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College of Science



Department of Mathematics

A uniform approach to the simple groups of Lie type

A Thesis Submitted in Partial Fulfillment of the Requirements for the
Master of Science Degree in Mathematics

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Thesis Discussion Committee Decision

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The Discussion Committee met on 24-Oct-23 In Microsoft Teams (Online). Upon discussing the student whose details are stated above in public, the discussion committee recommends the following:

The student is awarded a grade A- in the thesis subject.

- ☐ Without making any changes to the thesis.
- ☒ After making the modifications attached with this decision.

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Abstract

In this thesis we study the three simple Lie groups A_2 , B_2 , and G_2 . We explore the roots system of rank 2 for these groups and the associate Lie algebra. We present the structure of the Lie algebra related to these groups graphically.

We also explore the recent construction of Wilson to the groups B_2 and G_2 , where he defined these groups without the use of Lie algebra. He defined these groups using their roots as a spanning set for some vector spaces and defined three types of products on those vector spaces. He then defined the groups to be the linear automorphism that preserve these products.

In this thesis, the same approach will be applied to the simple case of A_2 , where many of the procedures in Wilson's paper will be naturally augmented to A_2 as these groups are all from the same family as simple groups of Lie type with root system of dimension 2.

Dedication

To my mother and father, thank you for comforting, teaching, feeding, and rising me.

Now I find myself standing on my own and I see knowledge flowing around me, and I have nothing in my hand but to say: “Lord, have mercy on them as they raised me when I was small, and increase them in goodness.”

And to my sweet family, and all those who supported me with kindness yesterday, today and in the future.

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Chapter 1

Introduction

- 1.1 Statement of the Problem
- 1.2 Justification of the Problem
- 1.3 Literature Review
- 1.4 Theoretical Framework
- 1.5 Methods and Procedures
- 1.6 Proposed Chapter Outlines
- 1.7 Background and History

Introduction

1.1 Statement of The Problem

Our thesis is about the new approach presented by Wilson [8] to describe the simple groups B_2 , G_2 , and F_2 without the use of Lie algebra while preserving all the conditions and features found in the Lie algebra description (as in Carter's book [2]).

Our goal in this thesis is to describe the root system A_2 in the same way that Wilson did in his article [8], i.e., without the use of Lie Algebra.

1.2 Justification of the Problem

The root system is essential in the theory of Lie groups and Lie algebras especially for the classification of semi simple Lie algebras. We want to expand our knowledge of the root system as much as possible, and since Wilson's work in regard of dimension 2 root system was done for the groups B_2 , G_2 , and F_2 . We would like to complete the picture for the groups A_2 by following the same approach in the paper (Wilson 2013 [8]).

1.3 Literature Review

Since the 60's, there is more research focusing into the root systems after discovering their relationship to semi-simple groups. Many articles have been written and still underway to this day in this field. The MathSciNet database mentions that there are about 300 search titles that include "the root system".

Just after 1960, Suzuki and Rimhak Ree were able to discover the famous groups: the Suzuki groups denoted by $Sz(2^{2n+1})$, ${}^2B_2(2^{2n+1})$, $suz(2^{2n+1})$, or $G(2^{2n+1})$, as an infinite family of groups of Lie type found that are simple for $n \geq 1$. In fact, Suzuki (1960) originally constructed the Suzuki groups as subgroups of $SL_4(F_2^{2n+1})$ generated by certain explicit matrices. Ree observed that the Suzuki groups were the fixed points of exceptional automorphisms of some symplectic groups of dimension 4, and used this to construct two further families of simple groups, called the Ree groups.

Within 40 years, studies continued on the structure and development of the semi-simple groups carried out by Ree and Suzuki. In 2000 Wilson presented a new structure of Ree groups of characteristic 3, and he included all calculations needed. With the same sequence, he found the method of simplifying the construction.

In 2009, Wilson published another article for new approach to the Suzuki group, and he worked the same way he built it in the first article. In 2013 Wilson [8] published an article in which he unified his approach to the Suzuki group and the two Ree groups based his construction on the rings of Gaussian integers, Eisenstein integers, and Hurwitz integers as motivation.

1.4 Theoretical Framework

In this proposal, the focus will be on the root system of dimension 2 for the simple groups A_2 , B_2 , and G_2 . The approach of defining the groups B_2 and G_2 , by Wilson in terms of inner/middle/outer roots and inner, outer, cross products will be used to describe a similar product and hence approach to define A_2 .

1.5 Methods and Procedures

Wilson approach used the Gaussian integers and their units to define the Suzuki group ${}^2B_2$, and the Eisenstein integers and their units in defining Ree group ${}^2G_2$ by assigning what he called inner/middle/outer roots. These roots will be the bases for some vector spaces in which he defined algebraic structure consisting of three product (inner, cross, and star). These vectors spaces with the given structures are isomorphic to the simple Lie algebra ${}^2B_2$, and ${}^2G_2$ (since they have the same automorphisms groups (Suzuki and Ree)). In this proposal, the same approach will apply to the simple case of A_2 , where many of the procedures above will be naturally augmented to A_2 as these groups are all from the same family as simple groups of Lie type with root system of dimension 2.

1.6 Proposed Chapter Outlines

Chapter 2: Introduction to Lie Algebra:

This is an introductory chapter where we will give a basic algebraic introduction to Lie algebras. Here we give a brief about classifications of Lie algebra of small dimensions. We give a summary for solvable and simple Lie algebra.

Chapter 3: Root System of Dimension 2:

In this chapter, we will define the root system of Lie algebra and do some computations to classify them with focus on the root system of dimension 2 and their associated Lie algebras and Lie groups.

Chapter 4: Wilson Approach to the Suzuki and Ree Groups:

Here we will give an introduction to the two research papers published by Wilson, in which he defined new structures for the Suzuki and Ree group without the use of Lie algebra. We will also show that Wilson's structure matches the Lie algebras introduced in Chapter 2.

Chapter 5: Construction of A_2 and ${}^2A_2$ without Lie Algebra:

In this chapter, we will show our work on describing A_2 in the same way as Wilson's description i.e., without using Lie algebra, while keeping all the restrictions and conditions set in the root system in terms of inner/middle/outer roots and inner, outer, cross products.

1.7 Background and History

Lie theory started with the work of Sophus Lie, who defined the so called Lie groups as certain transformation groups. His work continued further to develop the theory of Lie algebra. Currently, Lie groups and Lie algebras are fundamental tools to many area of mathematics and physics. We can get a complete classification of such Lie algebras, provided that we can classify the whole root systems. This was mostly done by Wilhelm Killing and Elie Cartan around 1880-1900. In the middle of the twentieth century, that is, in the 1940s and 1950s, Chevalley proved something similar for reductive algebraic groups. Later, a full description of the simple Lie groups (hence Lie Algebra) of dimension 2 ($A_1 \times A_1$, A_2 , B_2 , G_2) was given by classifying the root systems associated with these groups. Wilson through several publications ([8] for the Suzuki and [4] for the Ree and uniform approach in [6]) described the root systems without the use of Lie algebra structure.

Chapter 2

Introduction to Lie Algebra

2.1 Introduction

2.2 Definition of Lie algebra

2.3 Examples of Lie Algebras

2.4 Subalgebra and ideals of Lie Algebras

2.5 Homomorphisms of Lie Algebras

2.6 Construction of new ideals from old ones

2.7 Dimensions of Lie algebra

2.8 Root system

Introduction to Lie Algebra

2.1 Introduction

Lie theory is one of the very important topics, the merit of which appeared one century ago, and the scholar Suphas Lie had a great role in establishing that work. His work had a profound impact in mathematics and physics, and in our time Lie Algebra has become more practical in many special studies.

Here we will start by introducing a lot about fundamental concepts for Lie algebras.

2.2 Definition of Lie Algebras

Let F be a field. A Lie algebra L over F is a vector space together with a bilinear map called the Lie bracket.

$$\begin{aligned}[-, -]: L \times L &\rightarrow L, \\(a, b) &\mapsto [a, b]\end{aligned}$$

satisfying the following properties:

- $[a, a] = 0$ for all $a \in L$
- (Bilinearity) $[a + b, c + d] = [a, c] + [a, d] + [b, c] + [b, d]$
- (Bilinearity) $k[a, b] = [ka, b] = [a, kb]$
- (Jacobi Identity) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

For all $a, b, c, d \in L$ and $k \in F$

The Lie bracket $[a, b]$ is called the *commutator* of a and b .

From the previous conditions, we conclude something useful:

$$0 = [a + b, a + b] = [a, a] + [a, b] + [b, a] + [b, b]$$

But we know that $[a, a] = 0 = [b, b]$, so

$$0 = [a, b] + [b, a]$$

$$\text{i.e., } [a, b] = -[b, a], \text{ For all } a, b \in L[1]$$

2.3 Examples of Lie Algebras

Our first example of Lie algebra comes from what we learnt in Calculus III and in particular the cross product of three-dimensional vectors. One of the obvious and unambiguous ways to define a cross product is to use standard unit vector $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ for $\mathbb{R}^3$. Let

$$\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

then the cross-product $\mathbf{u} \times \mathbf{v}$ is defined as:

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Which can be interpreted using determinants

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

we explore some properties of the cross product that will be useful in the subsequent examples.

First the cross product is anticommutative, i.e.,

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

This results in any vector cross itself is 0, i.e.,

$$\mathbf{v} \times \mathbf{v} = 0$$

In addition, the cross product is bilinear, using the cross product we can illustrate our first example of Lie algebra as follows:

- 1) let $F = \mathbb{R}$, the vector product $(x, y) \mapsto x \wedge y$ defines a Lie algebra on $\mathbb{R}^3$, as follows: Let

$x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$

Then the wedge product is defined by

$$x \wedge y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$$

This example is a lot like cross product in which we use determinants to get the same result. Now we have to show that wedge product $\wedge$ is bilinear.

Let $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, and $c = (c_1, c_2, c_3)$ be vectors in $\mathbb{R}^3$

$$\begin{aligned} [a + b, c] &= (a_1 + b_1, a_2 + b_2, a_3 + b_3) \wedge (c_1, c_2, c_3) \\ &= ((a_2 + b_2) c_3 - (a_3 + b_3) c_2, (a_3 + b_3) c_1 - (a_1 + b_1) c_3, (a_1 + b_1) c_2 - \\ &\quad (a_2 + b_2) c_1) \\ &= (a_2 c_3 - a_3 c_2 + b_2 c_3 - b_3 c_2, a_3 c_1 - a_1 c_3 + b_3 c_1 - b_1 c_3, a_1 c_2 - \\ &\quad a_2 c_1 + b_1 c_2 - b_2 c_1) \\ &= (a_2 c_3 - a_3 c_2, a_3 c_1 - a_1 c_3, a_1 c_2 - a_2 c_1) + (b_2 c_3 - b_3 c_2, b_3 c_1 - b_1 c_3, b_1 c_2 - \\ &\quad b_2 c_1) = [a, c] + [b, c] \end{aligned}$$

Now let's get acquainted with the other feature of bilinearity, for $k \in F$ we have

$$\begin{aligned} k[a, b] &= k(a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) = (ka_2 b_3 - ka_3 b_2, \\ &\quad ka_3 b_1 - ka_1 b_3, ka_1 b_2 - ka_2 b_1) = k[a, b] = [ka, b] = [a, kb] \end{aligned}$$

2) Define a Lie bracket as $[a, b] = 0$, for all $a, b \in V$. This will always gives a 1-dimensional abelian Lie algebra (in the sense Lie bracket is commutative). In this case, the lie algebra structure is abelian on V . This example is trivial. Therefore, this example will not have a wide presence in our thesis.

- 3) Define $\mathfrak{gl}(V)$ as the set of all linear maps from V to V (endomorphisms). Which known as the **general linear algebra**. Let write the Lie bracket $[-, -]$:

$$[a, b] := a \circ b - b \circ a \quad \text{for } a, b \in \mathfrak{gl}(V)$$

Where $\circ$ denotes the composition of maps.

Here we will verify that the Jacobi identity holds.

$$\begin{aligned} [a, [b, c]] &= a \circ [b, c] - [b, c] \circ a \\ &= a \circ ((b \circ c) - (c \circ b)) - ((b \circ c) - (c \circ b)) \circ a \\ &= a \circ (b \circ c) - a \circ (c \circ b) - b \circ (c \circ a) + c \circ (b \circ a) \\ [b, [c, a]] &= b \circ [c, a] - [c, a] \circ b \\ &= b \circ ((c \circ a) - (a \circ c)) - ((c \circ a) - (a \circ c)) \circ b \\ &= b \circ (c \circ a) - b \circ (a \circ c) - c \circ (a \circ b) + a \circ (c \circ b) \\ [c, [a, b]] &= c \circ [a, b] - [a, b] \circ c \\ &= c \circ ((a \circ b) - (b \circ a)) - ((a \circ b) - (b \circ a)) \circ c \\ &= c \circ (a \circ b) - c \circ (b \circ a) - a \circ (b \circ c) + b \circ (a \circ c) \end{aligned}$$

Now, we should add all above lie brackets

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

- 4) Let $GL(n, F)$ be invertible matrices

We look at the set of invertible $n \times n$ matrices with entries in $F = \mathbb{R}$ or $\mathbb{C}$. so, we will define $GL_n(F)$, the invertible $n \times n$ matrices over F

$$[A, B] := AB - BA$$

Where AB is the usual product.

While the Lie bracket is defined on the vector space $M_n(F)$ of all $n \times n$ matrices with coefficients F , we notice that $GL_n(F)$ is not a subspace of $M_n(F)$. We can write the basis which consists of elementary matrices e_{ij}

for $1 \leq i, j \leq n$, where e_{ij} is the $n \times n$ matrix which has 0 in all the entries except the ij -th position which will be 1. The Lie bracket can be easily computed from the formula

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}$$

Where δ is the Kronecker delta $\delta_{ij}=1$ if $i = j$ and $\delta_{ij}=0$, otherwise.

2.4 Subalgebra and ideals of Lie Algebras

The Lie subalgebra can be defined as follows : Given a Lie algebra L and K a vector space such that $K \subseteq L$. We say that K is Lie subalgebra if for all $a, b \in K$ then $[a, b] \in K$.

An ideal I of a Lie algebra L is defined to be a subspace I of L such that $[a, b] \in I$ for all $a \in L$ and $b \in I$.

Notice that there is no need to differentiate between right and left ideals, since by definition $[a, b] = -[b, a]$.

And as we know through the basics of algebra, it is necessary that all the ideals are subalgebras, but not conversely.

Examples:

- 5) Special Linear algebra $\mathfrak{sl}(n, F)$, which is a subset of the vector space of $n \times n$ matrices that consists of all matrices of trace 0, i.e., the sum of the diagonal entries is zero.

The set $\mathfrak{sl}(n, F)$ is an ideal of $\mathfrak{gl}(n, F)$ and it has a basis consisted of e_{ij} for $i \neq j$ and $e_{ii} - e_{i+1, i+1}$ for $1 \leq i < n$.

Let check this for $n=2$. Let $A \in \mathfrak{gl}(n, F)$ and $B \in \mathfrak{sl}(n, F)$, so

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & -b_{11} \end{bmatrix} \\ [A, B] &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & -b_{11} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & -b_{11} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} - a_{12}b_{11} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} - a_{22}b_{11} \end{bmatrix} - \\ &\quad \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} - b_{11}a_{21} & b_{21}a_{12} - b_{11}a_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{12}b_{21} - b_{12}a_{21} & a_{11}b_{12} - b_{12}a_{22} - 2(a_{12}b_{11}) \\ 2(a_{21}b_{11}) + a_{22}b_{21} - b_{21}a_{11} & a_{21}b_{12} - b_{21}a_{12} \end{bmatrix} \in \\ &\quad \mathfrak{sl}(n, F) \end{aligned}$$

6) A matrix A is called upper triangular matrix if $a_{ij} = 0$ whenever $i > j$.

Let the upper triangular matrices in $\mathfrak{gl}(n, F)$ be denoted by $\mathfrak{b}(n, F)$. This is $\mathfrak{gl}(n, F)$'s Lie subalgebra. K must be a vector space of L , i.e, K is a vector space that is contained in L .

7) Assume now the strictly upper triangular matrices in $\mathfrak{gl}(n, F)$ which denoted by $\mathfrak{n}(n, F)$. Here if $a_{ij} = 0$ whenever $i \geq j$, then the matrix A is strictly upper triangular. It can be easily verified that $\mathfrak{n}(n, F)$ is an ideal of $\mathfrak{b}(n, F)$.

We check that in the case of $n=2$, Let $A \in \mathfrak{b}(n, F)$ and $B \in \mathfrak{n}(n, F)$

$$\begin{aligned} \text{Let } A &= \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & b_{12} \\ 0 & 0 \end{bmatrix} \\ [A, B] &= \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} 0 & b_{12} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & b_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_{11}b_{12} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & b_{12}a_{22} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_{11}b_{12} - b_{12}a_{22} \\ 0 & 0 \end{bmatrix} \in \mathfrak{n}(n, F) \end{aligned}$$

- 8) The Lie algebra L and $\{0\}$ are the improper and trivial ideals of L .
 9) The center of L defined by

$$Z(L) := \{a \in L: [a, b] = 0 \text{ for all } b \in L\}$$

We have already mentioned that $L = Z(L)$ it is the case if and only if L is commutative. But to prove that $Z(L) = \{0\}$, it takes more effort. [1]

2.5 Homomorphisms of Lie Algebras

Let L_1 and L_2 be two Lie algebras over a field F . The map $\varphi: L_1 \rightarrow L_2$ is called a homomorphism if it is a linear map over F and

$$\varphi([a, b]) = [\varphi(a), \varphi(b)] \text{ for all } a \text{ and } b \in L_1$$

But here it should be noted that the left part of the definition is related to L_1 and the right part is related to L_2 . The map φ is called isomorphism if φ is also bijective. One of the topics of great interest to us in homomorphism is the adjoint homomorphism which will be defined as follows.

Let L be a lie algebra, we define

$$\text{ad}: L \rightarrow \text{gl}(L)$$

by $(\text{ad } a)(b) := [a, b]$ for a and $b \in L$. This map is linear, we must make sure of the homomorphism.

$$\text{ad}([a, b]) = \text{ad } a \circ \text{ad } b - \text{ad } b \circ \text{ad } a \text{ for all } a, b \in L$$

This definition is in fact equivalent to Jacobi's identity. Let $c \in L$,

$$\text{ad}([a, b])(c) = (\text{ad } a \circ \text{ad } b)(c) - (\text{ad } b \circ \text{ad } a)(c)$$

$$[[a, b], c] = (\text{ad } a)([b, c]) - (\text{ad } b)([a, c])$$

$$[[a, b], c] = [a, [b, c]] - [b, [a, c]]$$

$$0 = [a, [b, c]] + [b, [c, a]] + [c, [a, b]]$$

2.6 Construction of new Ideals from old ones

It is well-known that ideals play important role in Lie algebra, as they do in ring theory, that is why it is useful to construct new ideals.

As we know that $I \cap J$ is subspace of L , we only have to make sure that if $a \in L$ and $b \in I \cap J$, then $[a, b] \in I \cap J$. In addition, $I + J$ is an ideal where $I + J := \{a + b : a \in I \text{ and } b \in J\}$. [1]

2.7 Dimensions of Lie algebra

Lie algebras are classified into several dimensions, which can be easily understood. In this subsection, we will look over Lie algebras of dimensions one, two, and three.

2.8 Root system

We now present the root system associated with any Lie algebra. We start first by an abstract definition of a root system:

Definition:

Assume that E is a real vector space of finite dimension along an inner product $(-, -)$. Let s_v be the reflection in the hyperplane normal to v for a non-zero vector $v \in E$. As a result, s_v fixes all elements y such that $(y, v) = 0$ and sends v to $-v$.

The equation of s_v can be written as follows:

$$s_v(x) = x - \frac{2(x,v)}{(v,v)}v \text{ for all } x \in E$$

and that the inner product is preserved by s_v that is,

$$(s_v(x), s_v(y)) = (x, y) \text{ for all } x, y \in E.$$

We shall write

$$\langle x, v \rangle := \frac{2(x, v)}{(v, v)}$$

Observing that only the first variable x is linear in the symbol $\langle x, v \rangle$. We are now able to describe root systems using this notation.

Definition:

A subset R of a real vector space E is a root system if it satisfies the following axioms.

- 1- R is finite, it spans E , and it does not contain 0
- 2- If $\alpha \in R$, then the only scalar multiples of α in R are $\pm\alpha$
- 3- If $\alpha \in R$, then the reflection s_α permutes the elements of R
- 4- If $\alpha, \beta \in R$ then $\langle \alpha, \beta \rangle$ is an integer

The elements of R are called roots.

Remark :

Assume R is a root system in the real inner-product space E . Let $\alpha, \beta \in R$ with $\beta \neq \pm\alpha$. Then

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}.$$

Remark :

Let $\alpha, \beta \in R$.

(a) If α and β are at a precisely obtuse angle, then $\alpha + \beta \in R$.

(b) If α and β have a precisely acute angle, and $(\beta, \beta) \geq (\alpha, \alpha)$, then

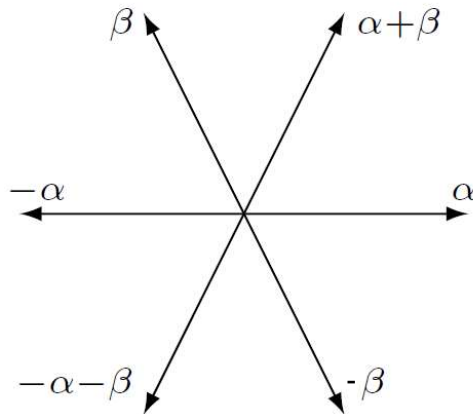
$$\alpha - \beta \in R.$$

Let consider some examples.

Examples:

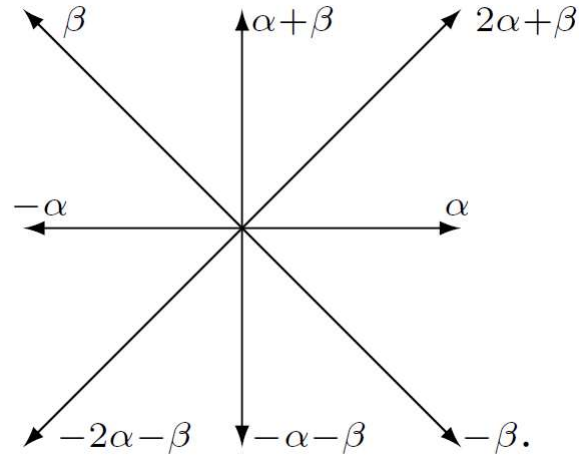
With the Euclidean inner product on $E = \mathbb{R}^2$. All root systems R in E will be calculated. Let α be the root with the shortest length. Since R spans E of dimension 2, it must contain some root $\beta \neq \pm\alpha$. By considering $-\beta$ if necessary, we may assume that β and α form an obtuse angle. Additionally, we can suppose that this angle θ is as large as it can be.

Case where $\theta = 2\pi/3$, then R contains the six roots shown below.



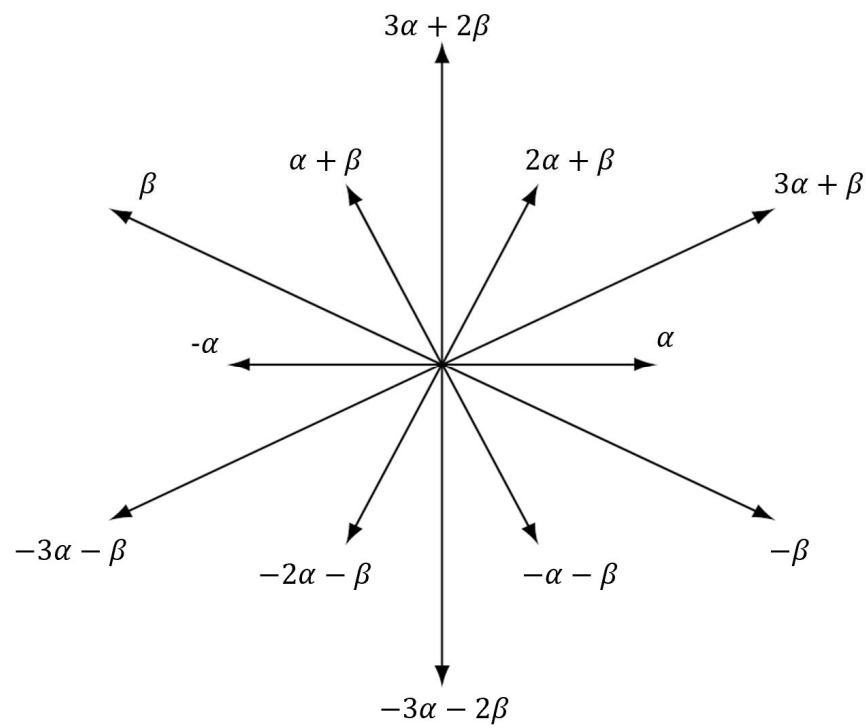
Under the action of the reflections $S_\alpha, S_\beta, S_{\alpha+\beta}$, one can verify that this set is closed. Because $S_{-\alpha} = S_\alpha$, we can verify the (R3) which will show that this is a root system. This root system is said to be of type A_2 . The dimension of the underlying space is 2 which is stated in the name of the root system.

Case where $\theta = 3\pi/4$. According to the previous remark, it shows that $\alpha + \beta$ is a root, and applying S_α to β shows that $2\alpha + \beta$ is a root, so R must contain the following.



This root system is said to be of type B_2 .

Case where $\theta = 5\pi/6$. The root system will look in the following way.



Chapter 3

Root system of Dimension 2

3.1 Introduction

3.2 The Lie algebra A_1

3.3 The Lie algebra A_2

3.4 The Lie algebra B_2

3.5 The Lie algebra G_2

Root Systems of Dimension 2

3.1 Introduction

The root system has many dimensions, and when talking about the root system of all kinds it will be very manifold. Therefore, we will only focus on the rank two root systems and their associated Lie algebras and Lie groups.

In this chapter, we will explore the root systems in the Euclidean space of dimension 2 and will show the associativity with the Lie Algebras. We will give a graphical description to the Lie bracket.

First, given that V is a vector space of finite dimension n with a basis $v_0, \dots, v_{n-1}$. We define the Lie algebra $E(V)$ as the collection of all linear mappings from V to V . The product in $E(V)$ is the composition of linear maps, or of matrix multiplication with respect to the basis $v_0, \dots, v_{n-1}$.

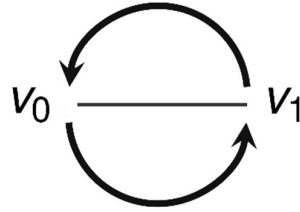
Next, the following definition uses elementary matrices and play a major role in approaching a bold description of the root system. Let the n^2 linear maps e_{ij} which maps $v_i \mapsto v_j$, and $v_k \mapsto 0$ for $k \neq i$. These maps forms a basis for $E(V)$. The operation $[a, b] = ab - ba$ on $E(V)$ satisfies the Lie algebra axioms as mentioned earlier. The vector space $E(V)$ together with the Lie bracket $[a, b]$ for $a, b \in E(V)$, defines a Lie algebra.

3.2 The Lie algebra A_1

First, let's start with a simple example, the Lie algebra A_1 . Let $n = 2$ and $V = \langle v_0, v_1 \rangle$.

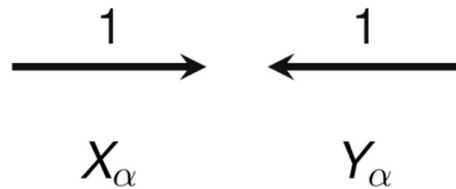
Define $X_\alpha = e_{01}$, which means that X_α will be a symbol for the basis map that maps v_0 to v_1 .

We will define also $Y_\alpha = e_{10}$ as its reverse i.e., this will be like the one before but here from v_1 to v_0 . This choice seems trivial at first, as there are not many basis vectors to allow the map to vary, but in the next examples, the selection will be of a pattern to ensure that the root system occurs as needed, and we find that Y always goes in the opposite direction to X .



This graph shows that any horizontal line from left to right means that the map takes v_0 to v_1 , meanwhile any horizontal line with an opposite direction which means from right to left, it takes v_1 to v_0

To make things look simple and uncomplicated, we do not need long writing so that the reader does not get lost. We will name X_α on the line that takes v_0 to v_1 , and as mentioned previously, Y always goes in the opposite direction to X , so Y_α will be named on the line that takes v_1 to v_0



Define H_α to be the Lie bracket of X_α and Y_α .

$$H_\alpha = [X_\alpha, Y_\alpha]$$

The map H_α will behave on the two basis vectors of $\{v_0, v_1\}$ as follows:

For v_0

$$\begin{aligned} H_\alpha(v_0) &= [X_\alpha, Y_\alpha](v_0) = X_\alpha Y_\alpha(v_0) - Y_\alpha X_\alpha(v_0) \\ &= e_{01}(e_{10}(v_0)) - e_{10}(e_{01}(v_0)) \\ &= e_{01}(0) - e_{10}(v_1) = 0 - v_0 = -v_0 \end{aligned}$$

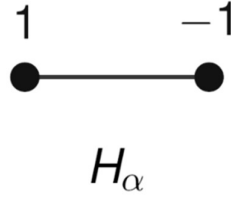
And this will be the first case which means that H_α maps v_0 to $-v_0$

For v_1

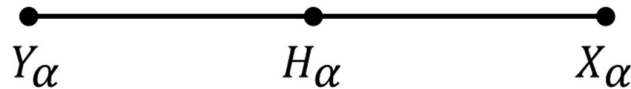
$$\begin{aligned} H_\alpha(v_1) &= [X_\alpha, Y_\alpha](v_1) = X_\alpha Y_\alpha(v_1) - Y_\alpha X_\alpha(v_1) \\ &= e_{01}(e_{10}(v_1)) - e_{10}(e_{01}(v_1)) \\ &= e_{01}(v_0) - e_{10}(0) = v_1 - 0 = v_1 \end{aligned}$$

Here we completed the two cases, thus H_α maps v_1 to v_1 . This can be written as $e_{11} - e_{00}$ or $H_\alpha = e_{00} - e_{11}$.

If we combine the two cases, we will get the result shown in the following figure.



To be clear and to have the image of the root system, we will calculate the adjoints: $\text{ad}(X_\alpha)$, $\text{ad}(Y_\alpha)$ and $\text{ad}(H_\alpha)$.



Let's calculate one of them, like we can start with H_α . We need to calculate $[H_\alpha, X_\alpha]$ and $[H_\alpha, Y_\alpha]$ only, because $[H_\alpha, H_\alpha] = 0$

We will calculate $[H_\alpha, X_\alpha]$ on v_0 is.

$$[e_{00} - e_{11}(e_{01}(v_0)) - e_{01}(e_{00} - e_{11}(v_0)) = -2v_1 = -2X_\alpha$$

and $[H_\alpha, X_\alpha]$ on v_1 is.

$$e_{00} - e_{11}(e_{01}(v_1)) - e_{01}(e_{00} - e_{11}(v_1)) = 0$$

So $[H_\alpha, X_\alpha] = -2X_\alpha$

And now we should calculate $[H_\alpha, Y_\alpha]$

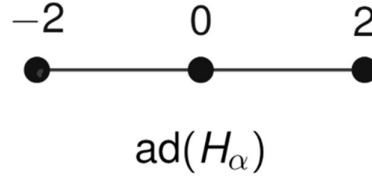
$[H_\alpha, Y_\alpha]$ on v_0 is

$$[H_\alpha, Y_\alpha] = e_{00} - e_{11}(e_{10}(v_0)) - e_{10}(e_{00} - e_{11}(v_0)) = 0$$

$[H_\alpha, Y_\alpha]$ on v_1 is

$$e_{00} - e_{11}(e_{10}(v_1)) - e_{10}(e_{00} - e_{11}(v_1)) = 2v_0 = 2Y_\alpha$$

So $[H_\alpha, Y_\alpha] = 2Y_\alpha$



So, the above figure shows that $\text{ad}(H_\alpha)$ maps X_α to $2X_\alpha$ and Y_α to $-2Y_\alpha$.

As for adjoint $\text{ad}(X_\alpha)$, $\text{ad}(Y_\alpha)$, we will leave the calculations to the reader, but we will give the final picture.

For $\text{ad}(X_\alpha)$

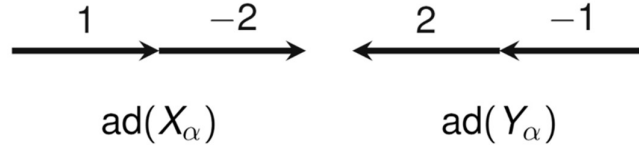
$$[X_\alpha, H_\alpha] = -2X_\alpha \text{ and } [X_\alpha, Y_\alpha] = H_\alpha$$

For $\text{ad}(Y_\alpha)$

$$[Y_\alpha, H_\alpha] = 2Y_\alpha \text{ and } [Y_\alpha, X_\alpha] = -H_\alpha$$

So the figure shows that $\text{ad}(X_\alpha)$ takes Y_α to H_α while it will take H_α to $-2X_\alpha$.

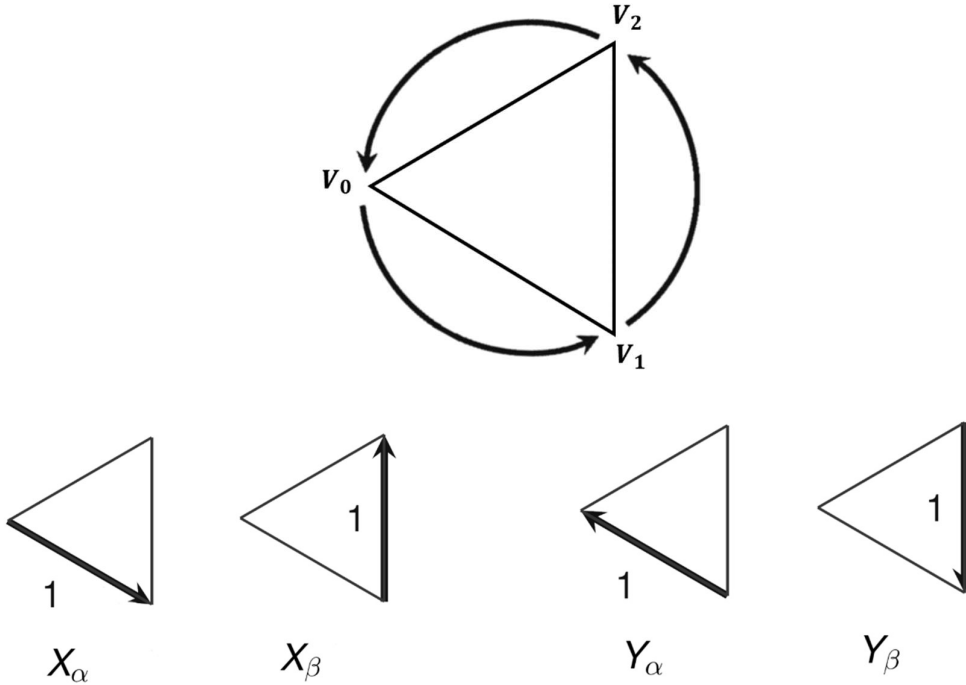
On the other side $\text{ad}(Y_\alpha)$ takes X_α to $-H_\alpha$ while it takes H_α to $2Y_\alpha$



3.3 The Lie algebra A_2

Next, we will review the main example in this thesis, and we define it using Lie algebra, but later in Chapter 5, we will give an alternative approach to define such algebra but without using Lie algebra structure.

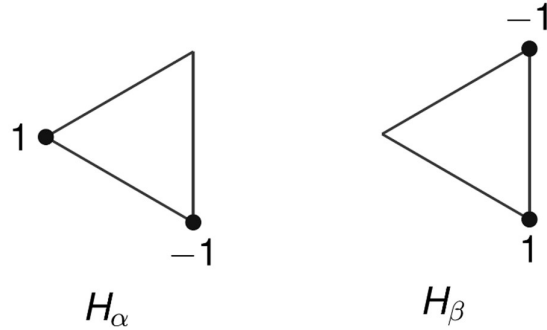
As with the case of A_2 , let $n = 3$ and $V = \langle v_0, v_1, v_2 \rangle$. We will define again $X_\alpha = e_{01}$ and $Y_\alpha = e_{10}$, but now we need a new path, and perhaps calling it a new name would be appropriate because it is related to a different base, so let $X_\beta = e_{12}$ and $Y_\beta = e_{21}$. Similarly, we write H_α and H_β and they are defined by $H_\alpha = [X_\alpha, Y_\alpha]$ and $H_\beta = [X_\beta, Y_\beta]$.



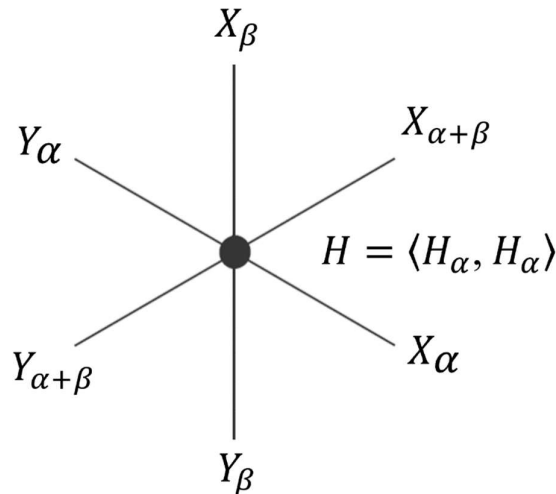
The image above indicates that X_α maps v_0 to v_1 , and X_β maps v_1 to v_2 . Again Y_α maps v_1 to v_0 , while Y_β maps v_2 to v_1 .

We will do the calculations for the actions of these maps in the three vectors that generate $V = \langle v_0, v_1, v_2 \rangle$. In this example and the examples that will come successively, we will need to make more calculations.

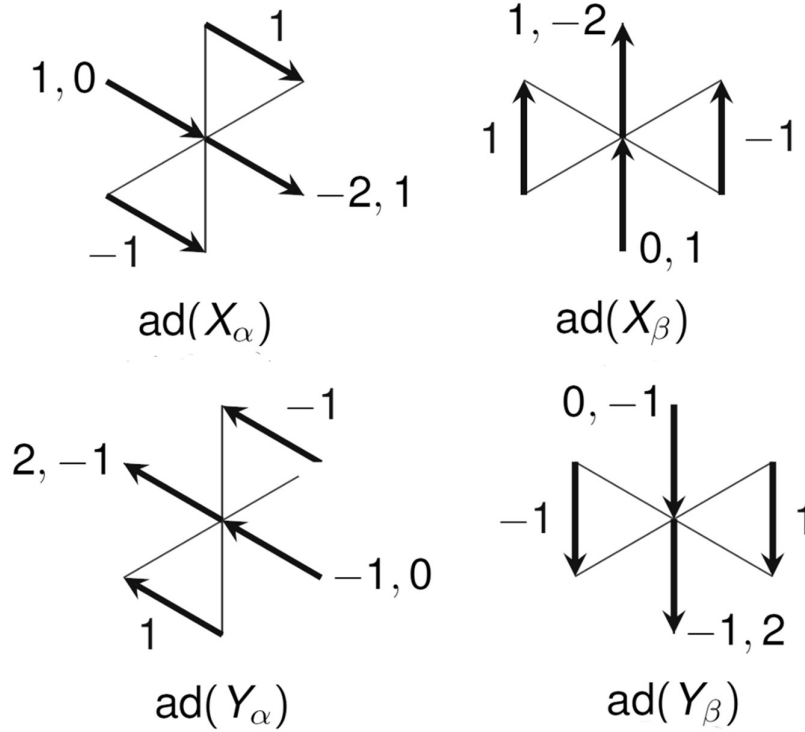
First, we notice that there will be no noticeable change in the action of H_α , nor in H_β . As $H_\alpha = [X_\alpha, Y_\alpha]$ will remain as in the previous example which is $= e_{00} - e_{11}$, and $H_\beta = [X_\beta, Y_\beta] = e_{11} - e_{22}$. we will get this result shown in the following figure.



Here it would be very appropriate to make a third path from v_0 to v_2 through composition between X_α and X_β , so we will write it in another form, which is $X_{\alpha+\beta}$ and the opposite path would be $Y_{\alpha+\beta}$ from v_2 to v_0



Calculating the adjoint will be very useful to have the full picture for the root system.



So the figure here shows the adjoint for $X_\alpha, Y_\alpha, X_\beta$, and Y_β

For example $\text{ad}(X_\alpha)$ will take Y_α to H_α while it will take H_α to $-2X_\alpha$, also the second number means that $\text{ad}(X_\alpha)$ cannot take Y_α to H_β but it will take H_β to X_α . The upper diagonal line is for $\text{ad}(X_\alpha)$ that's take X_β to $X_{\alpha+\beta}$ and the lower diagonal line for $\text{ad}(X_\alpha)$ that's take $Y_{\alpha+\beta}$ to $-Y_\beta$.

For now, we will concentrate on the adjoint of H_α and H_β . Trivial adjoint with itself yields $[H_\alpha, H_\alpha] = 0$. We will do next the action of $\text{ad}(H_\alpha)$ with X_α , Y_α , X_β , Y_β , $X_{\alpha+\beta}$ and $Y_{\alpha+\beta}$.

The actions on v_0

$$[H_\alpha, X_\alpha] = e_{00} - e_{11}(e_{01}(v_0)) - e_{01}(e_{00} - e_{11}(v_0)) = -v_1 - v_1 = -2v_1$$

$$\begin{aligned}
[H_\alpha, Y_\alpha] &= e_{00} - e_{11}(e_{10}(v_0)) - e_{10}(e_{00} - e_{11}(v_0)) = 0 \\
[H_\alpha, X_\beta] &= e_{00} - e_{11}(e_{12}(v_0)) - e_{12}(e_{00} - e_{11}(v_0)) = 0 \\
[H_\alpha, Y_\beta] &= e_{00} - e_{11}(e_{21}(v_0)) - e_{21}(e_{00} - e_{11}(v_0)) = 0 \\
[H_\alpha, X_{\alpha+\beta}] &= e_{00} - e_{11}(e_{02}(v_0)) - e_{02}(e_{00} - e_{11}(v_0)) = -v_2 \\
[H_\alpha, Y_{\alpha+\beta}] &= e_{00} - e_{11}(e_{20}(v_0)) - e_{20}(e_{00} - e_{11}(v_0)) = 0
\end{aligned}$$

The actions on v_1

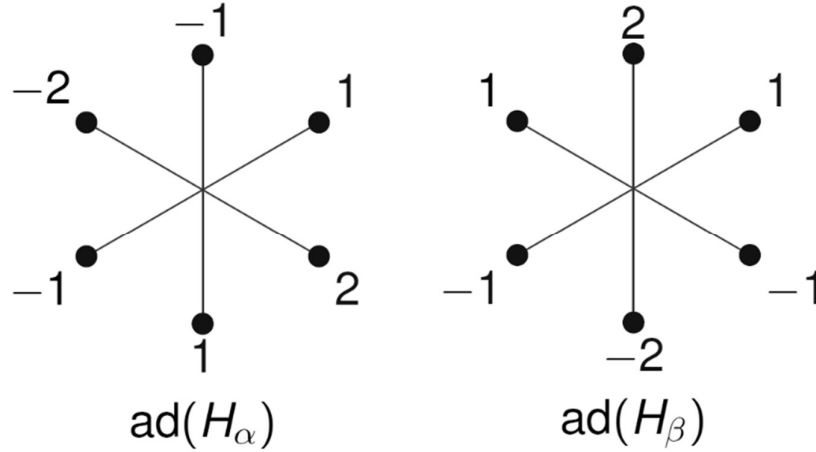
$$\begin{aligned}
[H_\alpha, X_\alpha] &= e_{00} - e_{11}(e_{01}(v_1)) - e_{01}(e_{00} - e_{11}(v_1)) = 0 \\
[H_\alpha, Y_\alpha] &= e_{00} - e_{11}(e_{10}(v_1)) - e_{10}(e_{00} - e_{11}(v_1)) = v_0 + v_0 = 2v_0 \\
[H_\alpha, X_\beta] &= e_{00} - e_{11}(e_{12}(v_1)) - e_{12}(e_{00} - e_{11}(v_1)) = v_2 \\
[H_\alpha, Y_\beta] &= e_{00} - e_{11}(e_{21}(v_1)) - e_{21}(e_{00} - e_{11}(v_1)) = 0 \\
[H_\alpha, X_{\alpha+\beta}] &= e_{00} - e_{11}(e_{02}(v_1)) - e_{02}(e_{00} - e_{11}(v_1)) = 0 \\
[H_\alpha, Y_{\alpha+\beta}] &= e_{00} - e_{11}(e_{20}(v_1)) - e_{20}(e_{00} - e_{11}(v_1)) = 0
\end{aligned}$$

The actions on v_2

$$\begin{aligned}
[H_\alpha, X_\alpha] &= e_{00} - e_{11}(e_{01}(v_2)) - e_{01}(e_{00} - e_{11}(v_2)) = 0 \\
[H_\alpha, Y_\alpha] &= e_{00} - e_{11}(e_{10}(v_2)) - e_{10}(e_{00} - e_{11}(v_2)) = 0 \\
[H_\alpha, X_\beta] &= e_{00} - e_{11}(e_{12}(v_2)) - e_{12}(e_{00} - e_{11}(v_2)) = 0 \\
[H_\alpha, Y_\beta] &= e_{00} - e_{11}(e_{21}(v_2)) - e_{21}(e_{00} - e_{11}(v_2)) = -v_1 \\
[H_\alpha, X_{\alpha+\beta}] &= e_{00} - e_{11}(e_{02}(v_2)) - e_{02}(e_{00} - e_{11}(v_2)) = 0 \\
[H_\alpha, Y_{\alpha+\beta}] &= e_{00} - e_{11}(e_{20}(v_2)) - e_{20}(e_{00} - e_{11}(v_2)) = v_0
\end{aligned}$$

So if we combine this we get

$$\begin{aligned}
[H_\alpha, X_\alpha] &= -2X_\alpha, [H_\alpha, Y_\alpha] = 2Y_\alpha, [H_\alpha, X_\beta] = X_\beta, [H_\alpha, Y_\beta] = -Y_\beta, \\
[H_\alpha, X_{\alpha+\beta}] &= -X_{\alpha+\beta} \text{ and } [H_\alpha, Y_{\alpha+\beta}] = Y_{\alpha+\beta}
\end{aligned}$$



3.4 The Lie algebra B_2

The group B_2 is an important turning point in this regard where some modifications will be made in the selection of basis elements, Therefore, we will first look at the pattern to which we are accustomed. Now for $n = 4$ and $V = \langle v_0, v_1, v_2, v_3 \rangle$, write $X_\alpha = e_{01}$, $X_\beta = e_{12}$, $X_\gamma = e_{23}$ and Y 's for the opposite paths. So let $Y_\alpha = e_{10}$, $Y_\beta = e_{21}$ and $Y_\gamma = e_{32}$. To connect all the paths, let $X_{\alpha+\beta} = e_{02}$, $X_{\beta+\gamma} = e_{13}$ and $X_{\alpha+\beta+\gamma} = e_{03}$, all together with $Y_{\alpha+\beta} = e_{20}$, $Y_{\beta+\gamma} = e_{31}$ and $Y_{\alpha+\beta+\gamma} = e_{30}$. Write $H_\alpha = [X_\alpha, Y_\alpha]$, $H_\beta = [X_\beta, Y_\beta]$ and $H_\gamma = [X_\gamma, Y_\gamma]$. $H_\alpha = [X_\alpha, Y_\alpha] = e_{00} - e_{11}$, $H_\beta = [X_\beta, Y_\beta] = e_{11} - e_{22}$ and $H_\gamma = [X_\gamma, Y_\gamma] = e_{22} - e_{33}$.

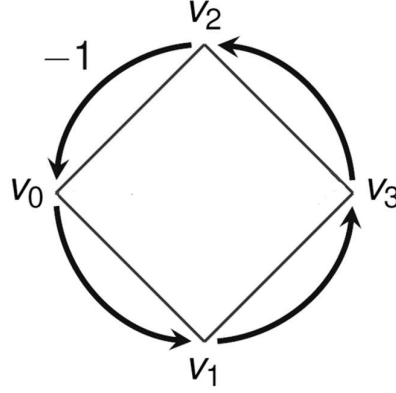
The first problem that will appear through this option is the presence of duplication of lines, in addition to the significant shortage in calculating the adjoint, for example let's take the adjoint for H_α .

$$\begin{aligned}
[H_\alpha, X_\alpha] &= -2X_\alpha, [H_\alpha, Y_\alpha] = 2Y_\alpha, [H_\alpha, X_\beta] = X_\beta, [H_\alpha, Y_\beta] = -Y_\beta, \\
[H_\alpha, X_\gamma] &= 0, [H_\alpha, Y_\gamma] = 0, [H_\alpha, X_{\alpha+\beta}] = -X_{\alpha+\beta}, [H_\alpha, Y_{\alpha+\beta}] = Y_{\alpha+\beta}, \\
[H_\alpha, X_{\beta+\gamma}] &= X_{\beta+\gamma}, [H_\alpha, Y_{\beta+\gamma}] = -Y_{\beta+\gamma}, [H_\alpha, X_{\alpha+\beta+\gamma}] = -X_{\alpha+\beta+\gamma},
\end{aligned}$$

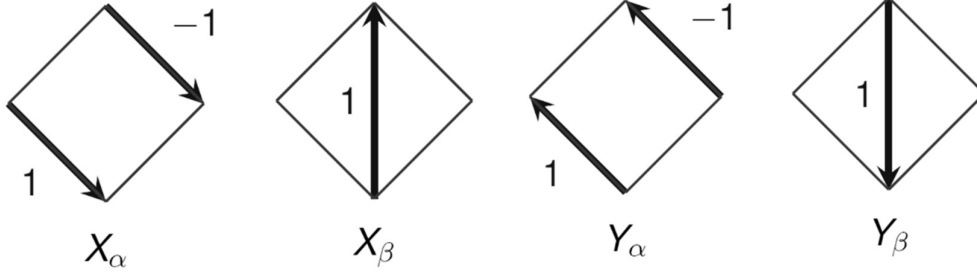
$$[H_\alpha, Y_{\alpha+\beta+\gamma}] = Y_{\alpha+\beta+\gamma}.$$

Therefore, there is no definite value and known direction for $[H_\alpha, X_\gamma]$ nor $[H_\alpha, Y_\gamma]$.

So instead, there will be a different pattern now.

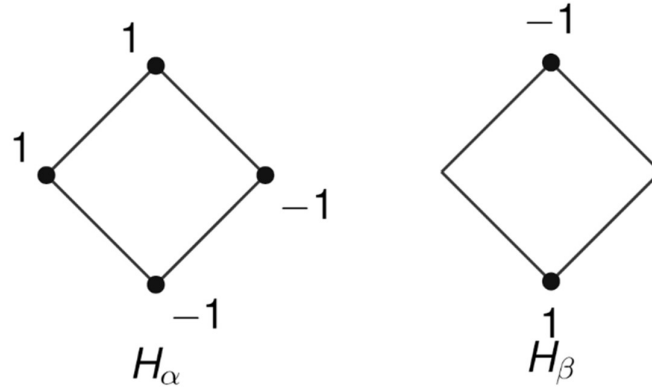


write $X_\alpha = e_{01} - e_{23}$, $X_\beta = e_{12}$, and $Y_\alpha = e_{10} - e_{32}$, $Y_\beta = e_{21}$.

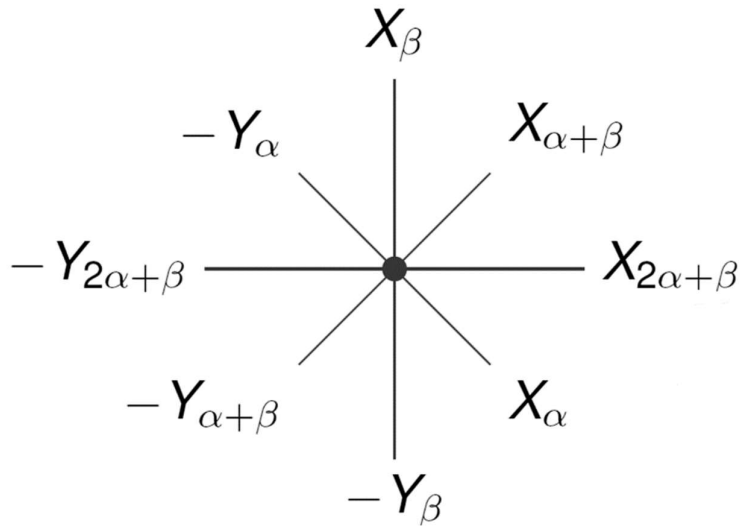


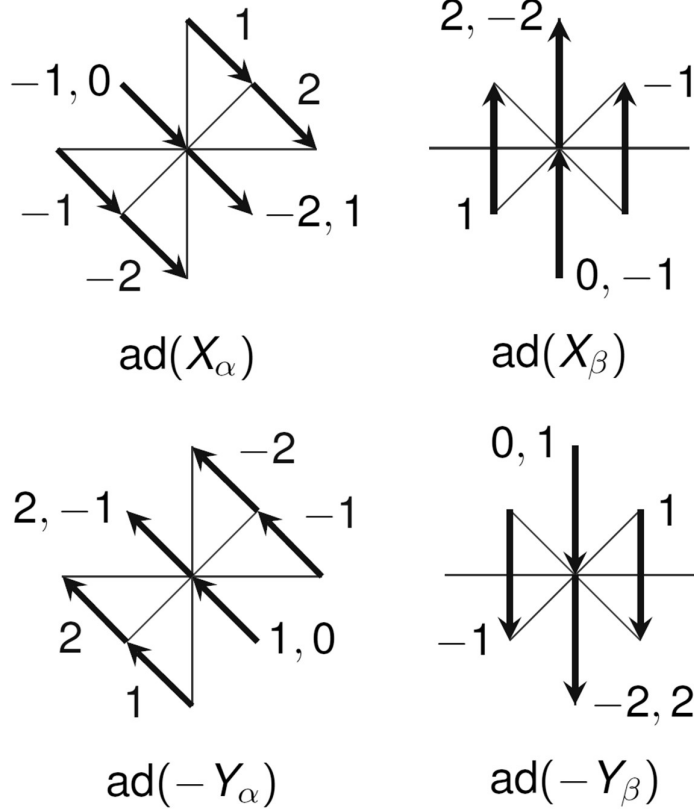
The image above indicates that X_α maps v_0 to v_1 and at the same time v_2 to $-v_3$, X_β maps v_1 to v_2 only. Again Y_α maps v_1 to v_0 and v_3 to $-v_2$, while Y_β maps v_2 to v_1 .

We define $H_\alpha = [X_\alpha, Y_\alpha] = e_{00} - e_{11} + e_{22} - e_{33}$, $H_\beta = [X_\beta, Y_\beta] = e_{11} - e_{22}$. We will get this result shown in the following figure.



There is no effect or benefit from the presence of a third base, on the contrary, this will lead to a duplication in the direction of the paths later. And, we will write two new paths, the first from v_0 to v_2 and from v_1 to v_3 through composition between X_α and X_β , which is $X_{\alpha+\beta} = e_{02} - e_{13}$ and the opposite path $Y_{\alpha+\beta} = e_{20} - e_{31}$ from v_2 to v_0 and from v_3 to v_1 . A second path in this case from v_0 to v_3 through composition between X_α and $X_{\alpha+\beta}$, it will be written as $X_{2\alpha+\beta} = e_{03}$ and finally we need the opposite $Y_{2\alpha+\beta} = e_{30}$ which is from v_3 to v_0 .





So, the figure above shows the adjoint for $X_\alpha, -Y_\alpha, X_\beta$, and $-Y_\beta$

For example $\text{ad}(X_\alpha)$ will take $-Y_\alpha$ to $-H_\alpha$ while it will take H_α to $-2X_\alpha$, also the second number means that $\text{ad}(X_\alpha)$ cannot take $-Y_\alpha$ to H_β but it takes H_β to X_α . The First upper diagonal line is for $\text{ad}(X_\alpha)$ that takes X_β to $X_{\alpha+\beta}$ and the second upper diagonal line is for $\text{ad}(X_\alpha)$ that takes $X_{\alpha+\beta}$ to $X_{2\alpha+\beta}$, Regarding the first lower diagonal line for $\text{ad}(X_\alpha)$ that takes $-Y_{2\alpha+\beta}$ to $Y_{\alpha+\beta}$ and the second lower diagonal line for $\text{ad}(X_\alpha)$ that takes $-Y_{\alpha+\beta}$ to $2Y_\beta$.

Now we will take the adjoint for H_α

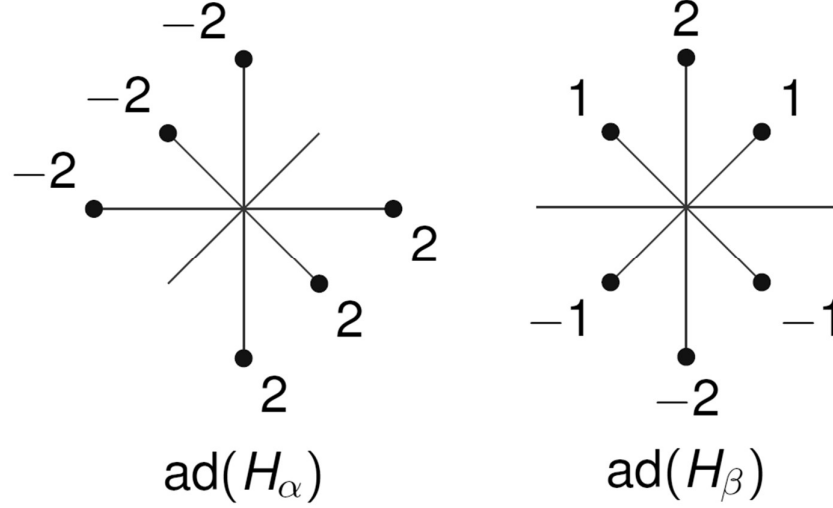
$$[H_\alpha, X_\alpha] = 2X_\alpha, [H_\alpha, Y_\alpha] = 2Y_\alpha, [H_\alpha, X_\beta] = 2X_\beta, [H_\alpha, Y_\beta] = -2Y_\beta,$$

$$[H_\alpha, X_{\alpha+\beta}] = 0, [H_\alpha, Y_{\alpha+\beta}] = 0, [H_\alpha, X_{2\alpha+\beta}] = 2X_{2\alpha+\beta},$$

$$\text{and } [H_\alpha, Y_{2\alpha+\beta}] = 2Y_{2\alpha+\beta}.$$

And the adjoint for H_β will be

$$[H_\beta, Y_\alpha] = -Y_\alpha, [H_\beta, X_\beta] = 2X_\beta, [H_\beta, X_{\alpha+\beta}] = X_{\alpha+\beta}, [H_\beta, Y_\beta] = 2Y_\beta, \\ [H_\beta, X_\alpha] = X_\alpha, [H_\beta, Y_{\alpha+\beta}] = Y_{\alpha+\beta}, [H_\beta, X_{2\alpha+\beta}] = 0, \text{ and } [H_\beta, Y_{2\alpha+\beta}] = 0.$$

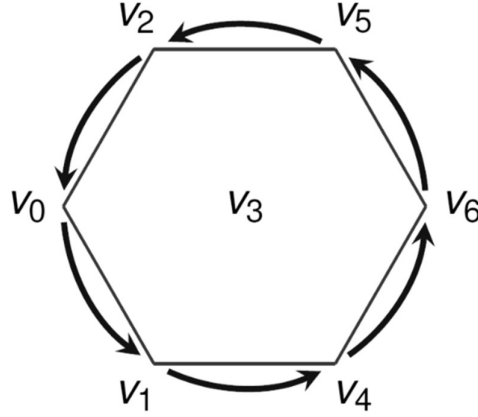


3.5 The Lie algebra G_2

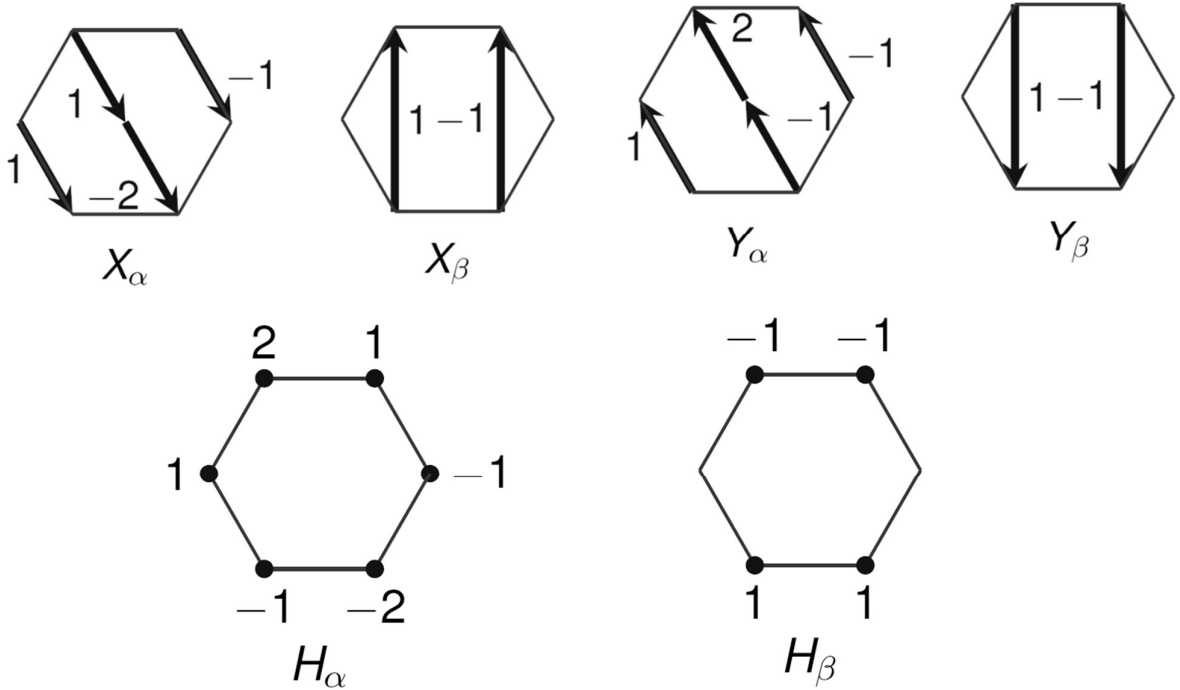
The G_2 is the last root system of dimension 2 that we will review, and now we expect the approach will be clear and unified. Although our approach has become more complicated, it is easy to predict what will turn out in this regard.

Now the basis will be more, and the angles between the roots will be less, so we have twelve roots, and the most appropriate way to generate them is through $n = 7$ and $V = \langle v_0, v_1, v_2, v_3, v_4, v_5, v_6 \rangle$. Thus, the plan is as in the previous examples to make the main diagonal lines for X_α and the vertical lines from top to bottom for X_β , and then we can move between the lines and generate internal roots to cover all the ways of this network. Here we will write $X_\alpha = e_{01} + e_{23} - 2e_{34} - e_{56}$, $X_\beta = -e_{12} + e_{45}$, $Y_\alpha = e_{10} + 2e_{32} - e_{43} - e_{65}$, $Y_\beta = -e_{21} + e_{54}$

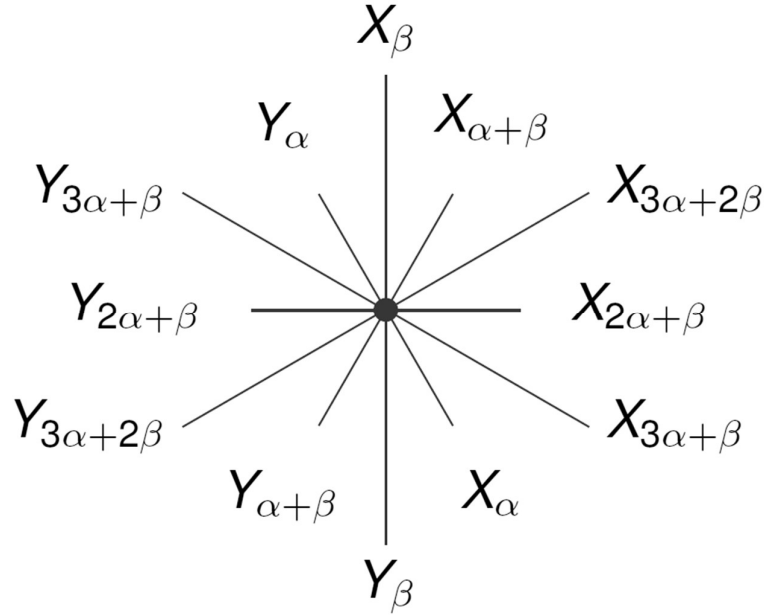
$$\begin{aligned}
, \quad X_{\alpha+\beta} &= -e_{02} - e_{13} - 2e_{35} - e_{46} \quad , \quad Y_{\alpha+\beta} = -e_{20} - e_{31} - 2e_{53} - e_{64} \quad , \\
X_{2\alpha+\beta} &= -e_{03} + 2e_{36} \quad , \quad Y_{2\alpha+\beta} = -e_{30} + 2e_{63} \quad , \quad X_{3\alpha+\beta} = 2e_{04} + 2e_{26} \quad , \\
Y_{3\alpha+\beta} &= 2e_{40} + 2e_{62} \quad , \quad X_{3\alpha+2\beta} = 2e_{05} - e_{16} \quad , \text{ and } Y_{3\alpha+2\beta} = 2e_{50} - e_{61}
\end{aligned}$$



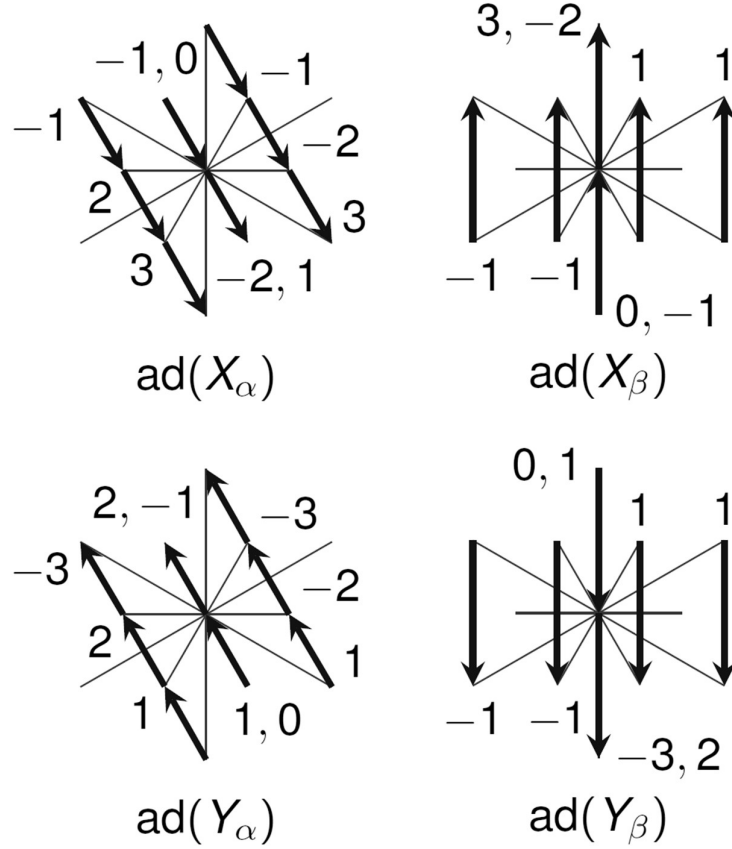
The image above indicates that X_α maps the diagonal lines v_0 to v_1 , v_2 to v_3 , v_3 to $-2v_4$, and v_5 to $-v_6$, while X_β maps the vertical lines v_1 to v_2 and v_4 to $-v_5$. Again Y_α maps the opposite diagonal line for X_α , while Y_β maps the opposite vertical line for X_β .



Since we have only two bases in this root system, we define $H_\alpha = [X_\alpha, Y_\alpha] = e_{00} - e_{11} + 2e_{22} - 2e_{44} + e_{55} - e_{66}$, $H_\beta = [X_\beta, Y_\beta] = e_{11} - e_{22} + e_{44} - e_{55}$. We will get the result shown in this figure.

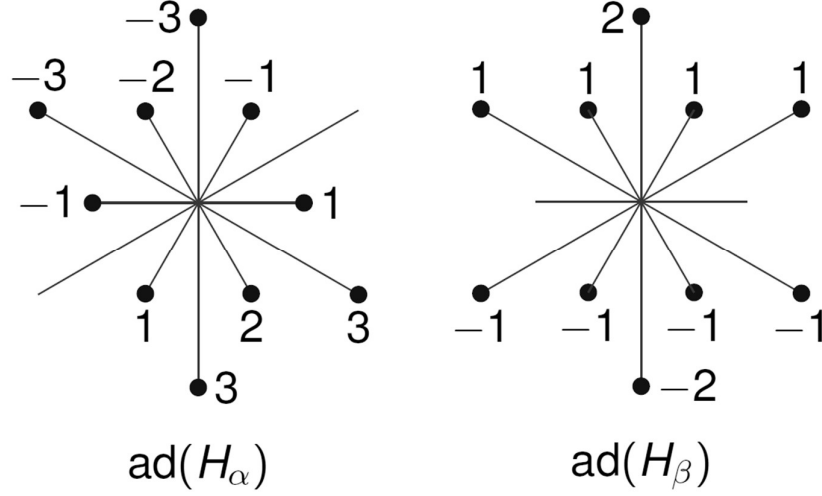


The Hexagon is very important to be this way, that's why in this case we need to do more composition between roots to have new roots instead of making new basis, and these lines must be parallel. For example: $X_{\alpha+\beta}$ will map v_0 to $-v_2$, and v_3 to $-2v_5$ as they have same slope. $X_{2\alpha+\beta}$ will map v_0 to $-v_3$ and v_3 to $2v_6$. Now we need also to define other angle from v_0 to v_5 and v_1 to v_6 , so write $X_{3\alpha+2\beta}$ maps v_0 to $2v_5$ and v_1 to $-2v_6$. Finally, v_0 to $2v_4$ and v_2 to $2v_6$ will be the composition between 3 alphas and one beta so we name it $X_{3\alpha+\beta}$. Now this will be sufficient that we will be able to see the root system that we want.



The figure above shows the adjoint for $X_\alpha, Y_\alpha, X_\beta$, and Y_β

As we did in B_2 , we will do one and the remaining will be the same. $\text{Ad}(X_\alpha)$ will take $Y_{3\alpha+\beta}$ to $-Y_{2\alpha+\beta}$, $Y_{2\alpha+\beta}$ to $2Y_{\alpha+\beta}$, and $Y_{\alpha+\beta}$ to $3Y_\beta$. In addition, the second diagonal line for $\text{Ad}(X_\alpha)$ that will take Y_α to $-H_\alpha$ while the second number means that it cannot take Y_α to H_β , but it will take H_β to X_α and H_α to $-2X_\alpha$. The third diagonal line is for $\text{ad}(X_\alpha)$ that takes X_β to $-X_{\alpha+\beta}$, $X_{\alpha+\beta}$ to $-2X_{2\alpha+\beta}$, and $X_{2\alpha+\beta}$ to $3X_{3\alpha+\beta}$



Finally, $\text{ad}(H_\alpha) = [H_\alpha, X_\alpha] = -2X_\alpha, [H_\alpha, X_\beta] = 3X_\beta, [H_\alpha, Y_\alpha] = 2Y_\alpha,$
 $[H_\alpha, Y_\beta] = -3Y_\beta, [H_\alpha, X_{\alpha+\beta}] = X_{\alpha+\beta}, [H_\alpha, Y_{\alpha+\beta}] = -Y_{\alpha+\beta},$
 $[H_\alpha, X_{2\alpha+\beta}] = -X_{2\alpha+\beta}, [H_\alpha, Y_{2\alpha+\beta}] = Y_{2\alpha+\beta}, [H_\alpha, X_{3\alpha+\beta}] = 3X_{3\alpha+\beta},$
 $[H_\alpha, Y_{3\alpha+\beta}] = -3Y_{3\alpha+\beta}$ and $[H_\alpha, X_{3\alpha+2\beta}] = 0$ and $[H_\alpha, Y_{3\alpha+2\beta}] = 0.$

And now we will do $\text{ad}(H_\beta)$

$$\begin{aligned} [H_\beta, X_\alpha] &= X_\alpha, [H_\beta, X_\beta] = -2X_\beta, \\ [H_\beta, Y_\alpha] &= -Y_\alpha, [H_\beta, Y_\beta] = -2Y_\beta, [H_\beta, X_{\alpha+\beta}] = -X_{\alpha+\beta}, \\ [H_\beta, Y_{\alpha+\beta}] &= Y_{\alpha+\beta}, [H_\beta, X_{2\alpha+\beta}] = 0, [H_\beta, Y_{2\alpha+\beta}] = 0, \\ [H_\beta, X_{3\alpha+\beta}] &= X_{3\alpha+\beta}, [H_\beta, Y_{3\alpha+\beta}] = -Y_{3\alpha+\beta} \text{ and } [H_\beta, X_{3\alpha+2\beta}] = X_{3\alpha+2\beta} \text{ and} \\ [H_\beta, Y_{3\alpha+2\beta}] &= -Y_{3\alpha+2\beta} \end{aligned}$$

Chapter 4

Wilson's Approach to the Suzuki and Ree Groups

4.1 Introduction

4.2 The Gaussian integers

4.3 The Eisenstein integers

Wilson's Approach to the Suzuki and Ree Groups

4.1 Introduction

Around 1960s, three families of simple Lie groups were extensively studied. Those groups are the Hermitian groups, Suzuki groups, and Ree groups. These families of groups can be constructed by various methods and approaches.[3]

The Suzuki group is a Lie group defined by [4] in 1960 and formed as a subset of 4×4 matrices. The Suzuki groups is denoted by $Sz(2^{2n+1})$.

The Ree group is a Lie group defined by Ree in 1960 and 1961 and is formed as a subset of 7×7 matrices over a finite field of characteristic 3. It is also formed as an automorphism of some dynkin's diagram that reverses the direction of multiple bonds.

As there are many approaches to study these groups, Wilson's [8] defined these groups in an easier and different way than the Lie group construction. In fact, Wilson defined these groups without the use of Lie Algebra. In this chapter we present Wilson's construction focusing on the small Ree group of type G_2 and the Suzuki groups B_2 (over a field of characteristic 2). It worth mentioning that in Wilson [8], we will not study the large Ree groups in this thesis due to the complication in the construction. [7]

In the next subsection, we recall the basic properties of the rings of Gaussian and Eisenstein integers.

4.2 The Gaussian integers

The Gaussian integers is the ring $\mathcal{G} = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$. The unit group of this Gaussian integer ring is $U := U(\mathcal{G}) = \{1, -1, i, -i\}$. These units will be used to construct the Suzuki group and can be thought as the short roots as describe in Chapter 2. The remaining roots will be formed from the set of non-units elements of smallest norm (in the complex number sense).

We will study in this thesis how the set $(1 + i)U$ generates the set of long roots in this case, and we will denote this set by L . In fact, this work will be uniform across other groups with some minor differences as we will do the changes whenever needed.

We will choose for both construction of the Suzuki group and Ree group, a linear map Φ from the short roots U to long roots L , which will be defined by

$$\Phi : z \rightarrow (1 + i)\bar{z}$$

This map squares to a scalar p , in this case it will square to 2.

Proposition 1 :

The Eigenspace for this map is one-dimensional with eigenvalues equal to $\pm\sqrt{2}$

Proof: From the definition for eigenvalue: $\Phi^2(z) = \lambda z$, we will have

$$\Phi^2(1) = \Phi(\Phi(1)) = \Phi((1 + i)\bar{1}) = \Phi(1 + i) = (1 + i)(\overline{1 + i}) = 2$$

$$\begin{aligned}
\Phi^2(-1) &= \Phi(\Phi(-1)) = \Phi((1+i)\overline{-1}) = \Phi(-1-i) = (1+i)\overline{(-1-i)} \\
&= 2(-1) \\
\Phi^2(i) &= \Phi(\Phi(i)) = \Phi((1+i)\bar{i}) = \Phi(1-i) = (1+i)\overline{(1-i)} = 2i \\
\Phi^2(-i) &= \Phi(\Phi(-i)) = \Phi((-1-i)(-\bar{i})) = \Phi(-1+i) = (1+i)\overline{(-1+i)} \\
&= 2(-i) .
\end{aligned}$$

Through the following calculations, there are two types of roots that will appear, and that will depend on the results of $r \cdot \Phi(r)$. With prior knowledge of the root system, we will have the following definition:

Definition:

- A root r is called an *inner root*, if $r \cdot \Phi(r) = \frac{-p}{2} = \frac{-2}{2} = -1$, as $p = 2$ in this case.
- A root r is called an *outer root*, if $r \cdot \Phi(r) = \frac{p}{2} = \frac{2}{2} = 1$

Let us classify the roots that we have, first we have outer roots when $r \in \{\pm 1\}$ because direct calculations yield:

$$r = 1 \rightarrow \langle 1, 0i \rangle \cdot \Phi(1) = \langle 1, 0i \rangle \cdot \langle 1, i \rangle = 1$$

$$r = -1 \rightarrow \langle -1, 0i \rangle \cdot \Phi(-1) = \langle -1, 0i \rangle \cdot \langle -1, -i \rangle = 1$$

Second, we have inner roots when $r \in \{\pm i\}$ as

$$r = i \rightarrow \langle 0, i \rangle \cdot \Phi(i) = \langle 0, i \rangle \cdot \langle 1, -i \rangle = -1$$

$$r = -i \rightarrow \langle 0, -i \rangle \cdot \Phi(-i) = \langle 0, -i \rangle \cdot \langle -1, i \rangle = -1$$

Next, we determine an ordering of the roots that is based on the map Φ . The ordering will result in the roots being either positive or negative, through an

appropriate vector, that works to discerns the roots, and this root must be proportional to the map Φ . The vector v_0 will not be a root, in this case Wilson takes $v_0 = 2 + i$, this does not necessarily mean that we should take this vector exactly, but Wilson chose it because he found it very suitable, and one can choose another vector that fulfills the following operations and does not affect the division of the plane, i.e., the mapping Φ preserves the ordering. We divide the plane as follows:

Definition :

- If $r \cdot v_0 > 0$, the root will be called *positive*,
- If $r \cdot v_0 < 0$, the root will be called *negative*.

Direct calculations classified the roots into positive roots and negative roots as follows:

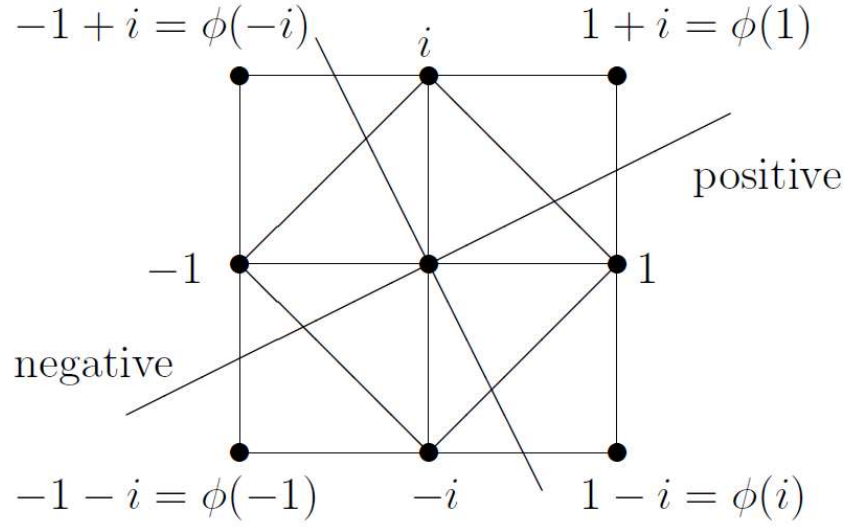
$$r = -1 \rightarrow \langle -1, 0 \rangle \cdot \langle -2, 1 \rangle = -2 < 0$$

$$r = -i \rightarrow \langle 0, -1 \rangle \cdot \langle 2, 1 \rangle = -1 < 0$$

$$r = i \rightarrow \langle 0, 1 \rangle \cdot \langle 2, 1 \rangle = 1 > 0$$

$$r = 1 \rightarrow \langle 1, 0 \rangle \cdot \langle 2, 1 \rangle = 2 > 0$$

We summarize all the discussion above in one picture as in the figure below:



The Vector space W

After defining the short/long roots, inner/outer roots, and positive/negative roots, we will define a vector space using these roots as follows:

Let U be an index set, together with a set Z of ‘zero’ elements (as augmentation) which is:

$$Z := \{0\} \text{ in the case } G$$

Define $I := U \cup Z$

Assuming the relation $\sum_{t \in Z} e_t = 0$. let $W = \langle e_t \mid \sum_{t \in Z} e_t = 0 \rangle$. Thus W is the vector space that has dimension 4. By writing W_4 , we will indicate the dimension. Denote e_r by $e(r)$ and $E(r, s, \dots)$ for the space created by the elements.

$\langle e_r, e_s, \dots \rangle$. We will use the ordering on the roots based on the map to list the vectors.

Next, we define three products on W . The first one is called the “inner” or “dot” product which is defined by a pair of short roots that add to zero. The second product is the "outer" or "cross" product, which is defined by pair of short roots which add up to another short root. The third product is the "middle" or "star" product, which is defined by a pair of short roots that add up to a long root. This will result in an algebraic structure consisting of a vector space and three different products.

Inner or dot product

The inner product $B : W \times W \rightarrow F$ is a symmetric bilinear form defined by $B(e_t, e_{-t}) = 1$ for $t \in U$, and $B(e_s, e_t) = 0$ otherwise. We write $v.w$ for $B(v, w)$

Outer or cross product

The outer product $M : W \times W \rightarrow W$ is an alternating bilinear product defined by pairs of short roots which sum to another short root, i.e., it has the property that $E(r) \times E(s) = E(r + s)$ whenever $r, s, r + s \in U$.

Trilinear form

A skew-symmetric trilinear form T is created when the inner and outer products are combined, and it is denoted by $T(u, v, w) = (u \times v).w$. The basis vectors make it simple to verify that this is cyclically symmetric. In fact, $T(e_r, e_s, e_t)$ has non-zero values at basis vectors when $r + s + t = 0$. Naturally, T is the zero form in the case of W_4 since the outer product is zero.

The middle product

The middle or star product is defined on the short roots whose sum is a long root, i.e., given r, s to be two short roots which sum is a long root, i.e., $t = \Phi^{-1}(r + s)$, define

$$e_r * e_s := e_t$$

With the condition,

$$e_r * e_{-r} + e_s * e_{-s} = e_t * e_{-t}$$

For W_4 , we have

$$e_1 * e_i = e_1$$

$$e_1 * e_{-i} = e_i$$

and images under negating subscripts.

4.3 The Eisenstein integers

The **Eisenstein** integers are the elements of the ring $\mathcal{E} = \mathbb{Z}[\omega]$ of complex numbers of the form $a + b\omega$, where a, b are integers and $\omega^2 + \omega + 1 = 0$, $\omega = \frac{-1+i\sqrt{3}}{2}$. This ring will be used to construct the Ree group.[5]

The unit group of this ring is $U := U(G) = \{1, -1, \omega, -\omega, \bar{\omega}, -\bar{\omega}\}$. These units will be the short roots. Then the remaining roots will be formed from the set of non-units as before. But here the set θU , where $\theta = \omega - \bar{\omega}$ will generate the set of long roots, and we will denote this set by L . Define the linear map by:

$$\Phi : z \rightarrow (1 - \bar{\omega})\bar{z}$$

Proposition 2 :

The map Φ squares to 3, i.e., $\Phi^2(z) = 3z$ with eigenspaces are 1-dimensional and eigenvalues to be $\pm\sqrt{3}$

Proof: Using $\bar{\omega}^2 = \omega$ and $\bar{\omega}\omega = 1$ as follows:

For $z = 1$

$$\begin{aligned}\Phi^2(1) &= \Phi(\Phi(1)) = \Phi((1 - \bar{\omega})\bar{1}) = \Phi(1 - \bar{\omega}) = (1 - \bar{\omega})(\overline{1 - \bar{\omega}}) = \\ &= (1 - \bar{\omega})(1 - \omega) = 2 - \omega - \bar{\omega} = 2 + \frac{1}{2} - \frac{\sqrt{3}}{2}i + \frac{1}{2} + \frac{\sqrt{3}}{2}i = 3(1)\end{aligned}$$

For $z = -1$

$$\begin{aligned}\Phi^2(-1) &= \Phi(\Phi(-1)) = \Phi((1 - \bar{\omega})\overline{-1}) = \Phi(-1 + \bar{\omega}) \\ &= (1 - \bar{\omega})(\overline{-1 + \bar{\omega}}) = (1 - \bar{\omega})(-1 + \omega) = -2 + \omega + \bar{\omega} \\ &= -2 - \frac{1}{2} + \frac{\sqrt{3}}{2}i - \frac{1}{2} - \frac{\sqrt{3}}{2}i = -3 = 3(-1)\end{aligned}$$

For $z = \omega$

$$\begin{aligned}\Phi^2(\omega) &= \Phi(\Phi(\omega)) = \Phi((1 - \bar{\omega})\bar{\omega}) = \Phi(\bar{\omega} - \omega) = (1 - \bar{\omega})(\overline{\bar{\omega} - \omega}) \\ &= (1 - \bar{\omega})(\omega - \bar{\omega}) = \omega - \bar{\omega} - \bar{\omega}\omega + \bar{\omega}^2 = 1 + 2\omega - \bar{\omega} \\ &= 1 + 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 3(\omega)\end{aligned}$$

For $z = -\omega$

$$\begin{aligned}\Phi^2(-\omega) &= \Phi(\Phi(-\omega)) = \Phi((1 - \bar{\omega})(-\bar{\omega})) = \Phi(\omega - \bar{\omega}) \\ &= (1 - \bar{\omega})(\overline{\omega - \bar{\omega}}) = (1 - \bar{\omega})(\bar{\omega} - \omega) = \bar{\omega} - \omega - \bar{\omega}^2 + \bar{\omega}\omega \\ &= -1 - 2\omega + \bar{\omega} = -1 - 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ &= -3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 3(-\omega)\end{aligned}$$

For $z = \bar{\omega}$

$$\begin{aligned}
\Phi^2(\bar{\omega}) &= \Phi(\Phi(\bar{\omega})) = \Phi((1 - \bar{\omega})(\bar{\omega})) = \Phi(\omega - 1) = (1 - \bar{\omega})(\overline{\omega - 1}) \\
&= (1 - \bar{\omega})(\bar{\omega} - 1) = 2\bar{\omega} - \omega - 1 \\
&= 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) - 1 = 3(\bar{\omega})
\end{aligned}$$

For $z = 1 - \bar{\omega}$

$$\begin{aligned}
\Phi^2(\bar{\omega}) &= \Phi(\Phi(-\bar{\omega})) = \Phi((1 - \bar{\omega})(-\bar{\omega})) = \Phi(1 - \omega) = (1 - \bar{\omega})(\overline{1 - \omega}) \\
&= (1 - \bar{\omega})(1 - \bar{\omega}) = -2\bar{\omega} + \omega + 1 \\
&= -2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 1 = 3(-\bar{\omega})
\end{aligned}$$

When we come to generate the rest of the roots from these units, we find that the calculations were somewhat simplified in the B_2 due to their number. But when it comes to G_2 , there are new roots that will appear, most notably the middle root, where it wasn't in the case of B_2 . Once again, inner product will help distinguish the roots generated. This time we have $p=3$, so we'll know root is outer if $r \cdot \Phi(r) = \frac{p}{2} = \frac{3}{2}$, inner if $r \cdot \Phi(r) = \frac{-3}{2}$ and **middle** if $r \cdot \Phi(r) = 0$

So, let's start with the calculation of the outer roots which are $r \in \{\pm 1\}$:

$$\begin{aligned}
r = 1 &\rightarrow r \cdot \Phi(r) = 1 \cdot (1 - \bar{\omega}) = \langle 1, 0 \rangle \cdot \langle \frac{3}{2}, -\frac{\sqrt{3}}{2} \rangle = \frac{3}{2} \\
r = -1 &\rightarrow r \cdot \Phi(r) = -1 \cdot (\bar{\omega} - 1) = \langle -1, 0 \rangle \cdot \langle -\frac{3}{2}, -\frac{\sqrt{3}}{2} \rangle = \frac{3}{2}
\end{aligned}$$

Then we have inner roots which are $r \in \{\pm\omega\}$ as follows:

$$\begin{aligned}
r = \omega &\rightarrow r \cdot \Phi(r) = \omega \cdot (\bar{\omega} - \omega) = \langle \frac{-1}{2}, \frac{\sqrt{3}}{2} \rangle \cdot \langle 0, -\sqrt{3} \rangle = \frac{-3}{2} \\
r = -\omega &\rightarrow r \cdot \Phi(r) = -\omega \cdot (\omega - \bar{\omega}) = \langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \rangle \cdot \langle 0, \sqrt{3} \rangle = \frac{-3}{2}
\end{aligned}$$

Here it will show us more roots, specifically another type, which is the middle root, it will be longer than the short root and shorter than the long root if

the roots are drawn together, we will define the following formula $r \cdot \Phi(r) = 0$ for the middle root as it occurs when $r \in \{\pm\bar{\omega}\}$,

$$r = \bar{\omega} \rightarrow r \cdot \Phi(r) = \bar{\omega} \cdot (\omega - 1) = \left\langle \frac{-1}{2}, -\frac{\sqrt{3}}{2} \right\rangle \cdot \left\langle \frac{-3}{2}, \frac{\sqrt{3}}{2} \right\rangle = 0$$

$$r = -\bar{\omega} \rightarrow r \cdot \Phi(r) = -\bar{\omega} \cdot (1 - \omega) = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle \cdot \left\langle \frac{3}{2}, -\frac{\sqrt{3}}{2} \right\rangle \geq 0$$

Now we will search for the appropriate vector to divide the plane into negative roots and positive roots so we can put an ordering of the roots with respect to the map Φ . Here Wilson had to approach the positive x-axis more and more in order to find a safe way to control the six roots, and the most appropriate order for the short roots would be $-1, \bar{\omega}, \omega, 0, -\omega, -\bar{\omega}, 1$ and for that Wilson determined the divided vector to be $v_0 = 4 - \bar{\omega}$. The order will be as we did in $\mathcal{B}_2$, by taking the product of the inner product

$$r = -1 \rightarrow \left\langle -1, 0 \right\rangle \cdot \left\langle \frac{9}{2}, \frac{\sqrt{3}}{2} \right\rangle = -\frac{9}{2} < 0$$

$$r = \bar{\omega} \rightarrow \left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle \cdot \left\langle \frac{9}{2}, \frac{\sqrt{3}}{2} \right\rangle = -3 < 0$$

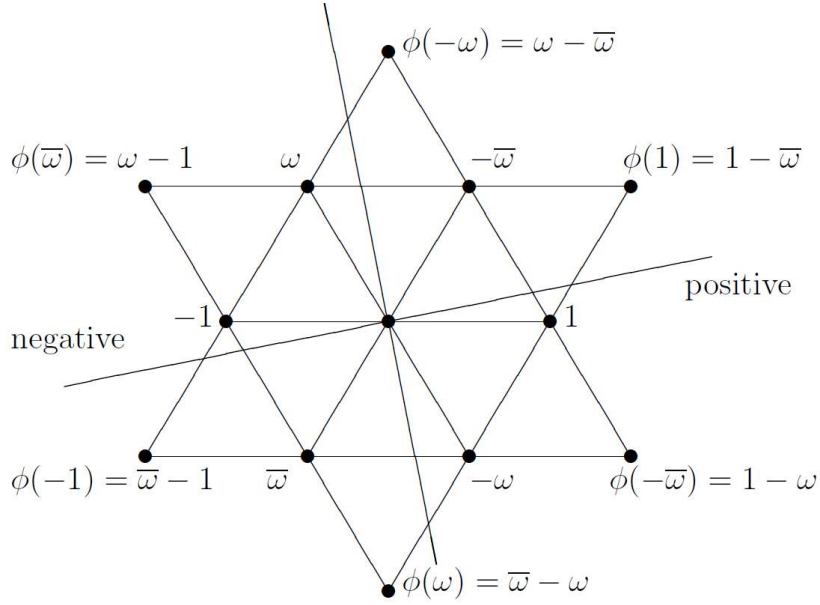
$$r = \omega \rightarrow \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle \cdot \left\langle \frac{9}{2}, \frac{\sqrt{3}}{2} \right\rangle = -\frac{3}{2} < 0$$

$$r = -\omega \rightarrow \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle \cdot \left\langle \frac{9}{2}, \frac{\sqrt{3}}{2} \right\rangle = \frac{3}{2} > 0$$

$$r = -\bar{\omega} \rightarrow \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle \cdot \left\langle \frac{9}{2}, \frac{\sqrt{3}}{2} \right\rangle = 3 > 0$$

$$r = 1 \rightarrow \left\langle 1, 0 \right\rangle \cdot \left\langle \frac{9}{2}, \frac{\sqrt{3}}{2} \right\rangle = \frac{9}{2} > 0$$

The above discussion is summarized in the figure below.



The Vector space W

Next, we define a vector space along with three products (inner, outer, and middle).

Assuming the relation $\sum_{t \in \mathbb{Z}} e_t = 0$, let W be the vector space over F span by the vectors e_t , for $t \in I$ satisfying the above relation. Note that W has a dimension of 7. We will write W_7 to indicate the dimension.

The inner product

As before, we define the inner product $B : W \times W \rightarrow F$ by $B(e_t, e_{-t}) = 1$ for $t \in U$ and $B(e_s, e_t) = 0$, otherwise. This gives a symmetric bilinear form

In the instance of W_7 , all such sums are obtained by moving one or more terms to the right side of the equation $1 + \omega + \bar{\omega} = 0$. There is a difficult concern about the signal because the characteristic is 3.

The Outer product

The outer product is defined by

$$\begin{aligned} e_1 \times e_\omega &= e_{\bar{\omega}} \\ e_{-1} \times e_{-\omega} &= e_{\bar{\omega}} \\ e_{-1} \times e_1 &= e_0 \\ e_1 \times e_0 &= e_1 \end{aligned}$$

and images under subscript multiplication by ω and $\bar{\omega}$. This product may be compared to the standard octonion product (modulo the center). [3 Section 4.5.2]

Trilinear form

As before, the trilinear form for W_7 is :

$$\begin{aligned} T(e_r, e_{\omega r}, e_{\bar{\omega} r}) &= 1, \\ T(e_r, e_{-r}, e_0) &= 1, \text{ for } r = 1, \omega, \bar{\omega} \end{aligned}$$

The middle or star product

For W_7 , we have

$$\begin{aligned} e_\omega \star e_{-1} &= e_{\bar{\omega}} \\ e_{\bar{\omega}} \star e_{-\omega} &= e_\omega \\ e_1 \star e_{-1} + e_{-\omega} \star e_\omega &= e_0 \\ e_{\bar{\omega}} \star e_{-\bar{\omega}} + e_{-1} \star e_1 &= e_0 \end{aligned}$$

and images under negating subscripts.

Wilson Construction of the Suzuki and Ree groups without Lie algebra tools:

Wilson's [8] defined an algebraic structure $\mathbb{W}$ to be the vector space W endowed with the three products (inner, outer, and middle) described above. Wilson's showed that this automorphism group for $\mathbb{W}$ is isomorphic to the Lie group of the Suzuki group and Ree group (hence a construction without using Lie algebra tools).

To end the chapter, we give the definition of the automorphism group of $\mathbb{W}$, which as expect should preserve the three products above as follows:

1. $u^g \cdot v^g = u \cdot v$ for all $u, v \in W$
2. $u^g \times v^g = (u \times v)^g$ for all $u, v \in W$; and
3. $u^g \star v^g = (u \star v)^g$ for all $u, v \in W$ which satisfy $u \cdot u = u \cdot v = v \cdot v = 0$ and $u \times v = 0$

Chapter 5

Construction of A_2 and ${}^2A_2$ without Lie Algebra

5.1 Introduction

5.2 The Gaussian integers

5.3 The Eisenstein integers

5.4 Conclusion of chapter 5

5.5 Conclusion

Construction of A_2 and ${}^2A_2$ without Lie Algebra

5.1 Introduction

The remaining case in Wilson's construction was for the root system of A_2 where the angle between the first and second root is $\frac{2\pi}{3}$. In this chapter of the thesis, we follow Wilson's construction for the case of A_2 by defining roots and vector spaces spanned by roots along with exploration of the three products described in Chapter 3.

5.2 Construction of A_2

First, we recall the basics of the root system A_2 . The system has three roots with angle between them equal to $\frac{2\pi}{3}$. If we choose the first root to be on the positive x-axis as a unit, then the second and the third root will be $(\cos \frac{2\pi}{3}, i \sin \frac{2\pi}{3})$ $(\cos \frac{4\pi}{3}, i \sin \frac{4\pi}{3})$, respectively.

The intrinsic information about this group is little compared to the groups in Chapter 3. However, we will continue using the similar operations to what Wilson did to complete the final picture.

Let's start with the ring $\mathcal{E} = \mathbb{Z}[\omega]$ of complex numbers of the form $a + b\omega$. The choice of this ring is due to the fact that the complete root system of A_2 is formed from the short roots of G_2 , and it would be a special case of it.

The unit group of this ring is $U := U(G) = \{1, \omega, \bar{\omega}\}$. These units form the short roots. From the set of non units of the lowest norm as before, the remaining roots must be created, but in this case, all the roots will be the same length. Denote the set of long roots by L .

The linear map Φ from the short roots U to the long roots L is define by

$$\Phi : z \rightarrow -z$$

Note that $\Phi^2(z) = z$ with eigenspaces are 1-dimensional and eigenvalues to be $\pm\sqrt{1}$

There is only one type of root that will appear here because of the result of inner product for r with $\Phi(r)$. According to this case $p = 2$. We will call the root r inner, if $r \cdot \Phi(r) = \frac{p}{2} = \frac{2}{2} = -1$,

Direct calculation yields the following:

$$r = 1 \rightarrow \langle 1, 0 \rangle \cdot \Phi(1) = \langle 1, 0 \rangle \cdot \langle -1, 0 \rangle = -1$$

$$r = -1 \rightarrow \langle -1, 0 \rangle \cdot \Phi(-1) = \langle -1, 0 \rangle \cdot \langle 1, -i \rangle = -1$$

$$r = \omega \rightarrow \langle \frac{-1}{2}, \frac{\sqrt{3}}{2} \rangle \cdot \Phi(\omega) = \langle \frac{-1}{2}, \frac{\sqrt{3}}{2} \rangle \cdot \langle \frac{1}{2}, \frac{-\sqrt{3}}{2} \rangle = -1$$

$$r = -\omega \rightarrow \langle \frac{1}{2}, \frac{-\sqrt{3}}{2} \rangle \cdot \Phi(-\omega) = \langle \frac{1}{2}, \frac{-\sqrt{3}}{2} \rangle \cdot \langle \frac{-1}{2}, \frac{\sqrt{3}}{2} \rangle = -1$$

$$r = \bar{\omega} \rightarrow \langle \frac{-1}{2}, \frac{-\sqrt{3}}{2} \rangle \cdot \Phi(\bar{\omega}) = \langle \frac{-1}{2}, \frac{-\sqrt{3}}{2} \rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = -1$$

$$r = -\bar{\omega} \rightarrow \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle \cdot \Phi(-\bar{\omega}) = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle \cdot \langle \frac{-1}{2}, \frac{-\sqrt{3}}{2} \rangle = -1$$

This shows that all roots are inner roots.

Now we will search for the appropriate vector to divide the plane between negative roots and positive roots just like we did in the previous chapter, and the most appropriate order for the short roots would be $-1, \bar{\omega}, \omega, 0, 1, -\omega, -\bar{\omega}$ by using $v_0 = 2 - \bar{\omega}$. The order will be as we did in $\mathcal{B}_2$, by taking the sign of the inner product

$$r = -1 \rightarrow \langle -1, 0 \rangle \cdot \langle \frac{5}{2}, \frac{\sqrt{3}}{2} \rangle = -\frac{5}{2} < 0$$

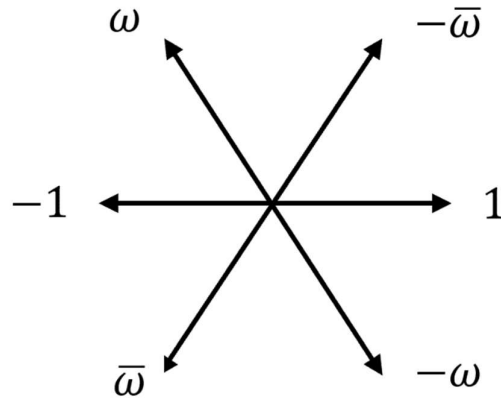
$$r = \bar{\omega} \rightarrow \langle -\frac{1}{2}, -\frac{\sqrt{3}}{2} \rangle \cdot \langle \frac{5}{2}, \frac{\sqrt{3}}{2} \rangle = -2 < 0$$

$$r = \omega \rightarrow \langle \frac{-1}{2}, \frac{\sqrt{3}}{2} \rangle \cdot \langle \frac{5}{2}, \frac{\sqrt{3}}{2} \rangle = -\frac{1}{2} < 0$$

$$r = -\omega \rightarrow \langle \frac{1}{2}, \frac{-\sqrt{3}}{2} \rangle \cdot \langle \frac{5}{2}, \frac{\sqrt{3}}{2} \rangle = \frac{1}{2} > 0$$

$$r = 1 \rightarrow \langle 1, 0 \rangle \cdot \langle \frac{5}{2}, \frac{\sqrt{3}}{2} \rangle = \frac{5}{2} > 0$$

$$r = -\bar{\omega} \rightarrow \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle \cdot \langle \frac{5}{2}, \frac{\sqrt{3}}{2} \rangle = 2 > 0$$



5.3 The Vector space W_3

Let I, U , and Z be as before and F is a field with $\text{char}(F)=2$

$$\text{Let } W = \langle e_t \mid \sum_{t \in Z} e_t = 0 \rangle$$

so, W has a dimension of 3.

As before we indicate the dimension by writing W_3 . To avoid making the notation unreadable, e_r , may be written as $e(r)$ for e_r , and $E(r, s, \dots)$ for $\langle e_r, e_s, \dots \rangle$.

The inner product

It is represented by the symmetric bilinear form $B : W \times W \rightarrow F$, which is also known as $B(v, w)$. $B(e_t, e_{-t}) = 1$ for $t \in U$ defines it, and in all other cases $B(e_s, e_t) = 0$ so $B(v, v) = 0$. Here we will have only three (six with symmetries) of nonzero products.

The outer product

The alternating bilinear product $M : W \times W \rightarrow W$ is the outer product. For $M(v, w)$. The outer product for W_3 is 0 since there are no pairs of short roots for W_3 that add to a short root. Given r, s to be short roots whose total is a long root, $t = \Phi^{-1}(-1)(r + s)$ is the definition of the middle product on the basis of short roots whose sum is a long root.

But in the case of W_3 we do not have short roots that sum up to long roots so again the product will be identically zero.

Following Wilson's construction as defined in Chapter 3, let the algebraic structure $\mathbb{W}$ to be the vector space W_3 endowed with the three products (inner, outer, and middle) of which two are identically zero as described above.

This structure will give the same structure (isomorphic sense) to the structure of A_2 with the usual Lie algebra/Lie group construction is via the automorphism group of $\mathbb{W}$. (hence a construction without using Lie algebra tools).

5.4 Conclusion of chapter 5

Following Wilson's construction and applying the same technique to what was done for the Suzuki and Ree Lie algebra, we defined the Hermitian Lie algebra in the same line without using the Lie algebra theory.

Here we come to the end of the topic, the most important topic of this work, and we may have fixed some of what happened to Wilson when he was working on B_2 , and G_2 development. I continued to work and used similar tools to get results very close to what Wilson achieved.

5.5 Conclusion

In this thesis, we focus on the Lie Algebra associated with root system of dimension 2, mainly the Lie Algebras A_2 , B_2 , and G_2 . We give an introduction to the bases of Lie Algebra theory of small dimensions and root systems associated to Lie algebra of dimension 2 including some examples and construction of these algebras representing them as homomorphisms.

We give an introduction to two research papers published by Wilson, in which the Suzuki and Ree groups (associated with Lie Algebras) are constructed without the use of Lie algebra theory. Wilson's idea was to define a new algebraic structure represented by vector space of dimension 4 (for Suzuki) and 7 for (Ree) along with three compatible products (inner, outer, and middle). The Suzuki/Ree group is defined later as the automorphism group that respects the three types of products. Wilson's showed that the resulting groups are isomorphism to the usual Suzuki and Ree groups.

The same approach was used in this thesis to deal with the case of A_2 , i.e., applying Wilson's structure to give an alternative way to describe A_2 without the use of Lie algebra by defining it as the automorphism group on a vector space of dimension 3 that respects the three products. It is worth noting that there are many alternatives to define A_2 without the use of Lie algebra but we choose to apply Wilson's construction to give a uniform approach to three groups as these are coming from root system to dimension 2.

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ملخص

في هذه الرسالة نناقش زمر لي الثلاث والتي لا تحتوي على زمر جزئية عادية. وستشمل المناقشة نظام الجذور من الرتبة الثانية لهذه الزمر.

ستركز هذه الدراسة على طريقة ويلسون في تعريف هذه الزمر باستخدام الفضاءات المولدة التي تحافظ على أنواع معينة من عمليات الضرب وبدون استخدام النظريات المتعلقة بجبريات لي. وسيتم استخدام نفس الطريقة لتعريف جميع زمر لي بنفس المنهجية المتبعة من ويلسون.

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أطروحة مقدمة كجزء من متطلبات الحصول على درجة الماجستير في الرياضيات

إعداد

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مملكة البحرين

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