dependent on a parameter involving cable geometry and elasticity. This parameter is considered very important from which the theory is proven to be converging to the cases of inextensible cable vibration and to the taut string vibration. For the antisymmetric in-plane motion the natural frequencies are independent of that parameter. The behavior of the frequency crossover between the symmetric and the antisymmetric modes is shown clearly. Ramberg and Griffin[28] studied the free vibrations of taut and slack cables experimentally where the tests have been conducted in both air and water for three different types of marine cables, namely double armored steel, uniline and seacon. For cables with sag to span ratios less than 1 : 8, they found that the natural frequencies and mode shapes are in good agreement with the linear theory presented by Irvine and Caughey[21]. They also confirmed the occurrence of frequency crossover that the linear theory has predicted. They have determined the fluid added mass and fluid damping at various vibration Reynolds numbers and they found that their results

agree well with the previous work[4]. Upon determining the values of the added mass and the fluid damping, they computed the natural frequencies of the double armor cable using the linear theory of Irvine and Caughey[21] and compared the results with the measured data in both air and water. They found good agreements for large cable tensions, but large discrepancy for low cable tensions. They express some doubt about the measured data points at low cable tensions. Henghold, Russell, and Morgan[18] used nonlinear finite elements procedure to study the free vibration of cables in three dimensions. They showed that the natural frequencies depend on the end conditions and the stiffness-to-weight ratio. For a suspended cable with same level supports their prediction of the crossover frequencies were in agreement with those found by Irvine and Caughey[21]. Examples of different end conditions and different stiffness-to-weight ratios are provided too. Irvine[22] extended the work of Irvine and Caughey[21] to cover an inclined cable hanging under self-weight and close to the chord. By transforming the axes so that one measure along the chord and the other on perpendicular to the chord, he was able to obtain expressions for the natural frequencies of the out-of-plane motion and the in-plane motions, which are similar to those obtained by Irvine and Caughey[21]. Furthermore, these expressions are more general and hence the cable with same level supports is a special case. His results are in agreement with those found by Henghold, et al.[18], but more accurate and in closed-form. Veletsos and Darbre[49] explained the salient features of the free vibration of parabolic inclined cables. They showed by mean of graphs the effect of changing the dimensionless parameter, which includes the geometry and elasticity of the cable, on the natural frequencies, the cable tension, and the extensional component of the potential energy. Also, approximate expressions for finding the natural frequencies of the symmetric modes are provided. In addition, they have given close-form expressions for some infinite series that contain natural frequencies and integrals

of the modes of vibration of the symmetric type. Triantafyllou[46] derived a general asymptotic solution to the dynamics of taut inclined cables. The total solution is a sum of a fast varying and a slow varying solutions compared with the static solution. In the case of a horizontal cable, he found a characteristic equation for the natural frequencies which reduces to the case of inelastic cable and to the case of taut wire that has transverse and elastic waves. In the case of inclined cables he found that, at some points of the cable characteristics, the modes are neither symmetric nor antisymmetric and instead they are hybrid modes, a mixture of symmetric and antisymmetric modes and no crossover points exist. The hybrid mode is as important as symmetric mode because it produces dynamic tension. Shih and Tadjbakhsh[37] studied the small amplitude vibrations of extensible cables that have same level supports and catenary shape. They adapted the Galerkin procedure to approximate the dynamic displacements of the cable under the free vibrations and obtained the natural frequencies. In their work, it is clear that both the symmetric and the antisymmetric modes are dependent on two parameters which are the geometry and elasticity parameters. A table of the natural frequencies of the symmetric and the antisymmetric modes for different cases of geometry and elasticity parameters are presented and they showed that the method converges when using only 12 terms in the series.

Hagedorn and Schafer[17] presented the nonlinear solution of the free vibrations of an elastic cable suspended between same level supports. First they obtained the eigenfunctions of the simplified linear problem and then used them in the Ritz-Galerkin method to reduce the problem to a single ordinary differential equation that contains quadratic and cubic nonlinearities in an unknown time function. They found an approximate solution to the nonlinear equation by using the Lindstedt perturbation technique. Their numerical example of the first normal modes showed considerable change in the frequencies as compared with the results obtained by the linear analysis.

A similar problem was studied by Luongo, Rega and Vestroni[24]. They considered the effect of the cable properties on the nonlinear modes. In particular, as the sag to span ratios increases the cable exhibits both softening and hardening spring behavior. The same behavior is true for increasing elasticity ratios. AL-Noury and Ali[1] studied the nonlinear vibration of a parabolic cable. The cable is assumed to vibrate in the transverse and vertical directions only. Discretization is achieved by applying the Galerkin procedure and terms up to cubic order were retained. They solved the resulting equations by the method of multiple scales. Furthermore, they introduced a detuning parameter that describe the nearness of the frequencies associated with vertical oscillations to those of the transverse oscillations and obtained the closely spaced modes. They also illustrated the softening and hardening spring behavior of the cable by mean of frequency-response curves. Takahashi and Konishi[43, 44] studied the nonlinear free vibrations of horizontal and inclined cables in three dimensions. Their analysis is valid for small and large sag to span ratios. In order to solve the nonlinear equations of motion, they used the Galerkin method in the space coordinate and harmonic balance in time coordinate and found two nonlinear algebraic equations. Similar observations regarding the softening and hardening spring behavior of the cable have been found.

Sergev and Iwan[36] found the experimental and theoretical natural frequencies and mode shapes of strings with attached masses. The string was considered as a sequence of segments connected at the attached masses. The theoretical solution is found by considering the following. Each segment is governed by the wave propagation equation and the displacement continuity and force balance at the nodes in addition to the fixed boundary conditions at the cable ends. The experimental results are obtained for two cases. The first one is a four equally spaced, equal masses, and the second case is a six unequally spaced, equal masses. In comparison, they have

found that the theoretical results agree with the experimental natural frequencies, node and antinode locations and the mode amplitudes. Rosenthal[30] used a new method to find the natural frequencies of slack cables with discrete masses. He used an iterative technique called the method of imaginary reaction to find the static configuration and Stodola's method for dynamics. He studied two cases: fixed-fixed cable and fixed-free or forced cable. The effect of the added mass was included in the analysis. He compared his results with those obtained by Henghold et al. for the case of no masses attached. He found good agreement at lower frequencies but rough estimates at higher frequencies. Another comparison was made with the results obtained by Sergev and Iwan[36] for six masses unequally spaced along the arc length of a taut cable. The results are in excellent agreements when using large number of integration intervals. The author has provided by means of a log-log plot the modal frequencies and the cable tension of the same problem for the complete range of sag to span ratio. The plot showed the phenomenon of frequency crossover and the behavior of the modes at large sag ratio. Cheng and Perkins[6] used the Hamilton's principle to derive the equations of motion of a sagged cable supporting a discrete mass. They assumed a parabolic shape for the static configuration and a linear analysis for dynamics in keeping with the linear theory presented by Irvine and Caughey[21] for bare cables. Therefore, they treated cables with small sag to span ratios only. They considered two asymptotic models. The first one neglected the tangential accelerations for both the cable and the attached mass. This model is valid for small masses only. The second model neglected only the tangential acceleration of the cable. This model is suitable for larger masses. The effect of the cable elasticity and the effect of the mass attached on the natural frequencies were shown by mean of graphs. Cheng and Perkins[5] extended their work of single mass to include an array of attached masses on a cable. They obtained a closed-form solution derived

from the linear theory of cable vibration. They compared their solution with the results obtained by Sergev and Iwan[36] and Rosenthal[30]. Their solution converged to the solution given by Sergev and Iwan[36] of a string supporting four equally spaced, equal masses and indistinguishable with the results of Rosenthal[30] for the example of six equal masses, unequally spaced in the range of sag to span ratio less than 1 : 8. The authors have shown that for a symmetric array of masses there exist symmetric and antisymmetric modes and for a large number of attached equally spaced, equal masses the solution converged to the modes of a bare cable. While for asymmetric array of masses there exist asymmetric modes only. Cheng and Perkins[7] studied the theoretical and experimental analysis of the forced response vibration of their previous work(Cheng and Perkins[5]). The dynamic displacements have been expressed in terms of the Green's function. They used the example of four equally spaced, equal masses discussed in their previous paper for comparison. The theoretical and experimental frequency response curves have similar pattern and some discrepancies arose because of the linear assumption in the asymptotic model. Yu[56] obtained closed-form solutions of the vibration of cable under complicated loads. He solved the more general case of cable vibration under both distributed and concentrated static loading, because he added load components that act in the horizontal direction(out-of-plane direction) along with vertical loadings. Therefore the cable lied in an inclined plane and the in-plane and out-of-plane equations of motions are now coupled. He modified the transfer matrix method that was used by the previously mentioned researchers, to consider a catenary static profile. As a special case his solution reduced to the first asymptotic model discussed by Cheng and Perkins[6] for a single mass and the absence of horizontal loading.

Sack[33] studied the steady state transverse oscillations of a string traveling at constant speed between two supports. The string is fixed at one support and given a

sinusoidal lateral vibrations at the other support. The tension in the string is assumed constant throughout. The oscillations without damping and with linear damping are considered. He found that the oscillations are similar to those of a fixed string except that the traveling string has an oscillation phase shift that varies from point to point. Swope and Ames[41] obtained the solution of a string as it is traversed and wound on a bobbin. The d'Alembert solution is obtained for an infinite string and the method of characteristics is used for a finite length string. In the infinite string case, the motion is governed by two waves that move with different wave speeds and have different shapes. For the finite string, two characteristic lines are obtained that are dependant on the initial conditions of the problem. An example is given to study the effect of changing the traverse velocity on the string response and the frequency of oscillations. Simpson[38] studied the oscillatory motions of an elastic shallow cable translating at constant speed. Two parameters are considered in the analysis a small sag to semi-span ratio and a large elasticity stiffness. An approximate solution based on a zero order is obtained. He presented different examples showing the frequency curves and mode shapes. The modes shape loses their symmetric and antisymmetric behavior and become distorted and complex. At some instances the frequency coalescence and mode reversion exist but the author couldn't justify the reason behind this phenomena. Triantafyllou[47] found the natural frequencies and mode shapes of translating cables moving with constant speed between the end supports. The cables considered are of small or large sags and horizontal or inclined. The equations of motions are derived along the tangential and normal directions and two different asymptotic solutions were obtained, one for small sag cables and the other one for large sag cables. He found that the frequency coalescence and mode reversion exist for small sag horizontal cables, while only mode reversion exists for inclined cables. For large sag cables, the asymptotic solution is derived based on the cable stiffness as

a large parameter. This solution converged within the first order to the results given by Saxon and Cahn[35] for a horizontal catenary in air and by Triantafyllou[48] for inclined catenary in water. Perkins and Mote[25] studied the same problem but in three-dimensions and for arbitrary cable sags. They used the Hamilton's principle to derive the nonlinear equations of motions in the tangential, normal and binormal directions. After linearization they applied the Galerkin method to obtain the solution. They found that an expansion of ten terms in the series is enough to represent the dynamic displacements and the comparison with the results of the linear theory of Irvine and Caughey[21] of a stationary horizontal cable showed good agreement over a wide practical range of the cable stiffness. As for the inclined stationary cables, the authors showed that there exist hybrid modes(or curve veerings), this was observed by Triantafyllou[48], instead of frequency crossover. They observed the same behavior for traveling cables which is contrary to the findings of Simpson[38] and of Triantafyllou[47]. Their final observation is that the cable becomes unstable only if the cable speed becomes infinite.

Wickert and Mote[50] studied the linear transverse vibration of an axially moving string-particle system transported between two fixed horizontal supports at low constant speed. The initial tension in the string was assumed very high so that the equilibrium configuration was straight and the dynamic tension was neglected. The equation of motion was derived from the Hamilton's principle and three different solutions are presented. For an infinitely long string the solution is given by d'Alembert solution that described the harmonic wave scattering in the form of reflected and transmitted waves due to the presence of the attached mass. For strings with finite lengths, an asymptotic solution for small mass ratios is found and the natural frequencies are given to the first order in closed-form. The characteristic equation is obtained from which the natural frequencies have to be found numerically. The

Galerkin method is used to describe the free response of the system. Only ten terms are used in the series approximation that described the dynamic displacement. Wickert and Mote[51] studied the response of a parabolic, stretched string moving axially at constant speed and coupled with a transported mass particle. The transverse displacements of both the string and the traveling mass are represented by the Green's function. The Volterra integral equation of the first kind for the interaction force between the string and the mass is found. In order to avoid the instability and divergence associated in the numerical integration of the Volterra equation of the first kind, the authors derived the Volterra equation with delay and solved it asymptotically by expanding the interaction force in power series of the mass ratio (mass of the particle divided by the mass of the string). Experiments have been conducted and the measurements are compared with the analytical results. Zhu and Mote[57] used the same solution technique of Wickert and Mote[51] for an axially moving string carrying a damped linear oscillator. Excitation at the left support is considered in addition to the loading described by Wickert and Mote[51].

Smith[40] considered a stretched stationary string that vibrates in the transverse direction only due to a mass particle moving along its span with a constant horizontal velocity component. The dynamic tension in the string is neglected because of the large initial strain. The general solution of the vibrating system is given by the superposition solutions of the static deflection, free vibration and the forced vibration due to the applied forces. The author reduced the problem to an integral equation with the interaction force between the string and the mass particle as an unknown. An approximate solution is found based on constant interaction force equals to the weight of the moving mass. This solution is valid for small velocity ratios much lower than $1/3$. Another approximate solution is found based on a massless string; i.e. the string inertia is neglected. The exact solution of the displacement of the moving mass have

been obtained for values of the velocity ratio greater than $1/3$, which means that the mass is sufficiently fast to exit the string before the initial disturbance reaches it after traversing the length of the string twice. Kanninen and Florence[23] considered two forces of the same magnitude traveling in opposite directions on a stretched infinite string. The forces are the result of the use of explosives located over the string. The Laplace transform is used for the solution. The velocity and displacement distributions for the supersonic and subsonic velocities are given. Flaherty[12] studied the effect of a moving force with varying speed on the transverse velocity of an infinite string. The solution of the wave propagation equation under the moving force was obtained by using the Laplace and Fourier transformations. The two cases of decreasing and increasing velocities of the moving force are considered. In both cases the velocities pass through the characteristic speed of the string. The results of the both cases showed that the singularity occurs at the momentary the velocity of the moving force passes through the characteristic speed. Yen and Tang[55] used the perturbation method to find the nonlinear vibration of an infinite elastic string supported by an elastic foundation due to a load moving in the vicinity of the critical speed of the linear string. Both the axial and the transverse displacements are expanded in power series of the dimensionless load intensity. Although, the solutions have been strictly given for the subcritical and supercritical velocities only, the authors have shown that each solution can be valid at the critical velocity and extended to the region of the other solution. An example is given which showed that solutions at the linear critical speed exist and for velocities greater than the critical speed the modes have two different amplitudes. Fryba[14] considered the vibration of a massless string subjected to a moving mass. The deflection of the string was represented by the Green's function. Then the problem was reduced to a nonhomogeneous differential equation for the deflection of the mass. The nonhomogeneous term contains a parameter that

relates the string tension and length and the moving mass and its speed. The solution is given for different cases of that parameter. Sagartz and Forrestal[34] studied the response of an infinitely long stretched string loaded by an accelerating force. The solution of the wave propagation equation that govern the motion of the string due to the applied force was obtained by employing the Laplace and Fourier transforms. An expression for the kink angle was provided and its location was discussed for an example of constant acceleration of magnitude equal to the wave speed of the string. Forrestal, Bickel and Sagartz[13] considered a slightly curved stretched cable carrying a mass moving with constant acceleration. The maximum speed sought for the mass was less than the propagation speed. They assumed that there are two kinds of moving forces acting on the cable. The first force is the weight of the mass and the second force is the reaction from the normal acceleration of the mass. The solution due to the latter force is given and added to the solution given by Sagartz and Forrestal[34] due to a moving force. The authors have given an expression to the kink angle and plots illustrating the deflection of the cable at several time values. A comparison of the theoretical and experimental results of the deflection of a selected point on the cable showed an accurate prediction. Rodeman, Longcope and Shampine[29] obtained the solution of an infinitely long stretched cable subjected to a mass particle moving at constant horizontal acceleration taken into consideration the unknown reaction force between the mass and the cable. The cable deflection is given by an integral form in term of the unknown reaction from which the delay differential equation is found for the deflection of the moving mass. The delay differential equation is solved numerically and asymptotically. The small parameter used in the asymptotic solution is the ratio of the moving mass inertia to the cable tension. These solutions have been used to obtain the cable response history. The graphical results showed that as the mass velocity passes through the wave speed to supersonic speed the cable deflection expe-

riences two opposite jumps which could not be predicted by the moving force model. D'Acunto[9] studied the free boundary problem of an infinitely long string subjected to a moving load. He considered that both the string vibration and the motion of the moving load are influenced by each other and they are unknowns. The author has reduced the problem to the one that describe the motion of the moving mass and he discussed the existence and uniqueness of the solutions. Wu and Chen[52] used the finite element method to study the dynamics of a suspended cable due to a moving load. They considered horizontal cables with sag to span ratios less than $1/8$. The updated Lagrangian formulation is used to derive the equations of motions of the cable structure. Then the finite element is used to discretize the displacements and obtained the property matrices which they solved by the Newmark direct integration scheme. For the spacial case of the free vibration of a bare cable, they found the natural frequencies and a comparison is made with results obtained by Henghold et al.[18] given in a form of table. They found a good agreement. The authors showed by mean of graphs the response of the dynamic shape to changing the cable stiffness ratio, the moving mass ratio and the nondimensional speed of the moving mass. Pierucci[26] studied the response of an infinite string due to a harmonic force moving at constant speed. The equation of motion is solved by Fourier transform and discussed the wave scattering in the string. He mentioned that there are waves on the string that are traveling to the left and to the right. When the force is moving at subsonic speed there is a wave traveling ahead of the force and a wave traveling to the left away from the force. At supersonic speed there are no traveling waves ahead of the force, instead all the waves move to the left. Tadjbakhsh and Wang[42] studied the vibrations of an inclined taut cable with a riding, accelerating mass. Their analysis is for the case of inextensible cables. Although, they used a catenary cable for the static analysis, their model is good for cables that have no minimum sag, i.e. the lower point of the cable

dose not go below the horizontal line that passes through the lower support. They used the Galerkin method to approximate the dynamic displacements of the cable in the normal and tangential directions of cable configuration. The basis functions used for the Galerkin procedure are the sine function for the normal displacement and the integral of the curvature of the cable and sine for the tangential displacement, while the coefficients of the series, which are dependent on time, are the same for the both displacements.

1.3 Thesis Organization

This thesis consists of five chapters and one appendix. The first chapter is an introduction to the thesis topic in which we mentioned the general application of cables with focus on the physical application of our thesis. A single structure element amazingly has tremendous amount of literature. We have cited those that might contribute to our study of cable vibrations directly or indirectly. We categorized our literature review according to the type of analysis and application. First we mentioned the theory of inextensible cable vibrations and its limitations. We, then, mentioned the linear and nonlinear theories of vibrations of elastic cables. We extended our literature review to cover the free vibrations of strings and elastic cables under stationary concentrated or distributed forces and attached masses. We also discussed the free vibrations of traveling strings and cables. The last category was the forced vibrations of strings and cables under moving forces or moving masses. In chapter two we discussed the static configuration of elastic cable under its own weight and the parabolic approximate. We also derived the equations of motions of an elastic cable carrying a mass particle moving under variable motion using Hamilton's principle. Those equations are nonlinear equations and their solution is postponed to chapter three. For the special cases of the free vibrations of bare cable and of cable with

attached mass, those equations are linearized and solved using separation of variables with harmonic functions to obtain the natural frequencies and mode shapes. In the case of a moving mass particle, the motion of the cable is not harmonic motion any more because of the chaos motion produced by the mass particle as it travels along the cable span and the use of separation of variables with harmonic motion is not feasible. Therefore, in chapter three, the Galerkin procedure is adapted for solution. The equations of motions are discretized using Fourier sine function and compactly supported wavelets as basis functions. Two models are provided. The first one is the linear model that is suitable for cables with small sags and for small moving mass particle with low speed. The second model is the nonlinear model that suitable to general cable configuration and moving mass characteristics. We used the the implicit direct integration methods associated with the Newton-Raphson method for the time integration scheme. In chapter four we gave numerical examples. First we proved that our method reproduce the exact natural frequencies of the linear theory of the free vibrations of bare cable and of cable with attached mass. Then we discussed the linear model and its extent of validity of capturing the dynamic behavior of the cable under a moving mass particle. We used the nonlinear model to compare with some work found in the literature. Chapter six is the conclusion. Appendix A is an introduction to wavelets and their application in solving partial differential equations.

CHAPTER II

STATICS AND DYNAMICS OF CABLES

2.1 Equilibrium Configuration

The dynamic quantities of any vibrating system or structure have to be measured from a reference state or configuration. Dynamic quantities such as the displacements of a vibrating cable are measured from an equilibrium configuration. This equilibrium configuration is the stretched catenary of elastic cables. An elastic cable that has known manufactured characteristics of length, mass density and cross sectional area, takes an unstretched catenary shape if suspended between two supports, but due to gravity it will stretch under its own weight and will take the equilibrium configuration which is called the stretched catenary shape. If the cable is inextensible there will only be the unstretched shape. However, for elastic cables the amount of stretching will depend on the modulus of elasticity. To obtain the stretched shape, reference has to be made with the unstretched shape. But this will complicate the vibration analysis and if someone is seeking a simplified solution, this type of formulation is undesired. Therefore, it is convenient to make reference to the stretched shape and assume that the amount of stretching has been taken into consideration. In other words, we assume that the stretched cable has known characteristics of length, mass density and cross sectional area. If we denote by $\bar{s}$ the unstretched spatial distance and by s the stretched spatial distance along the cable, we have the following relation

$$e = \frac{ds - d\bar{s}}{d\bar{s}} = \frac{H(\bar{s})}{E\bar{A}} \quad (2.1)$$

where e is the strain, $H(\bar{s})$ is the tension of the unstretched cable, E is the Young's modulus and $\bar{A}$ is the cross sectional area of the unstretched cable. For conservation of mass we have $\rho ds = \bar{\rho} d\bar{s}$, where $\bar{\rho}$ is the mass per unit length of the unstretched

cable and ρ is the mass per unit length of the stretched cable. This gives us

$$\rho = \bar{\rho} \frac{d\bar{s}}{ds} = \frac{\bar{\rho}}{e+1} \quad (2.2)$$

The length of the stretched cable, l , can be found from equation (2.1). Writing $ds/d\bar{s} = 1 + e$ and integrating we get

$$l = \int_0^{\bar{l}} (1 + e) d\bar{s} \quad (2.3)$$

where $\bar{l}$ is the length of the unstretched cable. The equations of equilibrium can be obtained by balancing the forces on an infinitesimal cable element or alternatively we can use equations (2.32) and (2.33). In these equations, reference is made to the stretched spatial variable s , but here we write them as a function of the unstretched spatial variable $\bar{s}$. Hence,

$$\frac{d}{d\bar{s}} \left\{ T \frac{dx}{d\bar{s}} \frac{d\bar{s}}{ds} \right\} \frac{d\bar{s}}{ds} = 0 \quad (2.4)$$

$$\frac{d}{d\bar{s}} \left\{ T \frac{dy}{d\bar{s}} \frac{d\bar{s}}{ds} \right\} \frac{d\bar{s}}{ds} + \bar{\rho} \frac{d\bar{s}}{ds} g = 0 \quad (2.5)$$

where g is the gravitational acceleration and $T(\bar{s})$ is the tension of the stretched cable. Note that equation (2.2) has been used in equation (2.5). Substituting for $d\bar{s}/ds = 1/(1 + e)$, the above equations become

$$\left\{ T \frac{x'}{1 + e} \right\}' = 0 \quad (2.6)$$

$$\left\{ T \frac{y'}{1 + e} \right\}' + \bar{\rho} g = 0 \quad (2.7)$$

where $()' = d/d\bar{s}$. The cable tension, $T(\bar{s})$, is given by the constitutive relation

$$T(\bar{s}) = EA e \quad (2.8)$$

where A is the cross sectional area of the stretched cable. Since the cable is made of compressible materials, such as steel, we will assume that EA is constant. The square of the arclength of an element of the cable at its stretched state can be written as

$$(ds)^2 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2$$

dividing the above equation by $(d\bar{s})^2$ we obtain the relation

$$x'^2 + y'^2 = (1 + e)^2 \quad (2.9)$$

Integrating equations (2.6) and (2.7) we get

$$Tx' = C(e + 1) \quad (2.10)$$

$$Ty' = (B - \bar{\rho} g \bar{s})(e + 1) \quad (2.11)$$

where C and B are constants of integration. Squaring both sides of the above equations and making use of equation (2.9) we get

$$\begin{aligned} T^2 &= C^2 + (B - \bar{\rho} g \bar{s})^2 \\ \therefore T &= \sqrt{C^2 + (B - \bar{\rho} g \bar{s})^2} \end{aligned} \quad (2.12)$$

Substituting equation (2.12) in equation (2.8) we can get an expression for e . Before doing this we will introduce the following parameters

$$\begin{aligned}\bar{s}^* &= \frac{\bar{s}}{\bar{l}}; x^* = \frac{x}{\bar{l}}; y^* = \frac{y}{\bar{l}}; T^* = \frac{T}{\bar{\rho} g \bar{l}} \\ \beta &= \frac{\bar{\rho} g \bar{l}}{EA}; \alpha = \frac{B}{C}; \sigma = \frac{\bar{\rho} g \bar{l}}{C}\end{aligned}\quad (2.13)$$

Therefore e becomes

$$e = \frac{\beta}{\sigma} T^* \quad (2.14)$$

Upon substituting for equation (2.8) and (2.14) in (2.10) and (2.11) we get

$$\begin{aligned}x^{*'} &= T^{*-1} \left(1 + \frac{\beta}{\sigma} T^* \right) \\ &= \frac{1}{\sqrt{1 + [\alpha - \sigma \bar{s}^*]^2}} + \beta/\sigma \\ y^{*'} &= (\alpha - \sigma \bar{s}^*) T^{*-1} \left(1 + \frac{\beta}{\sigma} T^* \right)^{-1} \\ &= \frac{\alpha - \sigma \bar{s}^*}{\sqrt{1 + [\alpha - \sigma \bar{s}^*]^2}} + \beta/\sigma\end{aligned}$$

where $()' = d/d\bar{s}^*$. Integrating the above equations we get

$$x^*(\bar{s}^*) = \frac{1}{\sigma} \left\{ \sinh^{-1} \alpha - \sinh^{-1} (\alpha - \sigma \bar{s}^*) + \beta \bar{s}^* \right\} \quad (2.15)$$

$$\bar{y}(\bar{s}^*) = \frac{1}{\sigma} \left\{ \sqrt{1 + \alpha^2} - \sqrt{1 + (\alpha - \sigma \bar{s}^*)^2} + \beta \bar{s}^* \right\} \quad (2.16)$$

The above equations satisfy the boundary conditions at $\bar{s}^* = 0$ which are $x^*(0) = y^*(0) = 0$. The unknowns σ and α can be found from the other boundary conditions

which are

$$\begin{aligned} x^*(1) &= b/\bar{l}; \text{ and} \\ y^*(1) &= h/\bar{l} \end{aligned} \tag{2.17}$$

Once these unknowns are found the new cable characteristics can be found and used to solve equations (2.32) and (2.33) as function of the stretched spatial variable s .

2.2 General Formulation

The equations of motions of a vibrating cable due to a moving mass will be derived from the Hamilton's principle. Consider a cable suspended between two points at any arbitrary levels. All dependent variables will be referenced to the stretched static shape described by the Cartesian coordinates $x(s)$ and $y(s)$ where s is the spatial variable measured along the cable length of the stretched configuration. The dynamic displacements in the longitudinal and transverse directions are described by $u(s, t)$ and $v(s, t)$, respectively, where t is time (see figure 2.1). Denoting by $s_o(t)$ the distance the moving mass has traveled along the cable length, its position vector $\hat{p}_o(t)$ can be represented as follows

$$\hat{p}_o(t) = [x(s_o) + u(s_o, t)] \hat{i} + [y(s_o) + v(s_o, t)] \hat{j}$$

where $\hat{i}$ and $\hat{j}$ are the fixed global unit vectors in the horizontal and vertical directions, respectively. For the general case of the characteristic motion of the moving mass, we assume that the mass has a local velocity $ds_o(t)/dt$ and a local acceleration $d^2s_o(t)/dt^2$ along the arc length of the cable with a direction tangent to the cable at the location of the moving mass. Then the velocity vector of the moving mass will be

$$\begin{aligned}
\hat{v}_o(t) &= \frac{d\hat{p}_o}{dt} \\
&= \left(\frac{\partial u(s_o, t)}{\partial t} + \frac{ds_o}{dt} \left(\frac{dx(s_o)}{ds_o} + \frac{\partial u(s_o, t)}{\partial s_o} \right) \right) \hat{i} \\
&\quad + \left(\frac{\partial v(s_o, t)}{\partial t} + \frac{ds_o}{dt} \left(\frac{dy(s_o)}{ds_o} + \frac{\partial v(s_o, t)}{\partial s_o} \right) \right) \hat{j}
\end{aligned} \tag{2.18}$$

Equation (2.18) and the velocity vector of the cable at any arbitrary point on the cable, which is $\partial u(s, t)/\partial t \hat{i} + \partial v(s, t)/\partial t \hat{j}$, can be used to obtain the kinetic energy of the system as follows

$$\begin{aligned}
K &= \frac{1}{2} \int_0^l \rho (\dot{u}^2 + \dot{v}^2) ds \\
&\quad + \frac{1}{2} \int_0^l M \delta(s - s_o(t)) \left((\dot{u} + \dot{s}_o(x' + u'))^2 + (\dot{v} + \dot{s}_o(y' + v'))^2 \right) ds
\end{aligned} \tag{2.19}$$

where dot means partial differentiation with respect to time, t , and the prime means partial differentiation with respect to the space variable s , l is the cable length, ρ is the mass of the cable per unit length, M is the mass of the moving particle and $\delta(s)$ is the Dirac delta. If dS is the current arc length of the cable, the Lagrangian strain $\varepsilon(s, t)$ becomes

$$\varepsilon(s, t) = \frac{dS - ds}{ds} = \sqrt{(x' + u')^2 + (y' + v')^2} - 1 \tag{2.20}$$

The potential energy can be written as

$$\Pi = \int_0^l \left(T \varepsilon + \frac{1}{2} EA \varepsilon^2 - \rho g v - M g \delta(s - s_o(t)) v \right) ds \tag{2.21}$$

where T is the cable static tension and it is a function of s , E is the modulus of elasticity and A is the cross sectional area of the cable. Using equation (2.19) and

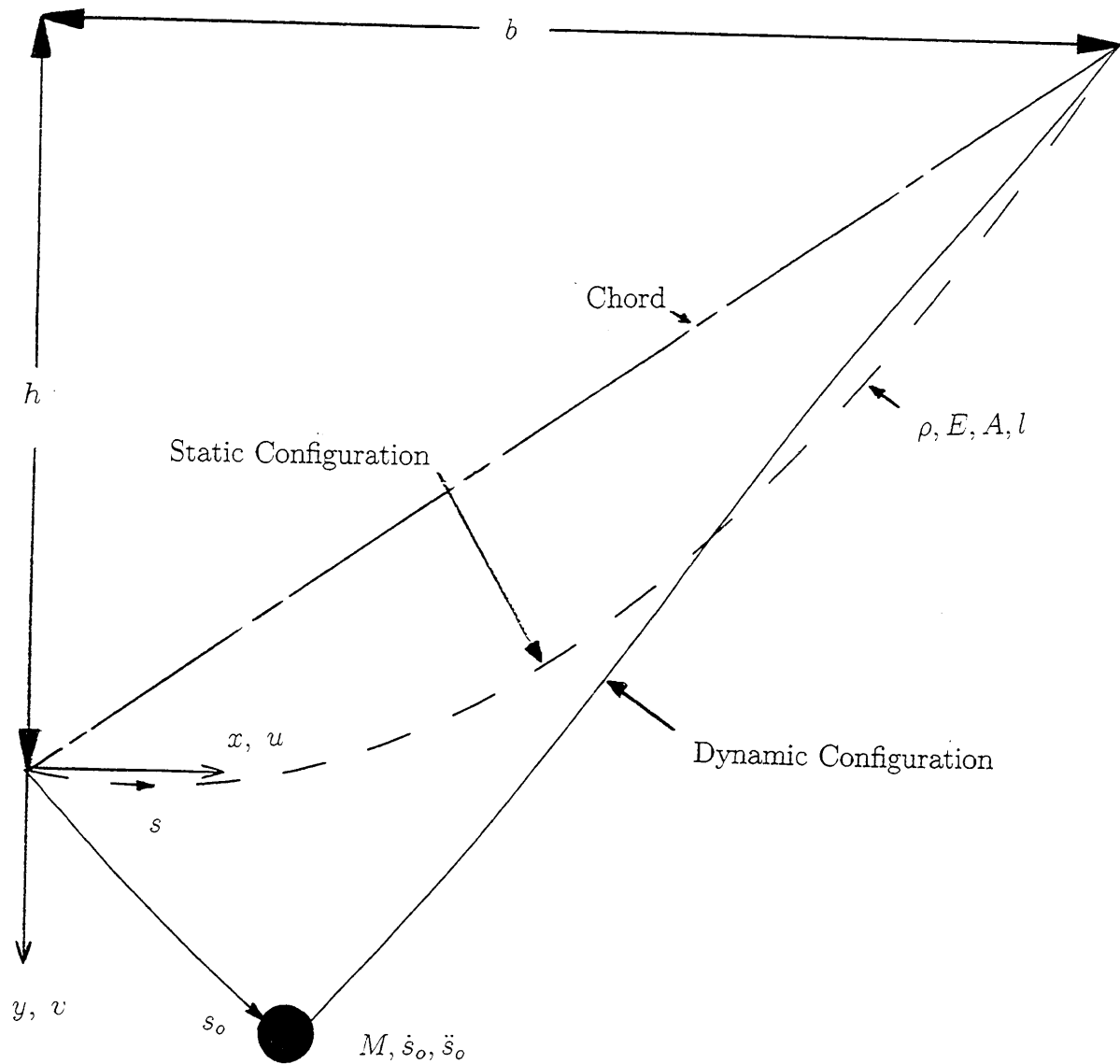


Figure 2.1: The Static and Dynamic Configurations

(2.21) in the Hamilton's principle

$$\delta \int_{t_1}^{t_2} (K - \Pi) dt = 0 \quad (2.22)$$

and performing the variation on u and v and integrating by parts lead to:

$$\begin{aligned} & \int_{t_1}^{t_2} \int_0^l \left\{ (\rho \ddot{u} + M \delta(s - s_o(t)) [\ddot{u} + 2 \dot{s}_o \dot{u}' + \dot{s}_o^2 (x'' + u'') + \ddot{s}_o (x' + u')]) \delta u \right. \\ & \quad + (\rho \ddot{v} + M \delta(s - s_o(t)) [\ddot{v} + 2 \dot{s}_o \dot{v}' + \dot{s}_o^2 (y'' + v'') + \ddot{s}_o (y' + v')]) \delta v \\ & \quad + \frac{T + EA \varepsilon}{\varepsilon + 1} [(x' + u') \delta u' + (y' + v') \delta v'] \\ & \quad \left. - [\rho + M \delta(s - s_o(t))] g \delta v \right\} ds dt = 0 \end{aligned} \quad (2.23)$$

Note that the boundary terms of integration have been set equal to zero, which are

$$\begin{aligned} & \int_0^l \{ [\rho \dot{u} + M \delta(s - s_o) (\dot{u} + \dot{s}_o (x' + u'))] \delta u \\ & \quad + [\rho \dot{v} + M \delta(s - s_o) (\dot{v} + \dot{s}_o (y' + v'))] \delta v \} \Big|_{t_1}^{t_2} ds = 0 \\ & \int_{t_1}^{t_2} M \delta(s - s_o) \{ [\dot{u} + \dot{s}_o (x' + u')] \delta u + [\dot{v} + \dot{s}_o (y' + v')] \delta v \} \Big|_0^l dt = 0 \end{aligned}$$

and the following relation has been used

$$\frac{\partial \delta(s - s_o)}{\partial t} + \dot{s}_o \frac{\partial \delta(s - s_o)}{\partial s} = 0$$

The boundary conditions of a fixed cable at its two ends are given by

$$\begin{aligned} u(0, t) &= v(0, t) = 0 \\ u(l, t) &= v(l, t) = 0 \end{aligned} \quad (2.24)$$

and the initial conditions are

$$\begin{aligned} u(s, 0) &= \frac{\partial u(s, 0)}{\partial t} = 0 \\ v(s, 0) &= \frac{\partial v(s, 0)}{\partial t} = 0 \end{aligned} \quad (2.25)$$

2.3 Boundary Value Problem

Before we propose a solution procedure for the problem described in the previous section, we will formulate the moving boundary value problem to gain some insights. Further integration by parts of the terms in equation (2.23) that contain $\delta u'$ and $\delta v'$, yields two governing equations and two interface conditions (or moving boundary conditions). For arbitrary δu and δv we get the following results after using equation (2.20)

Longitudinal component equation:

$$\left\{ \left[T + EA \left(\sqrt{(x' + u')^2 + (y' + v')^2} - 1 \right) \right] \frac{x' + u'}{\sqrt{(x' + u')^2 + (y' + v')^2}} \right\}' = \rho \ddot{u}$$

for $0 \leq s < s_o$ and $s_o < s \leq l$ (2.26)

Transverse component equation:

$$\left\{ \left[T + EA \left(\sqrt{(x' + u')^2 + (y' + v')^2} - 1 \right) \right] \frac{y' + v'}{\sqrt{(x' + u')^2 + (y' + v')^2}} \right\}' + \rho g = \rho \ddot{v}$$

for $0 \leq s < s_o$ and $s_o < s \leq l$ (2.27)

Equations (2.26) and (2.27) govern the motion of the cable over its span except

at the location of the mass. The limit terms of the integration by parts provide the necessary interface conditions at the mass location, $s = s_o$:

$$\begin{aligned} & \left(T + EA \left(\sqrt{(x' + u')^2 + (y' + v')^2} - 1 \right) \right) \frac{x' + u'}{\sqrt{(x' + u')^2 + (y' + v')^2}} \Big|_{s_o^-}^{s_o^+} \\ &= M \left(\ddot{u} + 2 \dot{s}_o \dot{u}' + \dot{s}_o^2 (x'' + u'') + \ddot{s}_o (x' + u') \right) \Big|_{s_o} \end{aligned} \quad (2.28)$$

$$\begin{aligned} & \left(T + EA \left(\sqrt{(x' + u')^2 + (y' + v')^2} - 1 \right) \right) \frac{y' + v'}{\sqrt{(x' + u')^2 + (y' + v')^2}} \Big|_{s_o^-}^{s_o^+} \\ &= M \left(\ddot{v} + 2 \dot{s}_o \dot{v}' + \dot{s}_o^2 (y'' + v'') + \ddot{s}_o (y' + v') \right) \Big|_{s_o} - M g \end{aligned} \quad (2.29)$$

Equations (2.28) and (2.29) show that the moving mass has three types of acceleration components, namely, the normal acceleration, the Coriolis acceleration and the centrifugal acceleration. To ensure that no separation of the cable occurs at the mass location two more conditions are needed. Those conditions are the continuity conditions which state that the cable displacement components are the same when approaching the mass from the left side or from the right side, hence

$$u(s_o^-, t) = u(s_o^+, t) \quad (2.30)$$

$$v(s_o^-, t) = v(s_o^+, t) \quad (2.31)$$

and the boundary conditions are as defined in equation (2.24). The nonlinear problem of a cable with a moving mass is now defined by the equations (2.26)-(2.31) and (2.24)-(2.25). This formulation is the most general for the vibration of fixed elastic cables.

Many limiting cases can be derived from it including the free vibration of bare cables if we put $M = 0$ and the free vibration of a cable with attached mass if we put $\ddot{s}_o = \dot{s}_o = 0$.

2.4 Static Configuration

The governing equations of the static shape of the bare cable can be found from equations (2.26) and (2.27) when the dynamic displacements vanish

$$\{T x'\}' = 0 \quad (2.32)$$

$$\{T y'\}' + \rho g = 0 \quad (2.33)$$

Notice that the following compatibility equation has been used

$$x'^2 + y'^2 = 1 \quad (2.34)$$

By referring to Figure 2.1, the boundary conditions for x and y are as follows

$$x(0) = 0, \quad x(l) = b \quad (2.35)$$

$$y(0) = 0, \quad y(l) = h \quad (2.36)$$

To get the catenary shape of the cable we first integrate equations (2.32) and (2.33) to get

$$T x' = T_o \quad (2.37)$$

$$T y' = -\rho g s + C \quad (2.38)$$

where C is a constant and T_o is the horizontal component of the cable static tension which is a constant. Squaring both sides of equation (2.37) and (2.38) then adding them together and using equation (2.34) we get

$$T(s) = \sqrt{T_o^2 + (C - \rho g s)^2} \quad (2.39)$$

Substituting equation (2.39) in equations (2.37) and (2.38) and integrating

$$\begin{aligned} x(s) &= \int_0^s \frac{T_o}{\sqrt{T_o^2 + (C - \rho g \zeta)^2}} d\zeta \\ &= \frac{l}{\bar{\sigma}} \left(\sinh^{-1} \bar{\alpha} - \sinh^{-1} \left(\bar{\alpha} - \bar{\sigma} \frac{s}{l} \right) \right) \end{aligned} \quad (2.40)$$

$$\begin{aligned} y(s) &= \int_0^s \frac{C - \rho g \zeta}{\sqrt{T_o^2 + (C - \rho g \zeta)^2}} d\zeta \\ &= \frac{l}{\bar{\sigma}} \left(\sqrt{1 + \bar{\alpha}^2} - \sqrt{1 + \left(\bar{\alpha} - \bar{\sigma} \frac{s}{l} \right)^2} \right) \end{aligned} \quad (2.41)$$

where the $\bar{\sigma} = \rho g l / T_o$ and $\bar{\alpha} = C / T_o$. Equation (2.40) satisfies the first equation of (2.35) and equation (2.41) satisfies the first equation of (2.36). The remaining unknown constants are $\bar{\sigma}$ and $\bar{\alpha}$ which can be found numerically after using the second equations of (2.35) and (2.36). For the horizontal cable, $h = 0$, and we get $\bar{\sigma} = 2\bar{\alpha}$ and a transcendental equation for $\bar{\alpha}$

$$\bar{\alpha} b/l = \sinh^{-1}(\bar{\alpha}) \quad (2.42)$$

Therefore, $\bar{\alpha}$ can be found if l and b are given.

For small sags the static configuration can be approximate by a parabolic shape. The uniform distribution of the cable weight is assumed to vary with x instead of

s. Small sags require high horizontal tension, T_o . Therefore if we expand equations (2.39)-(2.41) around $1/T_o$ and keep terms up to the first order we get

$$T = T_o \quad (2.43)$$

$$x = s \quad (2.44)$$

$$y(x) = \frac{C}{T_o} \left(x - \frac{\rho g}{2} x^2 \right) \quad (2.45)$$

For horizontal cables we get

$$y(x) = 4d \left(\frac{x}{b} - \left(\frac{x}{b} \right)^2 \right); \text{ and} \quad (2.46)$$

$$d = \frac{\rho g b^2}{8T_o} \quad (2.47)$$

where d is the sag of the cable. The length of the parabolic cable can be found from equation (2.34) when assuming that x is the independent variable. Thus,

$$\begin{aligned} l &= \int_0^b \sqrt{1 + (dy/dx)^2} dx \\ &= \int_0^b \sqrt{1 + \left(\frac{4d}{b} \right)^2 (1 - 2x/b)^2} dx \\ &= \frac{b^2}{8d} \left(\frac{4d}{b} \sqrt{1 + (4d/b)^2} + \sinh^{-1}(4d/b) \right) \end{aligned}$$

In parabolic approximation, if ρ and b are known only d , T_o or l needs to be specified to get the static shape. To compare the solutions of the static configurations consider a cable with a manufactured length $\bar{l} = 2000$ meters, $\beta = 1/5000$, $h = 0$ and $b = \bar{l}$. Substitute these values in equations (2.17) and then solve for α and σ numerically we get $\alpha = 0.0832279$ and $\sigma = 0.168833$. The new length can now be found from

equation (2.3)

$$\begin{aligned}
l &= \int_0^{\bar{l}} (1 + e) d\bar{s} \\
&= \bar{l} \int_0^1 (1 + \beta \bar{T}(\bar{s}^*)) d\bar{s}^* \\
&= \bar{l} \int_0^1 \left(\frac{1}{1 - \beta \sqrt{\sigma^{-2} + (\alpha - \bar{s}^*)^2}} \right) d\bar{s}^* \\
&= 1.00119\bar{l} \\
&= 2002.37 \text{ m}
\end{aligned}$$

Using $l = 1.00119\bar{l}$ and $b = \bar{l}$ in equation (2.42) we find $\bar{\alpha} = 0.0844416$ and $\bar{\sigma} = 2\bar{\alpha} = 0.1688832$. Since we have small sag here we can use the parabolic approximation. From equation (2.41) we set $d = y(l/2)$ for this example and use it in equation (2.46). Figure 2.2 shows the static shape obtained from the three formulations. It is clear from the figure that all the three solutions are in good agreement.

2.5 Linear Free Vibration of Elastic Bare Cables

The free vibration of an elastic cable can be obtained from the previous analysis when we let $M = 0$. Then equations (2.26) and (2.27) are the nonlinear governing equations of the free vibration. Neglecting the nonlinear terms in those two equations and substituting for the static equations (2.32) and (2.33), we obtain

$$\{T u' + (EA - T) x' (x' u' + y' v')\}' = \rho \ddot{u} \quad (2.48)$$

$$\{T v' + (EA - T) y' (x' u' + y' v')\}' = \rho \ddot{v} \quad (2.49)$$

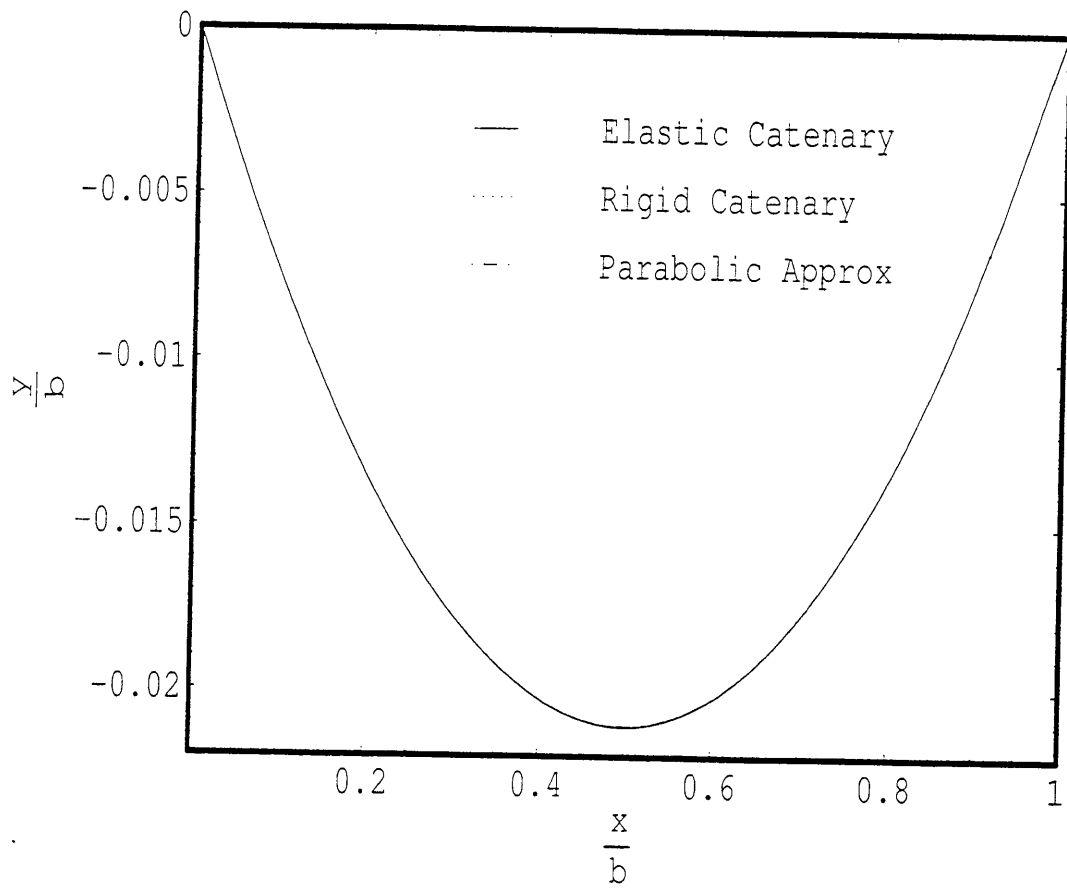


Figure 2.2: Static Configurations of a Cable Using Stretched, Unstretched and the Parabolic Formulations. $b = \bar{l} = 2000$, $\beta = \frac{1}{5000}$

Introducing the following dimensionless parameters

$$\begin{aligned}\bar{s} &= \frac{s}{l}, \bar{t} = t \left(\frac{T_o}{\rho l^2} \right)^{\frac{1}{2}}, \bar{x} = \frac{x}{l}, \bar{y} = \frac{y}{d}, \bar{u} = \frac{l}{d^2} u, \bar{v} = \frac{v}{d} \\ \bar{T} &= \frac{T}{T_o}, \epsilon = \frac{d}{l}, \lambda = \frac{T_o}{EA}, \hat{\alpha} = \frac{\bar{\alpha}}{\epsilon}, \hat{\sigma} = \frac{\bar{\sigma}}{\epsilon}\end{aligned}$$

where d is the sag of the cable. Notice that $\bar{y}$ and $\bar{v}$ are scaled with the sag d and the scaling parameter for $\bar{u}$ is found from the compatibility equation of an inextensible cable, that is $x'u' + y'v' = 0$. Using these parameters in equations (2.48) and (2.49)

$$\left\{ \bar{T} \bar{u}' + \left(\frac{1}{\lambda} - \bar{T} \right) \bar{x}' (\bar{x}' \bar{u}' + \bar{y}' \bar{v}') \right\}' = \ddot{\bar{u}} \quad (2.50)$$

$$\left\{ \bar{T} \bar{v}' + \epsilon^2 \left(\frac{1}{\lambda} - \bar{T} \right) \bar{y}' (\bar{x}' \bar{u}' + \bar{y}' \bar{v}') \right\}' = \ddot{\bar{v}} \quad (2.51)$$

where the prime denotes $\partial/\partial \bar{s}$ and the dot denotes $\partial/\partial \bar{t}$. It is convenient to choose d as $\bar{\sigma} l/8 = \rho g l^2/8T_o$, which represents the sag of a parabolic cable [20]. From equations (2.40) and (2.41) the first derivatives of x and y in their dimensionless form are

$$\bar{x}' = \frac{1}{\sqrt{1 + \epsilon^2 (\hat{\alpha} - \hat{\sigma} \bar{s})^2}} \quad (2.52)$$

$$\bar{y}' = \frac{\hat{\alpha} - \hat{\sigma} \bar{s}}{\sqrt{1 + \epsilon^2 (\hat{\alpha} - \hat{\sigma} \bar{s})^2}} \quad (2.53)$$

and the normalized tension is $\bar{T} = \sqrt{1 + \epsilon^2 (\hat{\alpha} - \hat{\sigma} \bar{s})^2}$ where for leveled supports we have $\hat{\sigma} = 2 \hat{\alpha} = 8$, and the static configuration depends on the choice of ϵ . If we now choose ϵ as an order of magnitude parameter, we see that $\bar{u}$ and $\bar{v}$ are of order of one. The value of λ , in most practical applications, is of second order of ϵ , meaning that $\lambda = k \epsilon^2$, where k is a constant. Keeping the terms of order one, equations (2.48) and

(2.49) reduce to the equations of motion derived for a parabolic cable

$$\{\bar{u}' + (\hat{\alpha} - \hat{\sigma} \bar{s}) \bar{v}'\}' = 0 \quad (2.54)$$

$$\left\{ \bar{v}' + \frac{1}{k} (\hat{\alpha} - \hat{\sigma} \bar{s}) (\bar{u}' + (\hat{\alpha} - \hat{\sigma} \bar{s}) \bar{v}') \right\}' = \ddot{\bar{v}} \quad (2.55)$$

Equation (2.54) has a very interesting explanation. From the one dimensional Hooke's law, the dynamic tension, say τ , of the cable is given by

$$\tau = EA\varepsilon \quad (2.56)$$

where E , A and ε are as defined before. Substituting equation (2.20) in equation (2.56) yields

$$\begin{aligned} \tau &= EA(\sqrt{(x' + u')^2 + (y' + v')^2} - 1) \\ &= EA(\sqrt{x'^2 + y'^2 + 2x'u' + 2y'v' + u'^2 + v'^2} - 1) \end{aligned}$$

Assuming that u' and v' are very small, we apply the binomial expansion on the square root term and retaining the linear terms only we get

$$\tau = EA(x'u' + y'v')$$

Putting the above equation in dimensionless form and using a parabolic approximation we get

$$\bar{\tau} = \frac{\tau}{\rho g l} = \frac{1}{\beta} (\bar{u}' + (\hat{\alpha} - \hat{\sigma} \bar{s}) \bar{v}') \quad (2.57)$$

Since β is constant, differentiating the above equation with respect to the spatial variable, s and comparing with equation (2.54), we see that equation (2.54) means that the dynamic tension is not function of the spatial variable, s , it is function of

time alone. Therefore,

$$\frac{d\bar{\tau}}{d\bar{s}} = 0 \quad (2.58)$$

To solve equations (2.54) and (2.55) we use the separable form $\bar{u} = U(\bar{s}) e^{i \bar{\omega} \bar{t}}$ and $\bar{v} = V(\bar{s}) e^{i \bar{\omega} \bar{t}}$ and then we substitute back to get

$$\{U' + (\hat{\alpha} - \hat{\sigma} \bar{s}) V'\}' = 0 \quad (2.59)$$

$$\left\{ V' + \frac{1}{k} (\hat{\alpha} - \hat{\sigma} \bar{s}) (U' + (\hat{\alpha} - \hat{\sigma} \bar{s}) V') \right\}' = -\bar{\omega}^2 V \quad (2.60)$$

Integrating equation (2.59)

$$U' + (\hat{\alpha} - \hat{\sigma} \bar{s}) V' = C_1 \quad (2.61)$$

where C_1 is a constant of integration. Substituting equation (2.61) in equation (2.60) we get

$$V'' - \frac{\hat{\sigma}}{k} C_1 = -\bar{\omega}^2 V$$

The solution of the above equation is

$$V = C_2 \cos \bar{\omega} \bar{s} + C_3 \sin \bar{\omega} \bar{s} + \frac{\hat{\sigma}}{k \bar{\omega}^2} C_1 \quad (2.62)$$

where C_2 and C_3 are constants of integrations. Substituting the above solution of V in equation (2.61) and performing a direct integration we obtain the solution of U

$$\begin{aligned} U = & \bar{s} C_1 - \left[(\hat{\alpha} - \hat{\sigma} \bar{s}) \cos \bar{\omega} \bar{s} + \frac{\hat{\sigma}}{\bar{\omega}} \sin \bar{\omega} \bar{s} \right] C_2 \\ & + \left[\frac{\hat{\sigma}}{\bar{\omega}} \cos \bar{\omega} \bar{s} - (\hat{\alpha} - \hat{\sigma} \bar{s}) \sin \bar{\omega} \bar{s} \right] C_3 + C_4 \end{aligned} \quad (2.63)$$

where C_4 is a constant of integration. Using equations (2.62) and (2.63) in the boundary conditions given in equation (2.24) we reach to the eigenvalue problem

$$\begin{bmatrix} 0 & -\hat{\alpha} & \hat{\sigma}/\bar{\omega} & 1 \\ \hat{\sigma}/k\bar{\omega}^2 & 1 & 0 & 0 \\ 1 & -(\hat{\alpha} - \hat{\sigma}) \cos \bar{\omega} - \frac{\hat{\sigma}}{\bar{\omega}} \sin \bar{\omega} & \frac{\hat{\sigma}}{\bar{\omega}} \cos \bar{\omega} - (\hat{\alpha} - \hat{\sigma}) \sin \bar{\omega} & 1 \\ \hat{\sigma}/k\bar{\omega}^2 & \cos \bar{\omega} & \sin \bar{\omega} & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.64)$$

For a nontrivial solution, the determinant in equation (2.64) must be equal to zero, i.e.

$$\sin \frac{\bar{\omega}}{2} \left[\sin \frac{\bar{\omega}}{2} - \frac{\bar{\omega}}{2} \left(1 - \frac{k\bar{\omega}^2}{\hat{\sigma}^2} \right) \cos \frac{\bar{\omega}}{2} \right] = 0 \quad (2.65)$$

Equation (2.65) is called a transcendental equation, from which we can obtain the natural frequencies of the bare cable. This equation has two sets of solutions, namely, if the first term is set equal to zero

$$\sin \frac{\bar{\omega}}{2} = 0 \quad (2.66)$$

the cable vibrates with antisymmetric modes. If the second term set equal to zero

$$\sin \frac{\bar{\omega}}{2} - \frac{\bar{\omega}}{2} \left(1 - \frac{k\bar{\omega}^2}{\hat{\sigma}^2} \right) \cos \frac{\bar{\omega}}{2} = 0 \quad (2.67)$$

the cable vibrates with symmetric modes.

Antisymmetric modes

From equation (2.66) the natural frequencies are

$$\begin{aligned} \bar{\omega} &= 2n\pi; n = 1, 2, \dots \\ \omega &= \bar{\omega} \sqrt{\frac{T_o}{\rho l^2}} \end{aligned} \quad (2.68)$$

Substituting equation (2.68) in equation (2.64) we have

$$\begin{aligned} C_1 &= C_2 = 0 \\ C_4 &= -\frac{\hat{\sigma}}{\bar{\omega}} C_3 \end{aligned}$$

Therefore, the mode shapes become

$$U(\bar{s}) = -C_3 \left\{ \frac{\hat{\sigma}}{\bar{\omega}} (1 - \cos \bar{\omega} \bar{s}) + (\hat{\alpha} - \hat{\sigma} \bar{s}) \sin \bar{\omega} \bar{s} \right\} \quad (2.69)$$

$$V(\bar{s}) = C_3 \sin(\bar{\omega} \bar{s}) \quad (2.70)$$

The antisymmetric modes produce a symmetric longitudinal displacement and antisymmetric transverse displacement. In the antisymmetric modes the dynamic tension is zero to the first order. From equation (2.57) the dynamic tension in its dimensionless form

$$\begin{aligned} \bar{\tau} &= \frac{1}{\beta} (\bar{u}' + (\hat{\alpha} - \hat{\sigma} \bar{s}) \bar{v}') \\ &= \frac{1}{\beta} (U' + (\hat{\alpha} - \hat{\sigma} \bar{s}) V') e^{i \bar{\omega} \bar{t}} \\ &= 0 \end{aligned}$$

In Figure 2.3, the maximum displacement of the longitudinal component occurs at the quarter span points and not at midspan. The displacement is a local minimum at midspan. In Figure 2.4, both the slope and displacement of the longitudinal component are zero at midspan. This pattern repeats itself for the higher longitudinal components.

Symmetric modes

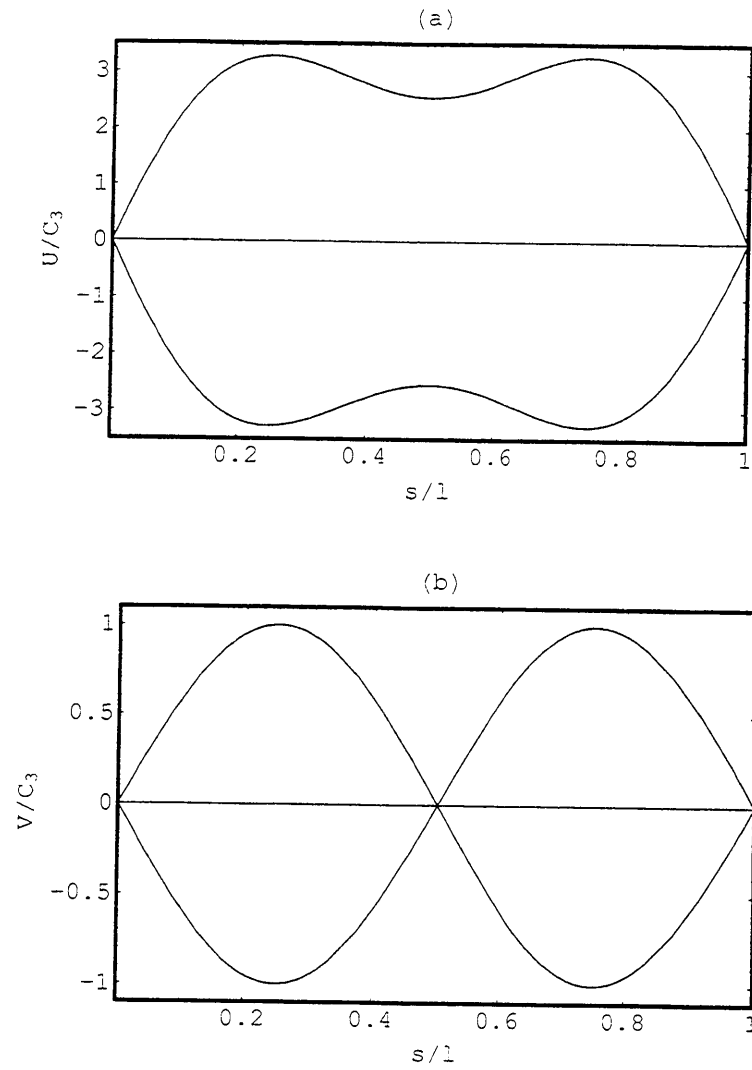


Figure 2.3: (a) Longitudinal Component and (b) Associated Vertical Component of the First Natural Frequency of the Antisymmetric Mode

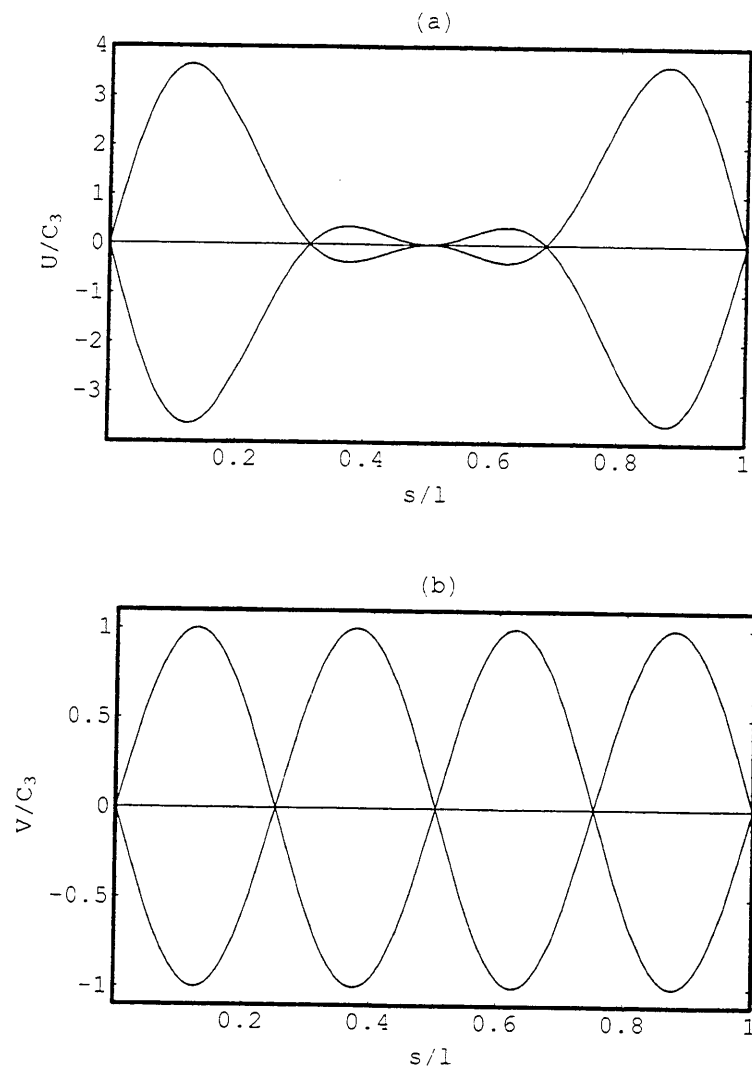


Figure 2.4: (a) Longitudinal Component and (b) Associated Vertical Component of the Second Natural Frequency of the Antisymmetric Mode

Symmetric modes correspond to antisymmetric longitudinal displacements and symmetric transverse displacements. The natural frequencies, $\bar{\omega}$, can be found from equation (2.67), that takes the following form after we divide by $\cos(\bar{\omega}/2)$

$$\tan\left(\frac{\bar{\omega}}{2}\right) = \frac{\bar{\omega}}{2} \left(1 - \frac{k}{16} \left(\frac{\bar{\omega}}{2}\right)^2\right) \quad (2.71)$$

Equation (2.71) shows that the symmetric modes depend on the parameter k , that constitute the cable geometry and elasticity, $k = \lambda/\epsilon^2$. A plot of equation (2.71) is shown in Figure 2.5 for different values of k . Substituting equation (2.71) in equation (2.64) we can obtain the mode shapes as follows

$$\begin{aligned} U(\bar{s}) = C_1 \left\{ \frac{k \bar{\omega}^2}{\hat{\sigma}} \bar{s} - (\hat{\alpha} - \hat{\sigma} \bar{s}) \left(1 - \tan \frac{\bar{\omega}}{2} \sin \bar{\omega} \bar{s} - \cos \bar{\omega} \bar{s}\right) \right. \\ \left. - \frac{\hat{\sigma}}{\bar{\omega}} \left(\bar{\omega} \bar{s} - \tan \frac{\bar{\omega}}{2} (1 - \cos \bar{\omega} \bar{s}) - \sin \bar{\omega} \bar{s}\right) \right\} \end{aligned} \quad (2.72)$$

$$V(\bar{s}) = C_1 \left\{ 1 - \tan \frac{\bar{\omega}}{2} \sin \bar{\omega} \bar{s} - \cos \bar{\omega} \bar{s} \right\} \quad (2.73)$$

Figure 2.5 and Figure 2.6 show that there are three cases. If $k > 16/\pi^2$, then the frequency of the first symmetric mode is less than the frequency of the first antisymmetric mode. The first symmetric, transverse modal component has no internal nodes along the span. If $k = 16/\pi^2$, then the frequency of the first symmetric mode is equal to the frequency of the first antisymmetric mode. This value of k gives the first cross-over point. The first symmetric, transverse modal component is tangential to the static cable profile at each support. If $k < 16/\pi^2$, then the frequency of the first symmetric mode is greater than the frequency of the first antisymmetric mode. The first symmetric, transverse modal component has two internal nodes along the span.

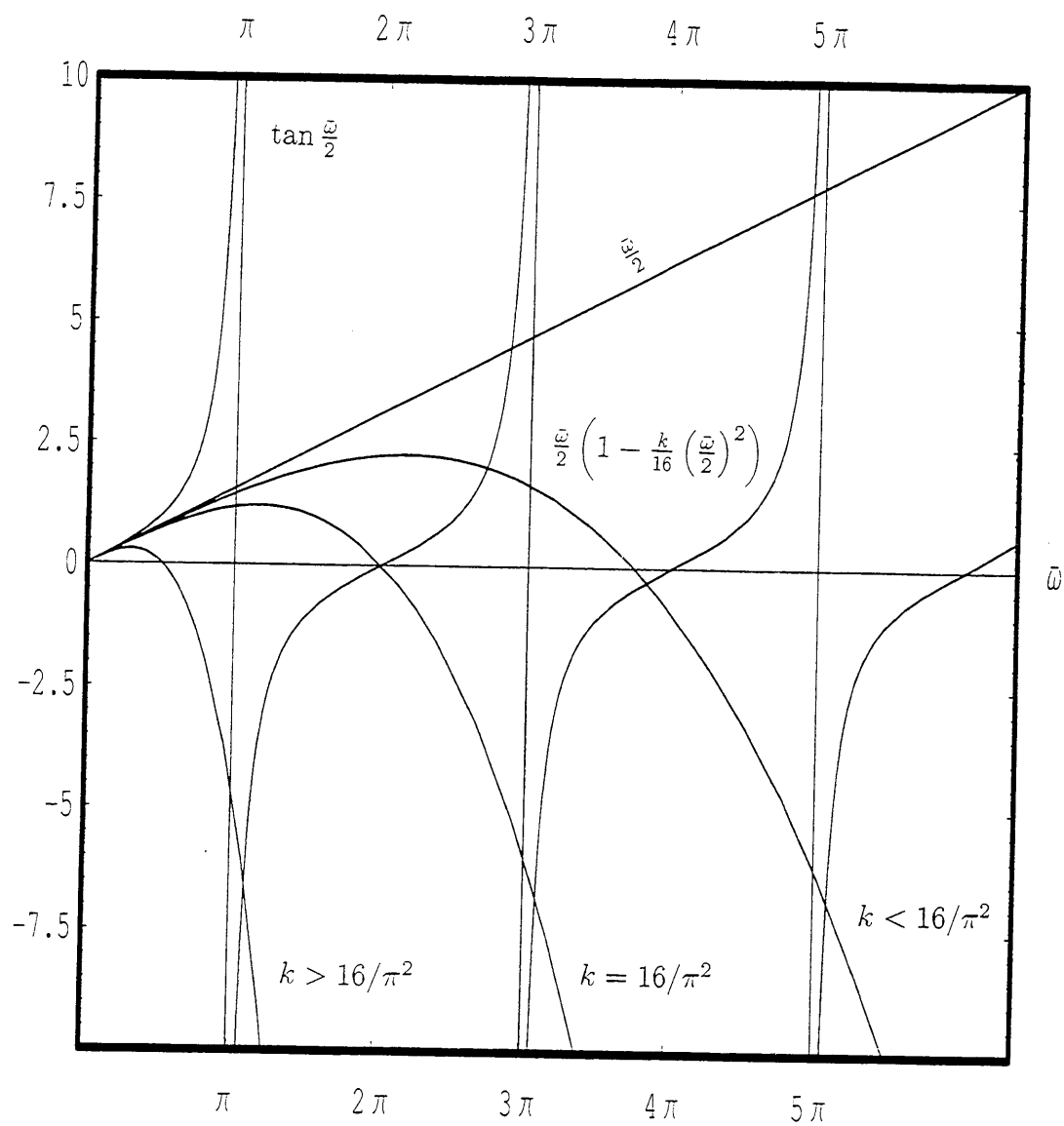


Figure 2.5: The Transcendental Equation of the Free Vibration of an Elastic Cable

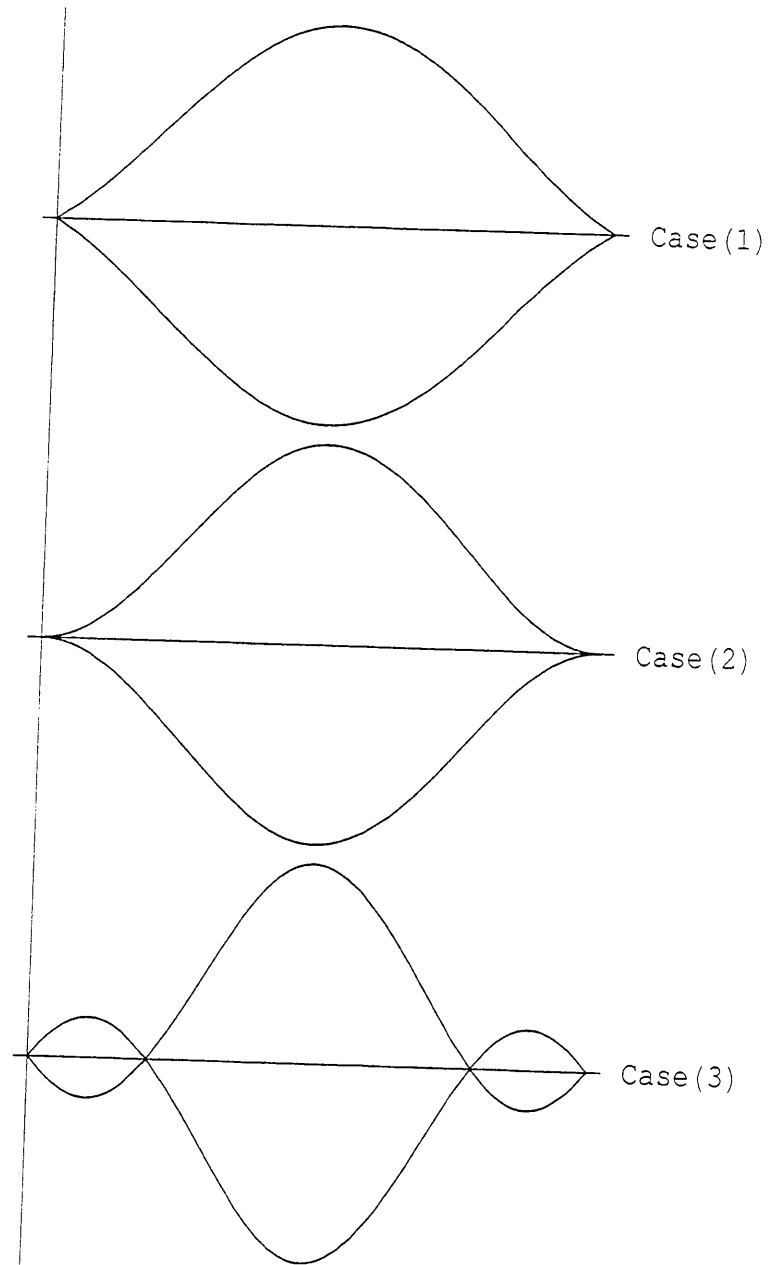


Figure 2.6: Symmetric Transverse Modes. Case (1): $k > 16/\pi^2$.
 Case (2): $k = 16/\pi^2$. Case (3): $k < 16/\pi^2$

2.6 Linear Free Vibration of an Elastic Cable with Mass Attached

2.6.1 Static Analysis. A cable with an attached mass can be considered as a cable that consists of two segments connected at the location of the mass and fixed at the other ends. Equations (2.32) and (2.33) apply to each segment individually, taking s_o as a constant, i.e. $\ddot{s}_o = \dot{s}_o = 0$. Therefore, the static problem can be stated as follows:

Equilibrium equations

$$(T x')' = 0 \quad 0 \leq s \leq s_o \quad \text{and} \quad s_o \leq s \leq l \quad (2.74)$$

$$(T y')' + \rho g = 0 \quad 0 \leq s \leq s_o \quad \text{and} \quad s_o \leq s \leq l \quad (2.75)$$

Boundary Conditions

$$x(0) = 0 \quad (2.76)$$

$$y(0) = 0$$

$$x(l) = b \quad (2.77)$$

$$y(l) = h$$

Interface Conditions at $s = s_o$

$$(T x')|_{s_o^-}^{s_o^+} = 0 \quad (2.78)$$

$$(T y')|_{s_o^-}^{s_o^+} = -M g \quad (2.79)$$

$$x(s_o^+) - x(s_o^-) = 0 \quad (2.80)$$

$$y(s_o^+) - y(s_o^-) = 0 \quad (2.81)$$

We will denote a subscript 1 for the cable segment on the left of the mass and a subscript 2 for the cable segment on the right of the mass. The solution of equations (2.74) and (2.75) that satisfy the conditions of equations (2.76)-(2.78) for each segment is given below

For $0 \leq s \leq s_o$

$$x_1(s) = \frac{l}{\bar{\sigma}} \left(\sinh^{-1}(\bar{\alpha}_1) - \sinh^{-1}\left(\bar{\alpha}_1 - \bar{\sigma} \frac{s}{l}\right) \right) \quad (2.82)$$

$$y_1(s) = \frac{l}{\bar{\sigma}} \left(\sqrt{1 + \bar{\alpha}_1^2} - \sqrt{1 + \left(\bar{\alpha}_1 - \bar{\sigma} \frac{s}{l}\right)^2} \right) \quad (2.83)$$

$$T_1(s) = T_o \sqrt{1 + \left(\bar{\alpha}_1 - \bar{\sigma} \frac{s}{l}\right)^2} \quad (2.84)$$

For $s_o \leq s \leq l$

$$x_2(s) = b - \frac{l}{\bar{\sigma}} \left(\sinh^{-1}(\bar{\alpha}_2) - \sinh^{-1}\left(\bar{\alpha}_2 - \bar{\sigma} \frac{l-s}{l}\right) \right) \quad (2.85)$$

$$y_2(s) = h + \frac{l}{\bar{\sigma}} \left(\sqrt{1 + \bar{\alpha}_2^2} - \sqrt{1 + \left(\bar{\alpha}_2 - \bar{\sigma} \frac{l-s}{l}\right)^2} \right) \quad (2.86)$$

$$T_2(s) = T_o \sqrt{1 + \left(\bar{\alpha}_2 - \bar{\sigma} \frac{l-s}{l}\right)^2} \quad (2.87)$$

where $\bar{\alpha}_1$, $\bar{\alpha}_2$ and $\bar{\sigma}$ are constants of integration that can be found from the conditions (2.79)-(2.81).

2.6.2 Linear Vibration Analysis. Using the same dimensionless parameters defined for the vibrations of a bare cable, and retaining first order terms only, the problem of the linear vibration of cable with an attached mass can be stated as

follows

Field equations

For $0 \leq \bar{s} \leq \bar{s}_o$

$$\{\bar{u}'_1 + (\hat{\alpha}_1 - \hat{\sigma} \bar{s}) \bar{v}'_1\}' = 0 \quad (2.88)$$

$$\left\{ \bar{v}'_1 + \frac{1}{k} (\hat{\alpha}_1 - \hat{\sigma} \bar{s}) (\bar{u}'_1 + (\hat{\alpha}_1 - \hat{\sigma} \bar{s}) \bar{v}'_1) \right\}' = \ddot{\bar{v}}_1 \quad (2.89)$$

For $\bar{s}_o \leq \bar{s} \leq 1$

$$\{\bar{u}'_2 - (\hat{\alpha}_2 - \hat{\sigma} (1 - \bar{s})) \bar{v}'_2\}' = 0 \quad (2.90)$$

$$\left\{ \bar{v}'_2 - \frac{1}{k} (\hat{\alpha}_2 - \hat{\sigma} (1 - \bar{s})) (\bar{u}'_2 - (\hat{\alpha}_2 - \hat{\sigma} (1 - \bar{s})) \bar{v}'_2) \right\}' = \ddot{\bar{v}}_2 \quad (2.91)$$

Interface Conditions $\bar{s} = \bar{s}_o$

$$\{\bar{u}'_2 - (\hat{\alpha}_2 - \hat{\sigma} (1 - \bar{s}_o)) \bar{v}'_2\} - \{\bar{u}'_1 + (\hat{\alpha}_1 - \hat{\sigma} \bar{s}_o) \bar{v}'_1\} = \lambda \mu \ddot{\bar{u}}_1 \quad (2.92)$$

$$\begin{aligned} & \left\{ \bar{v}'_2 - \frac{1}{k} (\hat{\alpha}_2 - \hat{\sigma} (1 - \bar{s}_o)) (\bar{u}'_2 + (\hat{\alpha}_2 - \hat{\sigma} (1 - \bar{s}_o)) \bar{v}'_2) \right\} \\ & - \left\{ \bar{v}'_1 + \frac{1}{k} (\hat{\alpha}_1 - \hat{\sigma} \bar{s}_o) (\bar{u}'_1 + (\hat{\alpha}_1 - \hat{\sigma} \bar{s}_o) \bar{v}'_1) \right\} = \mu \ddot{\bar{v}}_1 \end{aligned} \quad (2.93)$$

$$\bar{u}_1(\bar{s}_o) = \bar{u}_2(\bar{s}_o) \quad (2.94)$$

$$\bar{v}_1(\bar{s}_o) = \bar{v}_2(\bar{s}_o) \quad (2.95)$$

Boundary Conditions

$$\bar{u}_1(0, \bar{t}) = 0 \quad (2.96)$$

$$\bar{v}_1(0, \bar{t}) = 0 \quad (2.97)$$

$$\bar{u}_2(1, \bar{t}) = 0 \quad (2.98)$$

$$\bar{v}_2(1, \bar{t}) = 0 \quad (2.99)$$

where $\mu = M/\rho l$, $\bar{s}_o = s_o/l$, $\hat{\alpha}_1 = \bar{\alpha}_1/\epsilon$, $\hat{\alpha}_2 = \bar{\alpha}_2/\epsilon$, and $\hat{\sigma} = \bar{\sigma}/\epsilon$. Using $\bar{u}_1(\bar{s}) = \bar{U}_1(\bar{s})e^{i\bar{\omega}\bar{t}}$, $\bar{u}_2(\bar{s}) = \bar{U}_2(\bar{s})e^{i\bar{\omega}\bar{t}}$, $\bar{v}_1(\bar{s}) = \bar{V}_1(\bar{s})e^{i\bar{\omega}\bar{t}}$ and $\bar{v}_2(\bar{s}) = \bar{V}_2(\bar{s})e^{i\bar{\omega}\bar{t}}$, the solutions of equations (2.88)-(2.91) are

$$\begin{aligned} U_1(\bar{s}) = & \left[\bar{s} \left(1 - \frac{\hat{\sigma}^2}{k \bar{\omega}^2} \cos \bar{\omega} \bar{s} \right) \right. \\ & + \frac{\hat{\sigma}^2}{k \bar{\omega}^3} \left(1 + \frac{\hat{\alpha}_1}{\hat{\sigma}} \bar{\omega}^2 \left(\frac{\hat{\alpha}_1}{\hat{\sigma}} - \bar{s} \right) \right) \sin \bar{\omega} \bar{s} \left. \right] C_1 \\ & + \left[1 - \cos \bar{\omega} \bar{s} + \bar{\omega} \left(\frac{\hat{\alpha}_1}{\hat{\sigma}} - \bar{s} \right) \sin \bar{\omega} \bar{s} \right] D_1 \end{aligned}$$

$$V_1(\bar{s}) = \frac{\hat{\sigma}}{k \bar{\omega}^2} \left[1 - \cos \bar{\omega} \bar{s} - \frac{\hat{\alpha}_1}{\hat{\sigma}} \bar{\omega} \sin \bar{\omega} \bar{s} \right] C_1 - \frac{\bar{\omega}}{\hat{\sigma}} \sin \bar{\omega} \bar{s} D_1$$

$$\begin{aligned} U_2(\bar{s}) = & \left[-(1 - \bar{s}) \left(1 - \frac{\hat{\sigma}^2}{k \bar{\omega}^2} \cos(\bar{\omega}(1 - \bar{s})) \right) \right. \\ & - \frac{\hat{\sigma}^2}{k \bar{\omega}^3} \left(1 + \frac{\hat{\alpha}_2}{\hat{\sigma}} \bar{\omega}^2 \left(\frac{\hat{\alpha}_2}{\hat{\sigma}} - 1 + \bar{s} \right) \right) \sin(\bar{\omega}(1 - \bar{s})) \left. \right] C_2 \\ & + \left[1 - \cos(\bar{\omega}(1 - \bar{s})) + \bar{\omega} \left(\frac{\hat{\alpha}_2}{\hat{\sigma}} - 1 + \bar{s} \right) \sin(\bar{\omega}(1 - \bar{s})) \right] D_2 \end{aligned}$$

$$\begin{aligned} V_2(\bar{s}) = & \frac{\hat{\sigma}}{k \bar{\omega}^2} \left[1 - \cos(\bar{\omega}(1 - \bar{s})) - \frac{\hat{\alpha}_2}{\hat{\sigma}} \bar{\omega} \sin(\bar{\omega}(1 - \bar{s})) \right] C_2 \\ & + \frac{\bar{\omega}}{\hat{\sigma}} \sin(\bar{\omega}(1 - \bar{s})) D_2 \end{aligned}$$

which satisfy the boundary conditions (2.96)-(2.99). Substitution of the displacements in the interface conditions (2.92)-(2.95) leads to the transcendental equation of the system. For the special case of a centrally located mass the transcendental equation becomes

$$\sin\left(\frac{\bar{\omega}}{4}\right) \left(\bar{\omega}(-\hat{\sigma}^2 \lambda \mu + k(-4 + \lambda \mu \bar{\omega}^2)) \cos\left(\frac{\bar{\omega}}{4}\right) + 4\hat{\sigma}^2 \lambda \mu \sin\left(\frac{\bar{\omega}}{4}\right) \right)$$

$$\begin{aligned} & \left(-4\hat{\sigma}(-2\hat{\alpha}_1 + \hat{\sigma} + \hat{\sigma}\mu)\bar{\omega} + 2\bar{\omega}(-4\hat{\alpha}_1\hat{\sigma} + \hat{\sigma}^2 + 2\hat{\sigma}^2\mu + k\bar{\omega}^2) \cos\left(\frac{\bar{\omega}}{2}\right) \right. \\ & \left. + (4\hat{\sigma}^2 + 4\hat{\alpha}_1^2\bar{\omega}^2 - 4\hat{\alpha}_1\hat{\sigma}\bar{\omega}^2 + \hat{\sigma}^2\bar{\omega}^2 + \hat{\sigma}^2\bar{\omega}^2\mu - k\mu\bar{\omega}^4) \sin\left(\frac{\bar{\omega}}{2}\right) \right) = 0 \quad (2.100) \end{aligned}$$

where $\hat{\alpha}_2 = \hat{\alpha}_1$ in this case. Equation (2.100) shows that a centrally located mass has two types of antisymmetric modes

$$\sin \frac{\bar{\omega}}{4} = 0$$

$$\bar{\omega}(-\hat{\sigma}^2\lambda\mu + k(-4 + \lambda\mu\bar{\omega}^2)) \cos \frac{\bar{\omega}}{4} + 4\hat{\sigma}^2\lambda\mu \sin \frac{\bar{\omega}}{4} = 0$$

The first type is independent of any parameter whereas the second type is dependent on λ , k and μ . The symmetric mode is dependent on k and μ only. Figures 2.7, 2.8, and 2.9 show that all the frequencies of both symmetric and antisymmetric modes decrease as these parameters increase. The first frequencies of the symmetric and the antisymmetric modes decrease very rapidly for large values of μ and they are asymptotic to zero (see Figure 2.7). In Figure 2.8, the frequency of the second antisymmetric mode decreases very sharp at $\lambda = 0.01$ which is the case when $\lambda = \epsilon^2$ ($\epsilon = 0.1$ in this example). The parameter k has an important effect on the higher frequencies of the symmetric mode (see Figure 2.9).

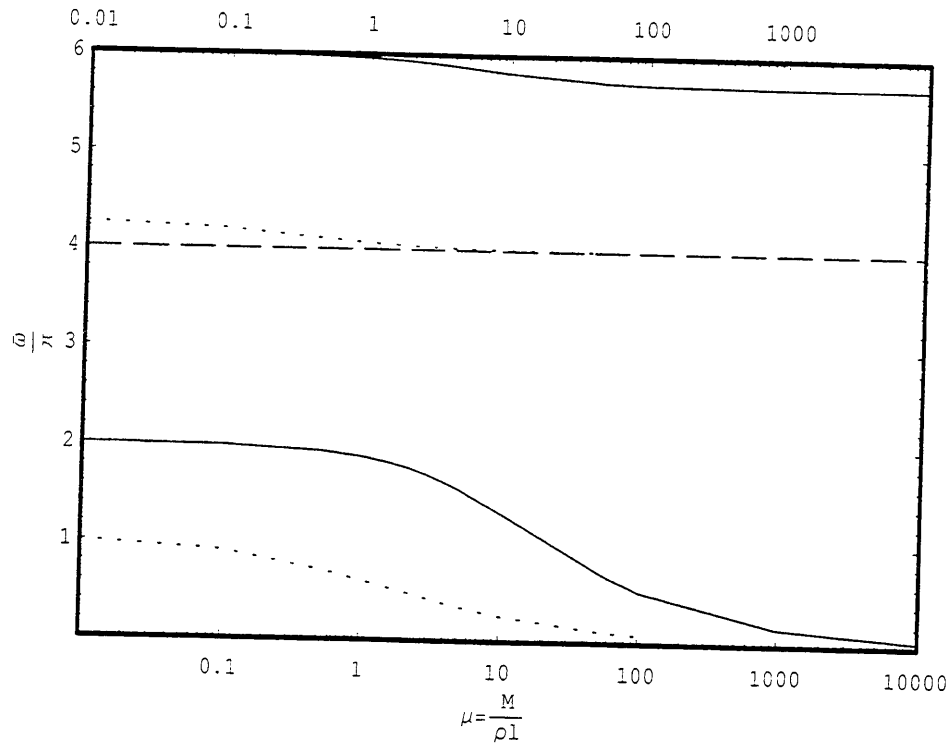


Figure 2.7: Symmetric (dotted) and Anti-symmetric (solid) Modes vs. a Centrally Located Mass. The Dashed Line is the First Type of the Antisymmetric Mode. For $k = 0.01$ and $\lambda = 0.0001$

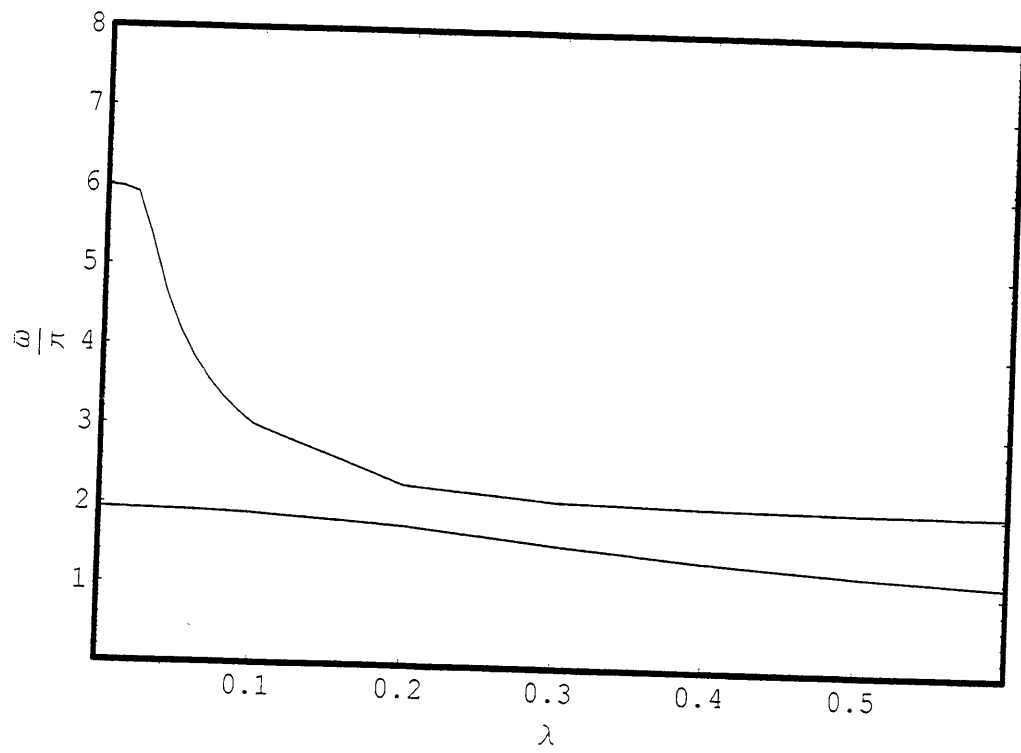


Figure 2.8: The First Two Anti-symmetric Modes vs. the Elastic Parameter, for $k = 0.01$ and $\mu = 0.5$

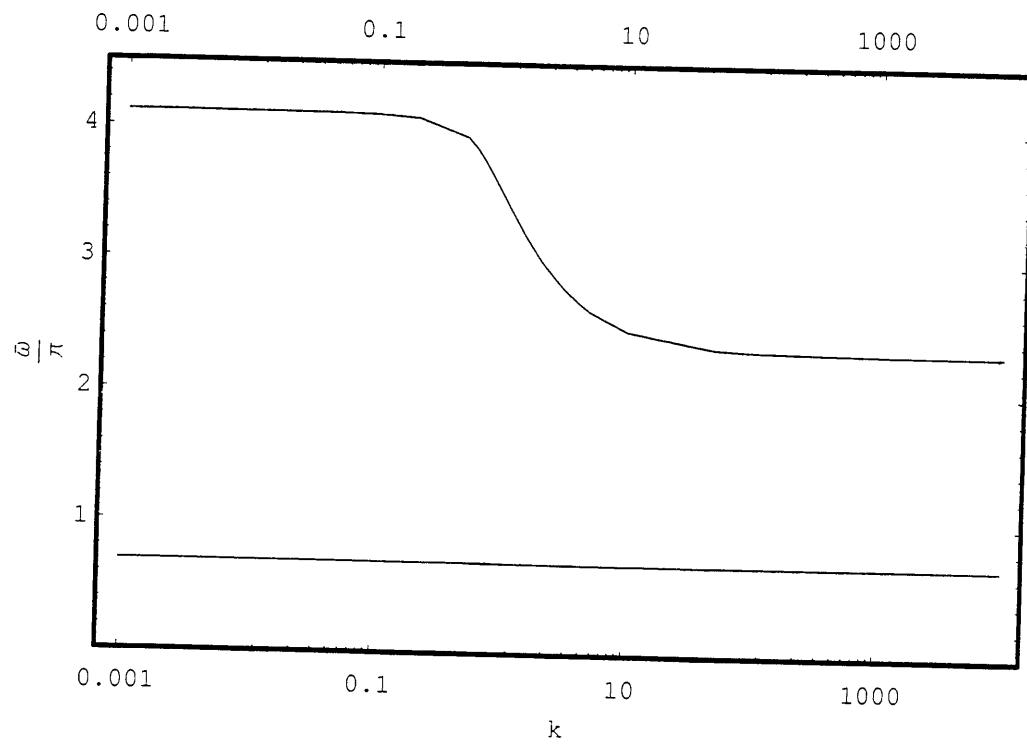


Figure 2.9: The First Two Symmetric Modes vs. the Parameter k , for $\mu=0.5$

CHAPTER III

SOLUTION USING THE GALERKIN PROCEDURE

Usually the difficulty in solving a moving mass problem involves treating the coupling terms[45] or the Coriolis acceleration terms which appear in the right hand side of equations (2.28) and (2.29). These terms have been neglected in most of the papers, that are mentioned in the literature review section, which deal with a moving mass on strings. In our problem we have an added difficulty which is the initial sag of the catenary cable. Therefore, the solution we adapt here for the vibrations of the cable due to a moving mass is the Galerkin procedure and iterative scheme for the numerical time integration.

3.1 Discretization

The Galerkin method is used to obtain approximate dynamic displacements of the cable. In the procedure of the Galerkin method, basis functions that satisfy the boundary conditions are used. Thus we assume series solutions for $u(s, t)$ and $v(s, t)$

$$u(s, t) = \sum_{m=1}^{\infty} U_m(t) \phi_m(s) \quad (3.1)$$

$$v(s, t) = \sum_{m=1}^{\infty} V_m(t) \phi_m(s) \quad (3.2)$$

where $\phi_m(s)$ is the basis function satisfying the boundary conditions given in equations (2.24). $U_m(t)$ and $V_m(t)$ are the time dependant functions for the longitudinal and transverse displacements, respectively. They are the only functions that admit variations, hence $\delta u = \phi_n \delta U_n$ and $\delta v = \phi_n \delta V_n$. Therefore, when substituting equations (3.1) and (3.2) in equation (2.23), sets of coupled ordinary differential equations can be obtained for arbitrary functions δU_n and δV_n . In the following equations, the

summation symbol has been omitted for brevity and the repeated indices indicate summation.

Longitudinal direction (for arbitrary δU_n)

$$\begin{aligned}
& \left(\rho \int_0^l \phi_n \phi_m ds + M \phi_n(s_o) \phi_m(s_o) \right) \ddot{U}_m(t) \\
& + 2M \dot{s}_o \phi_n(s_o) \phi'_m(s_o) \dot{U}_m(t) \\
& + M \phi_n(s_o) (\ddot{s}_o \phi'_m(s_o) + \dot{s}_o^2 \phi''_m(s_o)) U_m(t) \\
& + \int_0^l \frac{T + EA \varepsilon}{\varepsilon + 1} (x' + U_m \phi'_m) \phi'_n ds \\
& = -M \phi_n(s_o) (\ddot{s}_o x'(s_o) + \dot{s}_o^2 x''(s_o))
\end{aligned} \tag{3.3}$$

Transverse direction (for arbitrary δV_n)

$$\begin{aligned}
& \left(\rho \int_0^l \phi_n \phi_m ds + M \phi_n(s_o) \phi_m(s_o) \right) \ddot{V}_m(t) \\
& + 2M \dot{s}_o \phi_n(s_o) \phi'_m(s_o) \dot{V}_m(t) \\
& + M \phi_n(s_o) (\ddot{s}_o \phi'_m(s_o) + \dot{s}_o^2 \phi''_m(s_o)) V_m(t) \\
& + \int_0^l \left\{ \frac{T + EA \varepsilon}{\varepsilon + 1} (y' + V_m \phi'_m) \phi'_n - \rho g \phi_n \right\} ds \\
& = -M \phi_n(s_o) (\ddot{s}_o y'(s_o) + \dot{s}_o^2 y''(s_o) - g)
\end{aligned} \tag{3.4}$$

where $n, m = 1, 2, \dots, N$. N being the maximum number in the series. The right hand sides in the above equations represent the external forces due to the presence of the moving mass. Those forces are of two kinds, namely the weight of the moving mass which appears in equation (3.4) only and the forces due to the motion of the moving mass as centrifugal forces. For the sake of comparison, the force in equation (3.3) is smaller than the force in equation (3.4). Particularly, if the geometry of the cable is known such that the ratio of sag-to-span is small ($x'' = 0$, $x' = 1$ can be approximated) and if the mass is moving with constant speed, $\ddot{s}_o = 0$, the force in

(3.3) is zero.

In order to nondimensionalize the above equations we define the following dimensionless parameters

$$\begin{aligned} s^* &= \frac{s}{l}, \quad x^* = \frac{x}{l}, \quad y^* = \frac{y}{l}, \quad U^* = \frac{U}{l}, \quad V^* = \frac{V}{l}, \\ s_o^* &= \frac{s_o}{l}, \quad t^* = t \left(\frac{T_o}{\rho l^2} \right)^{\frac{1}{2}}, \quad \mu = \frac{M}{\rho l} \\ T^* &= \frac{T}{T_o}, \quad \lambda^* = \frac{EA}{T_o}, \quad \bar{\sigma} = \frac{\rho g l}{T_o} \end{aligned}$$

Introducing the above parameters in equations (3.3) and (3.4) to get in the longitudinal direction

$$\begin{aligned} & \left(\int_0^1 \phi_n \phi_m ds^* + \mu \phi_n(s_o^*) \phi_m(s_o^*) \right) \ddot{U}_m^*(t^*) \\ & + 2\mu \dot{s}_o^* \phi_n(s_o^*) \phi_m'(s_o^*) \dot{U}_m^*(t^*) \\ & + \mu \phi_n(s_o^*) \left(\ddot{s}_o^* \phi_m'(s_o^*) + \dot{s}_o^{*2} \phi_m''(s_o^*) \right) U_m^*(t^*) \\ & + \int_0^1 \frac{T^* + \lambda^* \varepsilon}{\varepsilon + 1} \left(x^{*'} + U_m^* \phi_m' \right) \phi_n' ds^* \\ & = -\mu \phi_n(s_o^*) \left(\ddot{s}_o^* x^{*'}(s_o^*) + \dot{s}_o^{*2} x^{*''}(s_o^*) \right) \end{aligned} \quad (3.5)$$

and in the transverse direction

$$\begin{aligned} & \left(\int_0^1 \phi_n \phi_m ds^* + \mu \phi_n(s_o^*) \phi_m(s_o^*) \right) \ddot{V}_m^*(t^*) \\ & + 2\mu \dot{s}_o^* \phi_n(s_o^*) \phi_m'(s_o^*) \dot{V}_m^*(t^*) \\ & + \mu \phi_n(s_o^*) \left(\ddot{s}_o^* \phi_m'(s_o^*) + \dot{s}_o^{*2} \phi_m''(s_o^*) \right) V_m^*(t^*) \\ & + \int_0^1 \left\{ \frac{T^* + \lambda^* \varepsilon}{\varepsilon + 1} \left(y^{*'} + V_m^* \phi_m' \right) \phi_n' - \bar{\sigma} \phi_n \right\} ds^* \\ & = -\mu \phi_n(s_o^*) \left(\ddot{s}_o^* y^{*'}(s_o^*) + \dot{s}_o^{*2} y^{*''}(s_o^*) - \bar{\sigma} \right) \end{aligned} \quad (3.6)$$

where the dot means differentiation with respect to t^* and prime means differentiation

with respect to s^* . The initial conditions, equations (2.25), transform to

$$\begin{aligned} U^*(0) &= \dot{U}^*(0) = 0 \\ V^*(0) &= \dot{V}^*(0) = 0 \end{aligned} \quad (3.7)$$

Equations (3.5), (3.6) and (3.7) define the discretized system of equations with the necessary initial conditions. In order to find the unknown functions $U^*(t^*)$ and $V^*(t^*)$, those equations have to be integrated over time starting at $t^* = 0$ to the moment the mass exit the cable at $s_o = 1$. Before making any attempt to do the time integration, it is more convenient to put equations (3.5), (3.6) in a matrix form as shown below

$$\mathbf{M} \ddot{\gamma} + \mathbf{C} \dot{\gamma} + \mathbf{K} \gamma + \mathbf{R}(\gamma) = \mathbf{F} \quad (3.8)$$

where

$$\gamma = \begin{Bmatrix} \mathbf{U} \\ \mathbf{V} \end{Bmatrix}; \quad \mathbf{U} = \begin{bmatrix} U_1^* & U_2^* & \cdots & U_N^* \end{bmatrix}^T \\ \mathbf{V} = \begin{bmatrix} V_1^* & V_2^* & \cdots & V_N^* \end{bmatrix}^T \quad (3.9)$$

$$\mathbf{M} = \begin{bmatrix} \bar{\mathbf{M}} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{M}} \end{bmatrix}; \quad \bar{M}_{nm} = \int_0^1 \phi_n \phi_m ds^* + \mu \phi_n(s_o^*) \phi_m(s_o^*) \quad (3.10)$$

$$\mathbf{C} = \begin{bmatrix} \bar{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{C}} \end{bmatrix}; \quad \bar{C}_{nm} = 2\mu \dot{s}_o^* \phi_n(s_o^*) \phi_m'(s_o^*) \quad (3.11)$$

$$\mathbf{K} = \begin{bmatrix} \bar{\mathbf{K}} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{K}} \end{bmatrix}; \quad \bar{K}_{nm} = \mu \phi_n(s_o^*) \left(\ddot{s}_o^* \phi_m'(s_o^*) + \dot{s}_o^{*2} \phi_m''(s_o^*) \right) \quad (3.12)$$

$$\mathbf{R}(\gamma) = \begin{Bmatrix} \mathbf{R}_x(\mathbf{U}, \mathbf{V}) \\ \mathbf{R}_y(\mathbf{U}, \mathbf{V}) \end{Bmatrix} \quad (3.13)$$

$$R_{x \ n} = \int_0^1 \frac{T^* + \lambda^* \varepsilon}{\varepsilon + 1} \left(x^{*'} + U_m^* \phi_m' \right) \phi_n' ds^* \quad (3.14)$$

$$R_{y \ n} = \int_0^1 \left\{ \frac{T^* + \lambda^* \varepsilon}{\varepsilon + 1} \left(y^{*'} + V_m^* \phi_m' \right) \phi_n' - \bar{\sigma} \phi_n \right\} ds^* \quad (3.15)$$

$$\mathbf{F} = \begin{Bmatrix} \mathbf{F}_x \\ \mathbf{F}_y \end{Bmatrix}; \quad \begin{aligned} F_{x \ n} &= -\mu \phi_n(s_o^*) \left(\ddot{s}_o^* x^{*'}(s_o^*) + \dot{s}_o^{*2} x^{*''}(s_o^*) \right) \\ F_{y \ n} &= -\mu \phi_n(s_o^*) \left(\ddot{s}_o^* y^{*'}(s_o^*) + \dot{s}_o^{*2} y^{*''}(s_o^*) - \bar{\sigma} \right) \end{aligned} \quad (3.16)$$

Equation (3.8) is the nonlinear model of the cable vibrations due to the moving mass represented in the standard matrix form. It is clear that the matrices (3.10)-(3.12) and the force vectors (3.16) are dependent on the mass location, s_o^* , consequently they are dependant on time, t^* . It means that those matrices and vectors have to be evaluated at each time step of the numerical integration. The size of the matrices in equation (3.8) is $2N \times 2N$ and the size of the submatrices in equations (3.10)-(3.12) is $N \times N$. The size of the vectors in equations (3.13)-(3.16) is $2N \times 1$ and the size of their subvectors is $N \times 1$, where N is as defined before.

3.2 Numerical Integration

Equations (3.8)-(3.16) and (3.7) will be integrated numerically using trapezoidal rule with the Newton-Raphson iterative method. Since trapezoidal rule is an implicit method we write equation (3.8) at $t^* = (n+1)\Delta t^*$

$$\mathbf{M}_{n+1} \ddot{\gamma}_{n+1} + \mathbf{C}_{n+1} \dot{\gamma}_{n+1} + \mathbf{K}_{n+1} \gamma_{n+1} + \mathbf{R}(\gamma_{n+1}) = \mathbf{F}_{n+1} \quad (3.17)$$

where n denotes $n\Delta t^*$ and Δt^* is the time step. The subscript on the matrices does not mean nonlinearities, it is to indicate that they are function of time, except $\mathbf{R}$ which is nonlinear because, it is a function of the displacements. In the trapezoidal

rule we have the following relations

$$\begin{aligned}\gamma_{n+1} &= \gamma_n + \frac{\Delta t^*}{2} (\dot{\gamma}_n + \dot{\gamma}_{n+1}) \\ \dot{\gamma}_{n+1} &= \dot{\gamma}_n + \frac{\Delta t^*}{2} (\ddot{\gamma}_n + \ddot{\gamma}_{n+1})\end{aligned}$$

which can be solved for $\dot{\gamma}_{n+1}$ and $\ddot{\gamma}_{n+1}$ as function of γ_{n+1} , hence

$$\dot{\gamma}_{n+1} = \frac{2}{\Delta t^*} (\gamma_{n+1} - \gamma_n) - \dot{\gamma}_n \quad (3.18)$$

$$\ddot{\gamma}_{n+1} = \frac{4}{\Delta t^{*2}} (\gamma_{n+1} - \gamma_n) - \frac{4}{\Delta t^*} \dot{\gamma}_n - \ddot{\gamma}_n \quad (3.19)$$

Substituting equations (3.18) and (3.19) in equation (3.17) and rearranging we get

$$\begin{aligned}\left[\frac{4}{\Delta t^{*2}} \mathbf{M}_{n+1} + \frac{2}{\Delta t^*} \mathbf{C}_{n+1} + \mathbf{K}_{n+1} \right] \gamma_{n+1} + \mathbf{R}(\gamma_{n+1}) &= \mathbf{F}_{n+1} \\ + \mathbf{M}_{n+1} \left\{ \frac{4}{\Delta t^{*2}} \gamma_n + \frac{4}{\Delta t^*} \dot{\gamma}_n + \ddot{\gamma}_n \right\} + \mathbf{C}_{n+1} \left\{ \frac{2}{\Delta t^*} \gamma_n + \dot{\gamma}_n \right\} &\end{aligned} \quad (3.20)$$

which is a system of nonlinear algebraic equations of γ_{n+1} . To render equation (3.20) compact we define the following matrices

$$\mathbf{E}_{n+1} = \frac{4}{\Delta t^{*2}} \mathbf{M}_{n+1} + \frac{2}{\Delta t^*} \mathbf{C}_{n+1} + \mathbf{K}_{n+1} \quad (3.21)$$

$$\mathbf{L}_{n+1} = \mathbf{F}_{n+1} + \mathbf{M}_{n+1} \left\{ \frac{4}{\Delta t^{*2}} \gamma_n + \frac{4}{\Delta t^*} \dot{\gamma}_n + \ddot{\gamma}_n \right\} + \mathbf{C}_{n+1} \left\{ \frac{2}{\Delta t^*} \gamma_n + \dot{\gamma}_n \right\} \quad (3.22)$$

and then we substitute the above matrices in equation (3.20)

$$\mathbf{E}_{n+1} \gamma_{n+1} + \mathbf{R}(\gamma_{n+1}) = \mathbf{L}_{n+1} \quad (3.23)$$

Notice, again, that the subscript in the matrices $\mathbf{E}$ and $\mathbf{L}$ does not mean nonlinearities. In other words, they are not function of γ_{n+1} , instead they are function of time. We

find the solution of equation (3.23) by applying the iterative technique of the Newton-Raphson. Therefore, we have

$$[\mathbf{E}_{n+1} + \mathbf{J}(\gamma_{n+1}^i)] \Delta \gamma_{n+1} = \mathbf{L}_{n+1} - \mathbf{E}_{n+1} \gamma_{n+1}^i - \mathbf{R}(\gamma_{n+1}^i) \quad (3.24)$$

$$\gamma_{n+1}^{i+1} = \gamma_{n+1}^i + \Delta \gamma_{n+1} \quad (3.25)$$

where

$$\mathbf{J}(\gamma_{n+1}^i) = \frac{\partial \mathbf{R}(\gamma_{n+1}^i)}{\partial \gamma_{n+1}^i} \quad (3.26)$$

$$= \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^T & \mathbf{J}_3 \end{bmatrix}; \text{ and} \quad (3.27)$$

$$\mathbf{J}_{1 \text{ } nm} = \int_0^1 \left\{ \frac{T^* + \lambda^* \varepsilon}{\varepsilon + 1} + \frac{\lambda^* - T^*}{(\varepsilon + 1)^3} (x^{*'} + U_k^* \phi_k')^2 \right\} \phi_n' \phi_m' ds^* \quad (3.28)$$

$$\mathbf{J}_{2 \text{ } nm} = \int_0^1 \frac{\lambda^* - T^*}{(\varepsilon + 1)^3} (x^{*'} + U_k^* \phi_k') (y^{*'} + V_k^* \phi_k') \phi_n' \phi_m' ds^* \quad (3.29)$$

$$\mathbf{J}_{3 \text{ } nm} = \int_0^1 \left\{ \frac{T^* + \lambda^* \varepsilon}{\varepsilon + 1} + \frac{\lambda^* - T^*}{(\varepsilon + 1)^3} (y^{*'} + V_k^* \phi_k')^2 \right\} \phi_n' \phi_m' ds^* \quad (3.30)$$

The criteria for convergence are

$$|\mathbf{L}_{n+1} - \mathbf{E}_{n+1} \gamma_{n+1}^i - \mathbf{R}(\gamma_{n+1}^i)| \leq TOL; \text{ and} \quad (3.31)$$

$$|\Delta \gamma_{n+1}| \leq TOL \quad (3.32)$$

where TOL is the allowable tolerance.

3.3 Simplified Linear Model

A simplified linear model can be deduced from the nonlinear model discussed in the previous section. The linear model will be valid for small mass ratio, μ , and

small mass speed. Although we might still use the catenary static shape, but we restrict the linear model for small cable sags. The displacements are considered small here. Consequently $U(t)$ and $V(t)$ are small quantities. Therefore, when expanding the nonlinear vector $\mathbf{R}(\gamma_{n+1}^i)$ the higher order terms will be neglected. Using binomial expansion in expanding the vectors given in equations (3.13)-(3.15) we get the following linear terms. From $\mathbf{R}_x$

$$\int_0^1 \left\{ \left((\lambda^* - T^*) (x^{*'})^2 + T^* \right) U_m^* + (\lambda^* - T^*) x^{*'} y^{*'} V_m^* \right\} \phi_m' \phi_n' ds^* \quad (3.33)$$

and from $\mathbf{R}_y$

$$\int_0^1 \left\{ (\lambda^* - T^*) x^{*'} y^{*'} U_m^* + \left((\lambda^* - T^*) (y^{*'})^2 + T^* \right) V_m^* \right\} \phi_m' \phi_n' ds^* \quad (3.34)$$

The static equations

$$\begin{aligned} \int_0^1 T^* x^{*'} \phi_n' ds^* &= 0 \\ \int_0^1 \left(T^* y^{*'} \phi_n' - \bar{\sigma} \phi_n \right) ds^* &= 0 \end{aligned}$$

have been substituted for in the expansions. The linear terms in equation (3.33) and (3.34), provide the linear stiffness matrix of the cable, which can be added to the matrix given in equation (3.12). Therefore, the linear model can be stated as follows

$$\mathbf{M} \ddot{\gamma} + \mathbf{C} \dot{\gamma} + \hat{\mathbf{K}} \gamma = \mathbf{F} \quad (3.35)$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{F}$ are as defined in equations (3.10), (3.11) and (3.16), respectively, and $\hat{\mathbf{K}}$ is

$$\hat{\mathbf{K}} = \begin{bmatrix} \bar{\mathbf{K}} + \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_2^T & \bar{\mathbf{K}} + \mathbf{K}_3 \end{bmatrix} \quad (3.36)$$

where

$$\mathbf{K}_{1nm} = \int_0^1 \left((\lambda^* - T^*) (x^{*'})^2 + T^* \right) \phi'_m \phi'_n ds^* \quad (3.37)$$

$$\mathbf{K}_{2nm} = \int_0^1 (\lambda^* - T^*) x^{*'} y^{*'} \phi'_m \phi'_n ds^* \quad (3.38)$$

$$\mathbf{K}_{3nm} = \int_0^1 \left((\lambda^* - T^*) (y^{*'})^2 + T^* \right) \phi'_m \phi'_n ds^* \quad (3.39)$$

and $\bar{\mathbf{K}}$ is defined in equation (3.12). To solve the equations of motions we use the trapezoidal rule as discussed in the previous section. First we write equation (3.35) at $t^* = (n+1) \Delta t^*$. Then we apply the relations given in equation (3.18) and (3.19) and rearranging to get a system of algebraic equations in the unknown γ_{n+1}

$$\hat{\mathbf{E}}_{n+1} \gamma_{n+1} = \mathbf{L}_{n+1} \quad (3.40)$$

where $\mathbf{L}_{n+1}$ is given in equation (3.22) and

$$\hat{\mathbf{E}}_{n+1} = \frac{4}{\Delta t^{*2}} \mathbf{M}_{n+1} + \frac{2}{\Delta t^*} \mathbf{C}_{n+1} + \hat{\mathbf{K}}_{n+1} \quad (3.41)$$

3.4 Fourier Representation

In this section, we use $\phi_n = \sin(n\pi s/l)$ as the basis function. Therefore, equation (3.1) and (3.2) become

$$u(s, t) = \sum_{m=1}^{\infty} U_m(t) \sin\left(\frac{m\pi s}{l}\right) \quad (3.42)$$

$$v(s, t) = \sum_{m=1}^{\infty} V_m(t) \sin\left(\frac{m\pi s}{l}\right) \quad (3.43)$$

The boundary conditions on the dynamic displacements u and v are satisfied by the above relations. We will use the normalized form of the above equations in the linear and nonlinear models.

3.4.1 Linear Model. The computational procedure of the Fourier linear model is shown in Table 3.1. The validity of the linear model requires that the sag-to-span ratio of the cable to be small. The ratio must obey the restriction given by Irvine and Caughey[20], that is the ratio should be 1 : 8 or less. For larger ratios the higher order terms will be as important as the catenary shape. Furthermore, the linear model is valid for small mass ratio and low mass speed.

For small sag-to-span ratios the parabolic cable approximation is reasonable, and in what follows, we derive the parabolic linear model to show that our formulation can lead to the known exact solution of the transverse vibration of strings. In dimensionless forms the equations (2.43)-(2.47), that describe a parabolic cable take the following form for a cable with same level supports.

$$\begin{aligned} x^{*'} &= 1 \\ y^{*'} &= 4d^*(1 - 2x^*) \\ T^* &= 1 \end{aligned}$$

where $d^* = d/l$. The mass location is also replaced by x_o^* . Therefore, the element matrices given in equations (3.10), (3.11), (3.16) and (3.36)-(3.39) will reduce to

$$\bar{M}_{nm} = \frac{1}{2}\delta_{nm} + \mu \sin(n\pi x_o^*) \sin(m\pi x_o^*) \quad (3.44)$$

$$\bar{C}_{nm} = 2\mu \dot{x}_o^* m\pi \sin(n\pi x_o^*) \cos(m\pi x_o^*) \quad (3.45)$$

$$F_{x\ n} = -\mu \ddot{x}_o^* \sin(n\pi x_o^*) \quad (3.46)$$

$$F_{y\ n} = -\mu \sin(n\pi x_o^*) \left\{ 4d^* \left(\ddot{x}_o^* (1 - 2x_o^*) - 2\dot{x}_o^{*2} \right) - \bar{\sigma} \right\} \quad (3.47)$$

$$\bar{K}_{nm} = \mu \sin(n\pi x_o^*) \left\{ \ddot{x}_o^* m\pi \cos(m\pi x_o^*) - \dot{x}_o^{*2} (m\pi)^2 \sin(m\pi x_o^*) \right\} \quad (3.48)$$

$$K_{1nm} = \frac{\lambda^*}{2} (n\pi)^2 \delta_{nm} \quad (3.49)$$

$$K_{2nm} = \begin{cases} 0 & ; n = m \\ 8d^* (\lambda^* - 1) nm \left(\frac{\sin^2(n+m)\frac{\pi}{2}}{(n+m)^2} + \frac{\sin^2(n-m)\frac{\pi}{2}}{(n-m)^2} \right) & ; n \neq m \end{cases} \quad (3.50)$$

$$K_{3nm} = \begin{cases} \frac{n^2\pi^2}{6} \left\{ 3 + (4d^*)^2 (\lambda^* - 1) \left(1 + \frac{6}{n^2\pi^2} \right) \right\} & ; n = m \\ 4nm (4d^*)^2 (\lambda^* - 1) \left\{ \frac{\cos^2(n+m)\frac{\pi}{2}}{(n+m)^2} + \frac{\cos^2(n-m)\frac{\pi}{2}}{(n-m)^2} \right\} & ; n \neq m \end{cases} \quad (3.51)$$

where δ_{nm} is the Kronecker delta. For the free vibration of a string we have $x_o^* = d^* = 0$ and the equations of motions decoupled as

$$\ddot{U}_m + \lambda^* (n\pi)^2 U_m = 0 \quad (3.52)$$

$$\ddot{V}_m + (n\pi)^2 V_m = 0 \quad (3.53)$$

Equation (3.52) is usually ignored, because the longitudinal displacement is very small. Equation (3.53) gives the solution of the transverse vibration of a string.

3.4.2 Nonlinear Model. The nonlinear solution is explained in the numerical integration section and in this section we will do the steps of the solution as shown in Table 3.2. Notice that we are not doing any approximation to the vector $\mathbf{R}(\gamma)$ nor to the jacobian matrix $\mathbf{J}$. For the static equilibrium we will use the catenary shape given by equations (2.40) and (2.41) after putting them into dimensionless form. The amount of stretching will be assumed known and obtained from the unstretched configuration.

Table 3.1: Computational Procedure of the Fourier Linear Model

1.	Form $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$. Calculated only once.
2.	At $t^* = 0(s_o^* = 0)$, we have $\mathbf{C} = \mathbf{F} = \bar{\mathbf{K}} = \mathbf{0}$, $\mathbf{M}$ is the identity matrix and $\hat{\mathbf{K}} \neq \mathbf{0}$.
3.	Using the initial conditions $\gamma(0) = \mathbf{0}$ and $\dot{\gamma}(0) = \mathbf{0}$, and Steps 1 and 2 in equation (3.35), we get $\ddot{\gamma}(0) = \mathbf{0}$.
4.	$n = 0$; $\gamma_0 = \dot{\gamma}_0 = \ddot{\gamma}_0 = \mathbf{0}$.
5.	Form $\mathbf{M}_{n+1}$, $\mathbf{C}_{n+1}$, $\hat{\mathbf{K}}_{n+1}$ and $\mathbf{F}_{n+1}$. Note that to form $\hat{\mathbf{K}}_{n+1}$, we first need to form $\bar{\mathbf{K}}_{n+1}$ and then use it with Step 1 in Equation (3.36).
6.	Form $\hat{\mathbf{E}}_{n+1}$ from equation (3.41).
7.	Form $\mathbf{L}_{n+1}$ from equation (3.22).
8.	Solve $\hat{\mathbf{E}}_{n+1} \gamma_{n+1} = \mathbf{L}_{n+1}$ for γ_{n+1} by forward- and back-substitution.
9.	Trapezoidal rule relations: Update velocity $\dot{\gamma}_{n+1}$ and acceleration $\ddot{\gamma}_{n+1}$ from equations (3.18) and (3.19).
10.	Output if desired; $n \leftarrow n + 1$, go to Step 5.

3.5 Wavelet Representation

The basis functions we adapt here are the anti-derivative wavelets that satisfy the boundary conditions of our problem. For description on wavelets and the anti-derivative wavelets see appendix A. The approximate solutions of the dynamic displacements are represented by

$$u_J(s, t) = \sum_{j=-1}^{J-1} \sum_{k \in D_j} U_{jk}(t) \Psi_{jk}(s) \quad (3.54)$$

$$v_J(s, t) = \sum_{j=-1}^{J-1} \sum_{k \in D_j} V_{jk}(t) \Psi_{jk}(s) \quad (3.55)$$

where $\Psi_{jk}(s)$ is the family of the anti-derivative wavelets given in equation (A.23), $j \geq 0$ and D_j is given in equation (A.24). The range of $\Psi_{jk}(s)$ in equation (A.23) is $0 \leq s \leq R$, but the range of the cable is $0 \leq s \leq l$ where l is the length of the cable. Therefore, we rewrite equation (A.23) in the form that is suitable for the cable

$$\begin{aligned} \Psi_{jk}(s) = & 2^{-j/2} \left[\chi_1 \left(2^j \frac{sR}{l} - k \right) - \chi_1(-k) \right. \\ & \left. - \frac{s}{l} (\chi_1(2^j R - k) - \chi_1(-k)) \right] \quad \text{for } 0 \leq s \leq l \end{aligned} \quad (3.56)$$

To make it convenient to work with wavelets, we need to change some of the dimensionless parameters that are given before. The ones that are not mentioned below are unchanged

$$\begin{aligned} s^* &= \frac{sR}{l}, \quad x^* = \frac{xR}{l}, \quad y^* = \frac{yR}{l}, \quad U_{jk}^* = \frac{U_{jk}R}{l}, \quad V_{jk}^* = \frac{V_{jk}R}{l}, \\ s_o^* &= \frac{s_o R}{l}, \quad \mu^* = \frac{MR}{\rho l}, \quad \bar{\sigma}^* = \frac{\rho g l}{T_o} R \end{aligned}$$

Carrying out the Galerkin procedure using the basis $\phi = \Psi_{jk}(s)$ and using the above dimensionless parameters we can derive the matrices of the system.

Table 3.2: Computational Procedure of the Fourier Nonlinear Model

1.	Initial Conditions: $\gamma(t^* = 0) = \mathbf{0}$ and $\dot{\gamma}(t^* = 0) = \mathbf{0}$ At $t^* = 0$ we have $s_o = 0$. Therefore $\mathbf{C} = \mathbf{F} = \bar{\mathbf{K}} = \mathbf{R} = \mathbf{0}$ and $\mathbf{M}$ is the identity matrix
2.	Using Step 1 in equation (3.8) we get $\ddot{\gamma}(t^* = 0) = \mathbf{0}$
3.	$n = 0$; $\gamma_0 = \dot{\gamma}_0 = \ddot{\gamma}_0 = \mathbf{0}$.
4.	Form the linear matrices $\mathbf{M}_{n+1}$, $\mathbf{C}_{n+1}$ and $\mathbf{K}_{n+1}$ using equations (3.44), (3.45) and (3.48).
5.	Form the force vector $\mathbf{F}_{n+1}$ using equation (3.16) and the dimensionless form of equations (2.40) and (2.41) for the static shape
6.	Form the matrices $\mathbf{E}_{n+1}$ and $\mathbf{L}_{n+1}$ using equations (3.21) and (3.22) respectively.
7.	Start the iteration using Newton Raphson method
8.	Set $i = 0$ and choose the initial points $\gamma_{n+1}^0 = \gamma_n$
9.	Form the vector $\mathbf{R}(\gamma_{n+1}^i)$ and the jacobian matrix $\mathbf{J}(\gamma_{n+1}^i)$ using equations (3.13)-(3.15) and (3.26)-(3.30) respectively.
10.	Solve equation (3.24) for $\Delta\gamma_{n+1}$ by forward- and back-substitution.
11.	Find γ_{n+1}^{i+1} from equation (3.25).
12.	Check for convergence: If $ \mathbf{L}_{n+1} - \mathbf{E}_{n+1}\gamma_{n+1}^i - \mathbf{R}(\gamma_{n+1}^i) \leq TOL$ and $ \Delta\gamma_{n+1} \leq TOL$ then go to next step else go to Step 16
13.	Set $\gamma_{n+1} = \gamma_{n+1}^{i+1}$
14.	Trapezoidal rule relations: Update velocity $\dot{\gamma}_{n+1}$ and acceleration $\ddot{\gamma}_{n+1}$ from equations (3.18) and (3.19) respectively.
15.	Output if desired; $n \leftarrow n + 1$, go to Step 4.
16.	$i \leftarrow i + 1$, go to Step 9.

3.5.1 Linear Model. For the linear model, equation (3.35), the matrices are given by

$$\bar{M}_{\varsigma\eta} = \int_0^R \Psi_{\varsigma} \Psi_{\eta} ds^* + \mu^* \Psi_{\varsigma}(s_o^*) \Psi_{\eta}(s_o^*) \quad (3.57)$$

$$\bar{C}_{\varsigma\eta} = 2\mu^* \dot{s}_o^* \Psi_{\varsigma}(s_o^*) \Psi'_{\eta}(s_o^*) \quad (3.58)$$

$$\bar{K}_{\varsigma\eta} = \mu^* \Psi_{\varsigma}(s_o^*) \left[\ddot{s}_o^* \Psi'_{\eta}(s_o^*) + \dot{s}_o^{*2} \Psi''_{\eta}(s_o^*) \right] \quad (3.59)$$

$$K_{1\varsigma\eta} = R^2 \int_0^R \left((\lambda^* - T^*) (x^{*'})^2 + T^* \right) \Psi'_{\varsigma}(s^*) \Psi'_{\eta}(s^*) ds^* \quad (3.60)$$

$$K_{2\varsigma\eta} = R^2 \int_0^R (\lambda^* - T^*) x^{*'} y^{*'} \Psi'_{\varsigma}(s^*) \Psi'_{\eta}(s^*) ds^* \quad (3.61)$$

$$K_{3\varsigma\eta} = R^2 \int_0^R \left((\lambda^* - T^*) (y^{*'})^2 + T^* \right) \Psi'_{\varsigma}(s^*) \Psi'_{\eta}(s^*) ds^* \quad (3.62)$$

$$F_{1\varsigma} = -\mu^* \Psi_{\varsigma}(s_o^*) \left(\ddot{s}_o^* x^{*'}(s_o^*) + \dot{s}_o^{*2} x^{*''}(s_o^*) \right) \quad (3.63)$$

$$F_{2\varsigma} = -\mu^* \Psi_{\varsigma}(s_o^*) \left[\ddot{s}_o^* y^{*'}(s_o^*) + \dot{s}_o^{*2} y^{*''}(s_o^*) - \bar{\sigma}^* \right] \quad (3.64)$$

where ς and η have been used to denote the double indices $\varsigma = (i, j)$ and $\eta = (m, n)$

For the computational procedure we first choose a working level J , then we can use Table 3.1, taking into consideration that the submatrices and subvectors are given in equations (3.57)-(3.64). Since the Daubechies wavelets are known only at dyadic points, the time step should be chosen so that s_o^* becomes a dyadic point.

3.5.2 Nonlinear Model. Using the anti-derivative wavelet given in equation (3.56) as a basis function and the dimensionless parameters given above, the matrices and vectors of the nonlinear system, equation (3.8), become

$$\bar{M}_{\varsigma\eta} = \int_0^R \Psi_{\varsigma} \Psi_{\eta} ds^* + \mu^* \Psi_{\varsigma}(s_o^*) \Psi_{\eta}(s_o^*) \quad (3.65)$$

$$\bar{C}_{\varsigma\eta} = 2\mu^* \dot{s}_o^* \Psi_{\varsigma}(s_o^*) \Psi'_{\eta}(s_o^*) \quad (3.66)$$

$$\bar{K}_{\varsigma\eta} = \mu^* \Psi_{\varsigma}(s_o^*) \left[\ddot{s}_o^* \Psi'_{\eta}(s_o^*) + \dot{s}_o^{*2} \Psi''_{\eta}(s_o^*) \right] \quad (3.67)$$

$$R_{x\varsigma} = R^2 \int_0^R \frac{T^* + \lambda\varepsilon}{\varepsilon + 1} \left(x^{*'} + U_{mn}^* \Psi'_{mn}(s^*) \right) \Psi'_{\varsigma}(s^*) ds^* \quad (3.68)$$

$$R_{y\varsigma} = \int_0^R \left\{ R^2 \frac{T^* + \lambda\varepsilon}{\varepsilon + 1} \left(y^{*'} + V_{mn}^* \Psi'_{mn}(s^*) \right) \Psi'_{\varsigma}(s^*) - \bar{\sigma}^* \Psi_{\varsigma}(s^*) \right\} ds^* \quad (3.69)$$

$$F_{x\varsigma} = -\mu^* \Psi_{\varsigma}(s_o^*) \left[\ddot{s}_o^* x^{*'}(s_o^*) + \dot{s}_o^{*2} x^{*''}(s_o^*) \right] \quad (3.70)$$

$$F_{y\varsigma} = -\mu^* \Psi_{\varsigma}(s_o^*) \left[\ddot{s}_o^* y^{*'}(s_o^*) + \dot{s}_o^{*2} y^{*''}(s_o^*) - \bar{\sigma}^* \right] \quad (3.71)$$

Although the integration scheme of the trapezoidal rule is unconditionally stable, it requires small time steps for better accuracy. We found that is the case when using the trapezoidal rule for the nonlinear model of the wavelets representation. Since we are using Daubechies wavelets, the time step has to be one of the dyadic points($k/2^j$) and a smaller time step means a larger computer memory. Therefore, we are going to use the Newmark method that has artificial damping to overcome the numerical noise resulting from the integration. The Newmark relations are given by

$$\gamma_{n+1} = \gamma_n + \Delta t \dot{\gamma}_n + \Delta t^2 \left[\left(\frac{1}{2} - \nu \right) \ddot{\gamma}_n + \nu \ddot{\gamma}_{n+1} \right] \quad (3.72)$$

$$\dot{\gamma}_{n+1} = \dot{\gamma}_n + \Delta t \left[(1 - \xi) \ddot{\gamma}_n + \xi \ddot{\gamma}_{n+1} \right] \quad (3.73)$$

where ν and ξ are the control parameters of stability and accuracy[8]. For implicit Newmark method we have the following conditions

$$\xi \geq 0.5, \text{ and } \nu \geq 0.25(0.5 + \xi)^2$$

We notice that the trapezoidal rule method is a special case of the Newmark method if we select $\nu = 1/4$ and $\xi = 1/2$. Defining the following constants

$$b_0 = \frac{1}{\nu \Delta t^2}; b_1 = \frac{\xi}{\nu \Delta t}; b_2 = \frac{1}{\nu \Delta t}; b_3 = \frac{1}{2\nu} - 1;$$

$$b_4 = \frac{\xi}{\nu} - 1; b_5 = \frac{\Delta t}{2} \left(\frac{\xi}{\nu} - 2 \right); b_6 = \Delta t(1 - \xi); b_7 = \xi \Delta t$$

the velocity and acceleration can be written as function of γ_{n+1} as follows

$$\ddot{\gamma}_{n+1} = b_0 [\gamma_{n+1} - \gamma_n] - b_2 \dot{\gamma}_n - b_3 \ddot{\gamma}_n \quad (3.74)$$

$$\dot{\gamma}_{n+1} = \dot{\gamma}_n + b_6 \ddot{\gamma}_n + b_7 \ddot{\gamma}_{n+1} \quad (3.75)$$

Applying the relations in equations (3.74) and (3.75) to equation (3.8) we obtain a system of equations similar to the one given in equation (3.23), but the matrices are redefined as given below

$$\mathbf{E}_{n+1} = b_0 \mathbf{M}_{n+1} + b_1 \mathbf{C}_{n+1} + \mathbf{K}_{n+1} \quad (3.76)$$

$$\begin{aligned} \mathbf{L}_{n+1} = & \mathbf{F}_{n+1} + \mathbf{M}_{n+1} \{b_0 \gamma_n + b_2 \dot{\gamma}_n + b_3 \ddot{\gamma}_n\} \\ & + \mathbf{C}_{n+1} \{b_1 \gamma_n + b_4 \dot{\gamma}_n + b_5 \ddot{\gamma}_n\} \end{aligned} \quad (3.77)$$

and when applying the iterative technique of the Newton-Raphson method yields similar equations as given in equations (3.24) and (3.25), but the submatrices of the jacobian matrix are given by

$$J_{1 \varsigma \eta} = R^2 \int_0^R \left\{ \frac{T^* + \lambda^* \varepsilon}{\varepsilon + 1} + \frac{\lambda^* - T^*}{(\varepsilon + 1)^3} \left(x^{*'} + U_{\varkappa}^* \Psi_{\varkappa}' \right)^2 \right\} \Psi_{\varsigma}' \Psi_{\eta}' ds^* \quad (3.78)$$

$$J_{2 \varsigma \eta} = R^2 \int_0^R \frac{\lambda^* - T^*}{(\varepsilon + 1)^3} \left(x^{*'} + U_{\varkappa}^* \Psi_{\varkappa}' \right) \left(y^{*'} + V_{\varkappa}^* \Psi_{\varkappa}' \right) \Psi_{\varsigma}' \Psi_{\eta}' ds^* \quad (3.79)$$

$$J_{3 \varsigma \eta} = R^2 \int_0^R \left\{ \frac{T^* + \lambda^* \varepsilon}{\varepsilon + 1} + \frac{\lambda^* - T^*}{(\varepsilon + 1)^3} \left(y^{*'} + V_{\varkappa}^* \Psi_{\varkappa}' \right)^2 \right\} \Psi_{\varsigma}' \Psi_{\eta}' ds^* \quad (3.80)$$

The computational procedure are given in Table 3.3.

Table 3.3: Computational Procedure of the Wavelet Nonlinear Model

1.	Select $\xi \geq 0.5$ and $\nu \geq 0.25(0.5 + \xi)$ and obtain $b_0, b_1, \dots, b_7$
2.	Initial Conditions: $\gamma(t^* = 0) = \mathbf{0}$ and $\dot{\gamma}(t^* = 0) = \mathbf{0}$ At $t^* = 0$ we have $s_o = 0$. Therefore $\mathbf{C} = \mathbf{F} = \bar{\mathbf{K}} = \mathbf{R} = \mathbf{0}$ and $\mathbf{M} \neq \mathbf{0}$.
3.	Using Step 2 in equation (3.8) we get $\ddot{\gamma}(t^* = 0) = \mathbf{0}$
4.	$n = 0$; $\gamma_0 = \dot{\gamma}_0 = \ddot{\gamma}_0 = \mathbf{0}$.
5.	Form the linear matrices $\mathbf{M}_{n+1}$, $\mathbf{C}_{n+1}$ and $\mathbf{K}_{n+1}$ using equations (3.65), (3.66) and (3.67).
6.	Form the force vector $\mathbf{F}_{n+1}$ using equation (3.70) and (3.71) and the dimensionless form of equations (2.40) and (2.41) for the static shape
7.	Form the matrices $\mathbf{E}_{n+1}$ and $\mathbf{L}_{n+1}$ using equations (3.76) and (3.77) respectively.
8.	Start the iteration using Newmark method
9.	Set $i = 0$ and choose the initial points $\gamma_{n+1}^0 = \gamma_n$
10.	Form the vector $\mathbf{R}(\gamma_{n+1}^i)$ and the jacobian matrix $\mathbf{J}(\gamma_{n+1}^i)$ using equations (3.68)-(3.69) and (3.78)-(3.80) respectively.
11.	Solve equation (3.24) for $\Delta\gamma_{n+1}$ by forward- and back-substitution.
12.	Find γ_{n+1}^{i+1} from equation (3.25).
13.	Check for convergence: If $ \mathbf{L}_{n+1} - \mathbf{E}_{n+1}\gamma_{n+1}^i - \mathbf{R}(\gamma_{n+1}^i) \leq TOL$ and $ \Delta\gamma_{n+1} \leq TOL$ then go to next step else go to Step 17
14.	Set $\gamma_{n+1} = \gamma_{n+1}^{i+1}$
15.	Newmark relations: Update velocity $\dot{\gamma}_{n+1}$ and acceleration $\ddot{\gamma}_{n+1}$ from equations (3.75) and (3.74) respectively.
16.	Output if desired; $n \leftarrow n + 1$, go to Step 5.
17.	$i \leftarrow i + 1$, go to Step 10.

CHAPTER IV

RESULTS AND DISCUSSIONS

4.1 Vibration of Bare Cable

The Galerkin solution of the linear free vibrations of elastic cables can be deduced from the element matrices of equations (3.44)-(3.51) using Fourier representation and equations (3.57)-(3.64) using wavelets representation. For bare cables we have $\mu = 0$ and the system of equations will reduce to an eigenvalue problem

$$\mathbf{M}\ddot{\gamma} + \hat{\mathbf{K}}\gamma = \mathbf{0} \quad (4.1)$$

To find the natural frequencies, we assume harmonic motion to be

$$\gamma = \mathbf{a} \sin(\bar{\omega} t^*) \quad (4.2)$$

where $\mathbf{a}$ denotes the eigenvector (mode shape) and $\bar{\omega}$ represents the normalized natural frequency of vibration. We substitute equation (4.2) in equation (4.1)

$$\left[-\bar{\omega}^2 \mathbf{M} + \hat{\mathbf{K}} \right] \mathbf{a} \sin(\bar{\omega} t^*) = \mathbf{0}$$

The coefficient of $\sin(\bar{\omega} t^*)$ must be zero, hence

$$\left[-\bar{\omega}^2 \mathbf{M} + \hat{\mathbf{K}} \right] \mathbf{a} = \mathbf{0}$$

or

$$\mathbf{M}^{-1} \hat{\mathbf{K}} \mathbf{a} = \Omega \mathbf{a} \quad (4.3)$$

where $\Omega = \bar{\omega}^2$, is the eigenvalue of the matrix $\mathbf{M}^{-1}\hat{\mathbf{K}}$ that can be found numerically using iterative algorithm[20]. For all the case studies in this chapter, we will refer to the following cable. A cable of length $l = 2002.37m$ that has level supports, $h = 0$, a span $b = 2000m$ and elastic parameter $\beta = \rho gl/EA = 1/5000$. From section 2.4, the static solution is given such that $\bar{\sigma} = 0.1688832$ and $\epsilon = d/l = \bar{\sigma}/8 = 0.021111$. The parameter λ then becomes $\lambda = T_o/EA = \beta/(8\epsilon) = 1.18425 \times 10^{-3}$ and $k = \lambda/\epsilon^2 = 2.6574$. According to equation (2.65), the linear theory of vibration of parabolic cables predicts that the first natural frequency is $\bar{\omega}^{(1)} = 5.34979$. The function that we used for wavelets representation is the anti-derivative wavelet of Daubechies' compactly supported wavelets of the order of $p = 4$. Figure 4.1 shows the first natural frequency as obtained from equation (4.3) for both Fourier and wavelets representations plotted against the number of terms, N , and compared with the result of the linear theory, $\bar{\omega}^{(1)} = 5.34979$. Both methods are converging very fast for a small number of terms. The mismatch between the linear theory and the Galerkin method is because they are different formulation and may be attributed to the fact that we have used the catenary analysis in the Galerkin method and a parabolic approximation in the linear theory. The minimum number of terms in wavelet representation is 10, because the minimum resolution is $J = -1$ that corresponds to 10 terms, which means that there are only ten nonzero wavelets at this level. The level $J = 0$ will correspond to only seven nonzero wavelets and therefore seven terms. But, since we are using multilevel wavelet representation, the total number of terms is the sum of the number of terms of the present level and all the terms of the previous levels back to the level $J = -1$. Therefore, if $J = 0$, the total number of terms will be the cumulative terms of $J = -1$ and $J = 0$, i.e. $N = 10 + 7 = 17$. If $J = 1$, then $N = 10 + 7 + 14 = 31$ and so on. The general equation for N is

$$N = 2^{J+1}R + R - p \quad (4.4)$$

Table 4.1: The First Six Natural Frequencies of a Bare Cable

Mode No.	Linear Theory, S_E	Fourier, S_F	$\left \frac{S_E - S_F}{S_E} 100 \right $	Wavelets, S_W	$\left \frac{S_E - S_W}{S_E} 100 \right $
1	5.34979	5.34809	0.0317	5.34848	0.0245
2	6.28319	6.27191	0.179	6.27187	0.180
3	9.57344	9.56742	0.0628	9.7343	1.680
4	12.5664	12.5608	0.0441	12.5755	0.0730
5	15.7354	15.732	0.0211	15.7455	0.0645
6	18.8496	18.846	0.0191	18.8589	0.0496

where $R = 2p - 1$ and p is the order of the wavelet. In order to compare between Fourier and wavelets solutions we need to use the same number of terms in each method. Using $N = 31$, Table 4.1 shows a good agreement between the exact and the approximated natural frequencies of the first six modes. Notice in Table 4.1 that the maximum percentage error of the Fourier solution is at the second mode, 0.179%, and then the error decreases as the mode number increases. While, the maximum percentage error of the wavelets solution is at the third mode, 1.680%, and then the error decreases as the mode number increases. The wavelets solution gives better estimate than the Fourier solution for the first mode and almost identical for the second mode, but for higher modes the Fourier solution gives better predictions.

4.2 Vibration of Cable with Attached Mass

For the same cable but with an attached mass of a mass ratio $\mu = M/\rho l = 0.1$ and centrally located, $\bar{s}_o = 0.5$ we have $\bar{\alpha}_1 = \bar{\alpha}_2 = 0.080568$ and $\bar{\sigma} = 0.14638$. In this case $\epsilon = 0.0182975$, $\lambda = 1.3663 \times 10^{-3}$ and $k = 4.081$. Substituting for the values of $\bar{\alpha}_1$, $\bar{\alpha}_2$ and $\bar{\sigma}$ in equations (2.82)-(2.87) we obtain the static shape of the catenary cable under the attached mass, M . Now we substitute for the static shape in equations (3.37)-(3.39) to obtain the stiffness matrix $\hat{\mathbf{K}}$ and an eigenvalue problem similar to equation (4.3) can be formed. When using the wavelets representation the

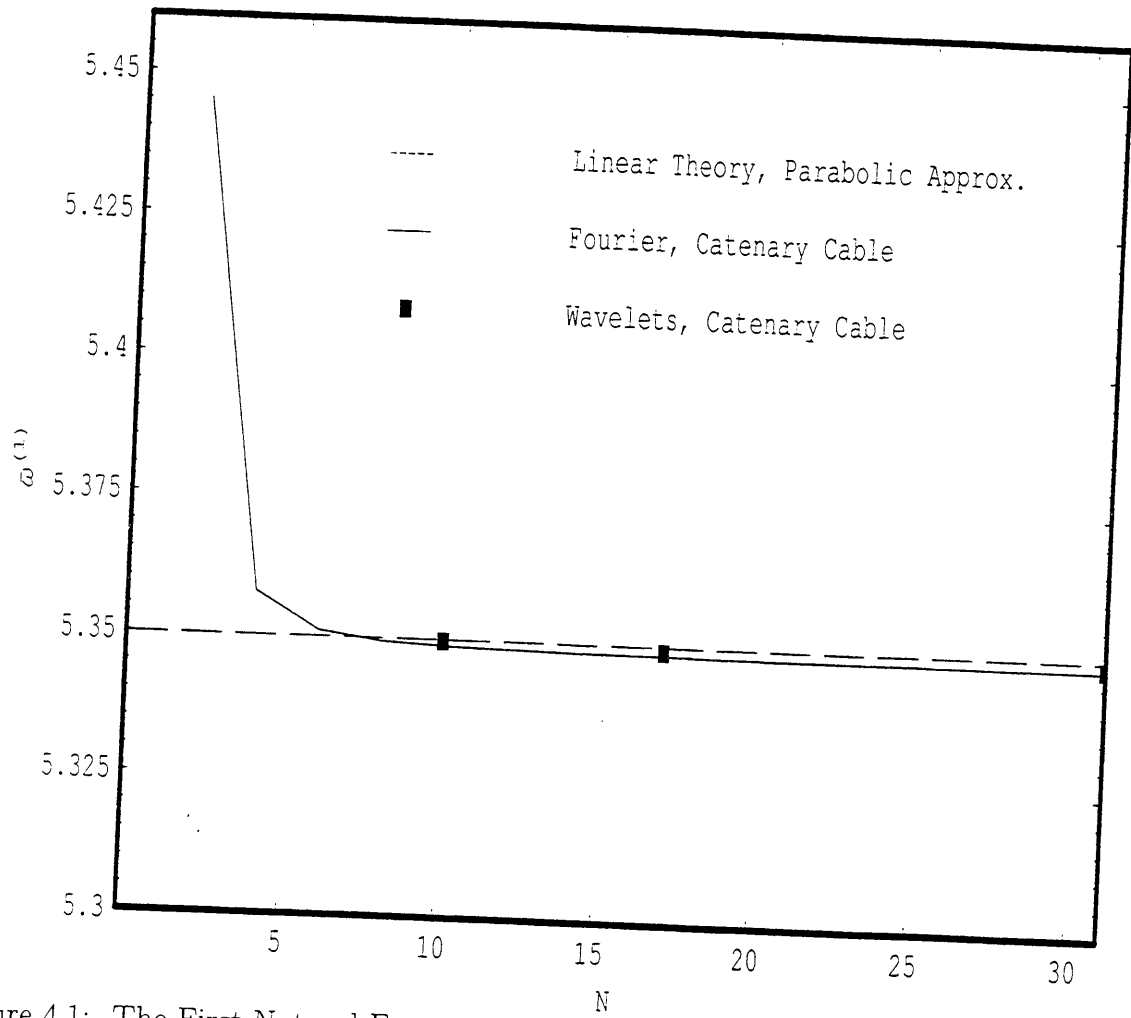


Figure 4.1: The First Natural Frequency vs. the Number of Terms, N , Using Fourier and Wavelets Representations

Table 4.2: The First Six Natural Frequencies of a Cable with an Attached Mass

Mode No.	Linear Theory, S_E	Fourier, S_F	$\left \frac{S_E - S_F}{S_E} 100 \right $	Wavelets, S_W	$\left \frac{S_E - S_W}{S_E} 100 \right $
1	4.658	4.65888	0.0189	4.65936	0.0293
2	6.28182	6.27915	0.0425	6.27694	0.0777
3	8.72413	8.74005	0.182	8.70911	0.172
4	12.5664	12.5762	0.0783	12.5803	0.111
5	14.4774	14.5433	0.455	14.5496	0.499
6	18.8491	18.8665	0.0924	18.868	0.101

appropriate dimensionless parameters have to be applied. Table 4.2 shows the first six natural frequencies of the above cable. In both Fourier and wavelets representations the number of terms used is $N = 31$. Note that to obtain the exact natural frequencies we use equation (2.100), where $\hat{\alpha}_1 = \hat{\alpha}_2 = \bar{\alpha}_1/\epsilon = 4.4$ and $\hat{\sigma} = \bar{\sigma}/\epsilon = 8$. In Table 4.2 the exact and approximate natural frequencies of the first six modes are shown. We see that both the Fourier and the wavelets solutions have minimum percentage error at the first mode, 0.0189% and 0.0293% respectively. The maximum percentage error for both solutions occur at the fifth mode, 0.455% and 0.499% respectively. We also notice that the Fourier solution gives better prediction than the wavelet solution for all modes.

4.3 Vibration of a Horizontal Cable Carrying a Moving Mass

Now we consider that the mass is moving at constant speed, $ds_o/dt = V_o$ starting from the left support of the cable. Thus, at $t = 0$, $s_o = 0$. In dimensionless form, the mass velocity becomes

$$\begin{aligned} \frac{ds_o^*}{dt^*} &= \frac{1}{l} \frac{ds_o}{dt} \frac{dt}{dt^*} \\ &= \frac{V_o}{l} \sqrt{\frac{\rho l^2}{T_o}} \end{aligned}$$

Table 4.3: Comparison between v_N for $N = 17, 31$ and 59

Mass Location, s_o/l	$\frac{v_{31}-v_{17}}{v_{31}} 100$	$\frac{v_{59}-v_{31}}{v_{59}} 100$
0.25	2.116	1.780
0.50	1.740	0.885
0.75	2.897	1.0394

$$\begin{aligned}
&= \frac{V_o}{\sqrt{gl}} \sqrt{\frac{\rho gl}{T_o}} \\
&= V_o^* \sqrt{\bar{\sigma}}
\end{aligned}$$

where $V_o^* = V_o/\sqrt{gl}$ and $\bar{\sigma} = \rho gl/T_o$. From the previous example given for the free vibrations of bare cable we have $\bar{\sigma} = 0.1688832$. For linear analysis to be valid V_o^* must be small, therefore we use $V_o^* = 0.25$ in this case. Figure 4.2 shows the path¹ of the moving mass along the cable span for different number of terms using Fourier representation. We see that the differences between the solutions obtained with $N = 17$, $N = 31$ and with $N = 59$ are not clear at this scale. Therefore, we compare those solutions by plotting only the dynamic displacements. Figure 4.3 shows the vertical component of the dynamic displacement in its dimensionless form, v/l for three time instants and for $N = 17, 31$ and 59 . We notice that the maximum difference is at the mass location. If we denote by v_N the vertical component of the dynamic displacement obtained using number of terms equal to N , then Table 4.3 shows that the difference between v_{31} and v_{59} is small, whereas the difference between v_{17} and v_{31} is relatively large. Therefore we will use $N = 31$ for the subsequent results.

In Figure 4.4 and Figure 4.5 we compare the Fourier solution with the wavelets solution at the resolution level $J = 1$ that corresponds to $N = 31$. Figure 4.4 shows the deflection of the moving mass as it travels along the cable span. We see that there is no discrepancy between the wavelets and the Fourier solutions. Figure 4.5 shows the

¹ The path figures correspond to the cable shape at which the static and dynamic quantities, such as x, y, u and v are evaluated at the mass location, $s = s_o^*$.

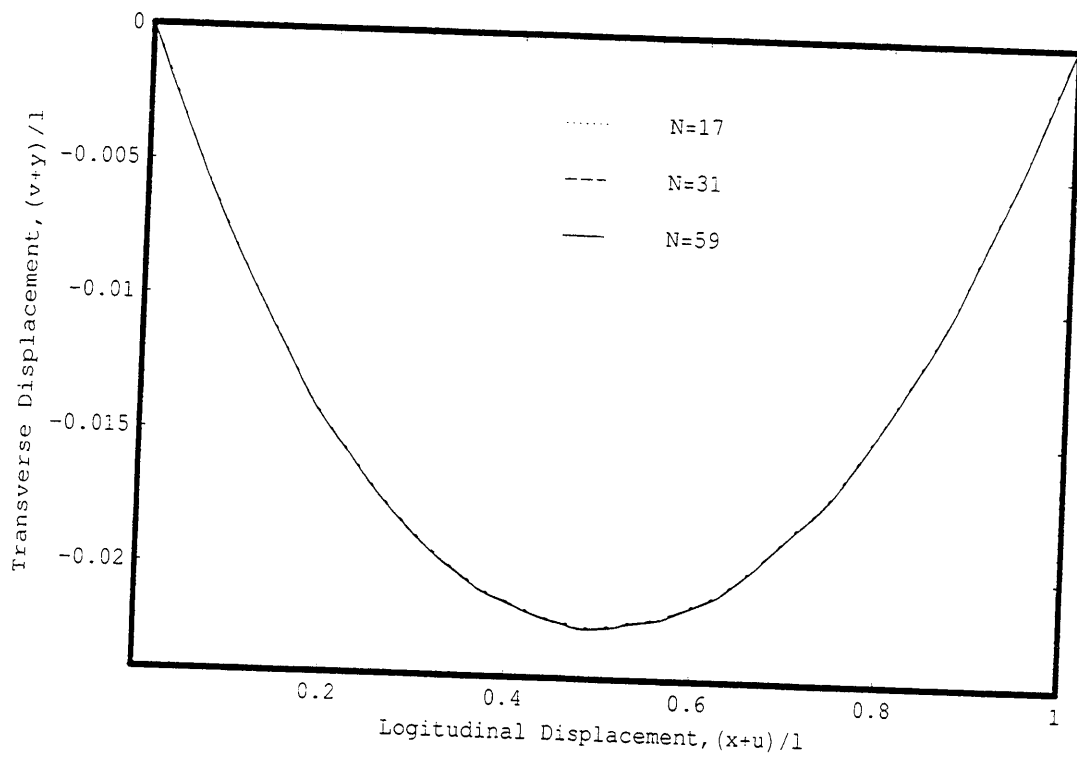


Figure 4.2: The Path of the Moving Mass Along the Span of the Cable Using Fourier Representation with $N = 17$, $N = 31$ and $N = 59$, for $\mu = 0.1$ and $V_o^* = 0.25$

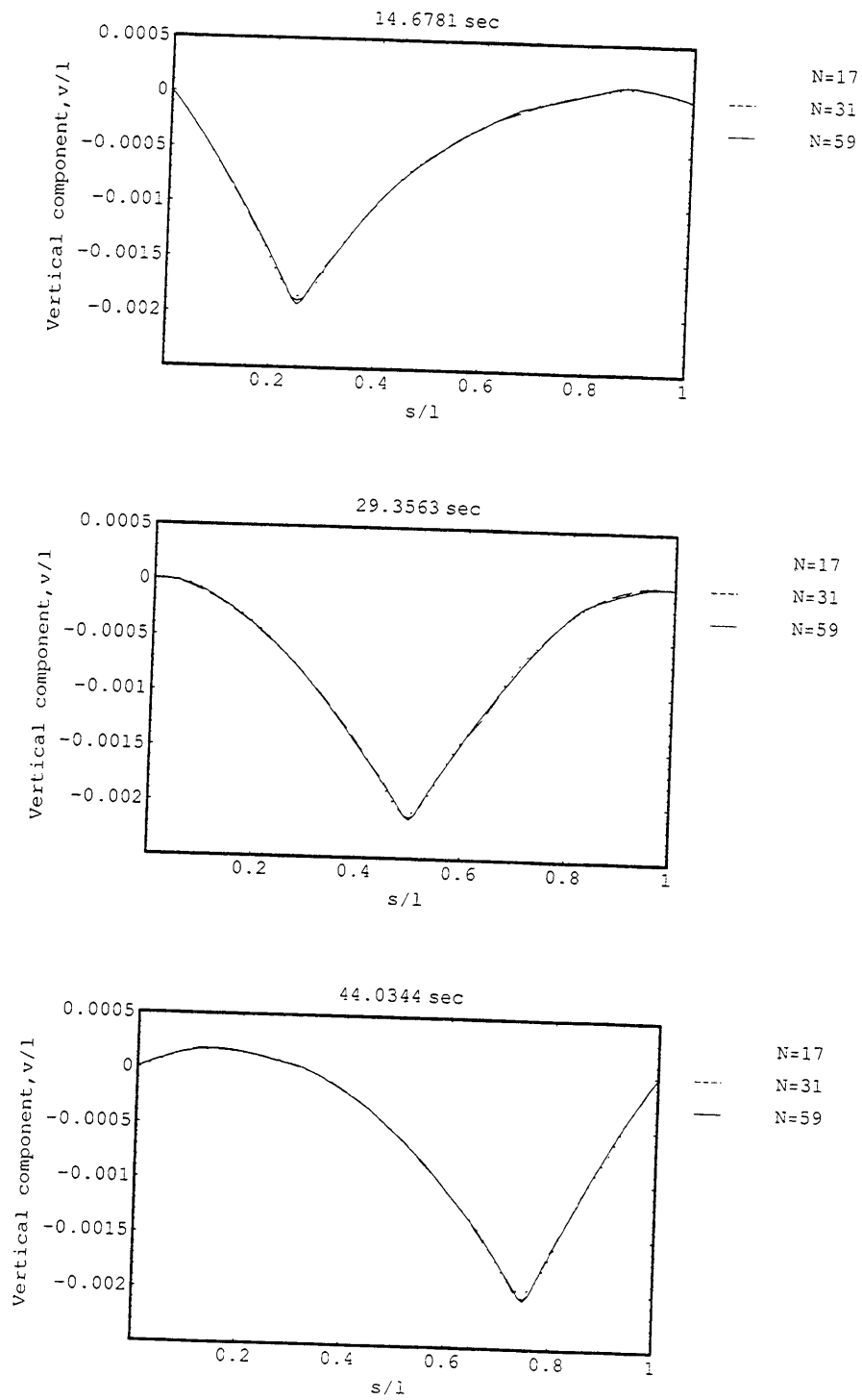


Figure 4.3: The Dynamic Shape of the Cable at Three Time Instants Using $N = 17$, 31 and 59 for $\mu = 0.1$ and $V_o^* = 0.25$

static and total tensions, static plus dynamic, of the cable at different time instants. In this figure the difference between the two solutions is clear. The Fourier solution shows oscillations at all times and along the entire cable span, while the wavelets solution at $t = 16.328$ second and $t = 32.656$ seconds shows horizontal straight lines with very small local oscillations at the position of the moving mass. At $t = 48.984$ seconds, the wavelets solution has oscillations but less than Fourier solution. To decide which solution is correct, recall from section 2.5, that the dynamic tension of the free vibration of cables is a function of time alone, see equations (2.57) and (2.58). We expect to notice similar behavior, since, that the example we are considering here has a small sag-to-span ratio, about 1 : 47. Therefore, the dynamic tension given by the wavelets solution is more accurate. The oscillations found in the Fourier solution can be attributed to the fact that the Fourier representation does not converge uniformly to such functions.

Figure 4.6 shows the Fourier representation of the dynamic configurations of the cable at selected time values. It is clear that the additional dynamic displacement is very small and the cable is more likely to vibrate under quasi-static motion. The reason is that the mass ratio is small. The Fourier linear and nonlinear solutions are also in good agreement as shown in Figure 4.7 when the mass is half way through the span. By increasing the mass ratio but keeping it small, the discrepancies between the linear and the nonlinear solutions will be relatively large. For illustration, for $\mu = 0.2$, Figure 4.8 shows that the linear solution will over estimate the dynamic displacements. We trust the nonlinear model, because we have used the full nonlinear terms that appear in the potential energy. More specifically, we did not approximate the Lagrangian strain, ϵ , in equations (3.14) and (3.15).

Now we compare our nonlinear model solution using Fourier representation with the finite element solution given by Wu et al.[52] who used the updated Lagrangian

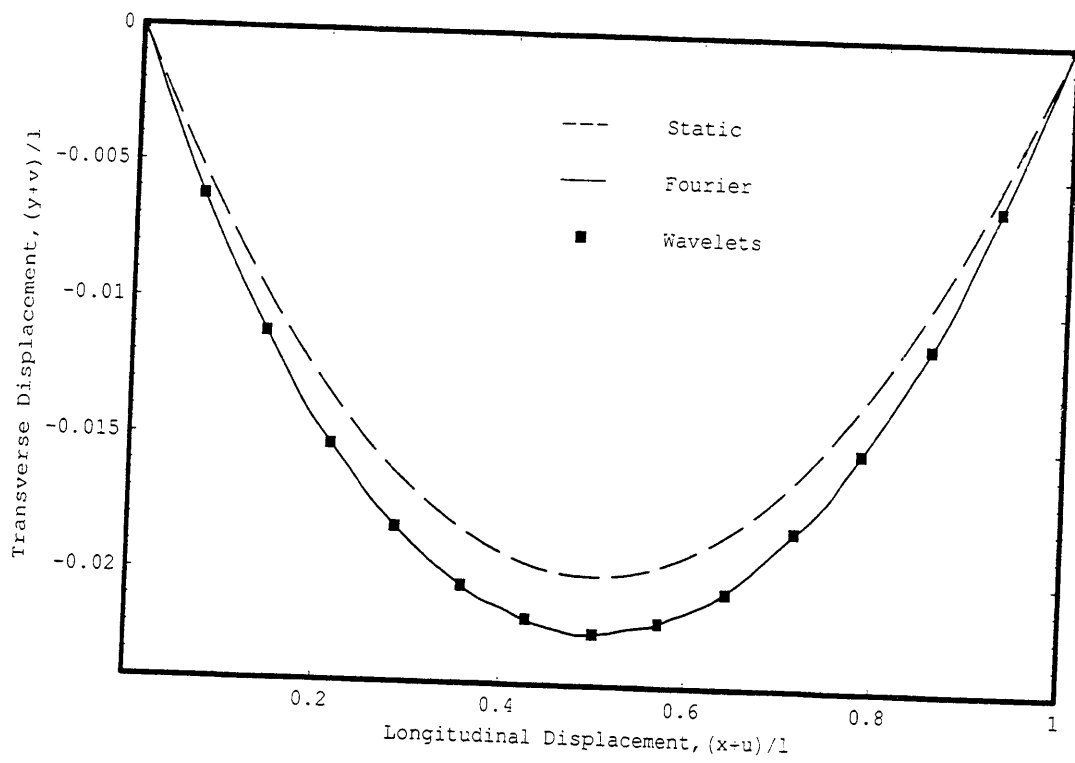


Figure 4.4: Comparison Between Fourier and Wavelets Solutions, Using $N = 31$, of the Path of the Moving Mass Along the Cable Length

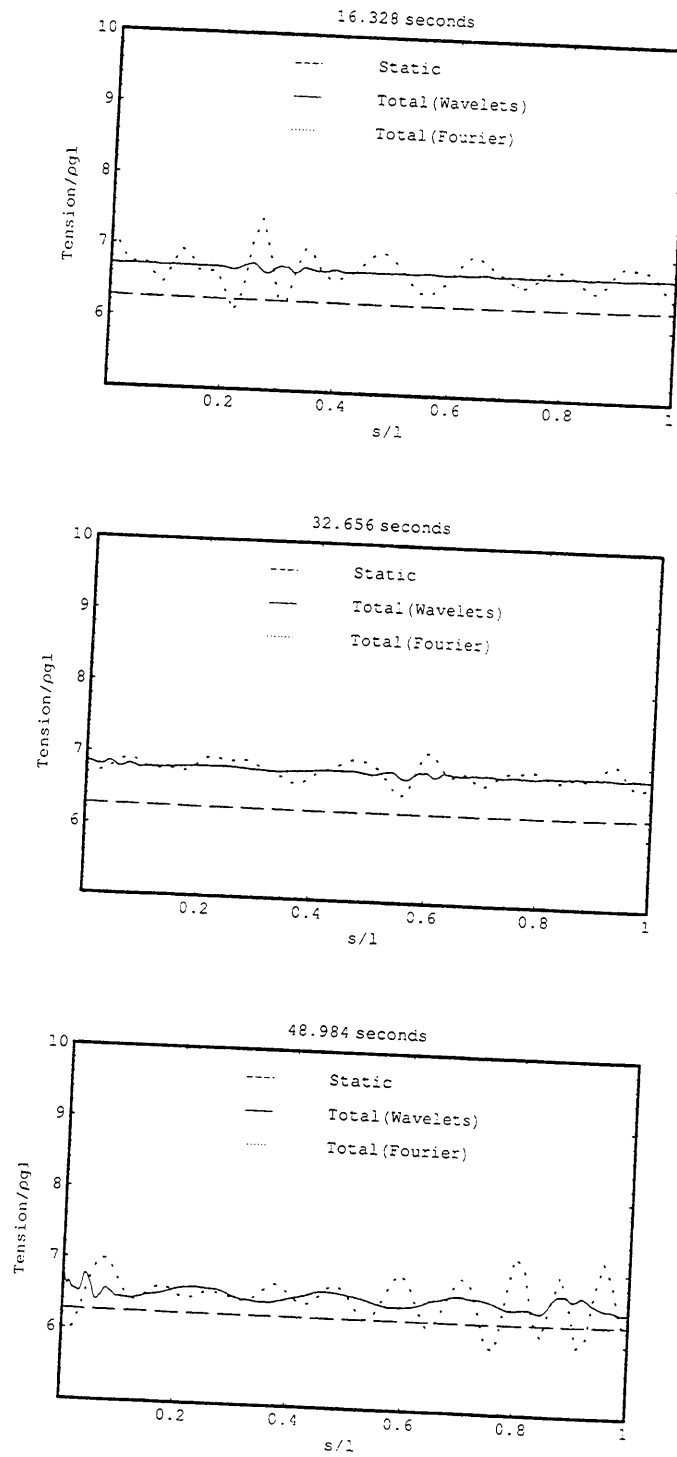


Figure 4.5: The Total Tension of the Cable at Three Time Instants: $t = 16.328$, 32.656 and 48.984 Seconds, Using Fourier and Wavelets Representations

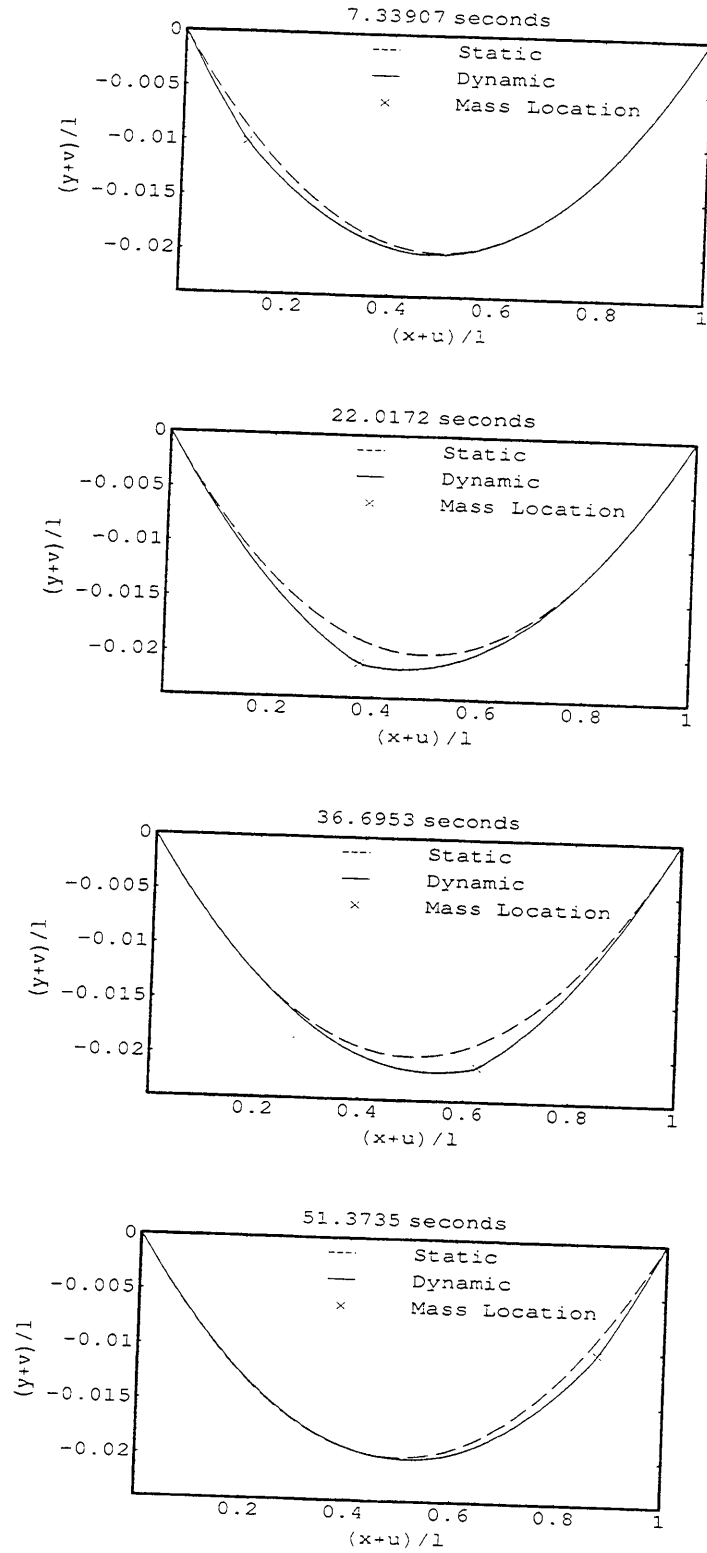


Figure 4.6: Dynamic Shape of the Cable at Selected Time Values. $\mu = 0.1$ and $V_o^* = 0.25$

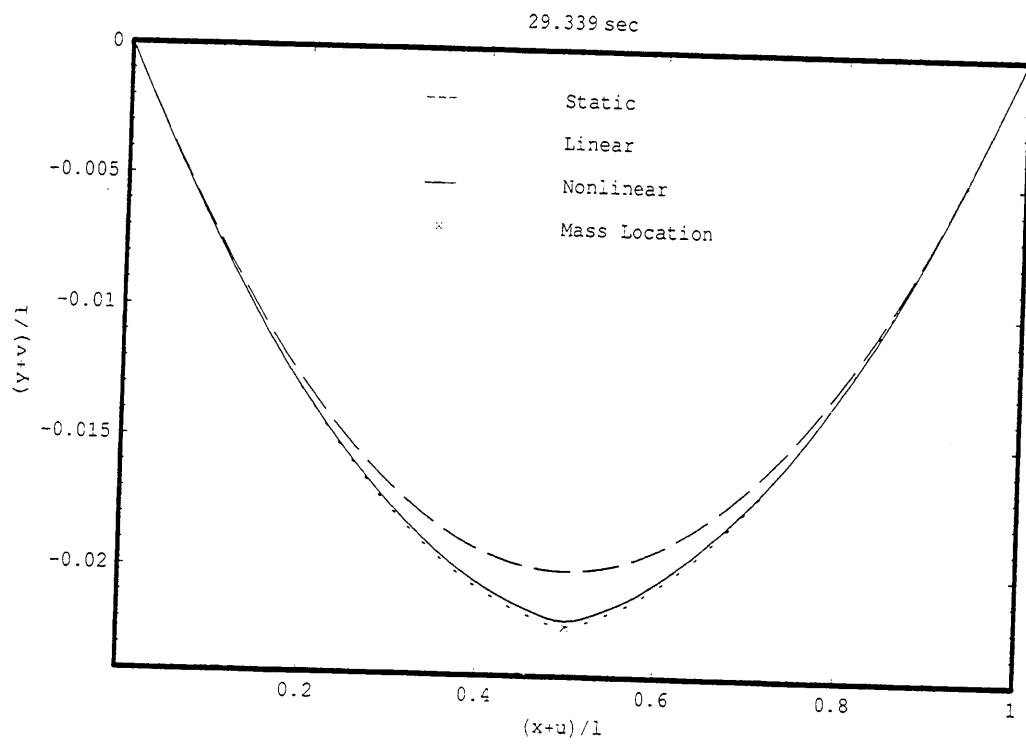


Figure 4.7: Comparison Between the Linear and Nonlinear Solutions When the Moving Mass is at the Middle of the Cable Span. $\mu = 0.1$ and $V_o^* = 0.25$

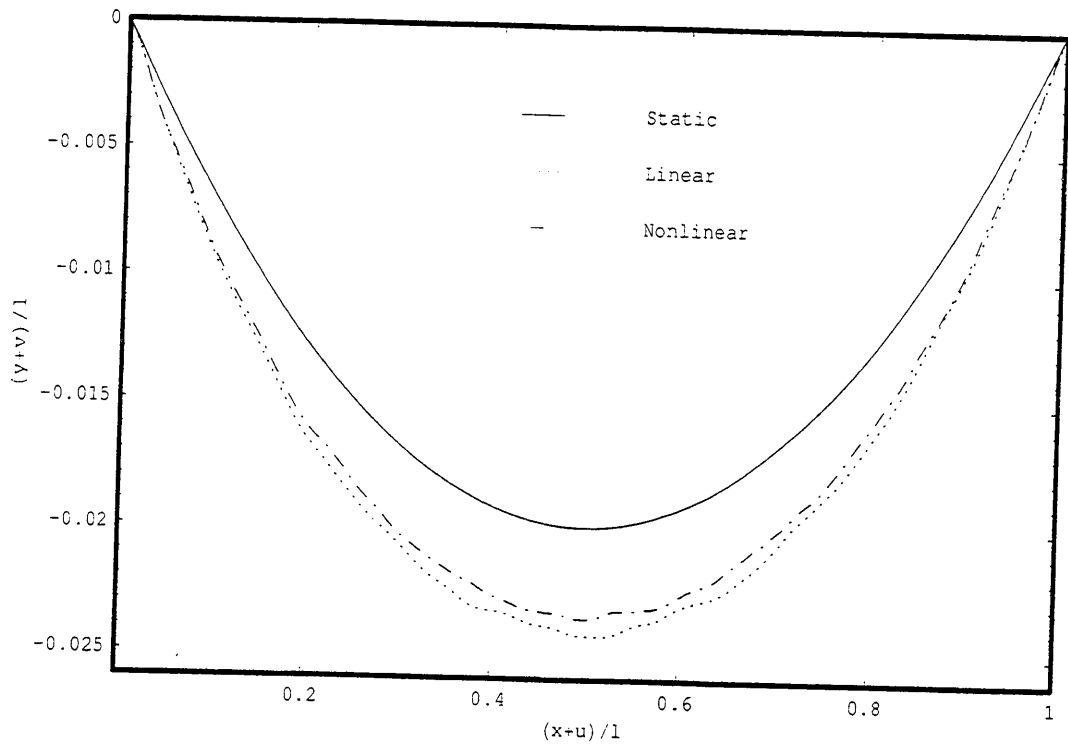


Figure 4.8: The Path of the Moving Mass Along the Span of the Cable Obtained by Linear and Nonlinear Analysis Using Fourier Representation with $N = 30$, for $\mu = 0.2$ and $V_o^* = 0.25$

formulation and Newmark direct integration scheme to obtain the dynamic configuration of cable carrying a moving mass. For $\mu = 1.0$ and $V_o^* = 1.0$, Figure 4.9 shows the path of the moving mass along the span of the cable using the nonlinear model with $N = 30$. Shown in the figure also the results from reference [52] for the same velocity ratio and for $\mu = 1.0$ and $\mu = 2.0$. From the figure we see that our solution, which is for $\mu = 1.0$, depicts the one for $\mu = 2.0$ of reference [52]. We do not doubt our solution, because we have shown in the beginning of this chapter that our approximate solution reproduce the exact natural frequencies of a bare cable and of a cable with an attached mass. Also, the authors in reference [52], reproduce, as a limiting case, the solution of the example given by Fryba[14], which is a force moving at constant speed on a string, but this is different than a moving mass because of the fact that a moving force does not have a coupling acceleration with the string. We notice in Figure 4.9 that there is a local maxima in our solution at $s_o/l = 0.655$ while their solution for $\mu = 2.0$ gives a local maxima at $s_o/l = 0.8$. This shift of the local maxima might be attributed to the fact that we have used the full nonlinear terms, while they used only the linear and quadratic terms. Therefore, we think that their mass ratio has a factor of two missing.

The dynamic configuration at selected time values is shown in Figure 4.10. As the mass enters the cable from the left support, a forward wave travels from left to right. It hits the right support and reflects back. It meets with the moving mass when the mass is half way of the cable span and some of it transmits through the mass and some of it get reflected by the mass. The wave that is reflected by the mass travels to the right of the mass and when it reaches the right support it gets reflected back again. The wave that gets transmitted through the mass travels to the left of the mass. This pattern of wave scattering repeats itself until the mass exits the cable at the right support.

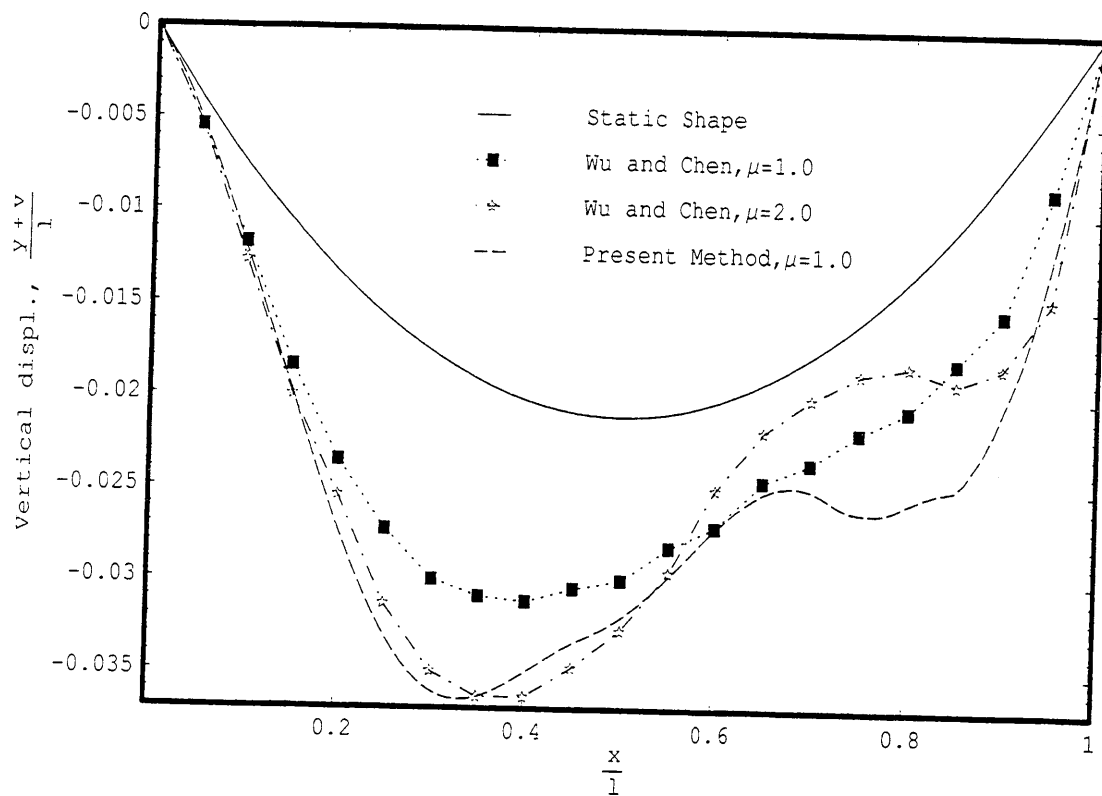


Figure 4.9: The Path of the Moving Mass Along the Span of the Cable Obtained by the Fourier Nonlinear Model for $\mu = 1.0$, and from Reference[52] for $\mu = 1.0$ and 2.0. The Velocity ratio is, $V_o^* = 1.0$

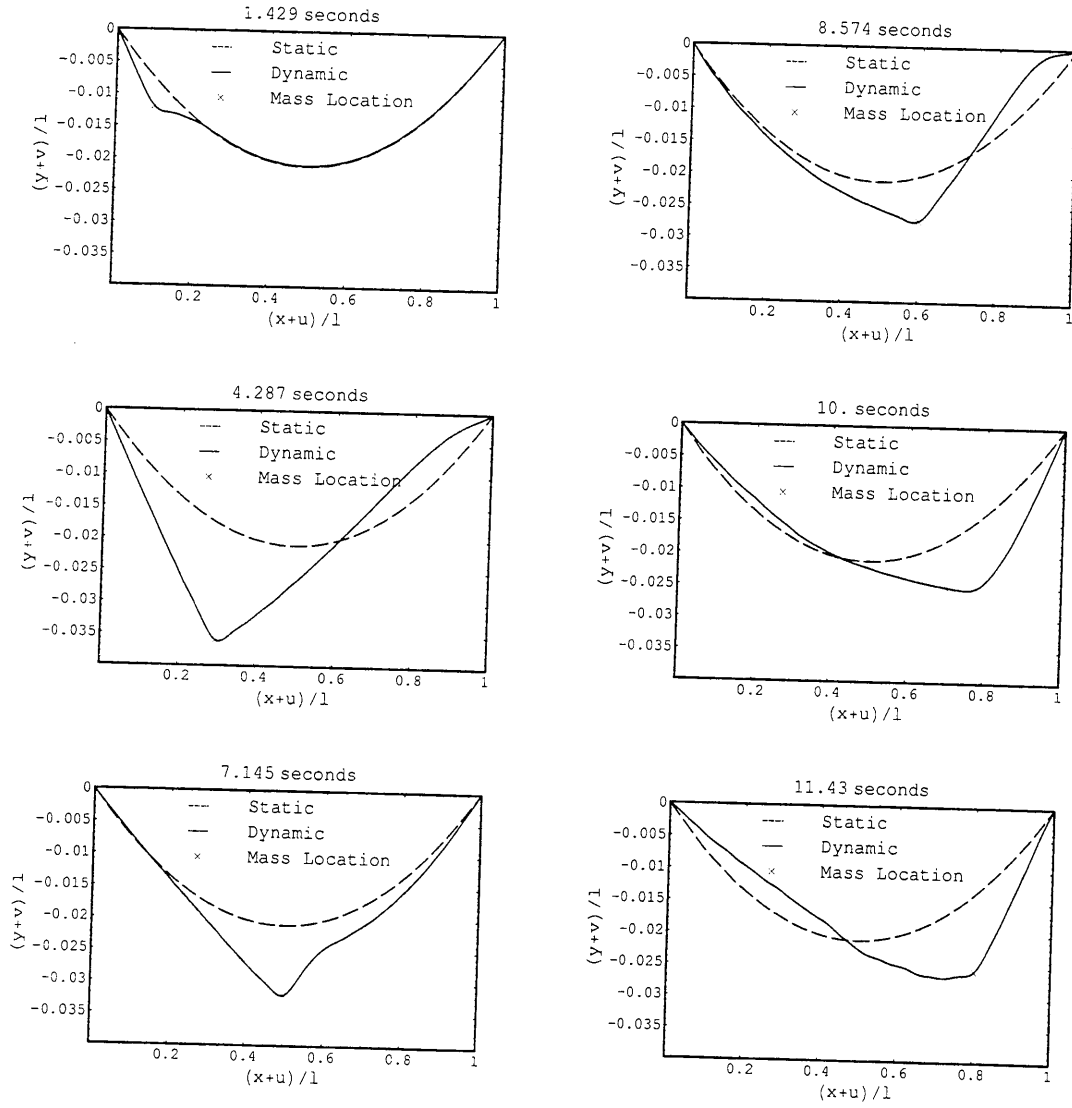


Figure 4.10: Dynamic Configuration at Different Time Instants. $\mu = 1.0$ and $V_o^* = 1.0$

In practice, it might be of interest to have the moving mass traveling along the cable with variable velocity and variable acceleration. There are many cases of this form, but we will consider the case in which the moving mass start from rest at the left support and moves with a constant acceleration to a desired maximum velocity when the mass reaches one third of the cable. Then the moving mass shall travel at that velocity until it reaches two thirds of the cable. Finally, the moving mass shall be brought to zero at the right support with constant deceleration. If we denote by $V_{\max}$, the maximum velocity attained by the moving mass, its dimensionless form is achieved by dividing by $\sqrt{T_o/\rho}$, hence

$$\begin{aligned} V_{\max}^* &= \frac{V_{\max}}{\sqrt{T_o/\rho}} \\ &= \frac{V_{\max}}{\sqrt{gl}} \sqrt{\frac{\rho gl}{T_o}} \\ &= \hat{V}_{\max} \sqrt{\bar{\sigma}} \end{aligned}$$

where $\hat{V}_{\max} = V_{\max}/\sqrt{gl}$. The variable acceleration in dimensionless form is given by

$$a_o^*(t^*) = \begin{cases} \frac{V_{\max}^*}{t_1^*} & ; 0 \leq t^* \leq t_1^* \\ 0 & ; t_1^* < t^* \leq t_2^* \\ -\frac{V_{\max}^*}{t_3^* - t_2^*} & ; t_2^* < t^* \leq t_3^* \end{cases} \quad (4.5)$$

where t_1^* is the time required by the moving mass to reach one third of the cable, t_2^* is the time required by the moving mass to reach two thirds of the cable and t_3^* is the total time required by the moving mass to reach the right support of the cable. The

variable velocity, V_o^* is achieved by integrating equation (4.5)

$$V_o^*(t^*) = \begin{cases} \frac{V_{\max}^* t^*}{t_1^*} & ; 0 \leq t^* \leq t_1^* \\ V_{\max}^* & ; t_1^* < t^* \leq t_2^* \\ \frac{V_{\max}^*}{t_3 - t_2} (t_3^* - t^*) & ; t_2^* < t^* \leq t_3^* \end{cases} \quad (4.6)$$

The position of the moving mass is achieved by integrating equation (4.6)

$$s_o^*(t^*) = \begin{cases} \frac{V_{\max}^* t^{*2}}{2t_1^*} & ; 0 \leq t^* \leq t_1^* \\ V_{\max}^* (t^* - \frac{t_1^*}{2}) & ; t_1^* < t^* \leq t_2^* \\ \frac{V_{\max}^*}{2t_1^*} (t^{*2} - (t^* - t_1^*)^2 - t_1^* \frac{(t^* - t_2^*)^2}{t_3^* - t_2^*}) & ; t_2^* < t^* \leq t_3^* \end{cases} \quad (4.7)$$

The values of t_1^* , t_2^* and t_3^* can be evaluated from their definition using equation (4.7) as follows: To find t_1^* , substitute $t^* = t_1^*$ in equation (4.7) to get

$$\begin{aligned} s_o^*(t_1^*) &= \frac{1}{3} = \frac{V_{\max}^* t_1^*}{2} \\ \therefore t_1^* &= \frac{2}{3V_{\max}^*} \end{aligned} \quad (4.8)$$

To find t_2^* , substitute $t^* = t_2^*$ and for t_1^* from equation (4.8) in equation (4.7) to get

$$\begin{aligned} s_o^*(t_2^*) &= \frac{2}{3} = V_{\max}^* (t_2^* - \frac{t_1^*}{2}) \\ \therefore t_2^* &= \frac{1}{V_{\max}^*} \end{aligned} \quad (4.9)$$

Finally, to find t_3^* , substitute $t^* = t_3^*$ and for t_1^* and t_2^* from equations (4.8) and (4.9) respectively, in equation (4.7) to get

$$\begin{aligned} s_o^*(t_3^*) &= 1 = \frac{V_{\max}^*}{2t_1^*} (t_3^{*2} - (t_3^* - t_1^*)^2 - t_1^* (t_3^* - t_2^*)) \\ \therefore t_3^* &= \frac{5}{3V_{\max}^*} \end{aligned} \quad (4.10)$$

Now the moving mass position, velocity and acceleration depend on the value of $V_{\max}^*$. As an example we will chose $\hat{V}_{\max} = 1.0$ and same cable configuration as the previous examples, i.e. $\bar{\sigma} = 0.1688832$, that leads to $V_{\max}^* = 0.41095$. Figure 4.11 shows the graphical representation of the moving mass motion. Under this motion the path of the moving mass along the cable is shown in Figure 4.12. We observe that the maximum deflection is at the middle of the cable length which is in contrary with the previous example of constant velocity and same mass ratio of this example, Figure 4.9, in which the maximum deflection occurs at around one-third of the cable length. We also observe that the maximum deflections in both examples are almost the same. The dynamic shape of the cable is shown in Figure 4.13 at different time instants.

4.4 Vibration of an Inclined Cable Carrying a Moving Mass

As a last example we consider an inclined cable carrying a moving mass particle that has a constant acceleration. The location of the mass in its dimensionless form will be

$$\begin{aligned}
 s_o^* &= \frac{a_o t^2}{2l} \\
 &= \frac{a_o}{2l} \frac{\rho l^2}{T_o} t^{*2} \\
 &= \frac{a_o^* \bar{\sigma}}{2} t^{*2}
 \end{aligned} \tag{4.11}$$

where $a_o^* = a_o/g$ and a_o is the constant acceleration of the moving mass. The length and the span of the cable will be taken same as the previous examples but supported at different elevations, $h = 90m$. The static configuration is shown in Figure 4.14. Using Fourier nonlinear model for $a_o^* = 1$ the path of the moving mass along the cable is given in Figure 4.15. The minimum point occurs at around $s_o/l = 0.8$. The dynamic shape and the total tension of the cable at selected time values are shown in

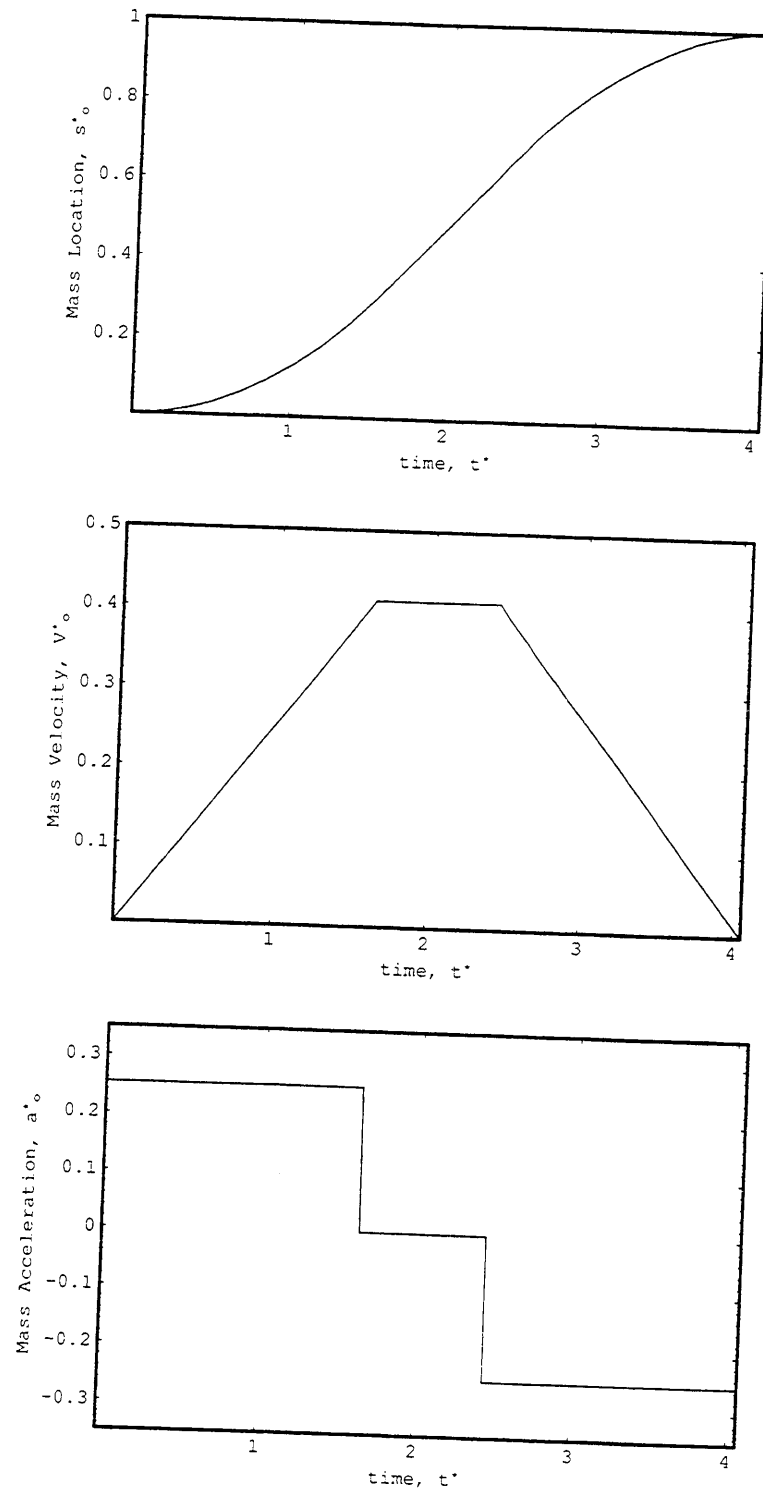


Figure 4.11: The Motion of the Moving Mass with Variable Velocity and Variable Acceleration with $V_{\max}^* = 0.41095$

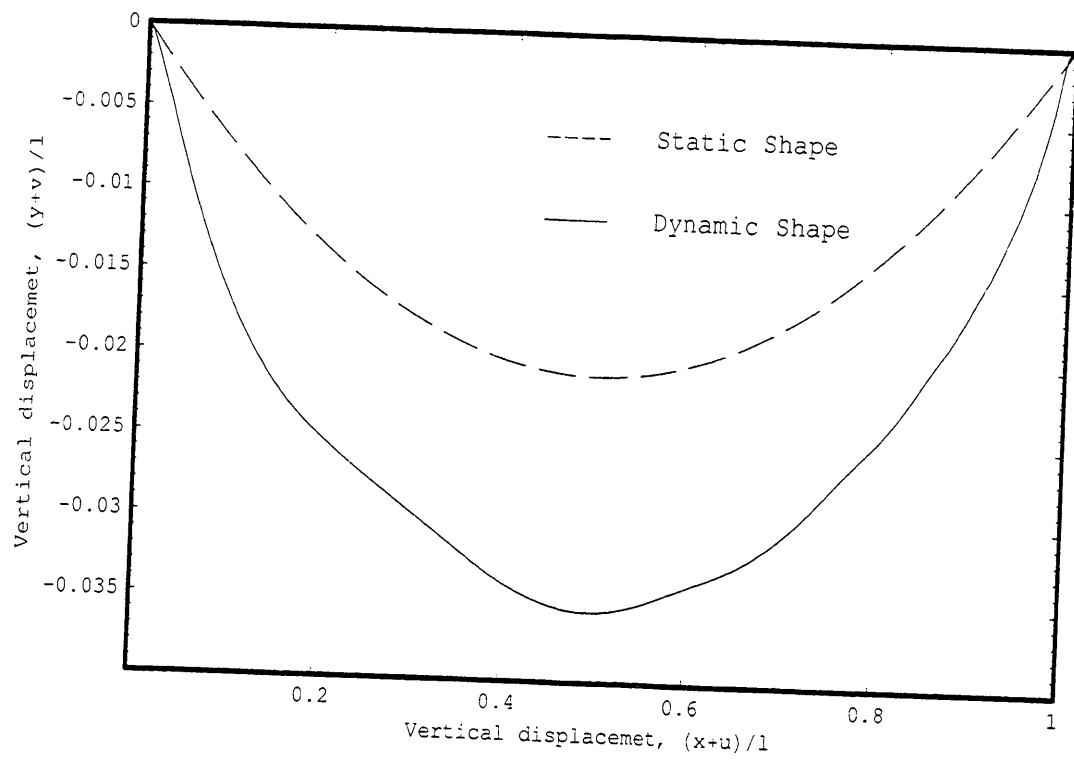


Figure 4.12: Path of the Moving Mass Along the Cable for a Variable Velocity and Variable Acceleration with Maximum Velocity of $\hat{V}_{\max} = 1.0$ and for $\mu = 1.0$

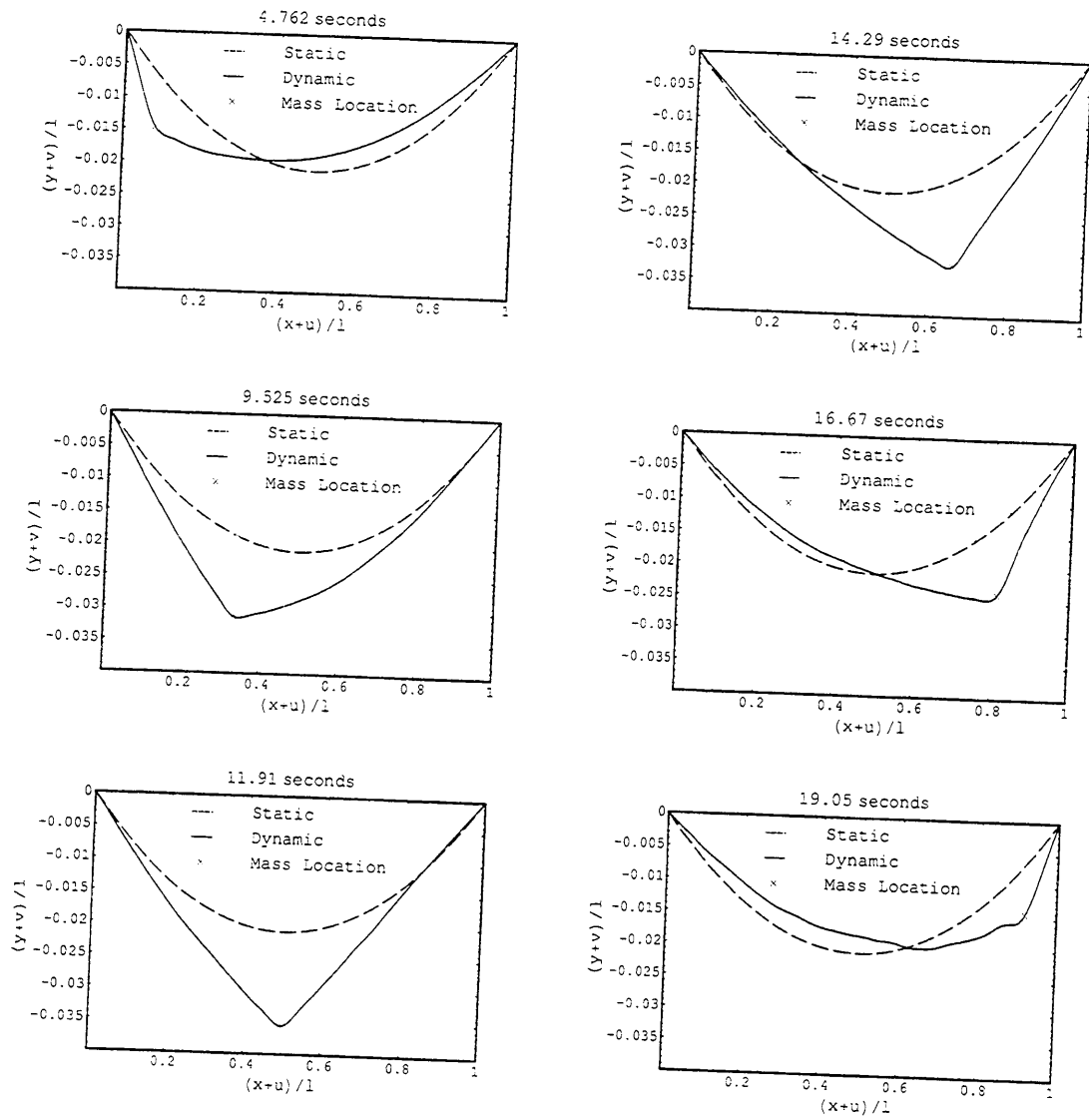


Figure 4.13: The Dynamic Shape of the Cable at Different Time Instants for a Moving Mass with Variable Velocity and Variable Acceleration with Maximum Dimensionless velocity $\hat{V}_{\max} = 1.0$ and $\mu = 1.0$

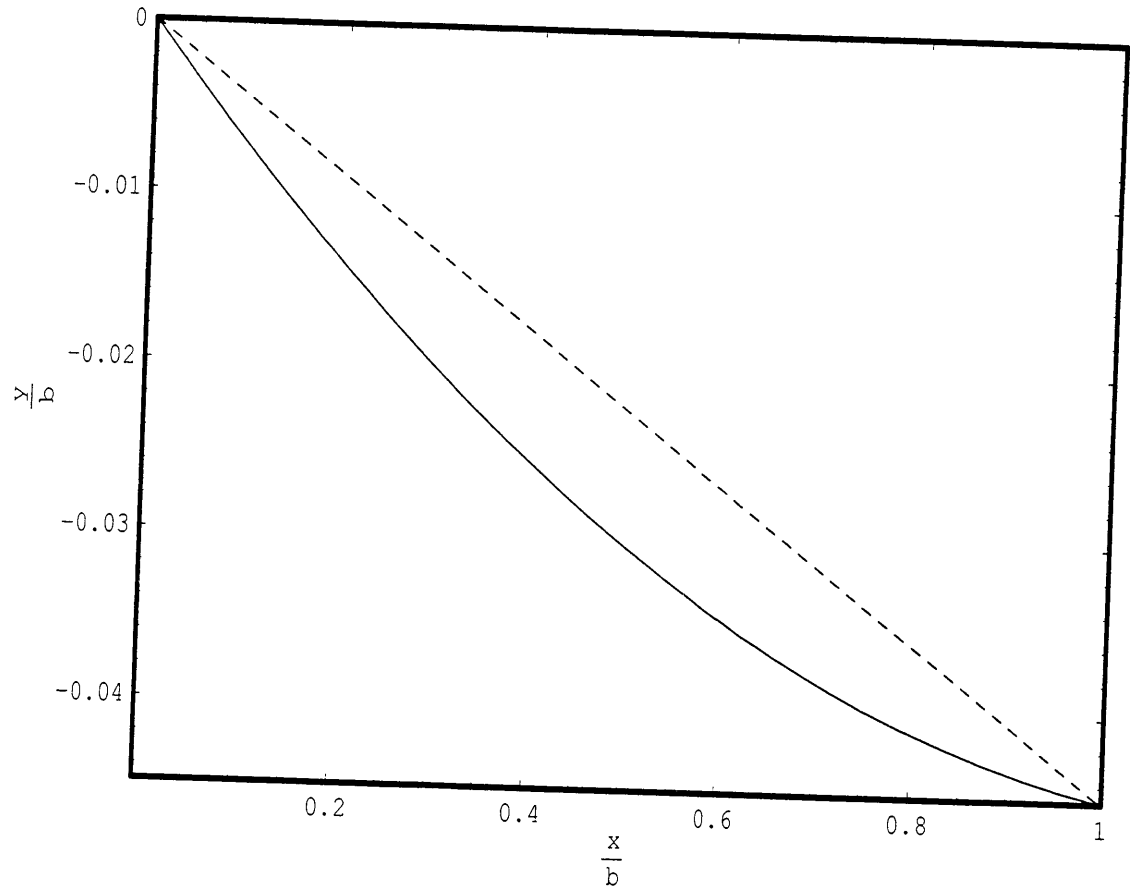


Figure 4.14: The Static Configuration

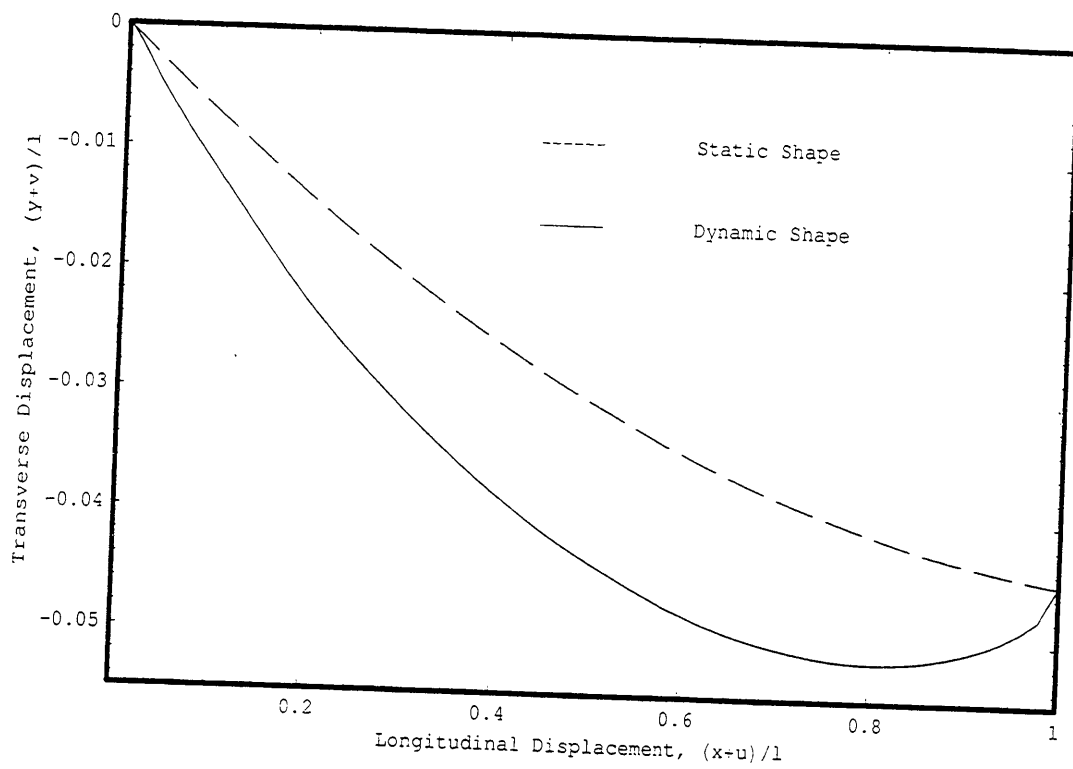


Figure 4.15: The Path of the Moving Mass Along the Cable Length for $\mu = 1.0$ and Acceleration ratio $a_o^* = 1.0$

Figure 4.16 using Fourier and wavelets nonlinear models. Both solutions are obtained using $N = 10$. We notice that the cable static tension is shown constant at this scale, because in this problem the equilibrium configuration was chosen to be very close to the chord. Regarding the cable displacement, we see that the discrepancies between Fourier and wavelets solutions are small. While, regarding the total cable tension, the Fourier solution has many oscillations more than the wavelets solution. In fact, as the moving mass travels along the cable length, the cable tension is discontinuous at the mass location. Away from the mass location the cable tension is slowly changing with the spatial variable, s and mostly changes with time, t . Therefore, the oscillations in the Fourier solution are due to the nonuniform convergence of the Fourier series to such functions and a larger value of N produces more oscillations.

The wavelets solution can be improved by taking larger value of N , but then, we have to use a smaller time step because it is related to the inverse of the maximum natural frequency of the system. Since s_o^* has to be a dyadic point we should select a time step such that when substituted in equation (4.11) we get

$$s_o^* = R \left(\frac{k}{2^\nu} \right)^2 \quad (4.12)$$

where ν is a positive integer and $k = 1, 2, \dots, 2^\nu$. Therefore, Ψ , Ψ' and Ψ'' need to be constructed at the smallest dyadic point appears in the numerical integration scheme which is $1/2^{2\nu+1}$ here. Then in the computer program, those functions have to be stored in a single column arrays of size of $R 2^{2\nu+1}$ each. The results shown in Figure 4.16 are obtained at $\nu = 9$. For $\nu \geq 10$ we exceed the computer stack limit and the computer program does not work.

To overcome this problem, we write the system of equations in terms of s_o^* using

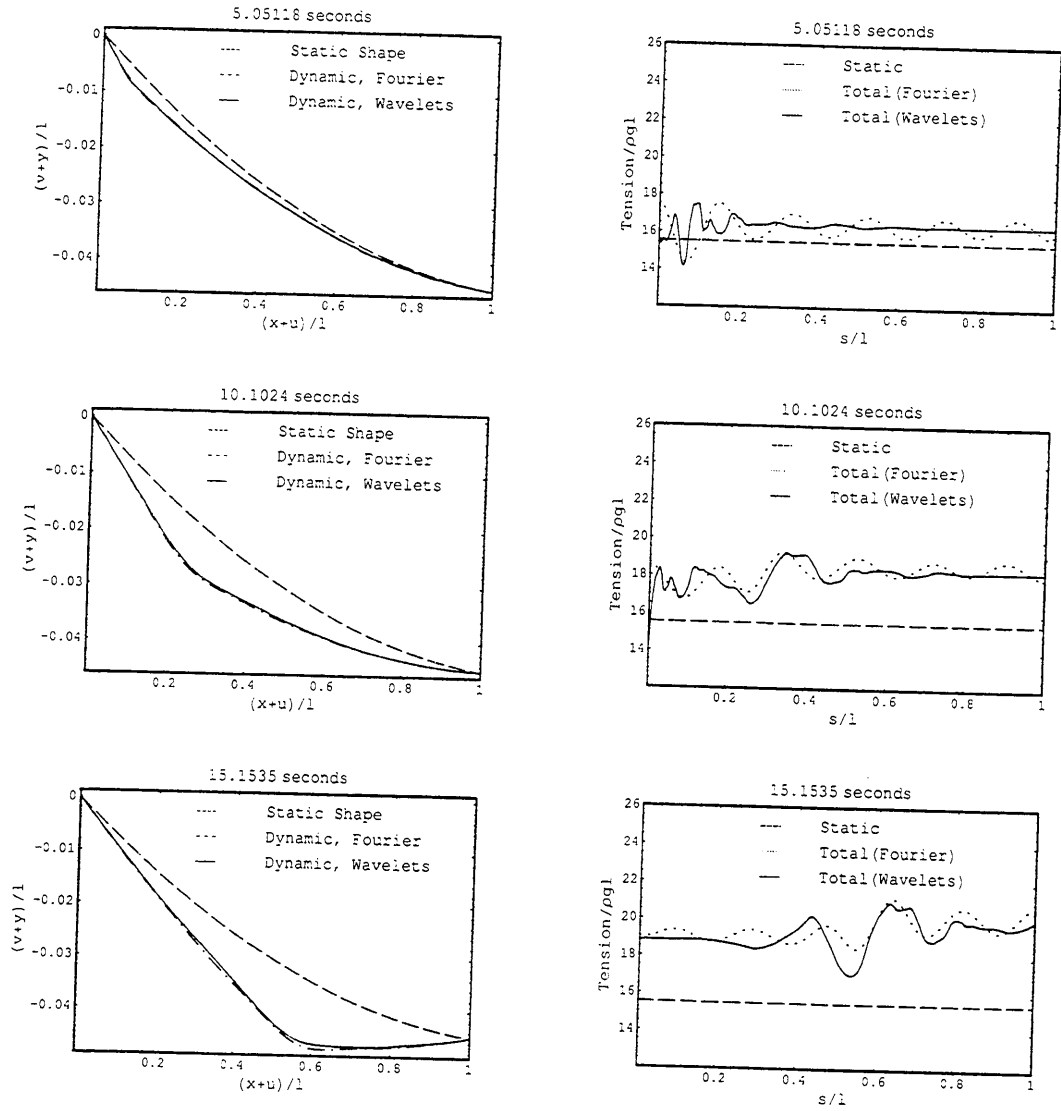


Figure 4.16: The Dynamic Shape and the Total Tension of the Cable at Three Time Instants for $\mu = 1.0$ and an Acceleration Ratio $a_o^* = 1.0$

the relation in equation (4.11). The time derivatives are transformed to

$$\begin{aligned}\frac{\partial(\cdot)}{\partial t^*} &= \frac{\partial(\cdot)}{\partial s_o^*} \frac{ds_o^*}{dt^*} \\ &= \dot{s}_o^* \frac{\partial(\cdot)}{\partial s_o^*}\end{aligned}\tag{4.13}$$

$$\begin{aligned}\frac{\partial^2(\cdot)}{\partial t^{*2}} &= \frac{\partial}{\partial s_o^*} \left(\dot{s}_o^* \frac{\partial(\cdot)}{\partial s_o^*} \right) \frac{ds_o^*}{dt^*} \\ &= \dot{s}_o^{*2} \frac{\partial^2(\cdot)}{\partial s_o^{*2}} + \dot{s}_o^* \frac{d\dot{s}_o^*}{ds_o^*} \frac{\partial(\cdot)}{\partial s_o^*} \\ &= \dot{s}_o^{*2} \frac{\partial^2(\cdot)}{\partial s_o^{*2}} + \ddot{s}_o^* \frac{\partial(\cdot)}{\partial s_o^*}\end{aligned}\tag{4.14}$$

where $\dot{s}_o^*$ and $\ddot{s}_o^*$ are function of s_o^* . Using equations (4.13) and (4.14) in equation (3.8) we obtain the following system of equations

$$\dot{s}_o^{*2} \mathbf{M} \frac{\partial^2 \gamma}{\partial s_o^{*2}} + [\ddot{s}_o^* \mathbf{M} + \dot{s}_o^* \mathbf{C}] \frac{\partial \gamma}{\partial s_o^*} + \mathbf{K} \gamma + \mathbf{R}(\gamma) = \mathbf{F}\tag{4.15}$$

where $\gamma = \gamma(s_o^*)$ and the vectors and matrices are as given before. By denoting the following matrices

$$\mathbf{M}^* = \dot{s}_o^{*2} \mathbf{M}$$

and

$$\mathbf{C}^* = \ddot{s}_o^* \mathbf{M} + \dot{s}_o^* \mathbf{C}$$

equation (4.15) becomes analogous to equation (3.8). The initial conditions are

$$\gamma(s_o^* = 0) = \mathbf{0}\tag{4.16}$$

$$\frac{\partial \gamma(s_o^* = 0)}{\partial s_o^*} = \mathbf{0}\tag{4.17}$$

and $\partial^2 \gamma(s_o^* = 0) / \partial s_o^{*2}$ is obtained from equation (4.15) at $s_o^* = 0$. However, at $s_o^* = 0$ all terms vanish including the term multiplying $\partial^2 \gamma / \partial s_o^{*2}$, because $\dot{s}_o^*(s_o^* = 0) = 0$.

Therefore, we divide both sides of equation (4.15) by $\dot{s}_o^{*2}$ and apply the L'Hopital's rule as follows

$$\begin{aligned} \mathbf{M}(0) \frac{\partial^2 \gamma(0)}{\partial s_o^{*2}} &= \lim_{\dot{s}_o^* \rightarrow 0} \frac{1}{\dot{s}_o^{*2}} \left\{ \mathbf{F} - [\ddot{s}_o^* \mathbf{M} + \dot{s}_o^* \mathbf{C}] \frac{\partial \gamma}{\partial s_o^*} - \mathbf{K} \gamma - \mathbf{R}(\gamma) \right\} \\ &= \frac{1}{2\ddot{s}_o^*} \frac{d\mathbf{F}(0)}{ds_o^*} - \frac{1}{2} \mathbf{M}(0) \frac{\partial^2 \gamma(0)}{\partial s_o^{*2}} \end{aligned}$$

hence,

$$\frac{\partial^2 \gamma(0)}{\partial s_o^{*2}} = \frac{1}{3\ddot{s}_o^*} \mathbf{M}^{-1}(0) \frac{d\mathbf{F}(0)}{ds_o^*} \quad (4.18)$$

The solution procedure outlined in Table 3.3 can be used to solve equation (4.15) subjected to the initial conditions (4.16)-(4.18) and replacing $\mathbf{M}$ and $\mathbf{C}$ by $\mathbf{M}^*$ and $\mathbf{C}^*$ respectively. The increment on the position of the moving mass is now achieved by the linear relation

$$s_o^* = k \Delta s_o^*$$

and the array size to store the wavelet becomes $R 2^{\nu+1}$, which is very much smaller than what was required in the previous formulations. For $N = 10$ the solution is shown in Figure 4.17. The time step used is $\Delta s_o^* = 2^{-12}$. By comparison Figure 4.17 shows less discrepancies between Fourier and wavelets solutions than Figure 4.16. For $N = 17$, Figure 4.18 shows that some oscillations are diminishing from the wavelet solution but not from Fourier solution. This support our argument that the cable tension behaves like a piecewise continuous function and that the Fourier solution does not converge to such functions.

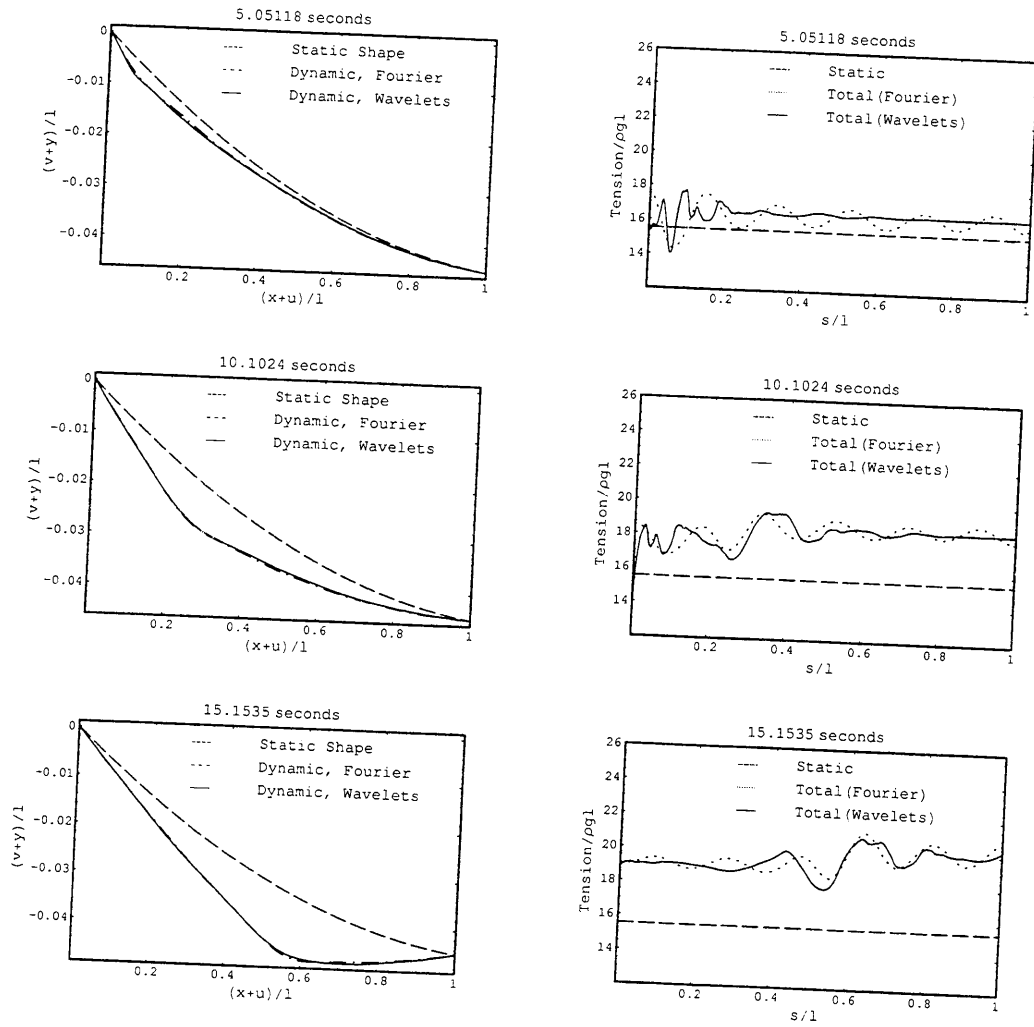


Figure 4.17: Same Label of Figure 4.16

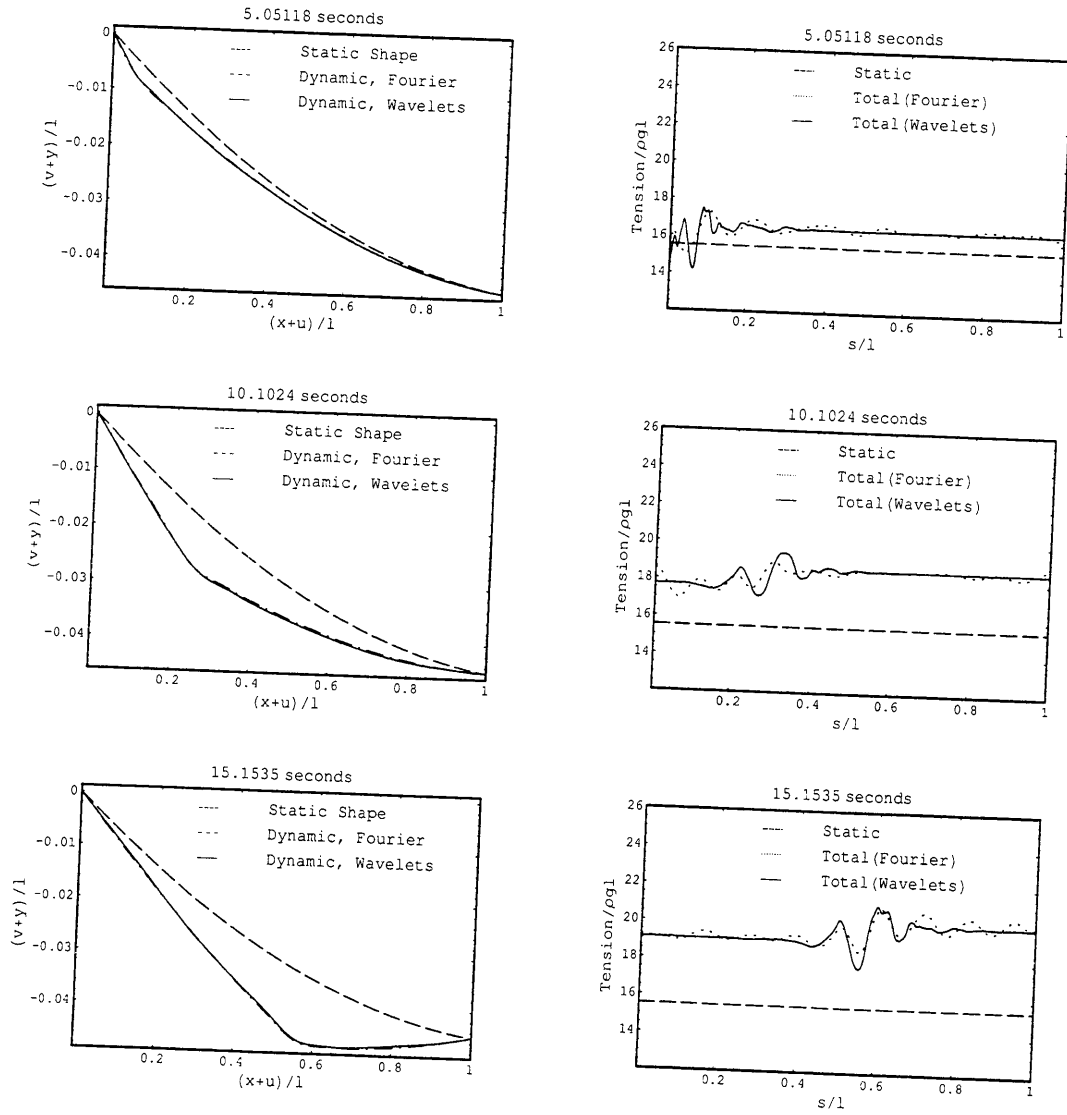


Figure 4.18: Same Label of Figure 4.16 But With $N = 17$

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

The vibration of a catenary elastic cable that is fixed at its supports and carrying a moving mass particle have been investigated. The general formulation have been derived using the Hamilton's principle. The cable configuration was not restricted to small sags and the moving mass particle was assumed to travel along the cable with general motion. The solution was obtained using the Galerkin procedure with two methods of representations. The first method is the Fourier representation with sine functions as the trial basis to remove the spatial dependence. The second method is the multilevel wavelets representation with anti-derivative of the compactly supported wavelets as the trial basis to remove the spatial dependence. Linear and nonlinear models have been obtained for both methods. The discretized equations are solved using implicit direct integration schemes and for the nonlinear models, the iterative technique of the Newton-Raphson method have been implemented. The results of the linear models of both Fourier and wavelets representations have been shown to converge to the results of the natural frequencies, obtained by the linear theory with parabolic approximation, of the free vibrations of bare cable and of the free vibrations of cable with attached mass for a number of terms $N = 31$. This number is obtained from equation (4.4) according to the resolution level $J = 1$, wavelet order $p = 4$ and wavelet range $R = 7$. The maximum percentage error found for the Fourier representation was 0.455% at mode five of the attached mass problem, while the maximum percentage error found for the wavelets representation was 1.68% at mode three of the bare cable problem. This comparison was for the first six modes only. In general, the results obtained from both methods are in good agreement with exact results.

For the problem of the moving mass along the cable length, we have studied

different cases. We have considered horizontal and inclined cables and a moving mass particle of small and large mass ratios with constant and variable velocities. For small mass ratios we have seen that the linear and nonlinear models are in good agreement. However, for large mass ratios the nonlinear model is more accurate. The comparison between the Fourier and wavelets representations showed that both methods are in good agreement when obtaining the path of the moving mass along the cable and when obtaining the cable displacements at different time instants. However, when obtaining the cable tension at different time instants, the wavelets solution was different than the Fourier solution. The Fourier solution showed many oscillations while the wavelets solution showed straight lines with localized oscillations at the location of the moving mass except when the mass approached the right support. We know that as the mass travels along the cable length the cable tension changes and it behaves as a piecewise continuous function. We also know that the Fourier solution does not converge uniformly when representing such functions and the Gibbs phenomenon occurs. In the wavelet solution, as we move to the higher levels with more terms the oscillations diminishes, as we have seen in Figures 4.17 and 4.18. Therefore, the cable tension obtained by the wavelets solution is more accurate, while the cable displacements obtained by the Fourier and wavelets solutions are in good agreement. The Fourier solution does not require very small time step to obtain accuracy and convergence. However, the wavelet solution does require a small time step for accuracy results and it requires many iterations in the nonlinear model to obtain convergence which makes the algorithm very slow.

Using our Fourier nonlinear model to compare with the finite element solution done by Wu et al.[52] for a horizontal cable carrying a mass moving at constant speed, we noticed that our solution of $\mu = 1.0$ captured their solution of $\mu = 2.0$. We believe that their solution has a factor of two missing. We also believe that our

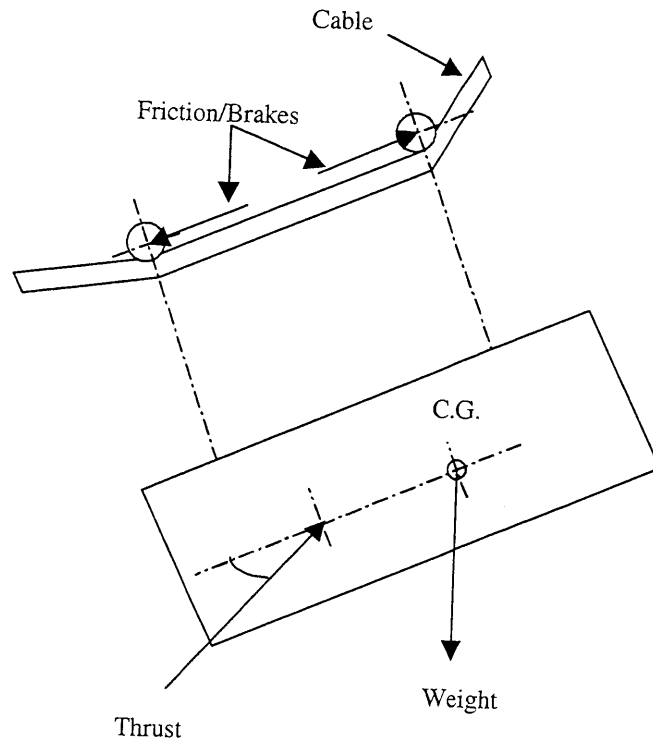


Figure 5.1: Schematic Diagram of a Trolley Suspended on a Cable

solution is more accurate because we have used the full nonlinear terms, while they used linear and quadratic terms only.

The moving mass considered in this thesis was assumed to be a lumped mass acting on the cable at a single point and its aerodynamic characteristics are ignored and the friction between the mass and the cable is also ignored. The cable bending moment has been ignored. This work can be extended by considering that the moving mass is a trolley acting on the cable at two points and has known moment of inertia and powered by a rocket with a thrust applied at a desired inclination see Figure 5.1. Brakes can be also included so that the trolley can be brought to rest before it reaches the second cable support.

APPENDIX A
WAVELET BASIS FUNCTIONS

APPENDIX A

WAVELET BASIS FUNCTIONS

A.1 Definition of Wavelets

A wavelet is a localized function on the real line $\mathbb{R}$ [10, 11, 19]. It occupies only an interval, say $[a, b]$, and outside this interval it is zero. This interval is determined by translation and dilation of the wavelet along the real line. If $\psi \in L^2[\mathbb{R}]$ is a wavelet then its translation and dilation is represented by

$$\psi_{jk}(x) = 2^{j/2} \psi(2^j x - k)$$

where $j, k \in \mathbb{Z}$ and $\mathbb{Z}$ is the set of all integers. The integer k is the parameter that is responsible of translation and the integer j is the parameter that is responsible of dilation. By changing k the wavelet will move along the real line $\mathbb{R}$ and by changing j the width of the wavelet will change. If j is greater than zero then the wavelet will be narrow and of high frequency. If j is less than zero the wavelet will be wide and of low frequency, see Figure A.1 (This is the Gaussian wavelet, we use it to show the property of dilation and translation only). In comparison with the Fourier transform we see that the wavelets transform is characterized by two parameters which are j and k , while Fourier transform is characterized by one parameter say n ,

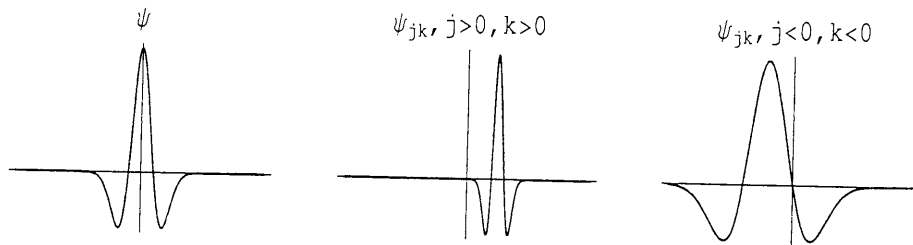


Figure A.1: Illustration of Dilation and Translation of Wavelets

in e^{inx} . Fourier transform is restricted to the use of sine and cosine functions only, whereas in wavelets transform there are infinitely many functions. Both transforms have the similarity of being powerful in representing functions that might be a signal, an image, a solution of partial differential equation, etc. But, the Fourier transform has a disadvantage in representing functions that are of singular nature and the Gibbs phenomenon is expected. The main advantage of wavelets is that they are local functions and the ability to overcome the Gibbs phenomenon is highly possible. Figure A.2 shows the Fourier and wavelet representations of the square-wave function. The Fourier representation shows many oscillation, while wavelet representation shows few oscillation at the boundaries only.

Wavelets are constructed from scaling functions, $\phi(x)$ that satisfy a recursion formula called two-scale relation,

$$\phi(x) = \sum_k a_k \phi(2x - k); \quad k \in \mathbb{Z} \quad (\text{A.1})$$

where $a_k, k \in \mathbb{Z}$ are called the filter coefficients. The values of the integer numbers are in the range $[-\infty, \infty]$. If k goes over a finite numbers of integers, then the scaling function has compact supports. Therefore wavelets are defined by

$$\psi(x) = \sum_k b_k \phi(2x - k); \quad k \in \mathbb{Z} \quad (\text{A.2})$$

Although wavelets were introduced in the beginning of the 1980s, the oldest wavelet is the Haar wavelets which has been known since 1910. Its scaling function is given by

$$\phi(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

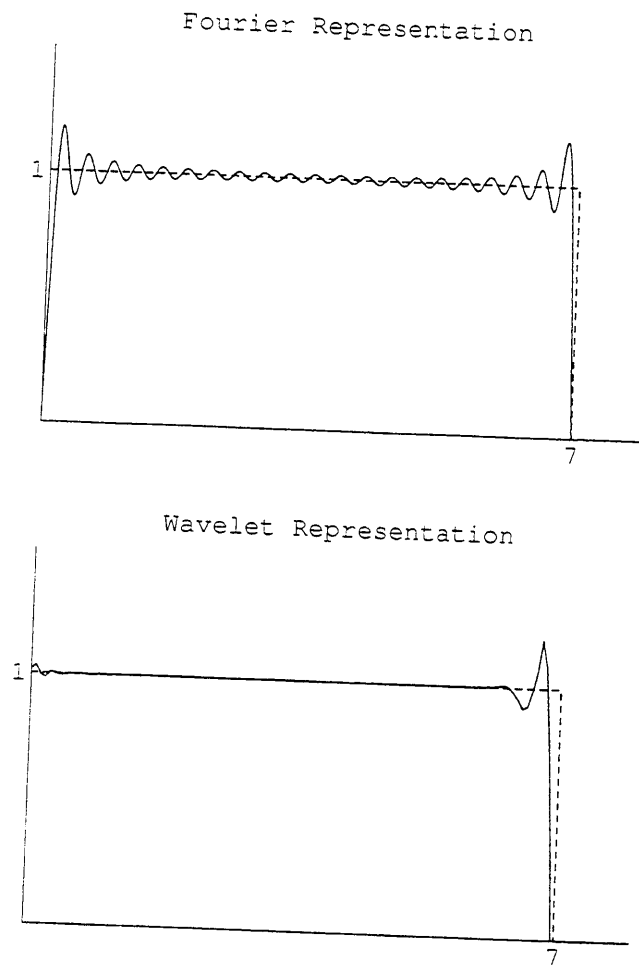


Figure A.2: Comparison Between Fourier and Wavelet Representations of the Square-Wave Function. In Both Methods the Number of Terms Used is 42

which satisfy the two-scale relation

$$\phi(x) = \phi(2x) + \phi(2x - 1)$$

and the wavelet is defined by

$$\psi(x) = \phi(2x) - \phi(2x - 1)$$

See figure A.3 for illustration.

A.2 Multiresolution Analysis

The multiresolution analysis consists of a sequence of closed subspaces V_j of $L^2(\mathbb{R})$ with the following properties:

1. $V_j \subset V_{j+1} \quad \forall j \in \mathbb{Z}$
2. $\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R})$ and $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$
3. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1} \quad \forall j \in \mathbb{Z}$

4. $f(x) \in V_j \Leftrightarrow f(x - 2^{-j}k) \in V_j \quad \forall k \in \mathbb{Z}$

There exists a scaling function $\phi(x) \in V_0$ so that

$$\phi_{jk} = 2^{j/2} \phi(2^j x - k)$$

where the set $\{\phi_{jk}, k \in \mathbb{Z}\}$ is an orthonormal basis for V_j . There also exists a sequence of subspace W_j that is orthogonal complement of V_j in V_{j+1}

$$V_{j+1} = V_j \oplus W_j$$

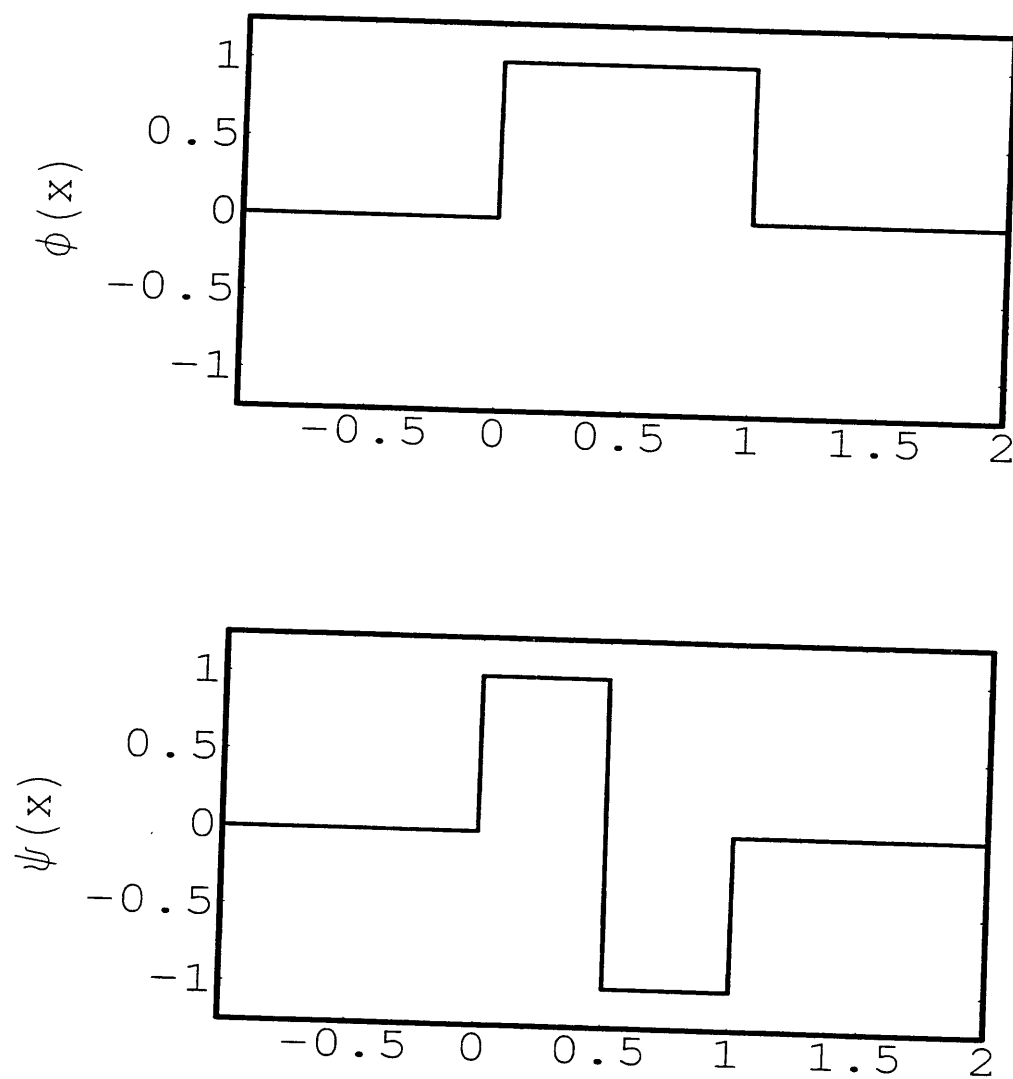


Figure A.3: Haar Scaling Function and Wavelet

this implies that the subspace V_j can be written as the direct sum

$$V_j = \bigoplus_{k > j} W_k$$

The family of functions $\{\psi_{jk} = 2^{j/2}\psi(2^j x - k), k \in \mathbb{Z}\}$ forms an orthonormal basis for W_j . The representation of a square integrable functions, i.e. $f(x) \in L^2(\mathbb{R})$ is

$$f(x) = \sum_{j,k \in \mathbb{Z}} c_{jk} \psi_{jk}(x)$$

where $c_{jk} = \langle f, \psi_{jk} \rangle$ is the inner product.

A.3 Compactly Supported Wavelets

When a wavelet has finitely many nonzero filter coefficients then it is called compactly supported wavelet. To construct such a wavelet it is sufficient to choose a scaling function ϕ with compact support, say $[0, 2p - 1]$, where p is a positive integer. The set of the filter coefficients that make ϕ nonzero is $\{a_k; k = 0, 1, \dots, 2p - 1\}$. Once this function is found the set of all translated and dilated wavelets will have compact supports. Consider the following properties:

The scaling function is normalized with unity

$$\int_{-\infty}^{\infty} \phi(x) dx = 1 \tag{A.3}$$

This formula will give us a condition on the filter coefficients when we substitute for

the two-scale relation (A.1)

$$\begin{aligned}
 \int_{-\infty}^{\infty} \phi(x) dx &= \int_{-\infty}^{\infty} \sum_{k=0}^{2p-1} a_k \phi(2x - k) dx \\
 &= \frac{1}{2} \sum_{k=0}^{2p-1} a_k \int_{-\infty}^{\infty} \phi(x) dx \\
 \therefore \sum_{k=0}^{2p-1} a_k &= 2
 \end{aligned} \tag{A.4}$$

The second property requires the scaling function be orthogonal to its translates, therefore,

$$\int_{-\infty}^{\infty} \phi(x) \phi(x - k) dx = \delta_{0k} \quad k \in \mathbb{Z} \tag{A.5}$$

where

$$\delta_{0k} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

Again, using the two scale relation we get another condition on the filter coefficients

$$\begin{aligned}
 \int_{-\infty}^{\infty} \phi(x) \phi(x - k) dx &= \delta_{0k} \\
 &= \int_{-\infty}^{\infty} \sum_{i=0}^{2p-1} a_i \phi(2x - i) \sum_{j=0}^{2p-1} a_j \phi(2x - 2k - j) dx \\
 &= \frac{1}{2} \sum_{i=0}^{2p-1} \sum_{j=0}^{2p-1} a_i a_j \int_{-\infty}^{\infty} \phi(x) \phi(x - j + i - 2k) dx \\
 \therefore \sum_{i=0}^{2p-1} a_i a_{i-2k} &= 2\delta_{0k} \quad k \in \mathbb{Z}
 \end{aligned} \tag{A.6}$$

The requirement on ψ is to be chosen so that it has vanishing moments to low degree polynomials. i.e.

$$\int_{-\infty}^{\infty} x^l \psi(x) dx = 0; \quad l = 0, 1, 2, \dots, p-1 \tag{A.7}$$

For a compactly supported wavelet, ψ is orthogonal to the scaling function ϕ and is

given by

$$\psi(x) = \sum_{k=2-2p}^1 (-1)^k a_{1-k} \phi(2x - k) \quad (\text{A.8})$$

and its interval is $[1 - p, p]$. Therefore, the last condition on the filter coefficients become

$$\sum_{k=0}^{2p-1} (-1)^k k^l a_k = 0; \quad l = 0, 1, 2, \dots, p-1 \quad (\text{A.9})$$

Equations (A.4), (A.6) and (A.9) can be used to obtain the filter coefficients for a given p and then form a compactly supported wavelets. If $p = 1$, this will result in the Haar wavelet. When $p \geq 2$ the Daubechies wavelets will be produced and they are called Daubechies $2p$. Once the filter coefficients are found, we can use the two scale relation and find the scaling function ϕ . We first substitute for x by integer values $\{x = k; k = 0, 1, \dots, 2p-1\}$ using the fact that $\phi(x) = 0$ for $x \leq 0$ and $x \geq 2p-1$, we obtain the eigenvalue problem corresponding to an eigenvalue of one,

$$\mathbf{A}\phi = \phi \quad (\text{A.10})$$

where

$$\begin{aligned} \mathbf{A} &= [a_{2j-k}]_{1 \leq j, k \leq 2p-2} \\ \phi &= [\phi(1) \phi(2) \dots \phi(2p-2)]^T \end{aligned}$$

For a unique solution the eigenvectors are normalized with the condition

$$\sum_{k=1}^{2p-2} \phi(k) = 1 \quad k \in \mathbb{Z} \quad (\text{A.11})$$

After finding ϕ at the integers, the two scale relation can be used at dyadic points,

which is $x = k/2^j$

$$\phi\left(\frac{k}{2^j}\right) = \sum_{i=0}^{2p-1} a_i \phi\left(\frac{k}{2^{j-1}} - i\right)$$

where $k = 1, 3, 5, \dots, 2^j(2p-1) - 1$ and $j = 1, 2, \dots$. For illustration, consider $p = 2$. Using equations (A.4), (A.6) and (A.9) we obtain the filter coefficients as

$$a_0 = \frac{1 + \sqrt{3}}{4}, a_1 = \frac{3 + \sqrt{3}}{4}, a_2 = \frac{3 - \sqrt{3}}{4}, a_3 = \frac{1 - \sqrt{3}}{4}$$

and equation (A.10) and (A.11) give us ϕ at integers

$$\phi(1) = \frac{1 + \sqrt{3}}{2}, \phi(2) = \frac{1 - \sqrt{3}}{2}$$

Figure A.4 shows the scaling function and its wavelet for selected values of p . Notice that the wavelets are shifted by $p - 1$. The construction of the derivatives of the compactly supported wavelets are achieved from the scaling function, also. For the n th derivative of $\phi(x)$, we differentiate the scaling function n times

$$\phi^{(n)}(x) = 2^n \sum_{k=0}^{2p-1} a_k \phi^{(n)}(2x - k); \quad n = 0, 1, \dots, p-1 \quad (\text{A.12})$$

This is the two scale relation of $\phi^{(n)}$ which is analogous to the two-scale relation of ϕ , equation (A.1). A similar procedure of constructing ϕ is followed here to construct $\phi^{(n)}$. The normalizing condition can be obtained from equation (A.7), as given by Beylkin[2] and Chen et al.[3]

$$\sum_{l=-\infty}^{\infty} l^n \phi(x - l) = x^n + \sum_{j=1}^n (-1)^j \binom{n}{j} M_j^\phi x^{n-j}, \quad n = 0, 1, \dots, p-1 \quad (\text{A.13})$$

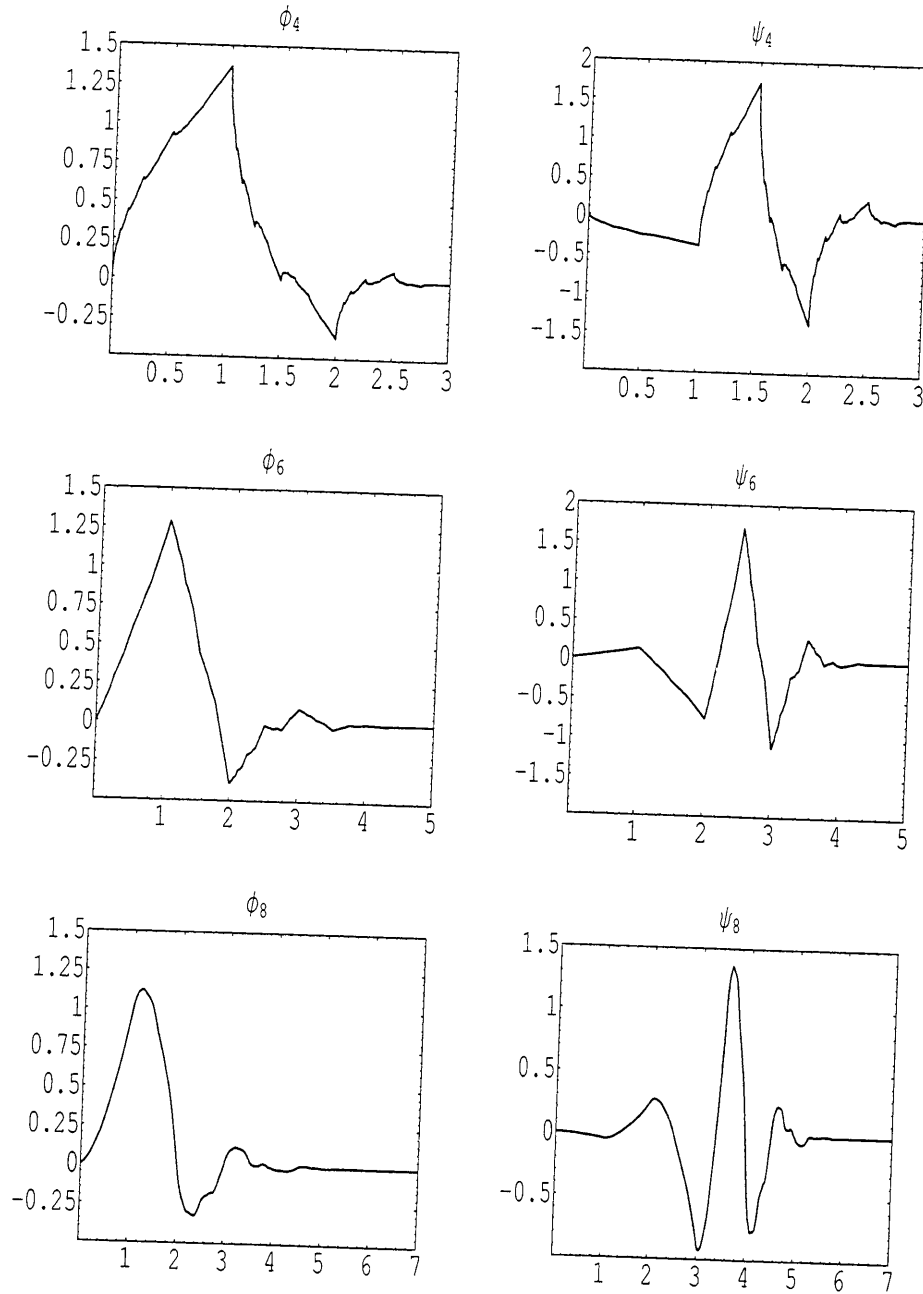


Figure A.4: Examples of Compactly Supported Scaling Functions and Wavelets

where

$$\begin{aligned} M_k^\phi &= \int_{-\infty}^{\infty} x^k \phi(x) dx \\ &= \frac{1}{2^{k+1} - 2} \sum_{i=0}^{2p-1} \sum_{l=1}^k \binom{k}{l} a_i i^l M_{k-l}^\phi \end{aligned}$$

with the initial condition $M_0^\phi = 1$. Differentiating equation (A.13) n times we obtain the normalizing condition for $\phi^{(n)}$

$$\sum_{k=1}^{2p-2} (-k)^n \phi^{(n)}(k) = n! \quad (\text{A.14})$$

Substituting for x by integer values in equation (A.14) we obtain the eigenvalue problem for $\phi^{(n)}$ corresponding to the eigenvalue of 2^{-n}

$$\mathbf{A}\phi = 2^{-n}\phi$$

where $\mathbf{A}$ is as defined before and $\phi = [\phi^{(n)}(1) \ \phi^{(n)}(2) \ \dots \ \phi^{(n)}(2p-2)]^T$. Notice that we have used $\phi^{(n)}(x) = 0$ for $x \leq 0$ and $x \geq 2p-1$. After obtaining the eigenvectors we use equation (A.14) to obtain $\phi^{(n)}$ at integers and then use the two scale relation (A.12) at dyadic points

$$\phi^{(n)}\left(\frac{k}{2^j}\right) = 2^n \sum_{i=0}^{2p-1} a_i \phi^{(n)}\left(\frac{k}{2^{j-1}} - i\right)$$

and the n th derivative of wavelets at dyadic points is readily obtained

$$\psi^{(n)}\left(\frac{k}{2^j}\right) = 2^n \sum_{k=2-2p}^1 (-1)^k a_{1-k} \phi^{(n)}\left(\frac{k}{2^{j-1}} - i\right)$$

As an example see Figure A.5 for the first derivative of Daubechies scaling function

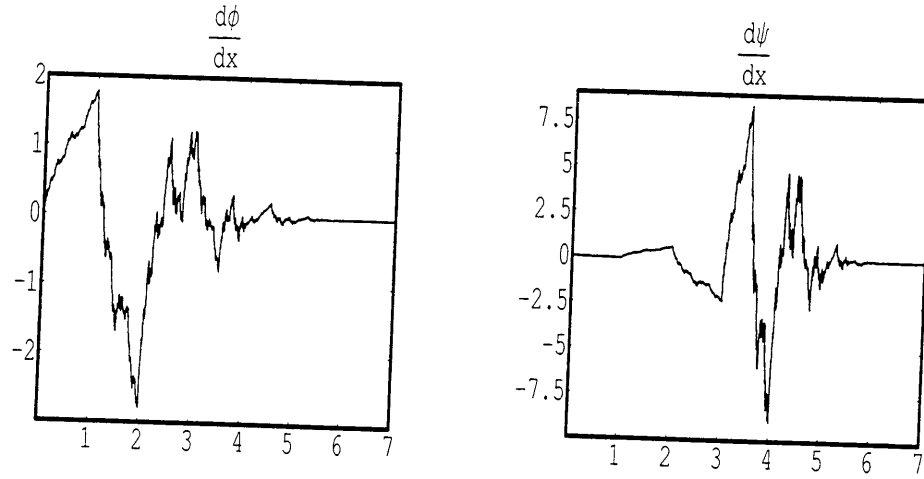


Figure A.5: The First Derivative of the Scaling Function and the Wavelet When $p = 4$ and wavelet when $p = 4$. The evaluation of the integrals of the scaling function can also be found from the two-scale relation (A.1). Consider the multiple integral of $\phi(x)$

$$\begin{aligned}\theta_n(x) &= \int_0^x \int_0^{y_n} \cdots \int_0^{y_2} \phi(y_1) dy_1 \cdots dy_{n-1} dy_n \\ &= \int_0^x \theta_{n-1}(y) dy\end{aligned}\tag{A.15}$$

where $\theta_n(x)$ is the n -tuple integrals of $\phi(x)$. When substituting the two scale relation of $\phi(x)$ in equation (A.15) we obtain the two scale relation for $\theta_n(x)$

$$\theta_n(x) = 2^{-n} \sum_{k=0}^{2p-1} a_k \theta_n(2x - k)\tag{A.16}$$

For $x \geq 2p - 1$, the integral $\theta_n(x)$ does not equal to zero, therefore we do not have an eigenvalue problem here. The values $\theta_n(2p - 1)$ for $n = 1, 2, \dots$ have to be found first in order to obtain $\theta_n(x)$ at dyadic points. This has been done by Chen et. al. [3]

and is given by

$$\theta_n(x) = \sum_{j=0}^{n-1} \frac{(x-2p+1)^j}{j!} \theta_{n-j}(2p-1); \text{ for } x \geq 2p-1 \quad (\text{A.17})$$

where $\theta_1(2p-1) = \int_0^{2p-1} \phi(y) dy = 1$. To find all $\theta_n(2p-1)$, $n = 2, 3, \dots$ we will consider equation (A.16) for $x \geq 2p-1$. Replace $\theta_n(2x-k)$ in equation (A.16) using equation (A.17) and then substitute for $x = 2p-1$ to obtain

$$\begin{aligned} \theta_n(2p-1) &= 2^{-n} \sum_{k=0}^{2p-1} a_k \sum_{j=0}^{n-1} \theta_{n-j}(2p-1) \frac{(2p-1-k)^j}{j!} \\ &= 2^{-n} \left(\theta_n(2p-1) \sum_{k=0}^{2p-1} a_k + \sum_{j=1}^{n-1} \sum_{k=0}^{2p-1} a_k \theta_{n-j}(2p-1) \frac{(2p-1-k)^j}{j!} \right) \end{aligned}$$

Using the property (A.4) in the above equation we get

$$\theta_n(2p-1) = \frac{1}{2^n - 2} \sum_{j=1}^{n-1} \sum_{k=0}^{2p-1} \frac{(2p-1-k)^j}{j!} a_k \theta_{n-j}(2p-1); \quad n = 2, 3, \dots \quad (\text{A.18})$$

The two-scale relation (A.16) now can be used to obtain $\theta_n(x)$; $x = 1, 2, \dots, 2p-2$. This yields

$$(\mathbf{I} - 2^{-n} \mathbf{A}) \boldsymbol{\theta}_n = \mathbf{c}$$

where $\mathbf{I}$ is the identity matrix, $\boldsymbol{\theta}_n = [\theta_n(1) \ \theta_n(2) \ \dots \ \theta_n(2p-2)]^T$, $\mathbf{A}$ is as defined before, and $\mathbf{c} = [c_1 \ c_2 \ \dots \ c_{2p-2}]^T$ where

$$c_i = 2^{-n} \sum_{\substack{2i-k \geq 2p-1 \\ k=0,1,\dots,2p-1}} a_k \theta_n(2i-k)$$

Once $\theta_n(x)$ is found at dyadic points, we can perform the n -tuple integrals of the

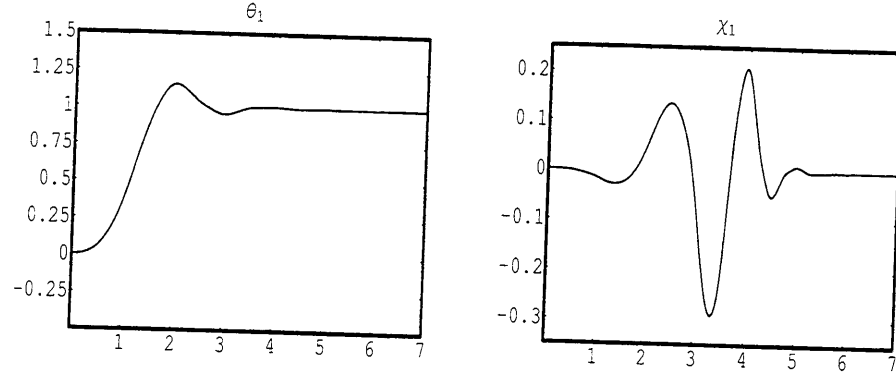


Figure A.6: First Integral of Daubechies Scaling and Wavelet Functions for $p = 4$ wavelets from equation (A.8)

$$\begin{aligned}
 \chi_n(x) &= \int_0^x \int_0^{y_n} \dots \int_0^{y_2} \psi(y_1) dy_1 \dots dy_{n-1} dy_n \\
 &= \sum_{k=2-2p}^1 (-1)^k a_{1-k} \int_0^x \int_0^{y_n} \dots \int_0^{y_2} \phi(2x-k) dy_1 \dots dy_{n-1} dy_n \\
 &= 2^{-n} \sum_{k=2-2p}^1 (-1)^k a_{1-k} \theta_n(2x-k)
 \end{aligned} \tag{A.19}$$

Figure A.6 shows the first integral of the Daubechies scaling and wavelet functions of $p = 4$.

A.4 Anti-Derivative Wavelets

The anti-derivative wavelets are derived by Xu and Shann[53] to smooth out the wavelets and to easily satisfy the boundary conditions of differential equations. They considered the anti-derivative of Daubechies compactly supported wavelets to represent functions in the Sobolev frames defined by

$H_0^1(\Omega) = \{v \in H^1(\Omega) | v(a) = v(b) = 0\}$ and $H_*^1(\Omega) = \{v \in H^1(\Omega) | v(a) = 0\}$ where $\Omega = (a, b)$, $-\infty < a < b < \infty$, and $v \in L^2(\mathbb{R})$. For $p \geq 1$, $R = 2p - 1$, $\Omega = (0, R)$ and ψ_{jk} is a wavelets of order p , they have the following lemma:

For $j \geq -1$ and $k \in I_j$, the functions

$$\Psi_{jk}(x) = \int_0^x \psi_{jk} ds - x \bar{\psi}_{jk}, \text{ for } 0 \leq x \leq R \quad (\text{A.20})$$

form a frame of $H_0^1(\Omega)$, and the functions

$$\Psi_{jk}(x) = \int_0^x \psi_{jk} ds, \text{ for } 0 \leq x \leq R \quad (\text{A.21})$$

form a frame of $H_*^1(\Omega)$, where

$$\bar{\psi}_{jk} = \frac{1}{R} \int_0^R \psi_{jk} ds \quad (\text{A.22})$$

The authors have used numerical integration to evaluate the integrals in equations (A.20)-(A.22), but we can use the algorithm given by Chen et al.[3] which is explained in the previous section. For our problem of the moving mass along a cable we need to use $H_0^1(\Omega)$ frame. Using equation (A.19) with $n = 1$, equation (A.20) becomes

$$\Psi_{jk}(x) = 2^{-j/2} \left[\chi_1(2^j x - k) - \chi_1(-k) - \frac{x}{R} (\chi_1(2^j R - k) - \chi_1(-k)) \right], \text{ for } 0 \leq x \leq R \quad (\text{A.23})$$

where $j \geq -1$ and

$$k \in D_j \iff \begin{cases} 1 - R \leq k \leq p - 1 & \text{if } j = -1 \\ p - R \leq k \leq 2^j R - p & \text{if } j \geq 0 \end{cases} \quad (\text{A.24})$$

Figure A.7 shows a comparison of the Daubechies wavelets and the anti-derivative wavelets of order $p = 4$ and $j = 1$.

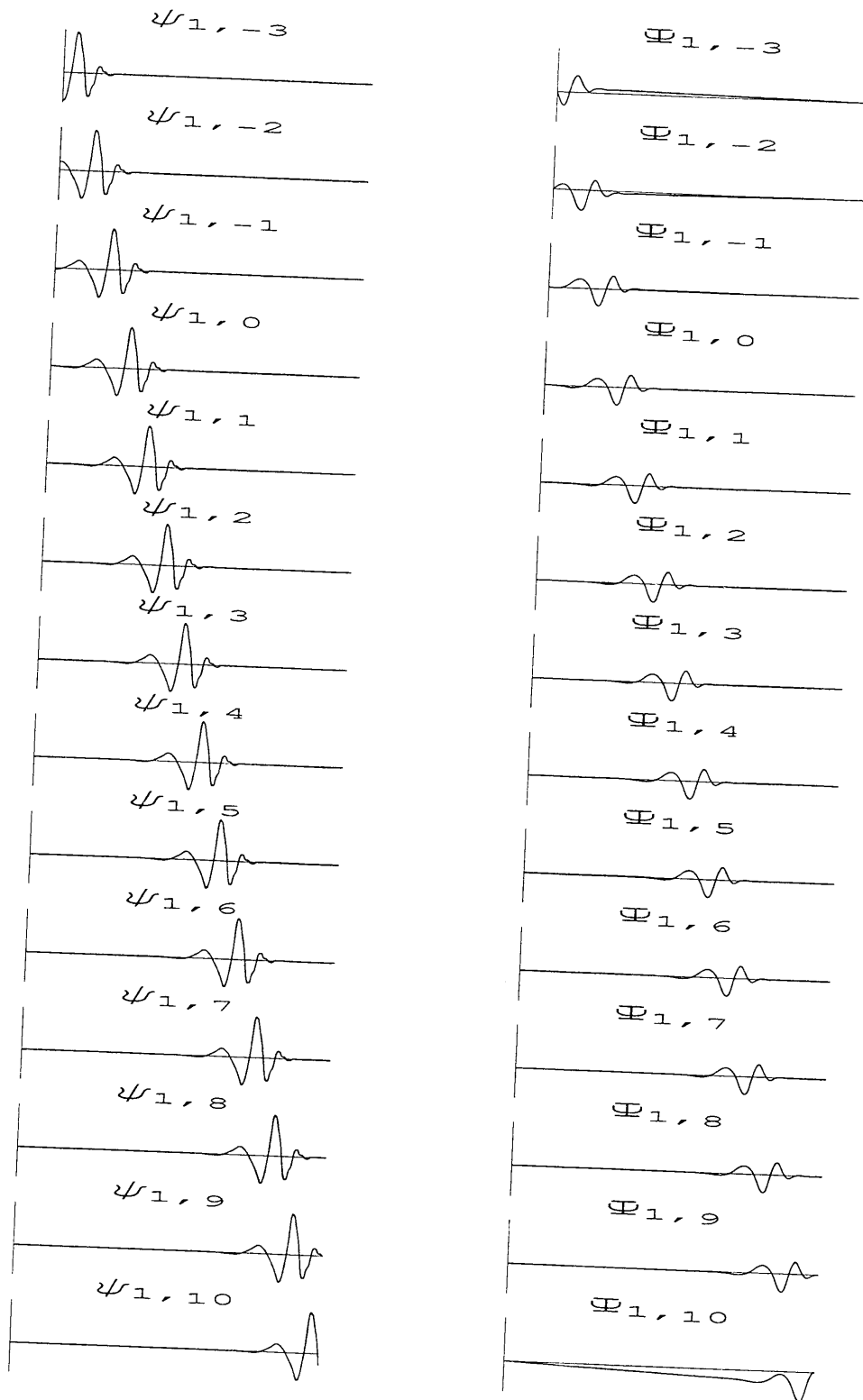


Figure A.7: Wavelets(Left) and Anti-Derivative Wavelets(Right) of Daubechies Wavelets of Order $p = 4$ and Level $j = 1$

A.5 Wavelet-Galerkin Solution of the Vibration of String

In this section our aim is to illustrate, by means of an example, the use of wavelets in solving partial differential equations. The trial basis functions that we are going to use will be the anti-derivative wavelets of the Daubechies compactly supported wavelets discussed in the previous section. This will guarantee the satisfaction of the fixed boundary conditions of the string. We shall derive the equation of motion from the Hamilton's principle as follows. Denoting by $w(x, t)$ as the string displacement, we write the kinetic energy and the potential energy of the string

$$K = \frac{1}{2} \int_0^l \rho \dot{w}^2 dx \quad (\text{A.25})$$

$$\Pi = \int_0^l T (\sqrt{1 + w'^2} - 1) dx \quad (\text{A.26})$$

where l is the length of the string, ρ is the mass density per unit length of the string and T is the tension of the string. The dot denote the differentiation with respect to time, t , and the prime denote differentiation with respect to the spatial variable, x . Using equation (A.25) and (A.26) in the Hamilton's principle

$$\begin{aligned} H &= \int_{t_1}^{t_2} (K - \Pi) dt \\ &= \int_{t_1}^{t_2} \int_0^l \left(\frac{1}{2} \rho \dot{w}^2 - T (\sqrt{1 + w'^2} - 1) \right) dx dt \end{aligned} \quad (\text{A.27})$$

Performing variation on the above equation

$$\begin{aligned} \delta H &= 0 \\ &= \delta \int_{t_1}^{t_2} \int_0^l \left(\frac{1}{2} \rho \dot{w}^2 - T (\sqrt{1 + w'^2} - 1) \right) dx dt \\ &= \int_{t_1}^{t_2} \int_0^l \left(\rho \dot{w} \delta \dot{w} - T \frac{w' \delta w'}{\sqrt{1 + w'^2}} \right) dx dt \end{aligned}$$

Integrating the first term by parts we get

$$\int_0^l \rho \dot{w} \delta w dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^l (\rho \ddot{w} \delta w + T \frac{w'}{\sqrt{1+w'^2}} \delta w') dx dt = 0$$

The first term vanishes. By considering that the string displacement is very small and consequently $w'^2 \ll 1$ we derive the linear vibration of the string as follows

$$\int_{t_1}^{t_2} \int_0^l (\rho \ddot{w} \delta w + T w' \delta w') dx dt = 0 \quad (\text{A.28})$$

Now we approximate the displacement using the anti-derivative of Daubechies wavelets. We assume that

$$w_J(x, t) = \sum_{j=-1}^{J-1} \sum_{k \in D_j} \alpha_{jk}(t) \Psi_{jk}(x) \quad (\text{A.29})$$

where $J \geq 0$, D_j is as defined in equation (A.24), and Ψ_{jk} is given by equation (A.23). Equation (A.29) constitutes a multilevel representation solution of the string displacement. By choosing an intermediate resolution level j where $0 \leq j \leq J$ we have

$$\begin{aligned} w_j(x, t) &= w_{j-1}(x, t) + \sum_{k \in D_{j-1}} \alpha_{j-1,k}(t) \Psi_{j-1,k}(x) \text{ for } 1 \leq j \leq J \\ w_0(x, t) &= \sum_{k \in D_{-1}} \alpha_{-1,k}(t) \Psi_{-1,k}(x) \end{aligned}$$

Define the following dimensionless parameters

$$x^* = \frac{xR}{l}, \quad w^* = \frac{w}{l}, \quad \alpha_{jk}^* = \frac{\alpha_{jk}}{l}, \quad t^* = t \sqrt{\frac{T}{\rho l^2}} \quad (\text{A.30})$$

Substituting equation (A.29) in equation (A.28) and making use of equation (A.30)

$$\sum_{m=-1}^{J-1} \sum_{n \in D_m} \int_{t_1^*}^{t_2^*} \int_0^R [\Psi_{ij}(x^*) \Psi_{mn}(x^*) \ddot{\alpha}_{mn}^* + R^2 \Psi'_{ij}(x^*) \Psi'_{mn}(x^*) \alpha_{mn}^*] \delta \alpha_{ij}^* dx dt = 0$$

For arbitrary $\delta \alpha_{ij}^*$ the above equation becomes

$$\mathbf{M} \ddot{\boldsymbol{\alpha}}^* + \mathbf{K} \boldsymbol{\alpha}^* = \mathbf{0} \quad (\text{A.31})$$

where

$$\begin{aligned} \boldsymbol{\alpha}^* &= [\alpha_1^* \ \alpha_2^* \ \dots \ \alpha_N^*] \\ \mathbf{M}_{\varsigma\eta} &= \int_0^R \Psi_{\varsigma}(x^*) \Psi_{\eta}(x^*) dx^* \end{aligned} \quad (\text{A.32})$$

$$\mathbf{K}_{\varsigma\eta} = R^2 \int_0^R \Psi'_{\varsigma}(x^*) \Psi'_{\eta}(x^*) dx^* \quad (\text{A.33})$$

where $\varsigma = (i, j)$ and $\eta = (m, n)$ have been used in the above equations. To solve equation (A.31) we apply the trapezoidal rule procedure. In the trapezoidal rule we have the following relations

$$\begin{aligned} \alpha_{n+1}^* &= \alpha_n^* + \frac{\Delta t^*}{2} (\dot{\alpha}_n^* + \dot{\alpha}_{n+1}^*) \\ \dot{\alpha}_{n+1}^* &= \dot{\alpha}_n^* + \frac{\Delta t^*}{2} (\ddot{\alpha}_n^* + \ddot{\alpha}_{n+1}^*) \end{aligned}$$

which can be solved for $\dot{\alpha}_{n+1}^*$ and $\ddot{\alpha}_{n+1}^*$ as function of α_{n+1}^* , hence

$$\dot{\alpha}_{n+1}^* = \frac{2}{\Delta t^*} (\alpha_{n+1}^* - \alpha_n^*) - \dot{\alpha}_n^* \quad (\text{A.34})$$

$$\ddot{\alpha}_{n+1}^* = \frac{4}{\Delta t^{*2}} (\alpha_{n+1}^* - \alpha_n^*) - \frac{4}{\Delta t^*} \dot{\alpha}_n^* - \ddot{\alpha}_n^* \quad (\text{A.35})$$

where Δt^* is the time step size. We write equation (A.31) at $t^* = \Delta t^*(n+1)$, where n is the step number, and then make use of equation (A.35)

$$\mathbf{M} \left[\frac{4}{\Delta t^{*2}} (\boldsymbol{\alpha}_{n+1}^* - \boldsymbol{\alpha}_n^*) - \frac{4}{\Delta t^*} \dot{\boldsymbol{\alpha}}_n^* - \ddot{\boldsymbol{\alpha}}_n^* \right] + \mathbf{K} \boldsymbol{\alpha}_{n+1}^* = \mathbf{0}$$

$$\begin{aligned} \mathbf{K}_{eff} \boldsymbol{\alpha}_{n+1}^* &= \mathbf{F}_{eff\ n} \\ \boldsymbol{\alpha}_{n+1}^* &= \mathbf{K}_{eff}^{-1} \mathbf{F}_{eff\ n} \end{aligned} \quad (\text{A.36})$$

where

$$\begin{aligned} \mathbf{K}_{eff} &= \left[\frac{4}{\Delta t^{*2}} \mathbf{M} + \mathbf{K} \right] \\ \mathbf{F}_{eff\ n} &= \mathbf{M} \left[\frac{4}{\Delta t^{*2}} \boldsymbol{\alpha}_n^* + \frac{4}{\Delta t^*} \dot{\boldsymbol{\alpha}}_n^* + \ddot{\boldsymbol{\alpha}}_n^* \right] \end{aligned}$$

Suppose the initial conditions are given such that

$$w^*(x^*, 0) = f(x^*)$$

$$\dot{w}^*(x^*, 0) = g(x^*)$$

then by using the normalized form of equation (A.29) in the above equations we get

$$\sum_{j=-1}^{J-1} \sum_{k \in D_j} \alpha_{jk}^*(0) \Psi_{jk}(x^*) = f(x^*) \quad (\text{A.37})$$

$$\sum_{j=-1}^{J-1} \sum_{k \in D_j} \dot{\alpha}_{jk}^*(0) \Psi_{jk}(x^*) = g(x^*) \quad (\text{A.38})$$

Multiplying equations (A.37) and (A.38) by $\Psi_{mn}(x^*)$ and integrating we

$$\boldsymbol{\alpha}^*(0) = \mathbf{M}^{-1} \mathbf{F} \quad (\text{A.39})$$

Table A.1: Wavelet-Galerkin Procedure Using Trapezoidal Rule Method

1.	Choose a working resolution level J .
2.	Form the matrices $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{K}_{eff}$.
3.	For given $g(x^*)$ and $h(x^*)$ find $\alpha^*(0)$, $\dot{\alpha}^*(0)$ and $\ddot{\alpha}^*(0)$ from equation (A.39)-(A.41).
4.	Set $n = 0$ and $\alpha_0^* = \alpha^*(0)$, $\dot{\alpha}_0^* = \dot{\alpha}^*(0)$ and $\ddot{\alpha}_0^* = \ddot{\alpha}^*(0)$
5.	Form the vector $\mathbf{F}_{n\ eff}$
6.	Solve $\alpha_{n+1}^* = \mathbf{K}_{eff}^{-1} \mathbf{F}_{n\ eff}$
7.	Update the velocity and acceleration of the string from equations (A.34) and (A.35).
8.	Output if desired, $n \leftarrow n + 1$, go to Step 5.

$$\dot{\alpha}^*(0) = \mathbf{M}^{-1} \mathbf{G} \quad (\text{A.40})$$

where $\mathbf{M}$ and α^* are as defined before, $\dot{\alpha}^*$ has a similar definition α^* , and $\mathbf{F}$ and $\mathbf{G}$ are

$$\begin{aligned} \mathbf{F}_m &= \int_0^R \Psi_{mn}(x^*) f(x^*) dx^* \\ \mathbf{G}_m &= \int_0^R \Psi_{mn}(x^*) g(x^*) dx^* \end{aligned}$$

Substituting $t^* = 0$ in equation (A.31) we obtain the initial acceleration of the string

$$\ddot{\alpha}^*(0) = -\mathbf{M}^{-1} \mathbf{K} \alpha^*(0) \quad (\text{A.41})$$

The procedure to obtain the solution of the string is shown in Table A.1.

Example:

Consider a string fixed between two points. Initially, the string shape is a triangle

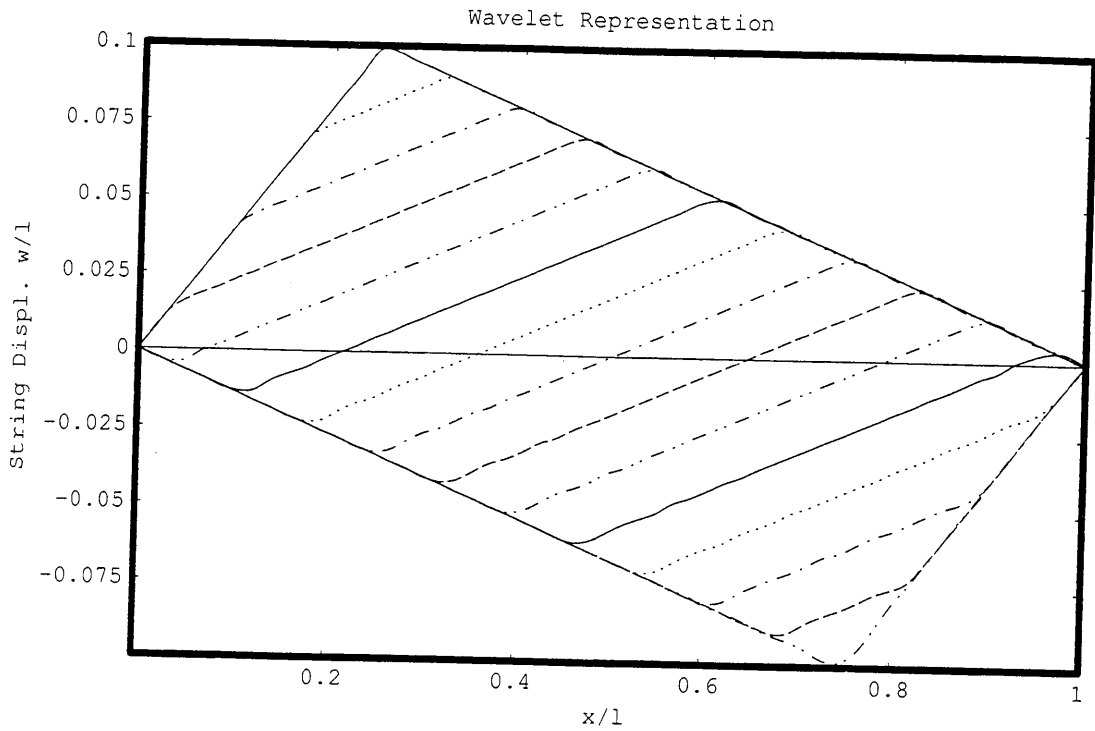


Figure A.8: The Dynamic Shape of the String at Different Time Instants

given by the following function

$$w^*(x^*, 0) = f(x^*) = \begin{cases} \frac{4}{10R}x^* & \text{if } x^* \leq \frac{R}{4} \\ -\frac{4}{30R}(x^* - R) & \text{if } \frac{R}{4} < x^* \leq R \end{cases}$$

The initial velocity is zero, $\dot{w}^*(x^*, 0) = g(x^*) = 0$. The string displacements are found using the wavelet-Galerkin procedure.

We can follow the procedure given in Table A.1 to obtain the solution. Selecting $J = 3$, the dynamic shape of the string at select time steps is show in Figure A.8.

At this point we mention that the matrix $\mathbf{M}$ is sparse. If we chose $J = 3$ Figure A.9 shows the density plot of $\mathbf{M}$ at that level.

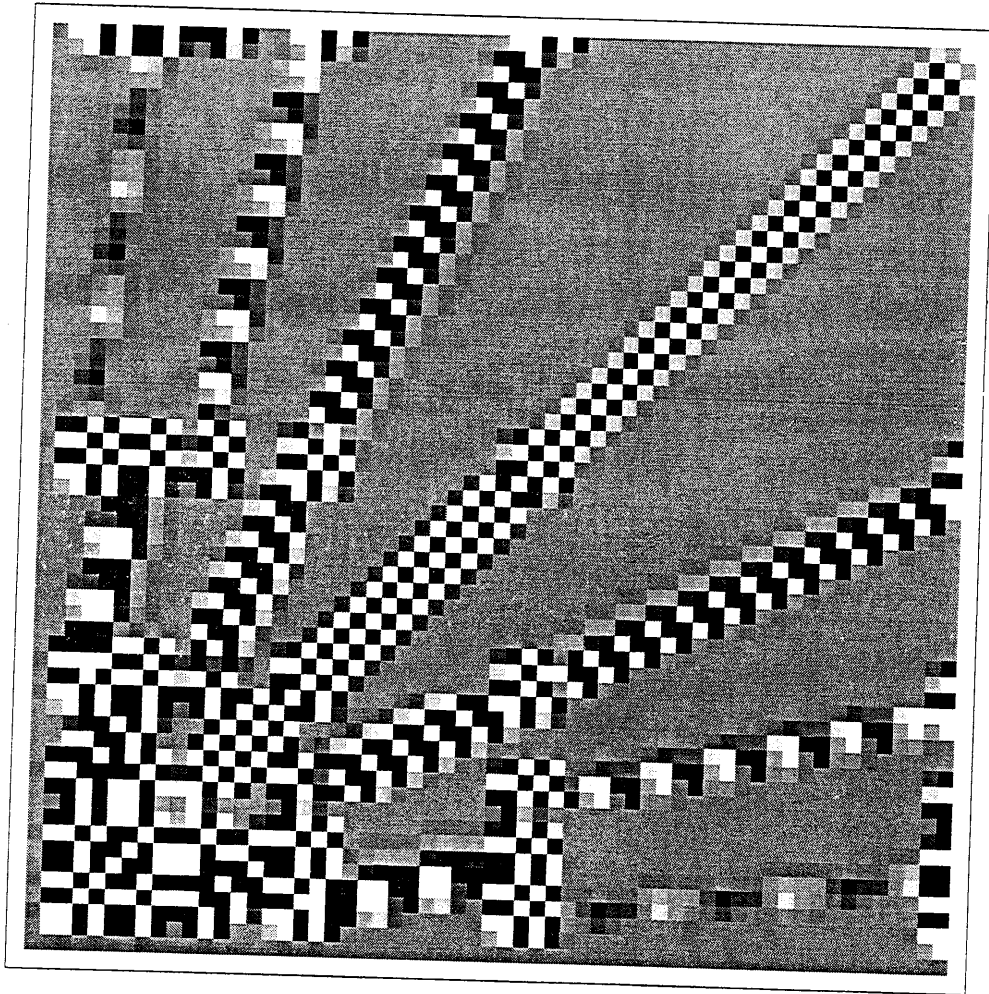


Figure A.9: The Density Plot of the Matrix M . Gray Areas are the Zero Elements

APPENDIX B
COMPUTER PROGRAMS

APPENDIX B

COMPUTER PROGRAMS

B.1 Nonlinear Model With Fourier Expansion

```

C -----
C      Vibration of a cable carrying a moving mass using
C      Fourier Sine representation
C      Nonlinear Case
C      METHOD:Trapezoidal Rule
C -----
      DOUBLE PRECISION mu,v,a1,a2
      INTEGER N,nvar
C-----Test No.0
C   PARAMETER (N=10,nvar=2*N)
C-----Test No.1
      PARAMETER (N=17,nvar=2*N)
C-----Test No.2
C   PARAMETER (N=31,nvar=2*N)
      INTEGER i,j,k,imax,dk
      DOUBLE PRECISION t,dt
      DOUBLE PRECISION y(nvar),dy(nvar),ddy(nvar)
      DOUBLE PRECISION t1,t2,t3
      COMMON /VARIA/ mu,v,a1,a2
      COMMON /TINTER/t1,t2,t3
      OPEN(41,FILE='Ndisf17_1.out',STATUS='UNKNOWN')
      OPEN(42,FILE='Nvelf17_1.out',STATUS='UNKNOWN')
      imax=2048
      dk=imax/128
C
C*****MOVING MASS PARAMETERS*****
C
C-----Example 1
C   mu=0.1
C   v=0.25
C-----Example 2
      mu=1.0
      v =0.5
C
C*****CABLE PARAMETERS*****
C
C-----Horizontal Cable
C   a1=0.08
C   a2=2.0*a1

```

```

C    a1=0.0844171
C    a2=2.0*a1
C-----Inclined Cable
C-----Case 1
C    a1=-0.0237144543964
C    a2= 0.0426806744603
c-----Case 2
C    a1=0.01316459995754039
C    a2=0.1165276127763643
C-----Case 3
      a1=0.077316785865
      a2=0.064602346992
C-----Constant Speed
C    v=v*sqrt(a2)
C    dt=1.0/(imax*v)
C-----Constant Acceleration(here v is acceleration)
      v=v*a2
      dt=sqrt(2.0/v)/imax
C-----Variable Acceleration
C    v=v*sqrt(a2)
C    t1=2.0/(3.0*v)
C    t2=1.0/v
C    t3=5.0/(3.0*v)
C    dt=t3/imax
C
      do 1 i=1,nvar
        y(i)=0.0
        dy(i)=0.0
        ddy(i)=0.0
1    continue
      k=dk
c    write(6,100) t,(y(j),j=1,nvar)
      write(41,100) t,(y(j),j=1,nvar)
      write(42,100) t,(dy(j),j=1,nvar)
      do 5 i=1,imax
        t=i*dt
        call trm(N,nvar,t,dt,y,dy,ddy)
        if(i.eq.k) then
c          write(6,100) t,(y(j),j=1,nvar)
            write(41,100) t,(y(j),j=1,nvar)
            write(42,100) t,(dy(j),j=1,nvar)
            k=k+dk
        endif
5    continue
100  FORMAT(200(1x,f15.6))
      CLOSE(UNIT=41)

```

```

CLOSE(UNIT=42)
STOP
END
C -----
SUBROUTINE mass(N,mu,So,M)
INTEGER N
DOUBLE PRECISION mu,So,M(N,N)
INTEGER i,j
DOUBLE PRECISION pi
PARAMETER(pi=3.141592653589793238463)
do 12 i=1,N
    do 11 j=i,N
        M(i,j)=2.0*mu*sin(i*pi*So)*sin(j*pi*So)
        if(i.eq.j) M(i,j)=M(i,j)+1.0
11    continue
12    continue
    return
END
C -----
SUBROUTINE coriol(N,mu,So,dSo,C)
INTEGER N
DOUBLE PRECISION mu,So,dSo,C(N,N)
INTEGER i,j
DOUBLE PRECISION pi
PARAMETER(pi=3.141592653589793238463)
do 12 i=1,N
    do 11 j=1,N
        C(i,j)=4.0*mu*dSo*j*pi*cos(j*pi*So)*sin(i*pi*So)
11    continue
12    continue
    return
END
C -----
SUBROUTINE stfnss(N,mu,So,dSo,ddSo,K)
INTEGER N
DOUBLE PRECISION mu,So,dSo,ddSo,K(N,N)
INTEGER i,j
DOUBLE PRECISION pi
PARAMETER(pi=3.141592653589793238463)
do 12 i=1,N
    do 11 j=1,N
        K(i,j)=2.0*mu*(ddSo*j*pi*cos(j*pi*So)-
*      (dSo*j*pi)**2*sin(j*pi*So))*sin(i*pi*So)
11    continue
12    continue
    return

```

```

END
C -----
SUBROUTINE force(N1,mu,So,dSo,ddSo,a1,a2,F)
INTEGER N1
DOUBLE PRECISION mu,So,dSo,ddSo,a1,a2,F(N1+N1)
INTEGER i
DOUBLE PRECISION pi,dx,ddx,dy,ddy
pi=3.141592653589793238463
do 11 i=1,N1
    F(i) =-2.0*mu*(ddSo*dx(a1,a2,So)+
*      dSo**2*ddx(a1,a2,So))*sin(i*pi*So)
    F(i+N1)=-2.0*mu*(ddSo*dy(a1,a2,So)+
*      dSo**2*ddy(a1,a2,So)-a2)*sin(i*pi*So)
11 continue
return
END
C -----
SUBROUTINE load(N,N2,a1,a2,y,L)
INTEGER N,N2
DOUBLE PRECISION a1,a2,y(N2),L(N2)
INTEGER i,j,k,NN
PARAMETER(NN=500)
DOUBLE PRECISION h,ld,sigma,du,dv,strain,tension,t,dx,dy,
* s,pi,A(0:NN),B(0:NN),simpson
sigma=5000.0
pi=3.141592653589793238463
h=1.0/NN
ld=sigma*a2
do 30 i=1,N
    do 20 j=0,NN
        s=float(j)*h
        t=sqrt(1.0+(a1-a2*s)**2)
        dx=1.0/t
        dy=(a1-a2*s)/t
        du=0.0
        dv=0.0
        do 10 k=1,N
            du=du+y(k)*k*cos(k*pi*s)
            dv=dv+y(k+N)*k*cos(k*pi*s)
10        continue
        du=pi*du+dx
        dv=pi*dv+dy
        strain=sqrt(du**2+dv**2)-1.0
        tension=(t+ld*strain)/(strain+1.0)
        A(j)=tension*du*cos(i*pi*s)
        B(j)=tension*dv*cos(i*pi*s)
    
```

```

20      continue
        L(i) =2.0*i*pi*simpson(NN,h,A)
        L(i+N)=2.0*(i*pi*simpson(NN,h,B)+a2*((-1.0)**i-1.0)/(i*pi))
c      write(6,*) i,L(i),L(i+N)
c      pause
30      continue
        return
        END
C -----
SUBROUTINE jacobn(N,N2,a1,a2,y,L)
INTEGER N,N2
DOUBLE PRECISION a1,a2,y(N2),L(N2,N2)
INTEGER i,j,jj,k,NN
PARAMETER(NN=500)
DOUBLE PRECISION h,ld,sigma,du,dv,strain,tens1,tens2,t,dx,dy,
*      s,pi,A(0:NN),B(0:NN),C(0:NN),cc,simpson
sigma=5000.0
pi=3.141592653589793238463
h=1.0/NN
ld=sigma*a2
do 40 i=1,N
    do 30 j=i,N
        do 20 jj=0,NN
            s=float(jj)*h
            t=sqrt(1.0+(a1-a2*s)**2)
            dx=1.0/t
            dy=(a1-a2*s)/t
            du=0.0
            dv=0.0
            do 10 k=1,N
                du=du+y(k)*k*cos(k*pi*s)
                dv=dv+y(k+N)*k*cos(k*pi*s)
10          continue
            du=pi*du+dx
            dv=pi*dv+dy
            strain=sqrt(du**2+dv**2)-1.0
            tens1=(t+ld*strain)/(strain+1.0)
            tens2=(ld-t)/(strain+1.0)**3
            cc=cos(i*pi*s)*cos(j*pi*s)
            A(jj)=(tens1+tens2*du**2)*cc
            B(jj)=tens2*du*dv*cc
            C(jj)=(tens1+tens2*dv**2)*cc
20        continue
        L(i,j) =2.0*i*j*pi**2*simpson(NN,h,A)
        L(i,j+N) =2.0*i*j*pi**2*simpson(NN,h,B)
        L(j,i+N) =L(i,j+N)

```

```

                L(i+N,j+N)=2.0*i*j*pi**2*simpson(NN,h,C)
c                write(6,*) L(i,j)
c                pause
30      continue
40      continue
      return
      END
C -----
      FUNCTION simpson(n,h,a)
      INTEGER i,n
      DOUBLE PRECISION simpson,a(0:n),h,s1,s2,s3
      s1=a(0)+a(n)
      s2=0.0
      s3=0.0
      do 11 i=1,n-1,2
          s2=s2+a(i)
11      continue
      do 12 i=2,n-2,2
          s3=s3+a(i)
12      continue
      simpson=(h/3.0)*(s1+4.0*s2+2.0*s3)
      return
      END
C -----
      FUNCTION f1(a1,a2,s)
      DOUBLE PRECISION f1,a1,a2,s
      DOUBLE PRECISION t,dx,sigma
      sigma=5000.0
      t=sqrt(1.0+(a1-a2*s)**2)
      dx=1.0/t
      f1=(sigma*a2-t)*dx**2+t
      return
      END
C -----
      FUNCTION f2(a1,a2,s)
      DOUBLE PRECISION a1,a2,f2,s
      DOUBLE PRECISION t,dx,dy,sigma
      sigma=5000.0
      t=sqrt(1.0+(a1-a2*s)**2)
      dx=1.0/t
      dy=(a1-a2*s)/t
      f2=(sigma*a2-t)*dx*dy
      return
      END
C -----
      FUNCTION f3(a1,a2,s)

```

```

DOUBLE PRECISION a1,a2,f3,s
DOUBLE PRECISION t,dy,sigma
sigma=5000.0
t=sqrt(1.0+(a1-a2*s)**2)
dy=(a1-a2*s)/t
f3=(sigma*a2-t)*dy**2+t
return
END

```

```

C -----
FUNCTION dx(a1,a2,s)
DOUBLE PRECISION a1,a2,s
DOUBLE PRECISION t,dx
t=sqrt(1.0+(a1-a2*s)**2)
dx=1.0/t
return
END

```

```

C -----
FUNCTION ddx(a1,a2,s)
DOUBLE PRECISION a1,a2,s
DOUBLE PRECISION t,ddx
t=sqrt(1.0+(a1-a2*s)**2)
ddx=a2*(a1-a2*s)/(t**3)
return
END

```

```

C -----
FUNCTION dy(a1,a2,s)
DOUBLE PRECISION a1,a2,s
DOUBLE PRECISION t,dy
t=sqrt(1.0+(a1-a2*s)**2)
dy=(a1-a2*s)/t
return
END

```

```

C -----
FUNCTION ddy(a1,a2,s)
DOUBLE PRECISION a1,a2,s
DOUBLE PRECISION t,ddy
t=sqrt(1.0+(a1-a2*s)**2)
ddy=-a2/(t**3)
return
END

```

```

C -----
SUBROUTINE ludcmp(a,n,np,indx,d)
C See Reference[27]
C -----
SUBROUTINE lubksb(a,n,np,indx,b)
C See Reference[27]

```

```

C -----
  SUBROUTINE MvMsM(t,v,So,dSo,ddSo)
  DOUBLE PRECISION t,v,So,dSo,ddSo
C-----Constant Velocity
C   So=v*t
C   dSo=v
C   ddSo=0.0
C-----Constant Acceleration
C   So=v*t**2/2.0
C   dSo=v*t
C   ddSo=v
C-----Nonconstant Speed
C   DOUBLE PRECISION t1,t2,t3
C   COMMON /TINTER/t1,t2,t3
C   if((t.ge.0.0).AND.(t.lt.t1)) then
C     So =v*t**2/(2*t1)
C     dSo =v*t/t1
C     ddSo=v/t1
C   elseif((t.ge.t1).AND.(t.lt.t2)) then
C     So =v*(t-t1/2.0)
C     dSo =v
C     ddSo=0.0
C   else
C     So =(v/(2.0*t1*(t3-t2)))*(t**2*(t3-t2)-(t-t1)**2*(t3-t2)-
C *      t1*(t-t2)**2)
C     dSo =v*(1.0-(t-t2)/(t3-t2))
C     ddSo=-v/(t3-t2)
C   endif
  return
  END
C -----
  SUBROUTINE trm(N,nvar,t,dt,y,dy,ddy)
  INTEGER N,nvar
  DOUBLE PRECISION t,dt,y(nvar),dy(nvar),ddy(nvar)
  DOUBLE PRECISION a1,a2,mu,v,tol
  INTEGER NMAX,NMAX2
C-----Test No.0
C   PARAMETER (NMAX=10,NMAX2=2*NMAX,tol=1.0e-6)
C-----Test No.1
C   PARAMETER (NMAX=17,NMAX2=2*NMAX,tol=1.0e-6)
C-----Test No.2
C   PARAMETER (NMAX=31,NMAX2=2*NMAX,tol=1.0e-6)
  DOUBLE PRECISION M(NMAX,NMAX),C(NMAX,NMAX),K(NMAX,NMAX),
*   F(NMAX2),E(NMAX,NMAX),Keff(NMAX2,NMAX2),Feff(NMAX2),
*   Err(NMAX2),yold(NMAX2),sum,sum1,sum2,sum3,sum4,So,dSo,ddSo
  DOUBLE PRECISION d

```

```

      INTEGER i,j,kk,indx(NMAX2)
      COMMON /VARIA/ mu,v,a1,a2
      do 10 i=1,nvar
        yold(i)=y(i)
10    continue
        call MvMsM (t,v,So,dSo,ddSo)
        call mass(N,mu,So,M)
        do 12 i=1,N-1
          do 11 j=i+1,N
            M(j,i)=M(i,j)
11      continue
12    continue
        call coriol(N,mu,So,dSo,C)
        call stfnss(N,mu,So,dSo,ddSo,K)
        do 14 i=1,N
          do 13 j=1,N
            E(i,j)=4.0*M(i,j)/dt**2+2.0*C(i,j)/dt+K(i,j)
13      continue
14    continue
        call force(N,mu,So,dSo,ddSo,a1,a2,F)
C-----Forming the forcing vector
        do 16 i=1,N
          sum1=0.0
          sum2=0.0
          sum3=0.0
          sum4=0.0
          do 15 j=1,N
            sum1=sum1+M(i,j)*(4.0*y(j)/dt**2+4.0*dy(j)/dt+ddy(j))
            sum2=sum2+M(i,j)*(4.0*y(j+N)/dt**2+4.0*dy(j+N)/dt+
*            ddy(j+N))
            sum3=sum3+C(i,j)*(2.0*y(j)/dt+dy(j))
            sum4=sum4+C(i,j)*(2.0*y(j+N)/dt+dy(j+N))
15      continue
          F(i)=F(i)+sum1+sum3
          F(i+N)=F(i+N)+sum2+sum4
16    continue
C
C  Newton Raphson Method
      do 100 kk=1,500
        call load(N,NMAX2,a1,a2,y,Feff)
        call jacobn(N,NMAX2,a1,a2,y,Keff)
        do 50 i=1,nvar-1
          do 40 j=i+1,nvar
            Keff(j,i)=Keff(i,j)
40      continue
50    continue

```

```

do 18 i=1,N
    sum1=0.0
    sum2=0.0
    do 17 j=1,N
        sum1=sum1+E(i,j)*y(j)
        sum2=sum2+E(i,j)*y(j+N)
        Keff(i,j)=Keff(i,j)+E(i,j)
        Keff(i+N,j+N)=Keff(i+N,j+N)+E(i,j)
17    continue
        Feff(i) =F(i)-sum1-Feff(i)
        Feff(i+N)=F(i+N)-sum2-Feff(i+N)
        Err(i) =Feff(i)
        Err(i+N) =Feff(i+N)
18    continue
        call ludcmp(Keff,nvar,NMAX2,indx,d)
C-----Solving Keff y = Feff
        call lubksb(Keff,nvar,NMAX2,indx,Feff)
        do 19 i=1,nvar
            y(i)=y(i)+Feff(i)
19    continue
        do 20 i=1,nvar
            if((abs(Err(i)).gt.tol).or.(abs(Feff(i)).gt.tol))
*                goto 100
20    continue
        goto 25
100 continue
        write(31,*) 'convergence not achieved'
        stop
C
C-----Trapezoidal Rule
25 do 30 i=1,nvar
        ddy(i)=4.0*(y(i)-yold(i))/dt**2-4.0*dy(i)/dt-ddy(i)
        dy(i)=2.0*(y(i)-yold(i))/dt-dy(i)
30 continue
    return
END

```

B.2 Nonlinear Model With Wavelets Expansion

```

C -----
C      Vibration of a cable carrying a moving mass using
C      Wavelets(D8) representation
C      Nonlinear Case
C      METHOD:Newmark Integration
C      Equations of motions are made function of mass location only
C -----

```

```

DOUBLE PRECISION mu,v,a1,a2
INTEGER p,R,NODES,LEVEL,base,JLEVEL,N,NN,nvar,RR,NDS
DOUBLE PRECISION h,hnn
PARAMETER(LEVEL=14,base=2**LEVEL,p=4,RR=2*p-1,NDS=RR*base)
DOUBLE PRECISION D(0:NDS),ID(0:NDS),dD(0:NDS),zt,at,b0,b1,b2,
*      b3,b4,b5,b6,b7
C-----D :is the Daubechies wavelet
C-----ID:is the anti-derivative wavelet of D
C-----dD:is the first derivative of the wavelet D
C-----Test No.1
C   PARAMETER (JLEVEL=-1)
C-----Test No.2
C   PARAMETER (JLEVEL=0)
C-----Test No.3
C   PARAMETER (JLEVEL=1)
C-----Test No.4
C   PARAMETER (JLEVEL=2)
C   PARAMETER (N=2**((JLEVEL+1)*RR+RR-p,nvar=2*N)
C   INTEGER i,j,k,dk,lpsize,indx(nvar)
C   DOUBLE PRECISION Ms(N,N),y(nvar),dy(nvar),ddy(nvar),So,dt,
C   * x,Keff(nvar,nvar)
C   COMMON /WAVLET/ D,ID
C   COMMON /DERWAV/ dD
C   COMMON /VARIA/ mu,v,a1,a2
C   COMMON /STIFF/ Ms
C   COMMON /DIMEN1/ R,NODES,h
C   COMMON /NWMARK/ zt,at,b0,b1,b2,b3,b4,b5,b6,b7
C   OPEN(31,FILE='NdW17_2.out',STATUS='UNKNOWN')
C   OPEN(32,FILE='NvW17_2.out',STATUS='UNKNOWN')
C
C*****MOVING MASS
C
C-----Example 1
C   mu=0.1
C   v =0.25
C-----Example 2
C   mu=1.0
C   v =1.0
C
C*****CABLE PARAMETERS
C
C-----Horizontal Cable
C
C-----Case 1
C   a1=0.08
C   a2=2.0*a1

```

```

C-----Case 2
C   a1=0.0844416
C   a2=2.0*a1
C
C-----Inclined Cable
C
C   a1=0.077316785865
C   a2=0.064602346992
C
C-----Wavelet D8
C   R   =RR
C   NODES =NDS
C   h   =1.0/base
C   NN   =R*2**8
C   hnn  =1.0/2.0**8
C   a2   =a2/R
C   mu   =mu*R
C   lpsize=R*2**12
C   dk=lpsize/4096
C   dt=float(R)/float(lpsize)
C   call wavelet(D,ID,dD)
C
C*****MOVING MASS MOTION
C
C-----Constant Velocity
C
C   v=v*R*sqrt(a2*R)
C
C-----Constant Acceleration(here v is acceleration)
C
C   v=v*a2*R**2
C
C-----Constants for the Newmark method
C
C   zt>=0.5; at>=0.25(0.5+zt)^2
C   zt=20.0
C   at=110.0
C   b0=1.0/(at*dt**2)
C   b1=zt/(at*dt)
C   b2=1.0/(at*dt)
C   b3=0.5/at-1.0
C   b4=zt/at-1.0
C   b5=0.5*dt*(zt/at-2.0)
C   b6=dt*(1.0-zt)
C   b7=zt*dt
C

```

```

      call masmat(Ms,N,p,R,JLEVEL)
      So=0.0
      call frczro(N,p,R,JLEVEL,mu,So,v,a1,a2,ddy)
      x=2.0/3.0
      do 13 i=1,N
        do 12 j=i,N
          Keff(i,j)=x*Ms(i,j)
          Keff(i+N,j+N)=x*Ms(i,j)
          Keff(i,j+N)=0.0
          Keff(j,i+N)=0.0
12      continue
13      continue
        do 15 i=1,nvar-1
          do 14 j=i+1,nvar
            Keff(j,i)=Keff(i,j)
14      continue
15      continue
        call ludcmp(Keff,nvar,nvar,indx,x)
        call lubksb(Keff,nvar,nvar,indx,ddy)
        do 1 i=1,nvar
          y(i)=0.0
          dy(i)=0.0
C      for constant velocity uncomment the line below
C      ddy(i)=0.0
1      continue
      k=dk
C      write(6,100) So,(y(j),j=1,nvar)
      write(31,100) So,(y(j),j=1,nvar)
      write(32,100) So,(dy(j),j=1,nvar)
      do 5 i=1,lpsize
        So=i*dt
        call newmrk(N,p,R,JLEVEL,nvar,So,y,dy,ddy)
C      write(6,100) So,(y(j),j=1,nvar)
        if(i.eq.k) then
          write(31,100) So,(y(j),j=1,nvar)
          write(32,100) So,(dy(j),j=1,nvar)
          k=k+dk
        endif
5      continue
100  FORMAT(200(1x,f15.6))
      CLOSE(UNIT=31)
      CLOSE(UNIT=32)
      STOP
      END
C -----
      SUBROUTINE wavelet(w,iw,dw)

```

```

C
C-----TO CONSTRUCT: w(x),d/dx(w),integral(w)
C
C
      INTEGER p,R,JJ,B,NDS
      PARAMETER(p=4,R=2*p-1,JJ=14,B=2**JJ,NDS=R*B)
      DOUBLE PRECISION A(0:NDS),w(0:NDS),dw(0:NDS),iw(0:NDS),
      *      c(0:R)
      INTEGER i,j,k,n1,n2,n3,n4,n5
      DOUBLE PRECISION s
C
C-----Wavelets coefficients
C
      c(0)=0.325803428051298382
      c(1)=1.01094571509182906
      c(2)=0.892200138246759699
      c(3)=-0.0395750262356446356
      c(4)=-0.264507167369039652
      c(5)=0.0436163004741772564
      c(6)=0.0465036010709817659
      c(7)=-0.0149869893303614709
C
C -----
C -----CONSTRUCT phi,psi-----
C -----
C
C-----phi at integer values(given)
C
      A(0) =0.0
      A(B) =1.0071699777256
      A(2*B)=-0.033836954052833927
      A(3*B)=0.039610462715903107
      A(4*B)=-0.011764358205726631
      A(5*B)=-0.001197957596177039
      A(6*B)=0.000018829413233547629
      A(7*B)=0.0
C
C-----calculate phi at dyadic points
C   See Reference[54]
C
C -----
C -----CONSTRUCT dphi,dpsi-----
C -----
C
C-----dphi at integer values(given); dphi:=A
C

```

```

A(0) =0.0
A(B) =1.776072007522192
A(2*B)=-2.785349397229546
A(3*B)=1.192452536632277
A(4*B)=-0.1313745151846724
A(5*B)=-0.05357102822023941
A(6*B)=0.001770396479992539
A(7*B)=0.0
C
C-----calculate dphi at dyadic points
C
  do 20 j=1,JJ
    n1=2**(JJ-j)
    n2=R*2**j
    do 19 k=1,n2,2
      n3=n1*k
      n4=2*n3
      s=0.0
      do 18 i=0,R
        n5=n4-i*B
        if((n5.gt.0).AND.(n5.lt.NDS)) s=s+2.0*c(i)*A(n5)
18      continue
        A(n1*k)=s
19    continue
20  continue
C
C-----calculate dpsi at dyadic points
C
  n1=(1-p)*B
  do 22 i=n1,p*B
    s=0.0
    n2=2*i
    do 21 k=1-R,1
      n3=n2-k*B
      if((n3.gt.0).AND.(n3.lt.NDS))
*      s=s+2.0*(-1)**k*c(1-k)*A(n3)
21    continue
    dw(i-n1)=s
22  continue
C
C -----
C -----CONSTRUCT iphi,ipsi-----
C -----
C
C-----iphi at integer values(given); iphi:=A
C

```

```

A(0) =0.0
A(B) =0.3799344099327083
A(2*B)=1.153381836328733
A(3*B)=0.954427917930807
A(4*B)=1.006365528562792
A(5*B)=1.000500936929562
A(6*B)=0.999996156872149
A(7*B)=1.0
C
C-----calculate iphi at dyadic points
C
  do 25 j=1,JJ
    n1=2**(JJ-j)
    n2=R*2**j
    do 24 k=1,n2,2
      n3=n1*k
      n4=2*n3
      s=0.0
      do 23 i=0,R
        n5=n4-i*B
        if(n5.gt.0) then
          if(n5.lt.NDS) then
            s=s+c(i)*A(n5)/2.0
          else
            s=s+c(i)/2.0
          endif
        else
          goto 23
        endif
      23      continue
      A(n1*k)=s
    24      continue
  25      continue
C
C-----calculate ipsi at dyadic points
C
  n1=(1-p)*B
  do 27 i=n1,p*B
    s=0.0
    n2=2*i
    do 26 k=1-R,1
      n3=n2-k*B
      if(n3.gt.0) then
        if(n3.lt.NDS) then
          s=s+(-1)**k*c(1-k)*A(n3)/2.0
        else

```

```

        s=s+(-1)**k*c(1-k)/2.0
    endif
    else
        goto 26
    endif
26    continue
    iw(i-n1)=s
27    continue
    return
    END
C -----
SUBROUTINE mass(N,p,R,JLEVEL,mu,So,Ms,M)
INTEGER N,p,R,JLEVEL
DOUBLE PRECISION mu,So,Ms(N,N),M(N,N)
INTEGER i,j,ii,jj,kmin,kmax,n1,n2,k1,k2,k11,k22
DOUBLE PRECISION psi,x1,x2,xo,xf,TINY
PARAMETER(TINY=1.0e-15)
do 14 i=-1,JLEVEL
    k1=kmin(i,p,R)
    k2=kmax(i,p,R)
    if(i.eq.-1) then
        n1=0
    else
        n1=2**i*R+R-p
    endif
    n1=n1+1
    do 13 j=k1,k2
        x1=mu*psi(i,j,R,So)
        if(abs(x1).gt.TINY) then
            do 12 ii=i,JLEVEL
                if(ii.eq.i) then
                    k11=j
                    k22=k2
                    n2=n1
                else
                    k11=kmin(ii,p,R)
                    k22=kmax(ii,p,R)
                    n2=2**ii*R+R-p
                    n2=n2+1
                endif
            do 11 jj=k11,k22
                x2=psi(ii,jj,R,So)
                M(n1,n2)=Ms(n1,n2)+x1*x2
                n2=n2+1
            11
        12
        13
    14
    continue
12    continue

```

```

        else
            do 10 n2=n1,N
                M(n1,n2)=Ms(n1,n2)
10         continue
            endif
            n1=n1+1
13     continue
14     continue
        return
        END
C -----
SUBROUTINE coriol(N,p,R,JLEVEL,mu,So,dSo,C)
INTEGER N,p,R,JLEVEL
DOUBLE PRECISION mu,So,dSo,C(N,N)
INTEGER i,j,ii,jj,kmin,kmax,k1,k2,k11,k22,n1,n2
DOUBLE PRECISION psi,dpsi,x1,x2,TINY,mu2v
PARAMETER(TINY=1.0e-15)
mu2v=2.0*mu*dSo**2
do 14 i=-1,JLEVEL
    k1=kmin(i,p,R)
    k2=kmax(i,p,R)
    if(i.eq.-1) then
        n1=0
    else
        n1=2**i*R+R-p
    endif
    n1=n1+1
    do 13 j=k1,k2
        x1=mu2v*psi(i,j,R,So)
        if(abs(x1).gt.TINY) then
            do 12 ii=-1,JLEVEL
                k11=kmin(ii,p,R)
                k22=kmax(ii,p,R)
                if(ii.eq.-1) then
                    n2=0
                else
                    n2=2**ii*R+R-p
                endif
                n2=n2+1
                do 11 jj=k11,k22
                    x2=dpsi(ii,jj,R,So)
                    C(n1,n2)=x1*x2
                    n2=n2+1
11                 continue
12             continue
                else

```

```

do 10 n2=1,N
    C(n1,n2)=0.0
10    continue
    endif
    n1=n1+1
13    continue
14    continue
    return
END
C -----
SUBROUTINE stfnss(N,p,R,JLEVEL,mu,So,dSo,ddSo,K)
INTEGER N,p,R,JLEVEL
DOUBLE PRECISION mu,So,dSo,ddSo,K(N,N)
INTEGER i,j,ii,jj,kmin,kmax,i1,i2,i11,i22,n1,n2
DOUBLE PRECISION psi,dpsi,ddpsi,x1,x2,x3,TINY
PARAMETER(TINY=1.0e-15)
do 14 i=-1,JLEVEL
    i1=kmin(i,p,R)
    i2=kmax(i,p,R)
    if(i.eq.-1) then
        n1=0
    else
        n1=2**i*R+R-p
    endif
    n1=n1+1
    do 13 j=i1,i2
        x1=mu*psi(i,j,R,So)
        if(abs(x1).gt.TINY) then
            do 12 ii=-1,JLEVEL
                i11=kmin(ii,p,R)
                i22=kmax(ii,p,R)
                if(ii.eq.-1) then
                    n2=0
                else
                    n2=2**ii*R+R-p
                endif
                n2=n2+1
                do 11 jj=i11,i22
                    x2=dpsi(ii,jj,R,So)
                    x3=ddpsi(ii,jj,So)
                    K(n1,n2)=x1*(ddSo*x2+dSo**2*x3)
                    n2=n2+1
11                continue
12            continue
            else
                do 10 n2=1,N

```

```

                                K(n1,n2)=0.0
10          continue
              endif
              n1=n1+1
13    continue
14  continue
      return
      END
C  -----
  SUBROUTINE force(N,p,R,JLEVEL,mu,So,dSo,ddSo,a1,a2,F)
  INTEGER N,p,R,JLEVEL
  DOUBLE PRECISION mu,So,dSo,ddSo,a1,a2,F(N+N)
  DOUBLE PRECISION dx,ddx,dy,ddy,psi,temp,x1,x2,y1,y2,fx,fy
  INTEGER i,j,kmin,kmax,k1,k2,n1
  n1=0
  x1=dx(a1,a2,So)
  x2=ddx(a1,a2,So)
  y1=dy(a1,a2,So)
  y2=ddy(a1,a2,So)
  fx=mu*(ddSo*x1+dSo**2*x2)
  fy=mu*(ddSo*y1+dSo**2*y2-a2*R**2)
  do 14 i=-1,JLEVEL
    k1=kmin(i,p,R)
    k2=kmax(i,p,R)
    do 13 j=k1,k2
      n1=n1+1
      temp=psi(i,j,R,So)
      F(n1) =-fx*temp
      F(n1+N)=-fy*temp
13    continue
14  continue
      return
      END
C  -----
  SUBROUTINE frczro(N,p,R,JLEVEL,mu,So,ddSo,a1,a2,F)
  INTEGER N,p,R,JLEVEL
  DOUBLE PRECISION mu,So,ddSo,a1,a2,F(N+N)
  DOUBLE PRECISION dx,dy,dpsi,temp,x1,y1,fx,fy
  INTEGER i,j,kmin,kmax,k1,k2,n1
  n1=0
  x1=dx(a1,a2,So)
  y1=dy(a1,a2,So)
  fx=mu*x1/2.0
  fy=mu*(y1-a2*R**2/ddSo)/2.0
  do 14 i=-1,JLEVEL
    k1=kmin(i,p,R)

```

```

        k2=kmax(i,p,R)
        do 13 j=k1,k2
            n1=n1+1
            temp=dpsi(i,j,R,So)
            F(n1) =-fx*temp
            F(n1+N)=-fy*temp
13      continue
14      continue
        return
        END
C -----
SUBROUTINE load(p,R,JLEVEL,N,NV,a1,a2,y,L)
INTEGER p,R,JLEVEL,N,NV
DOUBLE PRECISION a1,a2,y(NV),L(NV),TINY
INTEGER i,ii,j,jj,k,kmin,kmax,k1,k2,k11,k22,n1,n2,NN,R2,MX,pn
PARAMETER(NN=448,MX=114688,TINY=1.0e-15)
DOUBLE PRECISION ld,sigma,du,dv,strain,tension,t,dx,dy,s,s1,s2,
* s3,s4,A(0:NN),B(0:NN),simpson,psi,dpsi,h,D(0:MX),ID(0:MX),
* p11,p12,p13,p21,p22,p23,Wb(200),x1
COMMON /WAVLET/ D,ID
COMMON /WAVBAR/ Wb
sigma=5000.0
ld=sigma*a2*R
h=float(R)/float(NN)
R2=R**2
do 30 i=-1,JLEVEL
    p11=2.0**(i/2.0)
    p12=2.0**i
    p13=2.0**(-i/2.0)
    k1=kmin(i,p,R)
    k2=kmax(i,p,R)
    if(i.eq.-1) then
        n1=0
    else
        n1=2**i*R+R-p
    endif
    n1=n1+1
    do 35 ii=k1,k2
        x1=p13*Wb(n1)
        s=float(-ii)
        s2=ID(pn(s))
        do 20 j=0,NN
            s=j*h
C          s1=dpsi(i,ii,R,s)
C          s1=p11*D(pn(p12*s-ii))-x1
C          s3= psi(i,ii,R,s)

```

```

s3=p13*(ID(pn(p12*s-ii))-s2-s*Wb(n1))
if(abs(s1).gt.TINY) then
  t=sqrt(1.0+(a1-a2*s)**2)
  dx=1.0/t
  dy=(a1-a2*s)/t
  du=0.0
  dv=0.0
  n2=0
  do 25 jj=-1,JLEVEL
    k11=kmin(jj,p,R)
    k22=kmax(jj,p,R)
    p21=2**((jj/2.0)
    p22=2.0**jj
    p23=2**(-jj/2.0)
    do 10 k=k11,k22
      n2=n2+1
      s4=dpsi(jj,k,R,s)
      s4=p21*D(pn(p22*s-k))-p23*Wb(n2)
      du=du+y(n2)*s4
      dv=dv+y(n2+N)*s4
10      continue
25      continue
      du=du+dx
      dv=dv+dy
      strain=sqrt(du**2+dv**2)-1.0
      tension=(t+ld*strain)/(strain+1.0)
      A(j)=tension*du*s1
      B(j)=tension*dv*s1-a2*s3
    else
      A(j)=0.0
      B(j)=-a2*s3
    endif
20      continue
      L(n1) =R2*simpson(NN,h,A)
      L(n1+N)=R2*simpson(NN,h,B)
C      write(6,*) L(n1),L(n1+N)
      n1=n1+1
35      continue
30      continue
66      format(100(1x,f15.10))
      return
      END
C -----
SUBROUTINE jacobn(p,R,JLEVEL,N,NV,a1,a2,y,L)
INTEGER p,R,JLEVEL,N,NV
DOUBLE PRECISION a1,a2,y(NV),L(NV,NV),TINY

```

```

INTEGER NN,kmin,kmax,R2,MX
PARAMETER(NN=448,MX=114688,TINY=1.0e-15)
DOUBLE PRECISION ld,sigma,du,dv,strain,tens1,tens2,t,dx,dy,
*   s,A(0:NN),B(0:NN),C(0:NN),cc,simpson,dpsi,D(0:MX),ID(0:MX)
INTEGER i,ii,j,k,kk,k1,k2,k11,k22,k33,k44,n1,n2,n3,
*   ll,m,kll,mll,pn
DOUBLE PRECISION sum,p1,p2,p3,x1,x2,x3,p11,p22,p33,x11,x22,x33,
*   p41,p42,p43,xo,xf,xoo,xff,Wb(200),h
COMMON /WAVLET/ D,ID
COMMON /WAVBAR/ Wb
sigma=5000.0
ld=sigma*a2*R
R2=R**2
h=float(R)/float(NN)
do 911 j=-1,JLEVEL
    p1=2.0**(j/2.0)
    p2=2.0**j
    p3=2.0**(-j/2.0)
    kll=R*2**j-R
    if(j.eq.-1) then
        k1=1-R
        k2=p-1
        n1=0
    else
        k1=p-R
        k2=2**j*R-p
        n1=R*2**j+R-p
    endif
    n1=n1+1
do 912 k=k1,k2
    if(j.ge.1) then
        if((k.ge.0).AND.(k.le.kll)) then
            xo=k/p2
            xf=(k+R)/p2
        else
            xo=0.0
            xf=R
        endif
    else
        xo=0.0
        xf=R
    endif
    x2=p3*Wb(n1)
do 913 ll=j,JLEVEL
    p11=2.0**(ll/2.0)
    p22=2.0**ll

```

```

p33=2.0**(-ll/2.0)
mll=R*2**ll-R
if(ll.eq.j) then
    k11=k
    k22=k2
    n2=n1
else
    k11=p-R
    k22=2**ll*R-p
    n2=R*2**ll+R-p
    n2=n2+1
endif
do 914 m=k11,k22
    if(ll.ge.1) then
        if((m.ge.0).AND.(m.le.mll)) then
            xoo=m/p22
            xff=(m+R)/p22
        else
            xoo=0.0
            xff=R
        endif
    else
        xoo=0.0
        xff=R
    endif
    if((xo.lt.xff).OR.(xoo.lt.xf)) then
        x22=p33*Wb(n2)
        do 915 ii=0,NN
            s =ii*h
            x1 =D(pn(p2*s-k))
            x11=D(pn(p22*s-m))
            x3 =p1*x1-x2
            x33=p11*x11-x22
            cc =x3*x33
            if(abs(cc).gt.TINY) then
                t=sqrt(1.0+(a1-a2*s)**2)
                dx=1.0/t
                dy=(a1-a2*s)/t
                du=0.0
                dv=0.0
                n3=0
                do 11 i=-1,JLEVEL
                    k33=kmin(i,p,R)
                    k44=kmax(i,p,R)
                    p41=2**(i/2.0)
                    p42=(2.0)**i

```

```

                                p43=2**(-i/2.0)
                                do 10 kk=k33,k44
                                    n3=n3+1
                                    sum=p41*D(pn(p42*s-kk))
                                        -p43*Wb(n3)
                                    du=du+y(n3)*sum
                                    dv=dv+y(n3+N)*sum
10                                continue
11                                continue
                                du=du+dx
                                dv=dv+dy
                                strain=sqrt(du**2+dv**2)-1.0
                                tens1=(t+ld*strain)/(strain+1.0)
                                tens2=(ld-t)/(strain+1.0)**3
                                A(ii)=(tens1+tens2*du**2)*cc
                                B(ii)=tens2*du*dv*cc
                                C(ii)=(tens1+tens2*dv**2)*cc
                                else
                                    A(ii)=0.0
                                    B(ii)=0.0
                                    C(ii)=0.0
                                endif
915                                continue
                                L(n1,n2) =R2*simpson(NN,h,A)
                                L(n1,n2+N)=R2*simpson(NN,h,B)
                                L(n2,n1+N)=L(n1,n2+N)
                                L(n1+N,n2+N)=R2*simpson(NN,h,C)
c
                                else
                                    L(n1,n2)=0.0
                                    L(n1,n2+N)=0.0
                                    L(n2,n1+N)=0.0
                                    L(n1+N,n2+N)=0.0
                                endif
C
                                n2=n2+1
914                                continue
913                                continue
                                n1=n1+1
912                                continue
911                                continue
                                return
                                END
C -----
SUBROUTINE masmat(W,N,p,R,JLEVEL)

```

```

INTEGER N,p,R,JLEVEL
DOUBLE PRECISION W(N,N),h2
INTEGER ii,j,k,k1,k2,k11,k22,n1,n2,ll,m,pn
DOUBLE PRECISION sum,p1,p2,x1,x2,x3
DOUBLE PRECISION p11,p22,x11,x22,x33
DOUBLE PRECISION xo,xf,xoo,xff,simpson,Wb(200)
INTEGER NX,MX
PARAMETER (NX=1792,MX=114688)
DOUBLE PRECISION a(0:NX),D(0:MX),ID(0:MX)
COMMON /WAVLET/ D,ID
COMMON /WAVBAR/ Wb
h2=float(R)/float(NX)
do 904 j=-1,JLEVEL
  p1=2.0**(-j/2.0)
  p2=2.0**j
  if(j.EQ.-1) THEN
    k1=1-R
    k2=p-1
    n1=0
  else
    k1=p-R
    k2=2**j*R-p
    n1=R*2**j+R-p
  endif
  n1=n1+1
  do 905 k=k1,k2
    if(j.GE.1) THEN
      if((k.GE.0).AND.(k.LE.(R*2**j-R))) THEN
        xo=2.0**(-j)*k
        xf=2.0*(-j)*(k+R)
      else
        xo=0.0
        xf=R
      endif
    else
      xo=0.0
      xf=R
    endif
    sum=float(-k)
    x1=ID(pn(sum))
    if(j.NE.-1) THEN
      if((k.GE.0).AND.(k.LE.(R*2**j-R))) THEN
        x2=0.0
      else
        x2=(ID(pn(p2*R-k))-ID(pn(sum)))/R
      endif
    endif
  enddo
enddo

```

```

else
  x2=(ID(pn(p2*R-k))-ID(pn(sum)))/R
endif
Wb(n1)=x2
do 906 ll=j,JLEVEL
  p11=2.0**(-ll/2.0)
  p22=2.0**ll
  if(ll.EQ.j) THEN
    k11=k
    k22=k2
    n2 =n1
  else
    k11=p-R
    k22=2**ll*R-p
    n2=R*2**ll+R-p
    n2=n2+1
  endif
do 907 m=k11,k22
  if(ll.GE.1) THEN
    if((m.GE.0).AND.(m.LE.(R*2**ll-R))) THEN
      xoo=2.0**(-ll)*m
      xff=2.0**(-ll)*(m+R)
    else
      xoo=0.0
      xff=R
    endif
  else
    xoo=0.0
    xff=R
  endif
  if((xo.LT.xff).OR.(xoo.LT.xf)) THEN
    sum=float(-m)
    x11=ID(pn(sum))
    if(ll.NE.-1) THEN
      if((m.GE.0).AND.(m.LE.(R*2**ll-R))) THEN
        x22=0.0
      else
        x22=(ID(pn(p22*R-m))-ID(pn(sum)))/R
      endif
    else
      x22=(ID(pn(p22*R-m))-ID(pn(sum)))/R
    endif
  endif
do 908 ii=0,NX
  sum=ii*h2
  x3  =ID(pn(p2*sum-k))-x1-sum*x2
  x33 =ID(pn(p22*sum-m))-x11-sum*x22

```

```

          a(ii)=x3*x33
908      continue
          W(n1,n2)=p1*p11*simpson(NX,h2,a)
          else
          W(n1,n2)=0.0
          endif
          n2 = n2 + 1
907      continue
906      continue
          n1 = n1 + 1
905      continue
904      continue
          return
          END
C -----
FUNCTION kmin(j,p,R)
INTEGER kmin,j,p,R
if(j.eq.-1) then
    kmin=1-R
else
    kmin=p-R
endif
return
END
C -----
FUNCTION kmax(j,p,R)
INTEGER kmax,j,p,R
if(j.eq.-1) then
    kmax=p-1
else
    kmax=2**j*R-p
endif
return
END
C -----
FUNCTION psi(j,k,R,x)
INTEGER j,k,R
DOUBLE PRECISION psi,x
INTEGER MX,pn
PARAMETER(MX=114688)
DOUBLE PRECISION D(0:MX),ID(0:MX),p1,p2,s,s1
COMMON /WAVLET/ D,ID
p1=2.0**(-j/2.0)
p2=2.0**j
s=float(-k)
s1=ID(pn(s))

```

```

    psi=p1*(ID(pn(p2*x-k))-s1-x*(ID(pn(p2*R-k))-s1)/R)
    return
END
C -----
FUNCTION dpsi(j,k,R,x)
  INTEGER j,k,R
  DOUBLE PRECISION dpsi,x
  INTEGER MX,pn
  PARAMETER(MX=114688)
  DOUBLE PRECISION D(0:MX),ID(0:MX),p1,p2,p3,s,s1
  COMMON /WAVLET/ D,ID
  p1=2.0**(j/2.0)
  p2=2.0**j
  p3=2.0**(-j/2.0)
  s=float(-k)
  s1=ID(pn(s))
  dpsi=p1*D(pn(p2*x-k))-p3*(ID(pn(p2*R-k))-s1)/R
  return
END
C -----
FUNCTION ddpsi(j,k,x)
  INTEGER j,k
  DOUBLE PRECISION ddpsi,x
  INTEGER MX,pn
  PARAMETER(MX=114688)
  DOUBLE PRECISION dD(0:MX),p1,p2
  COMMON /DERWAV/ dD
  p1=2.0**(3.0*j/2.0)
  p2=2.0**j
  ddpsi=p1*dD(pn(p2*x-k))
  return
END
C -----
FUNCTION pn(x)
C See Reference[54]
C -----
FUNCTION simpson(n,h,a)
  INTEGER i,n
  DOUBLE PRECISION simpson,a(0:n),h,s1,s2,s3
  s1=a(0)+a(n)
  s2=0.0
  s3=0.0
  do 11 i=1,n-1,2
    s2=s2+a(i)
11 continue
  do 12 i=2,n-2,2

```

```

        s3=s3+a(i)
12  continue
    simpson=(h/3.0)*(s1+4.0*s2+2.0*s3)
    return
END
C  -----
    FUNCTION dx(a1,a2,s)
    DOUBLE PRECISION a1,a2,s
    DOUBLE PRECISION t,dx
    t=sqrt(1.0+(a1-a2*s)**2)
    dx=1.0/t
    return
END
C  -----
    FUNCTION ddx(a1,a2,s)
    DOUBLE PRECISION a1,a2,s
    DOUBLE PRECISION t,ddx
    t=sqrt(1.0+(a1-a2*s)**2)
    ddx=a2*(a1-a2*s)/(t**3)
    return
END
C  -----
    FUNCTION dy(a1,a2,s)
    DOUBLE PRECISION a1,a2,s
    DOUBLE PRECISION t,dy
    t=sqrt(1.0+(a1-a2*s)**2)
    dy=(a1-a2*s)/t
    return
END
C  -----
    FUNCTION ddy(a1,a2,s)
    DOUBLE PRECISION a1,a2,s
    DOUBLE PRECISION t,ddy
    t=sqrt(1.0+(a1-a2*s)**2)
    ddy=-a2/(t**3)
    return
END
C  -----
    SUBROUTINE ludcmp(a,n,np,indx,d)
C  See Reference[27]
C  -----
    SUBROUTINE lubksb(a,n,np,indx,b)
C  See Reference[27]
C  -----
    SUBROUTINE MvMsM(v,So,dSo,ddSo)
    DOUBLE PRECISION v,So,dSo,ddSo

```

```

C-----Constant Velocity
C   dSo=v
C   ddSo=0.0
C-----Constant Acceleration
      ddSo=v
      dSo =sqrt(2.0*v*So)
      return
      END
C   -----
      SUBROUTINE newmrk(N,p,R,JLEVEL,nvar,So,y,dy,ddy)
      INTEGER N,p,R,JLEVEL,nvar
      DOUBLE PRECISION So,y(nvar),dy(nvar),ddy(nvar)
      DOUBLE PRECISION a1,a2,mu,v,tol
      INTEGER NMAX,NMAX2
C-----Test No.1
C   PARAMETER (NMAX=10,NMAX2=2*NMAX,tol=1.0e-6)
C-----Test No.2
      PARAMETER (NMAX=17,NMAX2=2*NMAX,tol=1.0e-6)
C-----Test No.3
C   PARAMETER (NMAX=31,NMAX2=2*NMAX,tol=1.0e-6)
      DOUBLE PRECISION M(NMAX,NMAX),C(NMAX,NMAX),K(NMAX,NMAX),
*   F(NMAX2),E(NMAX,NMAX),Keff(NMAX2,NMAX2),Feff(NMAX2),
*   Err(NMAX2),yold(NMAX2),sum,sum1,sum2,sum3,sum4,dSo,ddSo
      DOUBLE PRECISION d,Ms(NMAX,NMAX),
*   zt,at,b0,b1,b2,b3,b4,b5,b6,b7,temp
      INTEGER i,j,kk,indx(NMAX2)
      COMMON /STIFF/ Ms
      COMMON /VARIA/ mu,v,a1,a2
      COMMON /NWMARK/ zt,at,b0,b1,b2,b3,b4,b5,b6,b7
      do 10 i=1,nvar
        yold(i)=y(i)
10    continue
        call MvMsM(v,So,dSo,ddSo)
        call mass(NMAX,p,R,JLEVEL,mu,So,Ms,M)
        do 12 i=1,N-1
          do 11 j=i+1,N
            M(j,i)=M(i,j)
11      continue
12    continue
        call coriol(NMAX,p,R,JLEVEL,mu,So,dSo,C)
        call stfnss(NMAX,p,R,JLEVEL,mu,So,dSo,ddSo,K)
        do 121 i=1,N
          do 122 j=1,N
            C(i,j)=C(i,j)+ddSo*M(i,j)
122      continue
121    continue

```

```

      d=dSo**2
      do 123 i=1,N
        do 124 j=i,N
          M(i,j)=d*M(i,j)
          M(j,i)=M(i,j)
124      continue
123      continue
        do 14 i=1,N
          do 13 j=1,N
            E(i,j)=b0*M(i,j)+b1*C(i,j)+K(i,j)
13      continue
14      continue
        call force(NMAX,p,R,JLEVEL,mu,So,dSo,ddSo,a1,a2,F)
C
C-----Forming the forcing vector
      do 16 i=1,N
        sum1=0.0
        sum2=0.0
        sum3=0.0
        sum4=0.0
        do 15 j=1,N
          sum1=sum1+M(i,j)*(b0*y(j)+b2*dy(j)+b3*ddy(j))
          sum2=sum2+M(i,j)*(b0*y(j+N)+b2*dy(j+N)+b3*ddy(j+N))
          sum3=sum3+C(i,j)*(b1*y(j)+b4*dy(j)+b5*ddy(j))
          sum4=sum4+C(i,j)*(b1*y(j+N)+b4*dy(j+N)+b5*ddy(j+N))
15      continue
          F(i)=F(i)+sum1+sum3
          F(i+N)=F(i+N)+sum2+sum4
16      continue
C-----Newton Raphson Method
C
      do 100 kk=1,500
        call load(p,R,JLEVEL,NMAX,NMAX2,a1,a2,y,Feff)
        call jacobn(p,R,JLEVEL,NMAX,NMAX2,a1,a2,y,Keff)
        do 50 i=1,nvar-1
          do 40 j=i+1,nvar
            Keff(j,i)=Keff(i,j)
40      continue
50      continue
        do 18 i=1,N
          sum1=0.0
          sum2=0.0
          do 17 j=1,N
            sum1=sum1+E(i,j)*y(j)
            sum2=sum2+E(i,j)*y(j+N)
            Keff(i,j)=Keff(i,j)+E(i,j)

```

```

      Keff(i+N,j+N)=Keff(i+N,j+N)+E(i,j)
17      continue
      Feff(i) =F(i)-sum1-Feff(i)
      Feff(i+N)=F(i+N)-sum2-Feff(i+N)
      Err(i) =Feff(i)
      Err(i+N) =Feff(i+N)
18      continue
      call ludcmp(Keff,nvar,NMAX2,indx,d)
C-----Solving Keff y = Feff
      call lubksb(Keff,nvar,NMAX2,indx,Feff)
      do 19 i=1,nvar
          y(i)=y(i)+Feff(i)
19      continue
      do 20 i=1,nvar
          if((abs(Err(i)).gt.tol).or.(abs(Feff(i)).gt.tol))
*          goto 100
20      continue
      goto 25
100 continue
      write(31,*) 'convergence not achieved'
      stop
C-----Newmark Relations
25  do 30 i=1,nvar
      temp =ddy(i)
      ddy(i)=b0*(y(i)-yold(i))-b2*dy(i)-b3*ddy(i)
      dy(i)=dy(i)+b6*temp+b7*ddy(i)
30  continue
      return
      END

```

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